SINGULAR PERTURBATIONS OF BLASCHKE PRODUCTS AND CONNECTIVITY OF FATOU COMPONENTS

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ABSTRACT. The goal of this paper is to study the family of singular perturbations of Blaschke products given by $B_{a,\lambda}(z) = z^3 \frac{z-a}{1-\bar{a}z} + \frac{\lambda}{z^2}$. We focus on the study of these rational maps for parameters a in the punctured disk $\mathbb{D}^*$ and $|\lambda|$ small. We prove that, under certain conditions, all Fatou components of a singularly perturbed Blaschke product $B_{a,\lambda}$ have finite connectivity but there are components of arbitrarily large connectivity within its dynamical plane. Under the same conditions we prove that the Julia set is the union of countably many Cantor sets of quasicircles and uncountably many point components.

1. INTRODUCTION

Given a Rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere, the Fatou set $\mathcal{F}(f)$ is defined as the set of points $z \in \widehat{\mathbb{C}}$ such that the family of iterates $\{f(z), f^2(z) = f(f(z)), ...\}$ is normal in some open neighbourhood of z. Its complement, the Julia set $\mathcal{J}(f)$, corresponds to the set of points where the dynamics is chaotic. The Fatou and the Julia sets are totally invariant under $f(z)$. The Fatou set is open and its connected components, known as Fatou components, map among themselves. The celebrated result of Sullivan [18] states that all Fatou components of rational maps are either periodic or preperiodic. Moreover, any cycle of periodic Fatou components has at least a critical point, i.e. a zero of $f'(z)$, somehow related to it (see [11]).

The aim of this paper is to study singular perturbations of a family of Blaschke products and analyse the structure of their dynamical plane. We focus on the special case for which

Fatou components of arbitrarily large connectivity appear. The study of singular perturbations of rational maps is a very active research field in holomorphic dynamics. They were used by McMullen to show the existence of buried Julia components for rational maps, i.e. connected components of the Julia set which do not intersect the boundary of any Fatou component. He described the nowadays called McMullen Julia sets, Julia sets consisting of Cantor sets of quasicircles (see [10]). A quasicircle is a Jordan curve which is the image of the unit circle by a quasiconformal map (c.f. [4]). To prove the existence of such Julia sets, he studied the singular perturbations of $p_n(z) = z^n$ given by

$$Q_{\lambda,n,d}(z) = z^n + \frac{\lambda}{z^d}, \tag{1}$$

where $\lambda \in \mathbb{C}$, $n \geq 2$ and $d \geq 1$. The map $p_n(z)$ has $z = 0$ and $z = \infty$ as superattracting fixed points of local degree n. However, the singular perturbation performed on $Q_{\lambda,n,d}$ transforms $z = 0$ on a preimage of ∞ and $n + d$ new critical points appear around it. McMullen showed that the Julia set $\mathcal{J}(Q_{\lambda,n,d})$ is a Cantor set of quasicircles if $|\lambda|$ is small enough and $1/n + 1/d < 1$ (see Figure 1 (a)). Afterwards, Devaney, Look and Uminsky [7] studied the different possible Julia sets of $Q_{\lambda,n,d}$ which may occur when all of the critical points belong to the basin of attraction of infinity. They proved that, in this situation, $\mathcal{J}(Q_{\lambda,n,d})$ can only be a Cantor set, a Cantor set of quasicircles or a Sierpinski curve (a homeomorphic image of a Sierpinski carpet). Since then, the family $Q_{\lambda,n,d}$ has been the object of study of several other papers (see e.g. [12], [15], [14]).

It arises as a very interesting topic to investigate McMullen-like Julia sets, i.e. Julia sets of rational maps which are somehow similar to the ones described by McMullen. They may be obtained by adding one or several poles to a polynomial $p_{n,c}(z) = z^n + c$ (see [3] and [9], respectively), where $n \geq 2$ and $c \in \mathbb{C}$ is such that $z = 0$ belongs to a superattracting cycle. In this scenario, the structure of the Julia set of the polynomial $p_{n,c}$ plays a very important role after the perturbation. The set $\mathcal{J}(p_{n,c})$ coincides with the boundary of the immediate basin of attraction of $z = \infty$, $A^*_{p_{n,c}}(\infty)$. After the perturbation, this remains as a component of the Julia set which moves continuously with respect to λ. Moreover, many preimages of it appear (see Figure 1 (b), (c) and (d)). Under certain conditions, this may lead to the existence of Cantor sets of closed curves instead of quasicircles (c.f. [9]).

In this paper we introduce a family of rational maps with McMullen-like Julia sets which present different dynamics than the one of the previously mentioned works. To do so, we consider singular perturbations of rational maps whose Julia set is the unit circle and which have free critical points, i.e. critical points which do not belong to a superattracting cycle. The existence of these free critical points allows the appearance of other types of dynamics in the Fatou set. More specifically, we study a family of singular perturbations of the Blaschke products

$$B_a(z) = z^3 \frac{z - a}{1 - \overline{a}z}, \tag{2}$$

where $a \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$. These Blaschke products have $z = 0$ and $z = \infty$ as superattracting fixed points. The basin of attraction of $z = 0$ is given by $A(0) = \mathbb{D}$ while the basin of attraction of $z = \infty$ is the Riemann sphere minus the closed unit disk, $A(\infty) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Consequently, the Julia set of these maps is the unit circle, $\mathcal{J}(B_a) = \mathbb{S}^1$. Moreover, they have
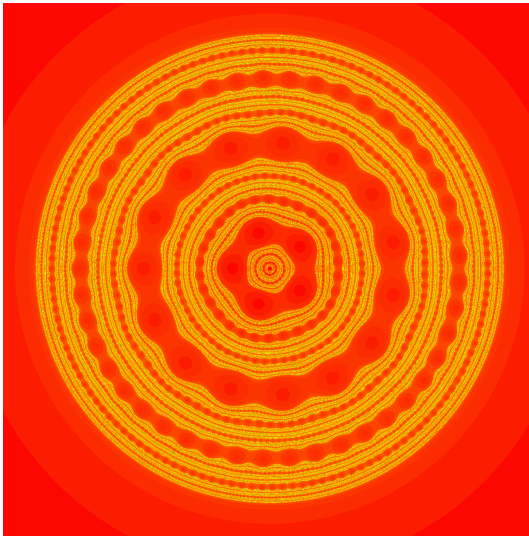
(a) A Julia set of quasicircles.
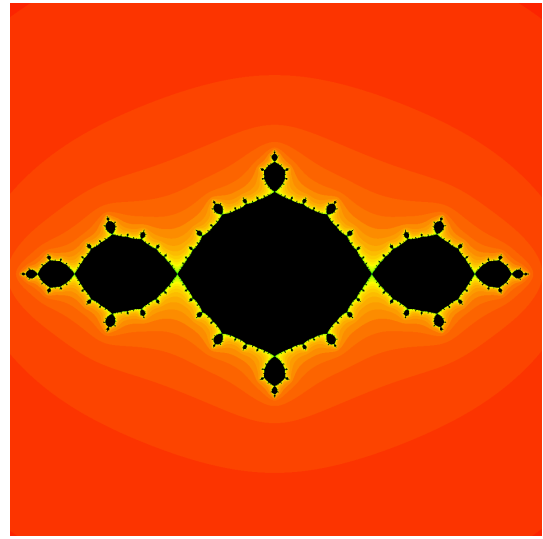
(b) The Basilica.
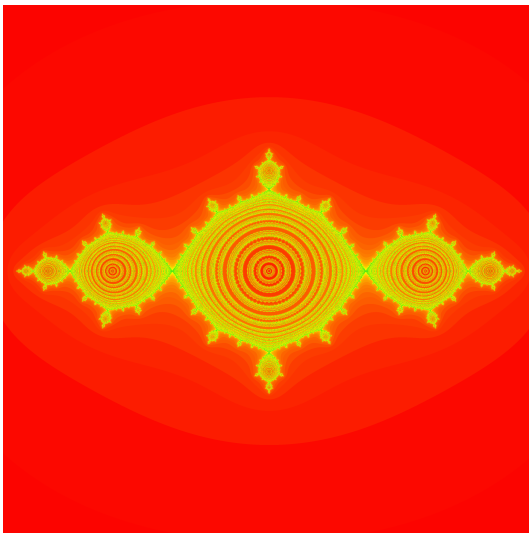
(c) A perturbed Basilica.

(d) Zoom in Figure (c).

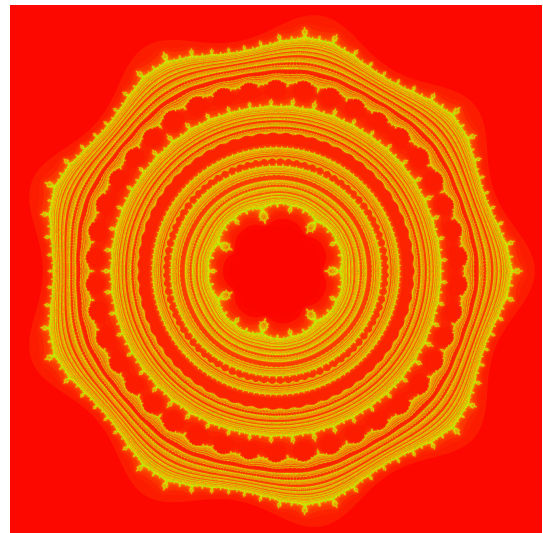FIGURE 1. Figure (a) corresponds to the dynamical plane of $Q_{\lambda,3,2}(z) = z^3 + \lambda/z^2$ for $\lambda = 10^{-4}$. Figure (b) corresponds to the dynamical plane of $p_{2,-1}(z) = z^2 - 1$, known as the Basilica. The map $p_{2,-1}$ has a period 2 superattracting cycle at $\{0, -1\}$. Figure (c) corresponds to the dynamical plane of $f(z) = z^2 - 1 + \lambda/(z^7(z + 1)^5)$ for $\lambda = 10^{-22}$. This map is a singular perturbation of the polynomial $p_{2,-1}$ which adds a pole at each point of the superattracting cycle. Figure (d) is a magnification of (c) around the point $z = 0$. The colours are as follows. We use a scaling from yellow to red to plot the basin of attraction of $z = \infty$. In Figure (b) we plot the basin of attraction of the cycle $\{0, -1\}$ in black. In the other figures we may observe an approximation of the Julia set in yellow.

two free critical points, $c_-(a) \in \mathbb{D}$ and $c_+(a) \in \mathbb{C} \setminus \overline{\mathbb{D}}$, which play an important role after the perturbation. See Section 2.1 for a more detailed introduction to their dynamics.

We consider singular perturbations of the Blaschke products B_a given by

$$B_{a,\lambda}(z) = z^3 \frac{z-a}{1-\overline{a}z} + \frac{\lambda}{z^2}, \tag{3}$$

where $\lambda \in \mathbb{C}$ and $a \in \mathbb{D}^*$. If $\lambda = 0$, they coincide with the Blaschke products $B_a(z)$ (Equation (2)). If $\lambda \neq 0$ then the point $z = 0$ becomes a pole and there appear 5 new critical points and 5 new zeros around it. The fixed point $z = \infty$ remains as a superattracting fixed point and there are two free critical points $c_+(a, \lambda)$ and $c_-(a, \lambda)$ which come from analytic continuation of the critical points $c_+(a, 0) = c_+(a)$ and $c_-(a, 0) = c_-(a)$ of $B_{a,0}(z) = B_a(z)$. For $|\lambda|$ small enough, the dynamics of $B_{a,\lambda}(z)$ around $z = 0$ are similar to the ones of $Q_{\lambda,3,2}$. Since the maps $Q_{\lambda,n,d}$ present McMullen Julia sets for $|\lambda|$ small enough if $1/n + 1/d < 1$ and we have that $1/3 + 1/2 < 1$, we may a priori expect some sort of McMullen-like Julia set for the maps $B_{a,\lambda}$. On the other hand, the free critical points allow the existence of other types of Julia sets than the ones described by Devaney, Look and Uminsky in [7]. However, the lack of symmetry of the critical points of $B_{a,\lambda}$ makes it much more difficult to investigate the family $B_{a,\lambda}$ than the family $Q_{\lambda,n,d}$ from a global point of view. We want to point out that similar results to the ones presented in this paper could be obtained if replacing the powers 3 and 2 in Equation (3) by naturals n and d such that $1/n + 1/d < 1$.

The goal of this paper is to study the dynamics of the family $B_{a,\lambda}$ for $a \in \mathbb{D}^*$ and $|\lambda|$ small, and to show that this family provides examples of new phenomena related to the connectivity of Fatou components. More precisely, it is known that any periodic Fatou component has connectivity 1, 2 or ∞ (c.f. [2]), while preperiodic Fatou components can have finite connectivity greater than 2. Beardon [2] introduced an explicit family of rational maps suggested by Shishikura with a Fatou component of finite connectivity greater than 2. This family was studied more deeply in [8]. The authors proved that if the parameter is small enough then there are Fatou components of connectivity 3 and 5. They also showed the existence of maps with a Fatou component of connectivity 9 and conjectured that for any given $n \in \mathbb{N}$, there exists a map within the family with a Fatou component of connectivity greater than n. Baker, Kotus and Lü [1] used a quasiconformal surgery procedure to show that, given any $n \in \mathbb{N}$, there exist rational and meromorphic maps with preperiodic Fatou components of connectivity n. Later on some explicit examples of rational maps with such properties were introduced (see [13] and [17]). However, the examples presented in [1], [13] and [17] use an increasing number of critical points to guarantee the existence of such multiply connected Fatou components. Therefore, the degree of the rational maps provided as examples grows with n. In Theorem A we prove that, fixed $a \in \mathbb{D}^*$, if $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $|\lambda|$ is mall enough, and $c_-(a, \lambda)$ belongs to the basin of attraction of $z = \infty$, $A(\infty)$, then there are only three possibilities: either every Fatou component has connectivity less or equal than two, or less or equal than three, or one can find components of arbitrarily large finite connectivity. Consequently, this family of degree 6 rational functions may contain maps with preperiodic Fatou components of arbitrarily large finite connectivity within a single dynamical plane. We do not know of any previous example having this property. In Figure 2 and Figure 3 we provide numerical evidence that all 3 cases of Theorem A take place for different parameters. A rigorous proof of the existence of parameters for which all three cases hold is work in progress and will appear in a manuscript which is currently under preparation.

Theorem A. *Fix $a \in \mathbb{D}^*$ and let $\lambda \in \mathbb{C}^*$. There exists a constant $\mathcal{C}(a)$ such that if $|\lambda| < \mathcal{C}(a)$ and $c_-(a, \lambda) \in A(\infty)$, then $c_-(a, \lambda)$ belongs to a connected component $\mathcal{U}_c$ of $A(\infty) \setminus A^*(\infty)$ and exactly one of the following holds.*

a) *The Fatou component $\mathcal{U}_c$ is simply connected. All Fatou components of $B_{a,\lambda}$ have connectivity 1 or 2 (see Figure 2 (a) and (b)).*
b) *The Fatou component $\mathcal{U}_c$ is multiply connected and does not surround $z = 0$. All Fatou components of $B_{a,\lambda}$ have connectivity 1, 2 or 3 (see Figure 2 (c) and (d)).*
c) *The Fatou component $\mathcal{U}_c$ is multiply connected and surrounds $z = 0$. All Fatou components of $B_{a,\lambda}$ have finite connectivity but there are components of arbitrarily large connectivity (see Figure 3).*

Figure 4 shows the parameter space of $B_{a,\lambda}$ for $a = 0.5$ and $|\lambda|$ small. We observe hyperbolic components of different connectivities which presumably correspond to the three different cases in Theorem A. More specifically, numerical exploration shows that the red annular regions which surround $\lambda = 0$ correspond to parameters for which statement *c)* holds, that the red simply connected regions correspond to parameters for which statement *a)* holds, and that the red annular regions which surround simply connected regions correspond to parameters for which statement *b)* holds. A detailed study of the parameter plane of the family for $a \in \mathbb{D}^*$ and $|\lambda|$ small is work in progress.

The third case of Theorem A provides dynamical planes with other interesting properties. Our next theorem shows that Cantor sets of quasicircles and uncountably many point components appear in the Julia set in these situations. Even if the result is similar to those in other examples in the literature (c.f. [3] and [9]), we shall see in the proof that the dynamics of these maps are quite different.

Theorem B. *Fix $a \in \mathbb{D}^*$ and let $\lambda \in \mathbb{C}^*$. Assume that $|\lambda| < \mathcal{C}(a)$, that $c_-(a, \lambda) \in A(\infty)$ and that $c_-(a, \lambda)$ lies in a multiply connected Fatou component which surrounds $z = 0$. Then the Julia set of $B_{a,\lambda}$ contains a countable union of Cantor sets of quasicircles and uncountably many point components.*

We want to remark that if statements *a)* or *b)* of Theorem A hold then the situation is essentially different. In this scenario, we can find annular Fatou domains bounded by quasicircles as described by McMullen in [10]. However, in this case they cannot lead to a Cantor set of quasicircles in the same way. This is because of the Fatou components which are homeomorphic to disks. They are to be found between McMullen's annuli together with other doubly (and triply) connected Fatou components (see Figure 2). This fact avoids McMullen's annuli to accumulate forming a Cantor set of quasicircles.

The paper is structured as follows. In Section 2 we describe the dynamics of the unperturbed Blaschke products B_a and investigate the local dynamics of the maps $B_{a,\lambda}(z)$ around $z = 0$ and the configuration of zeros and critical points (see Theorem 2.5). In Section 3 we study the Fatou components of the singularly perturbed Blaschke products and prove Theorem A as a restatement of Theorem 3.2. Finally, in Section 4 we investigate the Julia sets of the maps $B_{a,\lambda}$ proving Theorem B.
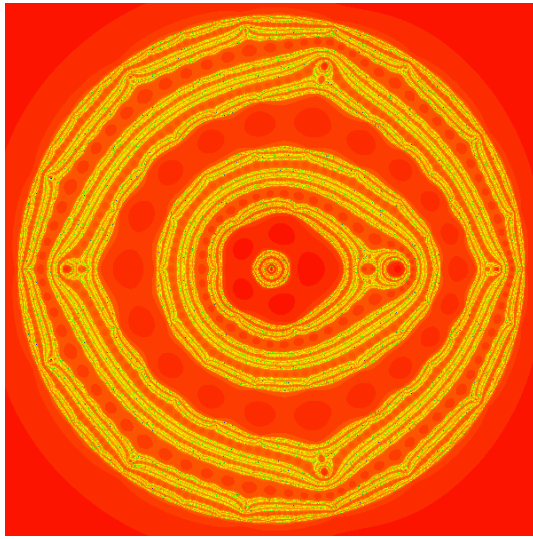
(a) Dynamical plane of a map $B_{a,\lambda}$ for which statement
a) of Theorem A holds.
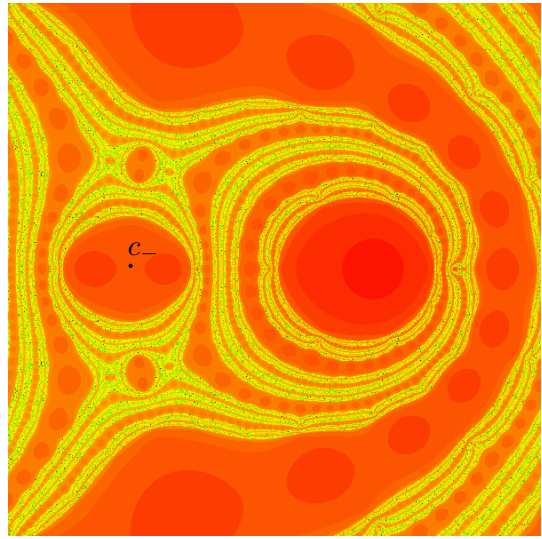
(b) Zoom in Figure (a).
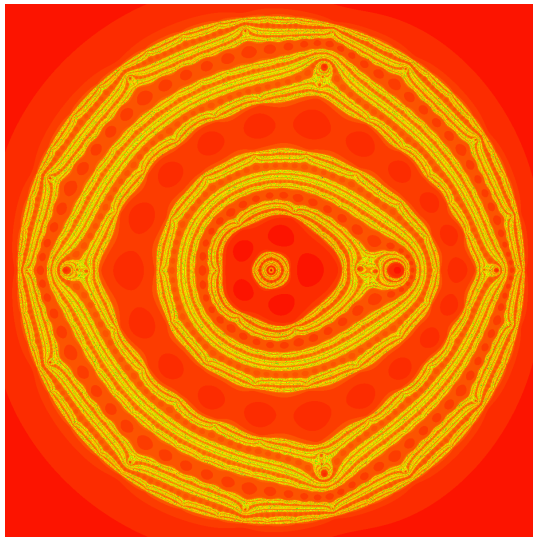
(c) Dynamical plane of a map $B_{a,\lambda}$ for which statement
b) of Theorem A holds.
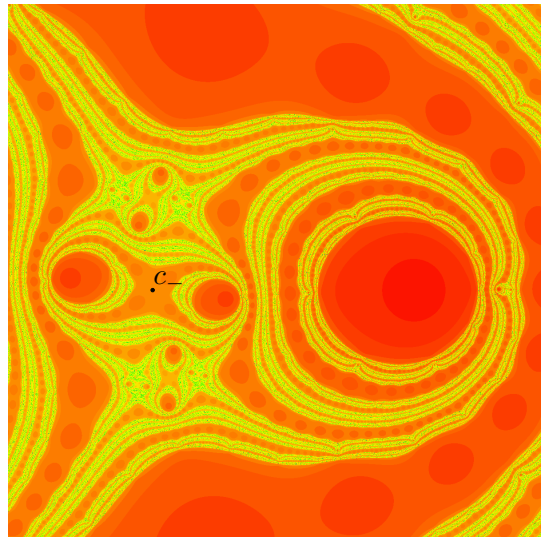
(d) Zoom in Figure (c).

FIGURE 2. Figures (a) and (b) represent the dynamical plane of the map $B_{a,\lambda}$ where $a = 0.5$ and $\lambda = 3.022 \times 10^{-5}$. Figures (c) and (d) represent the dynamical plane of the map $B_{a,\lambda}$ where $a = 0.5$ and $\lambda = 2.8 \times 10^{-5} + 8.4 \times 10^{-7}i$. These maps correspond to singularly perturbed Blaschke products for which statements a) and b) of Theorem A hold. The colours are as follows. We use a scaling from yellow to red to plot the basin of attraction of $z = \infty$. An approximation of the Julia set may be observed in yellow.
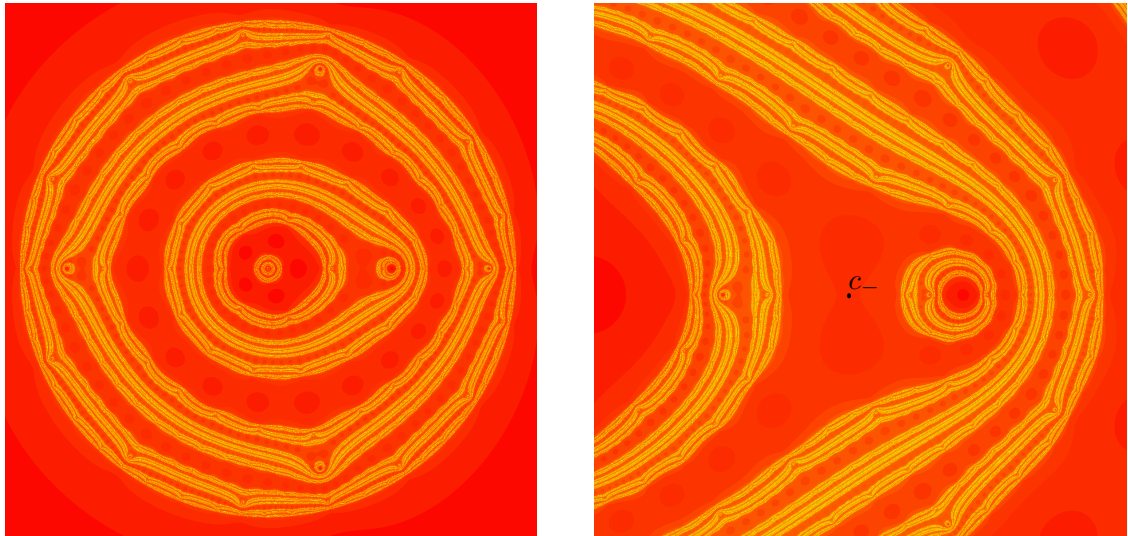
FIGURE 3. The left corresponds to the dynamical plane of the map $B_{a,\lambda}$ where $a = 0.5$ and $\lambda = 10^{-5}$. The right figure is a magnification of the left one. Statement *c)* of Theorem A holds for this map. Colours are as in Figure 2.
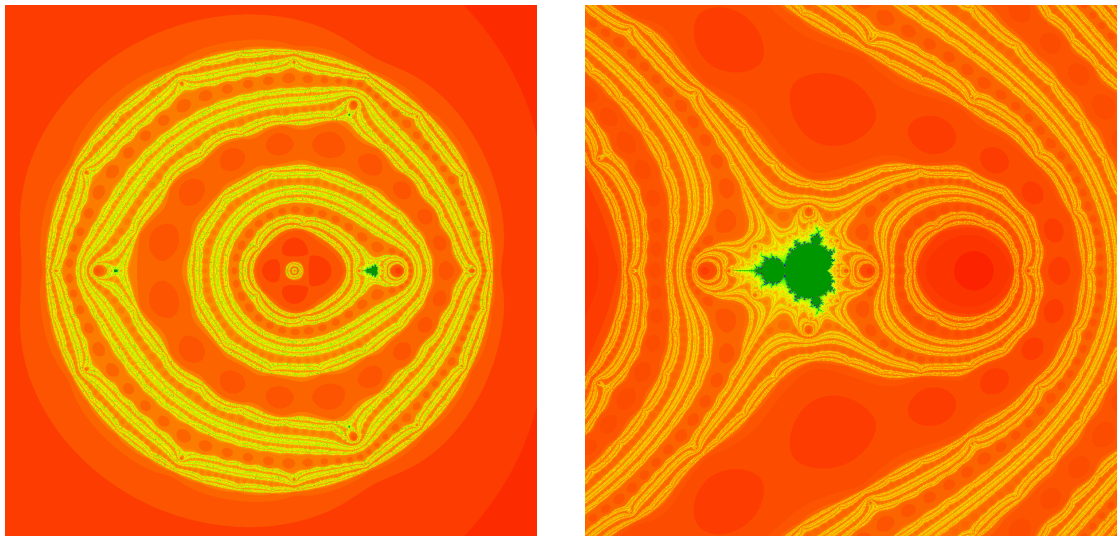
FIGURE 4. The left figure corresponds to the parameter space of the family $B_{a,\lambda}$ for $a = 0.5$, $\mathrm{Re}(\lambda) \in (-8.7 \times 10^{-5}, 7.3 \times 10^{-5})$ and $\mathrm{Im}(\lambda) \in (-8 \times 10^{-5}, 8 \times 10^{-5})$. The right figure is a magnification of the left one. The colours are as follows. We use a scaling from yellow to red to plot parameters such that $c_- \in A(\infty)$ and green otherwise.

2. Preliminaries

The goal of this section is to understand the local dynamics which take place near $z = 0$ for the singular perturbations of rational maps $B_{a,\lambda}(z)$ (Equation (3)). In section 2.1 we study the dynamics of the Blaschke products $B_a(z)$ (Equation (2)). In Section 2.2 we prove

Theorem 2.5, which is the main result of the preliminaries. It describes the local dynamics around $z = 0$ as well as the configurations of critical points of zeros. It also analyses the topology of the immediate basin of attraction of $z = \infty$, $A^*(\infty)$, and introduces notation which is used along the paper.

2.1. **Dynamics of the Blaschke products.** We consider the family of Blaschke products of the form $B_a(z) = z^3(z-a)/(1-\overline{a}z)$ where $z \in \widehat{\mathbb{C}}$ and $a \in \mathbb{D}^*$. The dynamics of these maps was studied in [5] and [6] for $a \in \mathbb{C}$. As all Blaschke products, they leave the unit circle $\mathbb{S}^1$ invariant. Moreover, $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3 and, therefore, are critical points of multiplicity 2. Given the fact that these rational maps have degree 4, they have 6 critical points counted with multiplicity. The other two critical points, denoted by $c_\pm$, are given by

$$c_\pm := c_\pm(a) := a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 \pm \sqrt{(|a|^2-4)(|a|^2-1)}\right).$$

If $a \in \mathbb{D}^*$, then the critical points satisfy $c_- \in \mathbb{D}$ and $c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Moreover, since the only pole $z_\infty = 1/\overline{a}$ does not lie in $\mathbb{D}$, we have that $B_a(\mathbb{D}) = \mathbb{D}$ and all points in $\mathbb{D}$ belong to the immediate basin of attraction of $z = 0$, $A^*(0)$. Analogously we also have $A^*(\infty) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We finish this first part of the preliminaries with Proposition 2.2, which will be used in the next subsection. During the proof we will use the Riemann-Hurwitz formula (c.f. [11, 16]), which can be stated as follows.

Theorem 2.1 (Riemann-Hurwitz formula)**.** *Let U and V be two connected domains of $\widehat{\mathbb{C}}$ of finite connectivity m_U and m_V and let $f : U \to V$ be a degree k proper map branched over r critical points counted with multiplicity. Then*

$$m_U - 2 = k(m_V - 2) + r.$$

Proposition 2.2. *Fixed $a \in \mathbb{D}^*$, there are analytic Jordan curves γ_0, γ_0^{-1}, γ_∞ and γ_∞^{-1} such that:*

a) *The curves γ_0 and γ_0^{-1} belong to $\mathbb{D}$ and surround $z = 0$. The curve γ_0^{-1} is mapped onto γ_0 with degree 4 under B_a. Moreover, we have $\gamma_0 \subset \operatorname{Int}(\gamma_0^{-1})$ and $z_0, c_- \in \operatorname{Int}(\gamma_0^{-1})$, where $\operatorname{Int}(\gamma_0^{-1})$ denotes the bounded component of $\mathbb{C} \setminus \gamma_0^{-1}$.*

b) *The curves γ_∞ and γ_∞^{-1} belong to $\mathbb{C} \setminus \overline{\mathbb{D}}$ and surround the unit disk $\mathbb{D}$. The curve γ_∞^{-1} is mapped onto γ_∞ with degree 4 under B_a. Moreover, we have $\gamma_\infty \subset \operatorname{Ext}(\gamma_\infty^{-1})$ and $z_\infty, c_+ \in \operatorname{Ext}(\gamma_\infty^{-1})$, where $\operatorname{Ext}(\gamma_\infty^{-1})$ denotes the unbounded component of $\mathbb{C} \setminus \gamma_\infty^{-1}$.*

Proof. We show how to obtain γ_0 and γ_0^{-1}. The curves γ_∞ and γ_∞^{-1} can be obtained in a similar way. Let U be the maximal domain of definition of the Böttcher coordinate of the superattracting fixed point $z = 0$ (see [11]). Then ∂U contains the critical point c_-. Moreover, there is an extra component V of $B_a^{-1}(B_a(U))$ attached to the critical point c_- which contains the zero z_0 (see Figure 5).

To finish the proof it is enough to take an analytic Jordan curve γ_0 surrounding $\overline{B_a(U)}$ such that $\gamma_0 \subset U \setminus \overline{B_a(U)}$. We want to show that any such curve satisfies that there is a unique component γ_0^{-1} of $B_a^{-1}(\gamma_0)$, which surrounds the set $\overline{U \cup V}$ and is mapped with degree 4 onto γ_0. Let A be the annulus bounded by γ_0 and $\mathbb{S}^1$ and let $A^{-1} = B_a^{-1}(A)$. Since $\mathbb{S}^1$ is completely invariant, A^{-1} consists of a unique connected component. It follows from the

Riemann-Hurwitz formula (Theorem 2.1) that A^{-1} is also an annulus since A contains no critical value (i.e. the image of a critical point) and, therefore, A^{-1} contains no critical point. The annulus A^{-1} is bounded by $\mathbb{S}^1$ and an analytic Jordan curve, say γ_0^{-1}, which is mapped onto γ_0 under B_a and surrounds $\overline{U \cup V}$. Given that $B_a|_{A^{-1}}$ is proper, B_a maps A^{-1} onto A with a certain degree d. Since $B_a|_{\mathbb{S}^1}$ has degree 4, we conclude this degree d is precisely 4 and that B_a maps γ_0^{-1} onto γ_0 with degree 4. Consequently, there can be no other preimage of γ_0. $\qquad\square$
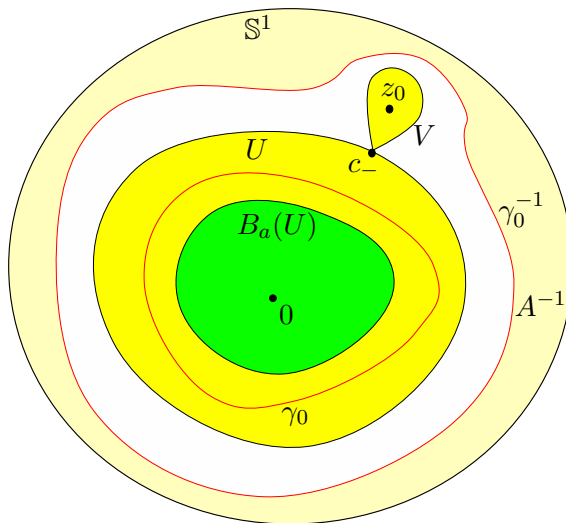

FIGURE 5. Scheme of the sets described in the proof of Proposition 2.2.

2.2. **Local dynamics of the singular perturbations near the origin.** To understand the global dynamics of the singularly perturbed maps $B_{a,\lambda}(z)$ (Equation (3)), it is important to control the orbits of the critical points and to know where the preimages of $z = 0$ and $z = \infty$ are. The goal of this section of the preliminaries is to locate them and use them to understand the local dynamics around $z = 0$ for $|\lambda|$ small (see Theorem 2.5).

As all degree 6 rational maps, the functions $B_{a,\lambda}$ have $2 \cdot 6 - 2 = 10$ critical points and 6 preimages of $z = 0$ and $z = \infty$, counted with multiplicity. The point $z = \infty$ is a superattracting fixed point of local degree 3, so it is a critical point of multiplicity 2 and a preimage of itself of multiplicity 3. Some of the other critical points, zeros and poles can be found by analytic continuation of the ones of $B_a(z)$ (Equation (2)). Indeed, if we fix $a \in \mathbb{D}^*$, then the maps $B_{a,\lambda}(z)$ depend analytically on λ for all z in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_\epsilon}$, where $\mathbb{D}_\epsilon$ denotes the disc centred at 0 with radius ϵ as small as desired. Therefore, we have two free critical points $c_+(a,\lambda)$ and $c_-(a,\lambda)$, a zero $z_0(a,\lambda)$ and a pole $z_\infty(a,\lambda) = 1/\overline{a}$ which are analytic continuation of the critical points c_+ and c_-, the zero z_0 and the pole z_∞ of the Blaschke product $B_a(z)$. We shall drop the dependence on a and λ of $c_+(a,\lambda)$, $c_-(a,\lambda)$, $z_0(a,\lambda)$ and $z_\infty(a,\lambda)$ whenever it is clear from the context.

There are 6 critical points, 5 zeros and 2 poles which are not to be found by analytic continuation of the ones of B_a. They appear in a small neighbourhood of $z = 0$. Because of the term λ/z^2 of $B_{a,\lambda}(z)$, the point $z = 0$ is a double preimage of infinity and a critical point of multiplicity 1. Therefore, there are only 5 zeros and 5 critical points whose location

we have not described yet. We may approximate the values of these zeros and critical points using the fact that they are fixed points of certain operators (c.f. [9]).

Proposition 2.3. *If we fix $a \in \mathbb{D}^*$, then $B_{a,\lambda}$ has 5 zeros of the form $\xi(\lambda/a)^{1/5} + o(\lambda^{1/5})$, where ξ denotes a fifth root of the unity and $o(\lambda^{1/5})$ is such that $\lim_{\lambda\to 0} |o(\lambda^{1/5})|/|\lambda^{1/5}| = 0$. Moreover, $B_{a,\lambda}$ has 5 critical points of the form $-\xi(2\lambda/3a)^{1/5} + o(\lambda^{1/5})$.*

Proof. We first prove the proposition for the zeros of $B_{a,\lambda}$. They are solutions of

$$z^3 \frac{z-a}{1-\overline{a}z} + \frac{\lambda}{z^2} = 0 \to z^5 = -\lambda \frac{1-\overline{a}z}{z-a}.$$

For $\lambda = 0$, the last equation has $z = 0$ as a solution of multiplicity 5. If we increase λ, by continuity, we will have 5 zeros near $z = 0$. These zeros are fixed points of the operators

$$T_\xi(z) = \xi \sqrt[5]{-\lambda \frac{1-\overline{a}z}{z-a}} = \xi \lambda^{1/5} \sqrt[5]{-\frac{1-\overline{a}z}{z-a}} = \xi \lambda^{1/5} R(z).$$

Notice that $R(z)$ does not depend on λ. We have 5 different choices for the operators T_ξ given by the choice of the fifth root of the unity, denoted by ξ. Fix any of the five choices. Then one of the five zeros which appear around zero is a fixed point of T_ξ, say $w_{\lambda,\xi}$. We can approximate $w_{\lambda,\xi}$ by $T(0) = \xi(\lambda/a)^{1/5}$. Hence, we have

$$|w_{\lambda,\xi} - \xi(\lambda/a)^{1/5}| = |T(w_{\lambda,\xi}) - T(0)| \leq \sup_{\eta \in [0,w_{\lambda,\xi}]} |T'(\eta)| \cdot |w_{\lambda,\xi} - 0|$$

$$= \sup_{\eta \in [0,w_{\lambda,\xi}]} |\lambda|^{1/5} |R'(\eta)| \cdot |w_{\lambda,\xi}|.$$

Moreover, we can find an upper bound of $|R'(\eta)|$, $\eta \in [0, w_{\lambda,\xi}]$, for $|\lambda|$ small enough. Indeed, the function $R(z)$ does not depend on λ and, if $|\lambda|$ is small enough, the points $z = a$ and $z = 1/\overline{a}$ are bounded away from the segment $[0, w_{\lambda,\xi}]$ given that $w_{\lambda,\xi} \to 0$ as $\lambda \to 0$. Therefore, there is $\epsilon > 0$ such that if $|\lambda| < \epsilon$, then $\sup_{\eta \in [0,w_{\lambda,\xi}]} |R'(\eta)| < C$, where C does not depend on λ. Finally,

$$\lim_{\lambda\to 0} \frac{|w_{\lambda,\xi} - \xi(\lambda/a)^{1/5}|}{|\lambda^{1/5}|} \leq \lim_{\lambda\to 0} C \cdot |w_{\lambda,\xi}| = 0.$$

The proof for the critical points is analogous using that they are solutions of the equation

$$3z^2 \frac{z-a}{1-\overline{a}z} + z^3 \frac{1-|a|^2}{(1-\overline{a}z)^2} - 2\frac{\lambda}{z^3} = 0$$

and fixed points of the operators

$$T_\xi(z) = \xi(2\lambda)^{1/5} \sqrt[5]{\frac{1-\overline{a}z}{3(z-a) + z\frac{1-|a|^2}{1-\overline{a}z}}},$$

where ξ denotes a fifth root of the unity. As before we can approximate the critical points by $T_\xi(0) = -\xi(2\lambda/3a)^{1/5}$. $\qquad\qquad\square$

The next proposition follows directly from the previous result using the expressions of the zeros and critical points. The proof is straightforward.

Proposition 2.4. *Fix $a \in \mathbb{D}^*$. Then, if $|\lambda|$ is small enough, the 5 critical points and the 5 zeros which appear around $z = 0$ belong to the annulus of inner radius $\left(\frac{|\lambda|}{2|a|}\right)^{1/5}$ and outer radius $\left(2\frac{|\lambda|}{|a|}\right)^{1/5}$.*

The main result of the preliminaries is the following theorem. It describes the local dynamics that takes place near $z = 0$ for $|\lambda|$ small. It also describes the topology of the immediate basin of attraction of infinity, $A^*(\infty)$.

Theorem 2.5. *Fix $a \in \mathbb{D}^*$ and let $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then, there is a constant $\mathcal{C}(a)$ such that if $|\lambda| < \mathcal{C}(a)$ the following hold:*

a) *The immediate basin of attraction of ∞, $A^*(\infty)$, is simply connected and $\partial A^*(\infty)$ is a quasicircle. Moreover, $A^*(\infty)$ is mapped with degree 4 onto itself and contains only a pole z_∞ and a critical point c_+ other than the superattracting fixed point $z = \infty$.*
b) *There is a simply connected neighbourhood T_0 of $z = 0$ which is mapped 2 to 1 onto $A^*(\infty)$.*
c) *There is an open annulus A_0 which contains 5 critical points and 5 preimages of $z = 0$ and is mapped 5 to 1 onto T_0.*
d) *The annular region in between A_0 and $A^*(\infty)$ contains a critical point c_- and a zero z_0. Moreover, the component D_0 of $B_{a,\lambda}^{-1}(T_0)$ in which z_0 lies is simply connected and is mapped with degree 1 onto T_0. Consequently, it does not contain the critical point c_-.*
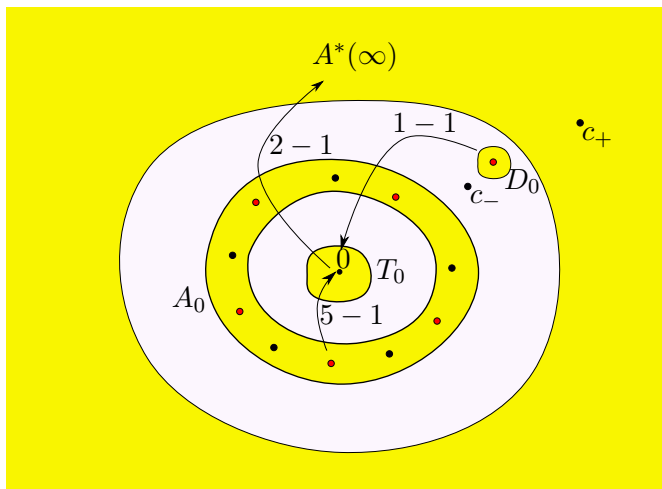
FIGURE 6. Scheme of the dynamics described in Theorem 2.5. We draw in red the preimages of zero and in black the critical points.

The conclusions of Theorem 2.5 are summarized in Figure 6. The proof of Theorem 2.5 is structured as follows. We first prove statement *a)*. Afterwards we give a criterion which guarantees that a point z belongs to the preimage T_0 of $A^*(\infty)$ (Lemma 2.6). Finally we use the previous results and Proposition 2.4 to prove the remaining statements of Theorem 2.5.

Proof of statement a) of Theorem 2.5. Let $a \in \mathbb{D}^*$. We prove that if $|\lambda|$ is small enough, then statement *a)* holds. Take the curves γ_0 and γ_∞ from Proposition 2.2. Then, it follows from continuity with respect to λ that if $|\lambda|$ is small enough then we can take connected components $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ of $B_{a,\lambda}^{-1}(\gamma_0)$ and $B_{a,\lambda}^{-1}(\gamma_\infty)$, respectively, such that the following hold:

- The sets $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ are analytic Jordan curves.
- The application $B_{a,\lambda}$ maps $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ onto γ_0 and γ_∞ with degree 4, respectively.
- The curves satisfy $\gamma_0 \subset \mathrm{Int}\left(\gamma_{0,\lambda}^{-1}\right)$ and $\gamma_\infty \subset \mathrm{Ext}\left(\gamma_{\infty,\lambda}^{-1}\right) \subset \mathrm{A}^*(\infty)$.
- There is no zero, pole or critical point in the annular region bounded by $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$. Moreover, $B_{a,\lambda}$ maps the annulus bounded by $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ onto the annulus bounded by γ_0 and γ_∞ with degree 4.
- We have the inclusion $\mathrm{Ext}\left(\gamma_{\infty,\lambda}^{-1}\right) \subset \mathrm{A}^*(\infty)$. Moreover, $\mathrm{Ext}\left(\gamma_{\infty,\lambda}^{-1}\right)$ contains the pole z_∞ and the critical point c_+ and no other than $z = \infty$.

Indeed, all of these properties were satisfied by the curves obtained in Proposition 2.2. By continuity, all of them are still satisfied as long as the perturbation is small enough.

We can now apply a standard quasiconformal surgery procedure to obtain a quasiregular map $F(z)$ which agrees with $B_{a,\lambda}(z)$ on the annulus bounded by $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ and is quasiconformally conjugate to $z \to z^4$ inside the regions $\mathrm{Int}\left(\gamma_{0,\lambda}^{-1}\right)$ and $\mathrm{Ext}\left(\gamma_{\infty,\lambda}^{-1}\right)$. In this surgery we use the curves γ_0, $\gamma_{0,\lambda}^{-1}$, γ_∞ and $\gamma_{\infty,\lambda}^{-1}$ to glue the dynamics of z^4 in $\mathrm{Int}\left(\gamma_{0,\lambda}^{-1}\right)$ and $\mathrm{Ext}\left(\gamma_{\infty,\lambda}^{-1}\right)$. We refer to [4] for an introduction to quasiconformal surgery and to [4, proof of Theorem 7.4] for the details on how to glue the dynamics of z^4. After the surgery we obtain a quasiregular map $F(z)$ such that the following hold.

- The map $F(z)$ is conformally conjugate to $z \to z^4$ inside the regions $\mathrm{Int}\left(\gamma_0^{-1}\right)$ and $\mathrm{Ext}\left(\gamma_\infty^{-1}\right)$ and has $z = 0$ and $z = \infty$ as superattracting fixed points of local degree 4.
- The map $F(z)$ has topological degree 4 and no other critical point than $z = 0$ and $z = \infty$ in the sense that it is bijective in a neighbourhood of any point other than $z = 0$ or $z = \infty$.
- The map $F(z)$ is conjugate to a holomorphic map $f(z)$ via a quasiconformal map $\varphi(z)$ that fixes 0 and ∞, i.e. $f(z) = \varphi^{-1} \circ F \circ \varphi(z)$.

Since the quasiregular map $F(z)$ has topological degree 4 and has $z = 0$ and $z = \infty$ as superattracting fixed points of local degree 4, then also does the holomorphic function $f(z)$. Therefore, $f(z)$ is necessarily of the form $f(z) = bz^4$ where $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consequently, the Julia set of $f(z)$ consists of a circle S (of radius $(1/|b|)^{1/3}$) which is the common boundary of the basins of attraction of $z = 0$ and $z = \infty$. The image under the quasiconformal map of this circle, $\varphi(S)$, necessarily belongs to the region bounded by $\gamma_{0,\lambda}^{-1}$ and $\gamma_{\infty,\lambda}^{-1}$ since its complement belongs, by construction, to the basins of attraction of $z = 0$ and $z = \infty$ under the quasiregular map $F(z)$.

We finish the proof noticing that, by construction, the immediate basin of attraction of ∞ under the quasiregular map F, $A_F^*(\infty)$, coincides with the immediate basin of attraction of ∞ under $B_{a,\lambda}$, $A_{B_{a,\lambda}}^*(\infty)$, and that $\partial A_F^*(\infty) = \varphi(S)$. Since $\varphi(S)$ is the image of a circle

under a quasiconformal map it is, by definition, a quasicircle. We conclude that $A^*_{B_{a,\lambda}}(\infty)$ is a simply connected domain bounded by a quascircle.

$\square$

The next lemma provides a criterion which guarantees that a point z belongs to a preimage of $A^*(\infty)$, for $|\lambda|$ small.

Lemma 2.6. *Let $a \in \mathbb{D}^*$ and $\lambda \in \mathbb{C}^*$. If $|\lambda|$ is small enough, then the following hold.*

a) We have the inclusion $\{z \in \mathbb{C};\ |z| > 2\} \subset A^*(\infty)$.
b) If $|z| < \left(\frac{|\lambda|}{3}\right)^{1/2}$ then $B_{a,\lambda}(z) \in A^*(\infty)$.

Proof. The first statement follows by continuity of $B_{a,\lambda}(z)$ with respect to λ and the fact that when $\lambda = 0$ the immediate basin of attraction of infinity consists of the complement of the closed unit disk (see Section 2.1).

Now assume that $|z| < \left(\frac{|\lambda|}{3}\right)^{1/2} < 1$. Then, using that $|z^3(z-a)/(1-\bar{a}z)| < 1$ and that $|\lambda|/|z|^2 > 3$ we have:

$$|B_{a,\lambda}(z)| = \left| z^3 \frac{z-a}{1-\bar{a}z} + \frac{\lambda}{z^2} \right| \geq \left| \left| \frac{\lambda}{z^2} \right| - \left| z^3 \frac{z-a}{1-\bar{a}z} \right| \right| > 3 - 1 = 2.$$

Hence, $|B_{a,\lambda}(z)| > 2$ and $B_{a,\lambda}(z) \in A^*(\infty)$ by statement *a)*. $\square$

We can now finish the proof of Theorem 2.5.

Proof of statements b), c) and d) of Theorem 2.5. By Proposition 2.4 we know that if $|\lambda|$ is small enough, then the round annulus A of inner radius $\left(\frac{|\lambda|}{2|a|}\right)^{1/5}$ and outer radius $\left(2\frac{|\lambda|}{|a|}\right)^{1/5}$ contains the 5 critical points and the 5 zeros which appear around $z = 0$ after the perturbation. We first show that, if $|\lambda|$ is small enough, then A is mapped under $B_{a,\lambda}$ into the Fatou component T_0 which contains the pole $z = 0$. Every point $w \in A$ has the form $w = b\lambda^{1/5}$ with $b \in \mathbb{C}$ such that $(1/2|a|)^{1/5} < |b| < (2/|a|)^{1/5}$. Using the Taylor expansion of $(z-a)/(1-\bar{a}z)$ around $z = 0$ we have

$$B_{a,\lambda}(z) = -az^3 + O(z^4) + \frac{\lambda}{z^2} \Rightarrow |B_{a,\lambda}(w)| \leq C|\lambda|^{3/5} + O(|\lambda|^{4/5}),$$

where C is a bounded constant which can be taken independently of b and λ. We know from Lemma 2.6 that if $|\lambda|$ is small enough and $|z| < \left(\frac{|\lambda|}{3}\right)^{1/2}$, then $B_{a,\lambda}(z) \in A^*(\infty)$. Since $|\lambda|^{3/5}$ tends to zero faster than $|\lambda|^{1/2}$, we can conclude that if $|\lambda|$ is small enough then $B_{a,\lambda}(A) \subset T_0$, where T_0 denotes the preimage of $A^*(\infty)$ that contains $z = 0$.

Up to this point we have studied the dynamics of all critical points and zeros of $B_{a,\lambda}$ but the critical point $c_- = c_-(a, \lambda)$ and the zero $z_0 = z_0(a, \lambda)$, which come from continuation of the critical point $c_-(a, 0)$ and the zero $z_0(a, 0)$ of $B_a(z)$ (Equation (2)). By continuity, if $|\lambda|$ is small enough we can take a curve γ such that it contains c_-, separates A and z_0, and

such that $B_{a,\lambda}(\gamma) \subset \mathbb{C} \setminus \{T_0 \cup A^*(\infty)\}$. This implies that c_- and z_0 belong to different Fatou components than A.

We can now finish the proof. Take $\mathcal{C}(a)$ to be a positive real constant depending on a such that if $|\lambda| < \mathcal{C}(a)$ then all previous considerations hold. Since $z = 0$ is a double preimage of ∞ and there are no other poles outside $A^*(\infty)$, we know that the preimage T_0 of $A^*(\infty)$ is mapped with degree 2 to $A^*(\infty)$ under $B_{a,\lambda}$. By application of the Riemann-Hurwitz formula (Theorem 2.1), we can conclude that T_0 is simply connected since it contains no other critical point than $z = 0$. This proves statement $b)$.

We denote by A_0 the Fatou component which contains the round annulus A. We know that it contains exactly 5 critical points and 5 zeros. Since it contains exactly 5 zeros, then A_0 is mapped onto T_0 with degree 5. Since T_0 is simply connected, by application of the Riemann-Hurwitz formula the set A_0 is doubly connected. This proves statement $c)$.

Finally, the annular region in between A_0 and $A^*(\infty)$ contains c_- and z_0 because of the choice of γ. Moreover, the connected component D_0 of $B_{a,\lambda}^{-1}(T_0)$ which contains z_0 is mapped with degree 1 onto T_0 since it only contains a preimage of $z = 0$ counting multiplicities. Consequently, it can not contain the critical point c_- and is simply connected. This proves statement $d)$.

$\square$

3. FATOU SET OF THE SINGULARLY PERTURBED BLASCHKE PRODUCTS: PROOF OF THEOREM A

The goal of this section is to study the Fatou components of the singular perturbation $B_{a,\lambda}$ in the case that $c_- \in A(\infty)$ and to prove Theorem A. The main result of the section is Theorem 3.2. Theorem A is a direct corollary of Theorem 3.2. Indeed, Theorem A is a simpler restatement of Theorem 3.2 which omits some technicalities.

The main ingredients for the proof of Theorem 3.2 are Theorem 2.5, the Riemann-Hurwitz formula (Theorem 2.1) and Proposition 3.1. This proposition describes the dynamics of $B_{a,\lambda}$ in the case that $|\lambda| < \mathcal{C}(a)$, that the orbit of c_-, $\mathcal{O}(c_-) = \{c_-, B_{a,\lambda}(c_-), B_{a,\lambda}^2(c_-), \cdots\}$, intersects the annulus A_0, and that c_- belongs to a Fatou component which surrounds $z = 0$ (see Figure 7).

Proposition 3.1. *Fix $a \in \mathbb{D}^*$ and let $\lambda \in \mathbb{C}^*$. If $|\lambda| < \mathcal{C}(a)$ and $\mathcal{O}(c_-)$ intersects the annulus A_0, then the critical point c_- lies in a triply connected Fatou component, say $\mathcal{U}_c$, which is eventually mapped onto A_0. Moreover, if $\mathcal{U}_c$ surrounds $z = 0$ then the following hold.*

- *The triply connected Fatou component $\mathcal{U}_c$ bounds a closed disk $\mathcal{V}_1$ which is mapped with degree 1 onto the closed disk bounded by $B_{a,\lambda}(\mathcal{U}_c)$, which contains T_0.*
- *The closed annular region $\mathcal{V}_2$ in between T_0 and A_0 is mapped with degree 2 onto the annular region in between T_0 and $A^*(\infty)$.*
- *The closed annular region $\mathcal{V}_3$ in between A_0 and $\mathcal{U}_c$ is mapped with degree 3 onto the annular region $\mathcal{W}_3$ in between T_0 and $B_{a,\lambda}(\mathcal{U}_c)$.*
- *The closed annular region $\mathcal{V}_4$ in between $\mathcal{U}_c$ and $A^*(\infty)$ is mapped with degree 4 onto the annular region $\mathcal{W}_4$ in between $B_{a,\lambda}(\mathcal{U}_c)$ and $A^*(\infty)$.*
- *The Fatou component $\mathcal{U}_c$ is contained in the region $\mathcal{W}_4$.*
- *The Fatou component $\mathcal{U}_c$ is mapped with degree 4 onto the annulus $B_{a,\lambda}(\mathcal{U}_c)$.*
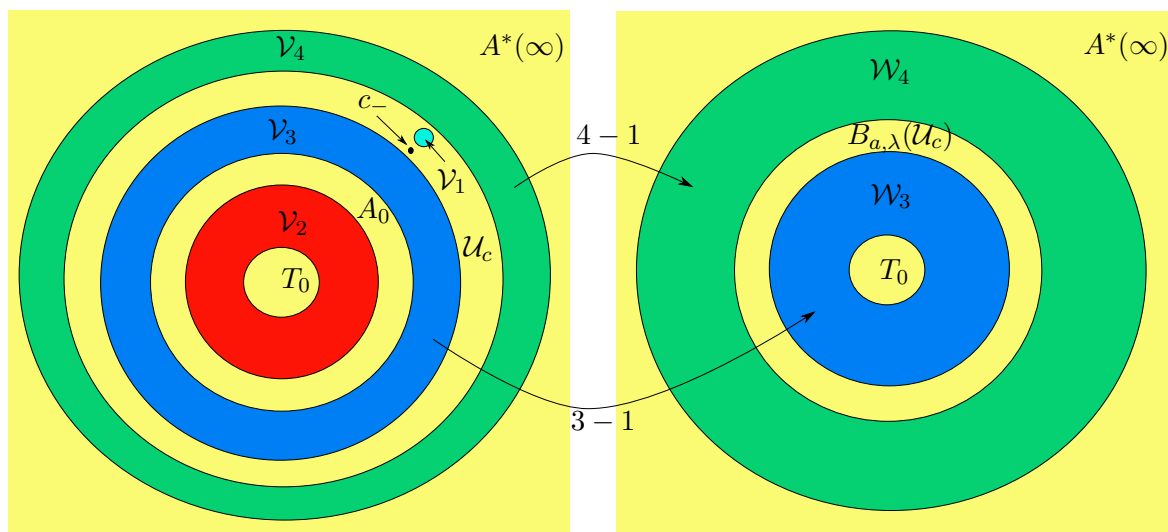
FIGURE 7. Summary of the dynamics of $B_{a,\lambda}$ described in Proposition 3.1. The triply connected region $\mathcal{U}_c$ is mapped with degree 4 onto the annulus $B_{a,\lambda}(\mathcal{U}_c)$. The green annular region $\mathcal{V}_4$ is mapped with degree 4 to the green annular region $\mathcal{W}_4$. The blue annular region $\mathcal{V}_3$ is mapped with degree 3 to the blue annular region $\mathcal{W}_3$. The pallid blue disk $\mathcal{V}_1$ is mapped with degree 1 to the region bounded by $B_{a,\lambda}(\mathcal{U}_c)$. The red region $\mathcal{V}_2$ is mapped with degree 2 to the full annular region bounded by $A^*(\infty)$ and T_0. Since $\mathcal{U}_c \subset \mathcal{W}_4$, $\mathcal{V}_4 \subset \mathcal{W}_4$ and either $B_{a,\lambda}(\mathcal{U}_c) = A_0$ or $B_{a,\lambda}(\mathcal{U}_c) \subset \mathcal{V}_3 \cup \mathcal{V}_2$.

Proof. It follows from the Riemann-Hurwitz formula (Theorem 2.1) that the preimage of an annulus A which contains no critical value (i.e. the image of a critical point) under a holomorphic map is another annulus A'. Therefore the iterated preimages of the annulus A_0 are also annuli until one of them, which we denote by $\mathcal{U}_c$, contains the critical point c_-. Since $\mathcal{U}_c$ contains only one critical point and is mapped to a doubly connected domain with a certain degree k, it follows again from the Riemann-Hurwitz formula that $\mathcal{U}_c$ has connectivity $k(2-2) + 1 + 2 = 3$. Therefore, $\partial \mathcal{U}_c$ consists of 3 connected components. Moreover, since these boundary components are eventually mapped onto $\partial A^*(\infty)$, which is a quasicircle that intersects no critical orbit, they are quasicircles.

Notice that the iterated preimages of A_0 either surround $z = 0$ or some iterated preimage of $z = 0$. Moreover, if a connected component V of $B_{a,\lambda}^{-n}(A_0)$ does not surround $z = 0$, then no component of $B_{a,\lambda}^{-1}(V)$ does. Now assume that $\mathcal{U}_c$ surrounds $z = 0$. Since the boundary curves of $B_{a,\lambda}(\mathcal{U}_c)$ also surround $z = 0$, we conclude that all boundary components of $\mathcal{U}_c$ surround $z = 0$ or its preimage z_0 or both. Indeed, since c_- lies in the annular region in between A_0 and $A^*(\infty)$, then so does $\mathcal{U}_c$ and the only zero that a connected component of $\partial \mathcal{U}_c$ can bound without surrounding $z = 0$ is z_0. On the other hand, at most one component can bound both $z = 0$ and z_0. Indeed, if two such boundary curves did, then the third one should lie in the annulus bounded by them but it could not surround neither z_0 nor $z = 0$. Similar arguments yield that at most one such component surrounds $z = 0$ and not z_0 and at most one of them surrounds z_0 and not $z = 0$. Therefore, $\mathcal{U}_c$ has 2 exterior boundary components, which are quasicircles that surround $z = 0$ (and consequently also the annulus A_0), and an interior boundary γ_1, which surrounds z_0 and not $z = 0$ (see Figure 7). We denote by γ_3 and γ_4 the exterior boundary components, where $\gamma_3 \subset \text{Int}(\gamma_4)$.

Since the closed disk $\mathcal{V}_1$ bounded by γ_1 contains no pole and no critical point, it is mapped with degree 1 onto the closed disk bounded by $B_{a,\lambda}(\mathcal{U}_c)$ under $B_{a,\lambda}$. Notice that $D_0 \subset \mathcal{V}_1$, where D_0 is the Fatou component described in Theorem 2.5 which contains the zero z_0.

Let $\mathcal{V}_4$ be the closed annular region in between $\mathcal{U}_c$ and $A^*(\infty)$. Since $B_{a,\lambda}|_{\mathcal{V}_4}$ is proper, $\mathcal{V}_4$ is mapped onto the closed annular region $\mathcal{W}_4$ in between $B_{a,\lambda}(\mathcal{U}_c)$ and $A^*(\infty)$ with a certain degree d which is accomplished on the boundaries. Given that $\partial A^*(\infty)$ is mapped with degree 4 onto itself we can conclude that this degree is 4. Analogously, since T_0 is mapped with degree 2 onto $A^*(\infty)$, we can conclude that the annular region $\mathcal{V}_2$ in between T_0 and A_0 is mapped with degree 2 onto the annular region in between T_0 and $A^*(\infty)$.

The domain $\mathcal{U}_c$ is contained inside the region $\mathcal{W}_4$ since otherwise the region $\mathcal{V}_4$ would be mapped into itself. Therefore, the points in $\mathcal{V}_4$ would never leave $\mathcal{V}_4$ under iteration. It would then follow from Montel's Theorem (see [11]) that the points in the interior of $\mathcal{V}_4$ would lie in a Fatou component which would not belong to $A(\infty)$, which is impossible since all critical points belong to $A(\infty)$.

The annular region $\mathcal{V}_3$ in between A_0 and $\mathcal{U}_c$ is mapped with degree 3 onto the region $\mathcal{W}_3$ in between T_0 and $B_{a,\lambda}(\mathcal{U}_c)$. Indeed, $B_{a,\lambda}(\mathcal{V}_3)$ has 6 preimages since $B_{a,\lambda}$ has degree 6 and it has 3 preimages missing since the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ contain 1 and 2 preimages of it, respectively.

Finally, the domain $\mathcal{U}_c$ is mapped with degree 4 onto $B_{a,\lambda}(\mathcal{U}_c)$ since its boundaries are mapped with degree 4. Indeed, γ_4 is mapped with degree 4 onto one of the boundary components of $B_{a,\lambda}(\mathcal{U}_c)$ while γ_1 and γ_3 are mapped onto the other boundary component with degree 1 and 3, respectively.

$\square$

Using the information provided by Theorem 2.5 we can study the Fatou set of $B_{a,\lambda}$ in the hyperbolic scenario for which the critical point c_- belongs to $A(\infty)$. Notice that A_0 and D_0 are the only preimages of T_0, which is the only preimage of $A^*(\infty)$. Therefore, a point in $A(\infty) \setminus (A^*(\infty) \cup T_0)$ is eventually mapped under iteration of $B_{a,\lambda}$ into A_0 or into D_0. The Fatou domains that we may obtain depending on whether c_- is eventually mapped into A_0 or D_0 can be very different and are studied in Theorem 3.2. In particular, Theorem 3.2 tells us that if c_- belongs to a preimage of A_0 which surrounds $z = 0$ then all Fatou components have finite connectivity but there are components of arbitrarily large connectivity.

Theorem 3.2. *Fix $a \in \mathbb{D}^*$ and let $\lambda \in \mathbb{C}^*$. Assume that $|\lambda| < \mathcal{C}(a)$ and that $c_- \in A(\infty)$. Then all Fatou components are bounded by quasicircles and exactly one of the following is satisfied.*

a) *The critical point c_- eventually falls under iteration into D_0 and lies in a simply connected Fatou component. All Fatou components are either simply or doubly connected (see Figure 2 (a) and (b)).*

b) *The critical point c_- eventually falls under iteration into A_0 and lies in a triply connected Fatou component $\mathcal{U}_c$ which does not surround $z = 0$. All Fatou components have connecivity 1, 2 or 3 (see Figure 2 (c) and (d)).*

c) *The critical point c_- eventually falls under iteration into A_0 and lies in a triply connected Fatou component $\mathcal{U}_c$ which surrounds $z = 0$. All Fatou components have finite connectivity but there are components of arbitrarily large connectivity (see Figure 3).*

Proof. Notice that all critical points belong to $A(\infty)$. Therefore, all Fatou components other than $A^*(\infty)$ are preimages of it. Since $\partial A^*(\infty)$ is a quasicircle and no critical orbit intersects it we can conclude that the boundaries of all Fatou component are also quasicircles.

Since $c_- \in A(\infty) \setminus (A^*(\infty) \cup T_0)$, the only preimage of $A^*(\infty)$ is T_0 and the only preimages of T_0 are A_0 and D_0, we can conclude that c_- is eventually mapped under iteration of $B_{a,\lambda}$ into D_0 or into A_0.

First we assume that $\mathcal{O}(c_-)$ intersects D_0. Since all Fatou components other than $A^*(\infty)$, T_0, D_0 or A_0 are eventually mapped under iteration onto A_0 or D_0 we have that all of them are either disks or annulus. Indeed, since $\mathcal{O}(c_-)$ does not intersect A_0 we obtain from the Riemann-Hurwitz formula (Theorem 2.1) that all iterated preimages of A_0 are annuli. On the other hand, it also follows from the Riemann-Hurwitz formula that at least two different critical points are required to map a multiply connected domain onto a disk. Hence, all iterated preimages of D_0 are disks.

Second we assume that $\mathcal{O}(c_-)$ intersects A_0. It follows from Proposition 3.1 that there is a triply connected domain $\mathcal{U}_c$ which contains the critical point c_-. As in the previous case, every Fatou component which is not eventually mapped under iteration into $\mathcal{U}_c$ is either a disk or an annulus. If we also assume that $\mathcal{U}_c$ does not surround $z = 0$ then all its iterated preimages are triply connected domains. To show that we can consider the filled disk $\mathcal{U}_c'$ which consists of the union of the points in $\mathcal{U}_c$ and the points contained in the bounded components of $\mathbb{C} \setminus \mathcal{U}_c$. Notice that all critical points are in $U = A^*(\infty) \cup T_0 \cup A_0 \cup \mathcal{U}_c$. Since $B_{a,\lambda}^{-n}(\mathcal{U}_c') \cap U = \emptyset$ for all $n \geq 1$, all branches of the inverse $B_{a,\lambda}^{-n}$ restricted to $\mathcal{U}_c'$ are conformal for all $n \in \mathbb{N}$. Consequently, all iterated preimages of $\mathcal{U}_c$ are also triply connected domains.

Finally we assume that $\mathcal{O}(c_-)$ intersects A_0 and that $\mathcal{U}_c$ surrounds $z = 0$. In that case, the domain $\mathcal{U}_c$ has a unique preimage in $\mathcal{V}_2$ and another one in $\mathcal{V}_3$ or $\mathcal{V}_4$, which also surround $z = 0$. This follows from the fact that every Jordan curve surrounding $z = 0$ and contained in the annular regions $\mathcal{W}_3$ or $\mathcal{W}_4$ has a unique preimage in the regions $\mathcal{V}_3$ or $\mathcal{V}_4$, respectively, and a unique preimage in $\mathcal{V}_2$. Let V be any of the two Fatou components which are preimages of $\mathcal{U}_c$ and surround $z = 0$. Since $B_{a,\lambda}$ restricted to $\mathcal{V}_i$ has degree i, it follows from the Riemann Hurwitz-formula that V has connectivity $i(3 - 2) + 2 > 3$, where $V \subset \mathcal{V}_i$. Repeating this process we obtain a sequence of iterated preimages of $\mathcal{U}_c$ which surround $z = 0$ and have connectivity $j(m - 2) + 2 > m$, where m is the connectivity of its image. In this way we obtain Fatou components of arbitrarily large connectivity (see Figure 3).

$\square$

4. Julia set of the singularly perturbed Blaschke products: proof of Theorem B

The goal of this section is to describe the Julia sets of the maps $B_{a,\lambda}$ in the case that statement *c)* of Theorem 3.2 holds and to prove Theorem B. To do so, we first show the existence of a Cantor set of quasicircles as described by McMullen in [10]. Afterwards we show how to obtain countably many copies of the original Cantor set of quasicircles using the dynamics of the map. Finally we use symbolic dynamics to show the existence of an uncountable number of point components in the Julia set.

Proof of Theorem B. We first show the existence of a Cantor set of quasicircles. Let A_0 be the annulus containing the 5 critical points and 5 zeros which appear around $z = 0$. It follows

from the structure of the dynamical plane shown in Proposition 3.1 that it has exactly two preimages which are Fatou components that surround $z = 0$. One of them lies in $\mathcal{V}_2$ and the other one lies in $\mathcal{V}_3 \cup \mathcal{V}_4 \cup \mathcal{U}_c$. We can iterate this process, obtaining at each step exactly two new Fatou components which surround $z = 0$ from every old component. These preimages are annuli, unless one of them is the triply connected Fatou component $\mathcal{U}_c$. In that case we repeat the process considering the annulus $A'_c = \mathcal{U}_c \cup \mathcal{V}_1$ instead. In this fashion we obtain a sequence of open annuli whose boundaries are quasicircles which surround $z = 0$ and belong to the Julia set. These boundaries together with its accumulation sets form a Cantor set of quasicircles in the same way as described by McMullen in [10].

We now see where the countable number of Cantor sets of quasicircles comes from. We know from Proposition 3.1 that the closed disk $\mathcal{V}_1$ is mapped with degree one onto the closed disk bounded by $B_{a,\lambda}(\mathcal{U}_c)$. This last disk contains the part of the previously described Cantor set of quasicircles that lies in $\mathcal{W}_3$, which is also a Cantor set of quasicircles. Since the map $B_{a,\lambda}$ restricted to $\mathcal{V}_1$ is a homeomorphism (it is conformal in a neighbourhood of $\mathcal{V}_1$), we conclude that $\mathcal{V}_1$ contains a Cantor set of quasicircles. To finish the argument it is enough to notice that the disk $\mathcal{V}_1$ has exactly 6 preimages and so does every set which is eventually mapped onto $\mathcal{V}_1$. Therefore, $\mathcal{V}_1$ has countably many iterated preimages which are also homeomorphic to $\mathcal{V}_1$ since they cannot bound any critical point and, therefore, $B_{a,\lambda}$ is conformal in a neighbourhood of them. Each of these disks contains a copy of the Cantor set in $\mathcal{V}_1$, which leads to the existence of a countable union of Cantor sets of quasicircles in the Julia set. Notice that these iterated preimages are not disjointed. Indeed, as we shall see in the remaining of the proof, many of these disks are contained in other ones.

The description that we have given of the Julia set up to this point is not complete. Indeed, it is known that periodic points are dense in the Julia set. However, the Cantor sets of quasicircles contained in the iterated preimages of $\mathcal{V}_1$ are eventually mapped into the original Cantor set of quasicircles and, therefore, contain no periodic point. The missing points of the Julia set are those whose orbit never leaves the union $\mathcal{D}$ of $\mathcal{V}_1$ and the disks which are eventually mapped onto it, $\mathcal{D} = \bigcup_{n=0}^{\infty} B_{a,\lambda}^{-n}(\mathcal{V}_1)$. To understand the dynamics of those points we will use symbolic dynamics. We begin by labelling the filled annulus $A'_c = \mathcal{U}_c \cup \mathcal{V}_1$ by 0, $A'_0 := A'_c$. Afterwards we label the iterated preimages of A'_c which surround $z = 0$ with the natural numbers, i.e. we denote every connected component of $\bigcup_{n=1}^{\infty} B_{a,\lambda}^{-n}(A'_c)$ in $\mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$ which surrounds $z = 0$ by A'_i with $i \in \mathbb{N}$. With these labels we can associate to every point z whose orbit never leaves $\mathcal{D}$ an itinerary $S(z) = (s_0, s_1, s_2, ...)$, where $s_j = k$ if $B_{a,\lambda}^j(z)$ belongs to A'_k. Notice that the itinerary depends on the chosen labelling, but this is not relevant.

It is now important to understand how the realizable itineraries, i.e. the itineraries which are realized by some point $z \in \mathcal{D}$, look like. The fact that if $A'_i \neq A'_0$ then $B_{a,\lambda}(A'_i)$ is another annulus, say $A'_{s(i)}$, implies that the index i must always be followed by the index $s(i)$. By Proposition 3.1 the disk $\mathcal{V}_1$, which is contained in $A'_c = A'_0$, is mapped onto $\mathcal{W}_3 \cup T_0$. Therefore, we conclude that the index 0 must be followed by an index which labels an annulus contained in $\mathcal{W}_3$. Again by Proposition 3.1, since $\mathcal{U}_c \subset \mathcal{W}_4$ we conclude that we cannot have two zeros in a row. Moreover, given any i, there is an $n = n(i)$ such that $s^n(i) = s \circ \overset{n)}{\cdots} \circ s(i) = 0$. Therefore, the index 0 appears infinitely many times in any realizable sequence $S(z)$.

We now discuss the simpler case where the sequence $S(z)$ is periodic. Without loss of generality we may assume that it begins with the index 0. Then we have $S(z) = (\overline{0, s_1, s_2, ..., s_n}) = (0, s_1, s_2, ..., s_n, 0, s_1, s_2, ..., s_n, ...)$. The first step is to understand the set of points which

follow the sequence $(0, s_1, s_2, ..., s_n, 0)$. Since $\mathcal{V}_1$ does not intersect any critical orbit, every branch of the inverse $B_{a,\lambda}^{-n}$ is conformal in a neighbourhood of $\mathcal{V}_1$ for all $n \in \mathbb{N}$. Given that $B_{a,\lambda}^n$ maps A'_{s_1} onto A'_0 with a certain degree $k \geq 2$, there are k disjoint preimages of $\mathcal{V}_1$ in A'_{s_1}. Since $B_{a,\lambda}$ maps $\mathcal{V}_1$ onto $T_0 \cup \mathcal{W}_3 \supset A'_{s_1}$ with degree 1, the set of points with itinerary $(0, s_1, s_2, ..., s_n, 0)$ corresponds to the disjoint union of k sets which are compactly contained in $\mathcal{V}_1$. Notice that we can take small annuli in the Fatou set which surround these k disks. Analogously, the set of points with itinerary $(0, s_1, s_2, ..., s_n, 0, s_1, s_2, ..., s_n, 0)$ correspond to k^2 disks compactly contained in the k previous ones. Indeed, every original disk contains k sub-disks, each of which is mapped univalently onto one of the original ones by $B_{a,\lambda}^{n+1}$. As before, these new disks are surrounded by small annuli in the Fatou set. Repeating this process, at the step p we obtain k^p closed disks which are compactly contained in the k^{p-1} disks of the previous step and are surrounded by small annuli in the Fatou set. Standard arguments of complex dynamics yield that these disks shrink to a Cantor set. We conclude that the set of points with itinerary $S(z) = (\overline{0, s_1, s_2, ..., s_n})$ corresponds to Cantor set of points. Moreover, since each of these points is surrounded by arbitrarily small annuli in the Fatou set we conclude that they are point components of the Julia set.

The general case derives analogously. The only difference is that every time that we come back to the index 0, the number of disks compactly contained in the previous ones varies depending on the subsequence. For each realizable sequence we obtain a nested sequence of disks which will also converge to a Cantor set of points which are point components of the Julia set (c.f. [3]).

$\square$

References

[1] I. N. Baker, J. Kotus and Y. N. Lü, Iterates of meromorphic functions. III. Preperiodic domains, *Ergodic Theory Dynam. Systems*, **11** (1991), 603–618.

[2] A. F. Beardon, *Iteration of rational functions*, vol. 132 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1991, Complex analytic dynamical systems.

[3] P. Blanchard, R. L. Devaney, A. Garijo and E. D. Russell, A generalized version of the McMullen domain, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **18** (2008), 2309–2318.

[4] B. Branner and N. Fagella, *Quasiconformal surgery in holomorphic dynamics*, vol. 141 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2014.

[5] J. Canela, N. Fagella and A. Garijo, On a family of rational perturbations of the doubling map, *J. Difference Equ. Appl.*, **21** (2015), 715–741.

[6] J. Canela, N. Fagella and A. Garijo, Tongues in degree 4 Blaschke products, *Nonlinearity*, **29** (2016), 3464–3495.

[7] R. L. Devaney, D. M. Look and D. Uminsky, The escape trichotomy for singularly perturbed rational maps, *Indiana Univ. Math. J.*, **54** (2005), 1621–1634.

[8] J. Y. Gao and G. Liu, On connectivity of Fatou components concerning a family of rational maps, *Abstr. Appl. Anal.*, Art. ID 621312, 7.

[9] A. Garijo, S. M. Marotta and E. D. Russell, Singular perturbations in the quadratic family with multiple poles, *J. Difference Equ. Appl.*, **19** (2013), 124–145.

[10] C. McMullen, Automorphisms of rational maps, in *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, vol. 10 of Math. Sci. Res. Inst. Publ., Springer, New York, 1988, 31–60.

[11] J. Milnor, *Dynamics in one complex variable*, vol. 160 of Annals of Mathematics Studies, 3rd edition, Princeton University Press, Princeton, NJ, 2006.

[12] M. Morabito and R. L. Devaney, Limiting Julia sets for singularly perturbed rational maps, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **18** (2008), 3175–3181.

[13] J. Y. Qiao and J. Y. Gao, The connectivity numbers of Fatou components of rational mappings, *Acta Math. Sinica (Chin. Ser.)*, **47** (2004), 625–628.

[14] W. Y. Qiu, P. Roesch, X. G. Wang and Y. C. Yin, Hyperbolic components of McMullen maps, *Ann. Sci. Éc. Norm. Supér. (4)*, **48** (2015), 703–737.

[15] W. Y. Qiu, X. G. Wang and Y. C. Yin, Dynamics of McMullen maps, *Adv. Math.*, **229** (2012), 2525–2577.

[16] N. Steinmetz, The formula of Riemann-Hurwitz and iteration of rational functions, *Complex Variables Theory Appl.*, **22** (1993), 203–206.

[17] M. Stiemer, Rational maps with Fatou components of arbitrary connectivity number, *Comput. Methods Funct. Theory*, **7** (2007), 415–427.

[18] D. Sullivan, Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains, *Ann. of Math. (2)*, **122** (1985), 401–418.