The curve shortening flow with density of a spherical curve in codimension two *

Francisco Viñado-Lereu †

Abstract

In the present paper we carry out a systematic study about the flow of a spherical curve by the mean curvature flow with density in a 3-dimensional rotationally symmetric space with density $(M^3_w,\, g_w,\, \xi)$ where the density ξ decomposes as sum of a radial part φ and an angular part ψ. We analyse how either the parabolicity or the hyperbolicity of $(M^3_w,\, g_w)$ condition the behaviour of the flow when the solution goes to infinity.

Keywords Mean curvature flow - Manifolds with density
Mathematics Subject Classification 53C44 - 35R01

1 Introduction

A n+1-dimensional *manifold with density* (M, g, ξ) is a Riemannian manifold $(M,\, g)$ and a function $\xi : M \to \mathbb{R}$. In this type of manifold we may calculate the *weighted volume* or *volume with density* of the k-dimensional immersed submanifolds $\iota : P^k \to M^{n+1}$ as:

$$V_\xi(P) := \int_P e^{\xi\circ\iota}\, dv_{g_P}, \tag{1}$$

where $g_P \equiv \iota^\star g$ is the induced metric over the manifold P by the immersion ι. We shall denote by $dv_{\xi,\,P}$, dv_ξ or $e^\xi\, dv_{g_P}$ the volume element associated to a density.

In this context we have a natural generalization of the mean curvature vector of a submanifold as the negative L^2-gradient of the k-dimensional functional of volume with density. We shall call to this vector field *mean curvature vector with density* and it shall be denoted by $\vec{H}_\xi$. It has the form:

$$\vec{H}_\xi := \vec{H} - \left(\nabla^M \xi\right)^\perp, \tag{2}$$

*This is a post-peer-review, pre-copyedit version of an article published in Journal of Evolution Equations. The final authenticated version is available online at: http://dx.doi.org/10.1007/s00028-020-00620-y

†ORCID ID: 0000-0002-5828-6756.

Research partially supported by Universitat Jaume I research project UJI-B2018-35 and by MINECO research project MTM2017-84851-C2-2-P. The author has been supported by a postdoctoral grant from *Plan de promoción de la investigación de la Universitat Jaume I del año 2018 Acción 3.2.* POSDOC-A/2018/32 - grupo 041.

where $\vec{H}$ is the mean curvature vector of the submanifold and $\left(\nabla^M \xi\right)^\perp$ is the orthogonal projection of the gradient $\nabla^M \xi$ of ξ in $(M,\, g_M)$ onto the normal bundle. In the particular case where $k = n = 1$, which is the case of curves on a surface, we shall change $\vec{H}$ by $\vec{k}$ the geodesic curvature vector and we shall denote by $\vec{k}_\xi$ the new vector field, we shall call this vector field the *geodesic curvature vector with density.*

This fact motivates us to study the following flow:

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,\, t) &= \left(\vec{H}_\xi\right)_{F(p,\, t)}, \\ F(p,\, 0) &= F_0(p), \end{cases} \tag{3}$$

where $F_0 : P^k \to M^{n+1}$ is a k-dimensional immersed submanifold. This is the analogue of the mean curvature flow in the context of the geometry with density. This flow is called *the mean curvature flow with density* (ξMCF for short). In the particular case where $k = 1$, the case of curves, this problem is also called *the curve shortening flow with density.* Some works performed in this context are [19, 3, 15, 16]; let us remark that these authors did not necessarily use this name for the flow. Other authors had indirectly explored this problem to study the mean curvature flow of submanifolds with some symmetries [13, 20]. All these works were done for hypersurfaces ($k = n$).

Given a n-dimensional immersed submanifold $\iota : P^n \to M^{n+1}$ with $n \geq 2$ we shall define the *mean curvature with density* as:

$$H_\xi := H - g_M(\nabla^M \xi,\, N),$$

where H is the mean curvature of the immersion, N is a unit normal field to the hypersurface and we use the following sign convention:

$$AX = -\nabla^M_X N,\; \alpha(X,\, Y) = g_M(\nabla^M_X Y,\, N)\, N = g_M(AX,\, Y)\, N,$$
$$h(X,\, Y) = g_M(\alpha(X,\, Y),\, N),\; H = trA = \sum_{i=1}^n g_M(A(e_i),\, e_i),$$

with A as the Weingarten map, α as the tensorial second fundamental form, h as the scalar second fundamental form, ∇^M as the Levi-Civita connection on M and $\{e_i\}_{i=1}^n$ a local orthonormal frame of the hypersurface. If $n = 1$ we define the *geodesic curvature with density* as:

$$k_\xi := k - g_M(\nabla^M \xi,\, N),$$

where k is the geodesic curvature of the immersed curve. Given an immersion such that $H_\xi = 0$ or $k_\xi = 0$, depending on the dimension, we shall call this immersion ξ-*minimal.*

In the present paper we work on the manifold with density $(M_w^3,\, g_w,\, \xi)$ that we describe below. Let $(M_w^3,\, g_w)$ be a 3-dimensional smooth rotationally symmetric space:

$$\begin{aligned} M_w^3 &\equiv [0,\, \infty) \times \mathbb{S}^2, \\ g_w &\equiv \pi^\star dr^2 + (w \circ \pi)^2 \sigma^\star g_{\mathbb{S}^2}, \end{aligned} \tag{4}$$

where $w : [0,\, \infty) \to \mathbb{R}$, $w = w(r)$, is a smooth map such that $w|_{(0,\, \infty)} > 0$ and $w(0) = 0$, $g_{\mathbb{S}^2}$ is the metric over the 2-sphere with Gauss curvature equal to one and $\pi : M_w^3 \to [0,\, \infty)$, $\sigma : M_w^3 \to \mathbb{S}^2$ are the natural projections. As the manifold is smooth, we may prove that the function w satisfies:

$$w'(0) = 1$$

(see pag. 179-183 in [22]). Also, $o \equiv \{0\} \times \mathbb{S}^2$ is a pole for the Riemannian manifold $(M_w^3,\, g_w)$. Regarding the density, let $\xi : M_w^3 - \{o\} \to \mathbb{R}$ be a smooth map such that:

$$\xi(x) = \varphi \circ \pi(x) + \psi \circ \sigma(x), \tag{5}$$

with $\varphi \in C^\infty((0,\, \infty))$ and $\psi \in C^\infty(\mathbb{S}^2)$. We note that ξ is not defined in the pole o.

In this manifold we are able to study the evolution of a closed smooth embedded spherical curve by the mean curvature flow with density. We shall denote by $\mathcal{A}$ the set of curves that is given by

$$\mathcal{A} := \bigcup_{r \in (0,\, \infty)} \mathcal{A}_r$$

where

$$\mathcal{A}_r := \left\{ \gamma : \mathbb{S}^1 \to M_w^3 \mid \gamma \text{ is a smooth embedded curve such that } \pi(\operatorname{Im} \gamma) = \{r\} \right\}$$

for all $r \in (0,\, \infty)$.

Actually, the aim of this paper is to make a systematic study of the following problem:

If we consider the Riemannian manifold with density $(M_w^3,\, g_w,\, \xi)$ given by (4) and (5) then, in this space, we may study the following initial value problem:

$$\left\{ \begin{array}{rcl} \dfrac{\partial}{\partial t}\gamma(p,\, t) &=& (\overrightarrow{H}_\xi)_{\gamma(p,\, t)}, \\ \gamma(\cdot,\, 0) &=& \gamma_0 \in \mathcal{A}, \end{array} \right. \tag{6}$$

where $(\overrightarrow{H}_\xi)_{\gamma(\cdot,\, t)}$ denotes the mean curvature vector with density of the curve $\gamma(\cdot,\, t)$.

Other previous works in the field about curve shortening flow in codimension greater than or equal to two are [14, 24]. In these two works the ambient manifold is compact and the authors assume some restrictions on the initial curve to guarantee that the maximal time of the solution is infinite. In [14] the initial curve is a ramp and in [24] the initial curve is a graphical curve. Also in [14] we can find some more general results.

Regarding the ξMCF, we note that most of the results in the literature about ξMCF concern radial densities in the Euclidean space [19, 3, 15], thus $(M^3_w,\, g_w,\, \xi)$ is a good manifold to keep increasing the understanding of this type of problems. On the other hand, the choice of $\mathcal{A}$ as set of initial condition for the problem (6) is motived because this family of curves is good enough to guarantee that if the solution has finite maximal time then the curve collapses to a point, moreover, the property of being embedded is preserved throughout the flow. In general, when we study the evolution of a curve in codimension greater than or equal to two, the properties above described are false. An interesting problem would be to look for other families of curves with these good properties in codimension greater than or equal to two.

It is equally important to remark that these types of flows (6) had been indirectly and partially studied in the 3-dimensional Euclidean space without density ($\xi = 0$). This situation was studied to understand the mean curvature flow of lagrangian spherical surfaces in $\mathbb{C}^2$ in [4]. This fact is motivated by the link between the mean curvature flow and the ξMCF, which is explicitly detailed in the article [16].

In our situation we may look the geodesic spheres

$$S_r := \{p \in M^3_w |\, \pi(p) = r\}$$

as Riemannian manifolds with density $(S_r,\, g_{S_r},\, \psi)$ with $g_{S_r} \equiv w^2(r)\, g_{\mathbb{S}^2}$ and $\psi \equiv \psi \circ \sigma$, with a slight overuse in notation about the density. Further, when considering $\gamma \in \mathcal{A}_r$ as a curve in S_r, we denote it by $\widetilde{\gamma}$.

With the selection of a spherical curve as initial condition, it becomes natural to consider the following problem:

Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\widetilde{\gamma} : \mathbb{S}^1 \times [0,\, \widetilde{T}) \to S_{r_0}$, $\widetilde{\gamma} = \widetilde{\gamma}(p,\, \widetilde{t})$, be a smooth function such that:

$$\begin{cases} \dfrac{\partial}{\partial \widetilde{t}} \widetilde{\gamma}(p,\, \widetilde{t}) & = & (\overrightarrow{k}_{S_{r_0},\, \psi})_{\widetilde{\gamma}(p,\, \widetilde{t})}, \\ \widetilde{\gamma}(\cdot,\, 0) & = & \widetilde{\gamma}_0, \end{cases} \tag{7}$$

where $\overrightarrow{k}_{S_{r_0},\, \psi}$ denotes the geodesic curvature vector with density of the curve in the Riemannian manifold $(S_{r_0},\, g_{S_{r_0}},\, \psi)$. This problem is included in the theory [1, 2, 17], as well as the behaviour when the curve collapses to a point is included in the work [25] and the behaviour when the solution exists for all time is in [15].

In the present work, we study in which way we could use the results known for (7) to understand the solution of (6). On the other hand, we also note that the problem (6) is a generalization of the problem (7).

In [15] it was shown that in the study of the ζMCF of a curve in the Euclidean plane $\mathbb{R}^2$ with a radial density $\zeta = \zeta(r)$, a very relevant fact is that the sign and the zeros of the function $r \mapsto \dfrac{1}{r} + \zeta'(r)$ determine the dynamics of the solution. Analogously, in our situation the dynamics of the solution are influenced by the function:

$$\begin{aligned} \vec{B} : M_w^3 - \{o\} &\longrightarrow TM_w^3 \\ p &\longmapsto \vec{B}(p) := \Big(- \frac{w'(\pi(p))}{w(\pi(p))} - \varphi'(\pi(p)) \Big) \partial_r|_p, \end{aligned} \tag{8}$$

whose scalar version is

$$\begin{aligned} B : (0, \infty) &\longrightarrow \mathbb{R} \\ r &\longmapsto B(r) := \frac{w'(r)}{w(r)} + \varphi'(r), \end{aligned} \tag{9}$$

that is $\vec{B}(p) = -B(\pi(p))\partial_r|_p$. This scalar version generalizes the function given in [15].

We note that the function (9) has the following interpretation: Let $(\mathbb{R}^2,\, g_w,\, \varphi)$ be the Riemannian manifold where $g_w := dr^2 + w^2(r)\, g_{\mathbb{S}^1}$ and $\varphi = \varphi \circ \pi$ is the density that appears in (5). Then, $B(r)$ is the geodesic curvature with density φ of the C_r circle centered at the origin whose radius is r, if we consider $-\partial_r$ as unit normal vector to C_r. Therefore, if $B(r) = 0$, this means that the circle C_r is φ-minimal. For this reason, given a geodesic sphere S_r such that $B(r) = 0$ we shall say that it is *B-minimal*.

Once proven the existence and uniqueness of the solution for the problem (6), we get that, if the maximal time of the solution of (6) is finite and the solution is bounded, then:

Theorem A. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M_w^3$ be the unique maximal solution for the initial value problem (6) with γ_0 as initial condition. If the solution is bounded and the maximal time T is finite, then, the curve collapses to a point $p \in M_w^3$ and*

i) if $p \neq o$ the curve collapses to a spherical round point in the geodesic sphere $S_{\pi(p)}$.

ii) if $p = o$ and there exists $\widetilde{\varphi} \in C^1([0,\, \infty))$ such that $\widetilde{\varphi}|_{(0,\, \infty)} = \varphi$ then, a blow-up centered at this point gives a limit flow by the ψMCF in $(S_{r_0},\, g_{S_{r_0}},\, \psi)$ that C^∞-subconverges, after a reparametrization of the curves, to a closed ψ-minimal curve.

The next important goal is to study the behaviour of the flow when the solution exists for all time and it is bounded:

Theorem B. *Let $\gamma_0 \in \mathcal{A}$ and let $\gamma : \mathbb{S}^1 \times [0, T) \to M^3_w$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition. If the solution exists for all t, that is, $T = \infty$, it is bounded and it does not go to the pole o, then the flow C^∞-subconverges, after a reparametrization of the curves $\gamma(\cdot,\, t)$, to a closed ψ-minimal spherical curve contained in the B-minimal geodesic sphere $S_{\lim_{t\to\infty} \pi(Im\, \gamma(\cdot,\, t))}$.*

However, if $\widetilde{\varphi} \in C^1([0,\, \infty))$ with $\widetilde{\varphi}|_{(0,\, \infty)} = \varphi$ exists and the flow reaches the pole o, then the maximal time is finite. But, generally, if the function φ does not have a C^1-extension to the pole o, it is possible that the flow collapses to the pole for $T = \infty$.

On the other hand, the B-minimal geodesic spheres are barriers for the flow. Thus, given a spherical curve between two B-minimal geodesic spheres then, the flow is contained between these two spheres. Taking this into account, it is not difficult to find situations in which the solution is bounded.

We note that in the previous works [14, 24] the ambient manifold is compact so the solution always is bounded. However, in our setting it is not compact and allows the solution to go off to infinity. A required condition here is that the function B is negative for all $r \in [r_0,\, \infty)$ with $r_0 = \pi(\gamma_0)$ where $\gamma_0 \in \mathcal{A}$ is the initial condition of (6).

In this last situation, the key for understanding the behaviour of the solution is given by a relation between the area of the geodesic spheres S_r and the function B. More accurately, the behaviour of the flow at infinity is given by the integral:

$$\int_{r_0}^{\infty} \frac{1}{B(r)\; Area(S_r)}\, dr \tag{10}$$

If this integral converges or diverges then, the behaviour of the flow is different.

Here, we need two new definitions. Let $(M,\, g_M)$ be a Riemannian manifold, we say that $(M,\, g_M)$ is *non-parabolic* or *hyperbolic* if it admits a non-constant positive superharmonic function. Otherwise, we say that $(M,\, g_M)$ is *parabolic*. The parabolicity and hyperbolicity of a smooth rotationally symmetric space are characterized by the divergence or convergence, respectively, of the integral (see Prop. 3.1 in [11]):

$$\int_{\rho}^{\infty} \frac{1}{Area(S_r)}\, dr$$

Therefore, we take up the last case, if:

$$\int_{r_0}^{\infty} \frac{1}{B(r)\; Area(S_r)}\, dr \sim \int_{r_0}^{\infty} \frac{1}{Area(S_r)}\, dr \tag{11}$$

the behaviour of the flow is given by the parabolicity or hyperbolicity of the manifold M^3_w. Following this trend, the result that we have obtained in this way is:

Theorem C. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M^3_w$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition. If the solution is not bounded:*

a) If M^3_w is parabolic and $\liminf_{r\to\infty} B(r)$ is finite then, the flow topologically subconverges to $\gamma_\infty : \mathbb{S}^1 \to [0,\, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty,\, \chi(p))$, where $\chi : \mathbb{S}^1 \to \mathbb{S}^2$ is a smooth embedded closed ψ-minimal curve in $(S_{r_0},\, g_{S_{r_0}},\, \psi)$.

b) If M^3_w is hyperbolic and $\limsup_{r\to\infty} B(r) \neq 0$ then, the flow either

- *topologically converges to $\gamma_\infty : \mathbb{S}^1 \to [0, \infty] \times \mathbb{S}^2, p \mapsto (\infty, \widetilde{\gamma}(p, \widetilde{T}))$,*
- *or topologically converges to a point $p_\infty \in \mathbb{S}_\infty \equiv \{\infty\} \times \mathbb{S}^2 \subset [0,\, \infty] \times \mathbb{S}^2$ in the infinite radius sphere,*

where $\widetilde{\gamma}$ is the solution of (7) with initial condition $\widetilde{\gamma}_0$ and $\widetilde{T}$ shall be defined in (19).

Let us remind that to achieve the situation of the last theorem, having an unbounded solution, we need that $B|_{[r^\star,\, \infty)} < 0$ for some $r^\star \in (0,\, \infty)$.

The case b) of the last theorem is satisfied in the Euclidean space $\mathbb{R}^3$ with a Gaussian density $\xi(x) = e^{-\mu^2 r^2/2}$. This case shall be fully studied in the section 6 expanding our understanding of the Gaussian mean curvature flow [3, 15].

In the paper [16] the link between MCF and the ξMCF was detailed. This relation gives us an equivalence between the flows that in our situation is the following: the ξMCF $\gamma : \mathbb{S}^1 \times [0,\, T) \to M^3_w$ of a curve $\gamma_0 \in \mathcal{A}$ is equivalent to the MCF $F : N \times [0,\, T) \to \widehat{M}$ of a submanifold $F_0 : N \to \widehat{M}$ in the (m+3)-dimensional smooth Riemannian manifold $(\widehat{M},\, \widehat{g})$ given by:

$$\begin{aligned} \widehat{M} &:= M^3_w \times Q = [0,\, \infty) \times \mathbb{S}^2 \times Q, \\ \widehat{g} &:= \widehat{\pi}^\star g_w + \Big(\frac{e^{\xi\circ\widehat{\pi}}}{Vol_{g_Q}(Q)}\Big)^{2/m} \widehat{\sigma}^\star\, g_Q, \end{aligned} \tag{12}$$

where $(Q,\, g_Q)$ is a m-dimensional smooth compact Riemannian manifold and $\widehat{\pi} : \widehat{M} \to M^3_w$, $\widehat{\sigma} : \widehat{M} \to Q$ are the natural projections. The initial submanifold F_0 is a (m+1)-dimensional smooth submanifold such that

$$F_0 : N = \mathbb{S}^1 \times Q \to \widehat{M},\ F_0(\alpha,\, q) := (\gamma_0(\alpha),\, q). \tag{13}$$

Therefore, this work leads us to expand our understanding about the MCF of submanifolds in codimension two. Some works in this area are [5, 23].

This paper is structured in the following way. In section 2, we introduce the basic results about the curve shortening flow with density on a surface. In section 3, we give the existence and uniqueness result for solutions to the problem (6) and we show the relation between the solutions of (6) and (7). In section 4, we analyse the situation where the solution is bounded. In section 5, we study the situation where the solution is not bounded. Finally, in section 6, we carry out a detailed study of the problem (6) with the Gaussian density in the 3-dimensional Euclidean space.

2 Preliminaries

2.1 Curve shortening flow with density

Let $(\overline{M}, \overline{g}, \psi)$ be a 2-dimensional smooth Riemannian manifold with density and let $\gamma_0 : \mathbb{S}^1 \to \overline{M}$ be a smooth curve. Then we shall call the *curve shortening flow with density* (ψMCF) of γ_0 the solution $\gamma : \mathbb{S}^1 \times [0, T) \to \overline{M}$ of the problem:

$$\left\{ \begin{array}{rcl} \dfrac{\partial}{\partial t}\gamma(p, t) & = & \left(\overrightarrow{k}_{\psi}\right)_{\gamma(p, t)}, \\ \gamma(\cdot, 0) & = & \gamma_0, \end{array} \right. \tag{14}$$

where $\vec{k}_\psi$ is the geodesic curvature vector with density of the curve.

In the particular case where the density is $\psi = 0$, then the problem is the classic curve shortening flow widely studied [8, 7, 10, 9, 6].

As already remarked in [15], we may use the theory of S. Angenent [1, 2] to guarantee the existence and uniqueness of solution for the problem (14). This theory, together with the later work of Oaks [17], allows us to formulate the following theorem:

Theorem 1. *[1, 2, 17] Let $\gamma_0 : \mathbb{S}^1 \to \overline{M}$ be a simple C^2 curve. Then, there exists a unique solution to (14) with initial condition γ_0. Moreover, the solution either collapses to a point on $\overline{M}$ in finite time or exists for infinite time.*

This theorem was already written for this flow in the article [15]. Moreover, we can use the work of Xi-Ping Zhu [25] to obtain that:

Theorem 2. *(Proof Cor. 4.2 in [25]) Let $\gamma_0 : \mathbb{S}^1 \to \overline{M}$ be a simple C^2 curve. Then, if the solution to (14) with initial condition γ_0 has finite maximal time, the solution collapses to a round point.*

The case in which the solution exists for all time was studied in [15] by V. Miquel and the author in a Riemannian manifold with density with the following properties in the region where the curve moves. Here, K is the

Gauss curvature of the surface $(\overline{M},\, \overline{g})$ and $\overline{\nabla}$ is the covariant derivative of $(\overline{M},\, \overline{g})$:

$$i) \quad \begin{array}{l} |\overline{\nabla}^j K| \leq C_j,\ |\overline{\nabla}^j \psi| \leq P_j,\ 0 < E \leq e^{\psi} \leq D, \\ \text{for some constants } C_j,\ P_j,\ E,\ D;\ j = 0,\ 1,\ 2, \cdots \end{array} \tag{15}$$

$ii)$ The isoperimetric profile $\mathcal{I}$ is a well defined continuous function which satisfies $\lim\limits_{a \to a_0} \mathcal{I}(a) = 0$ implies $a_0 = 0$. (16)

The authors obtained the following theorem:

Theorem 3. *[15] Let $(\overline{M},\, \overline{g},\, \psi)$ be an orientable 2-Riemannian manifold with density satisfying (16). Let $\gamma(\cdot,\, t)$ be a solution of the ψMCF (14) with initial condition an embedded curve $\gamma_0 : \mathbb{S}^1 \to \overline{M}^2$. If this solution exists for every $t \in [0,\, \infty)$, and $\gamma(\mathbb{S}^1,\, t)$ is contained in a fixed compact domain U where the conditions (15) are satisfied, then there is a reparametrization $\widetilde{\gamma}(\cdot,\, t)$ of $\gamma(\cdot,\, t)$ such that there is a sequence $\{\widetilde{\gamma}(\cdot,\, t_k)\}_{k\in\mathbb{N}}$, $t_k \to \infty$, which C^m-converges to a closed ψ-minimal curve of $\overline{M}^2$ for every $m \in \mathbb{N}$.*

In our particular case (7) the Riemannian manifold with density is $(S_{r_0}, g_{S_{r_0}}, \psi)$. This manifold is compact and the density is smooth thus, we have that the properties (15) and (16) are satisfied, so we may use Theorem 3 in our situation.

Other important properties that may be obtained from the work of S. Angenent [1, 2] are:

Theorem 4. (Preservation of the embedded property) *Let $\gamma_0 : \mathbb{S}^1 \to \overline{M}$ be a smooth embedded curve and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to \overline{M}$ be the solution of (14) with initial condition γ_0 then, $\gamma(\cdot,\, t) : \mathbb{S}^1 \to \overline{M}$ is embedded for all $t \in [0,\, T)$.*

Theorem 5. (Comparison principle) *Let $\gamma_1 : \mathbb{S}^1 \times [0,\, T_1) \to \overline{M}$, $\gamma_2 : \mathbb{S}^1 \times [0,\, T_2) \to \overline{M}$ be solutions of the initial value problem (14) such that $\gamma_1(\cdot,\, 0)$ and $\gamma_2(\cdot,\, 0)$ are immersed curves. If $Im\, \gamma_1(\cdot,\, 0) \cap Im\, \gamma_2(\cdot,\, 0) = \emptyset$ then $Im\, \gamma_1(\cdot,\, t) \cap Im\, \gamma_2(\cdot,\, t) = \emptyset$ for all $t \in [0,\, \min\{T_1,\, T_2\})$.*

3 The flow

In this section we prove the existence and uniqueness of the solution of the problem (6). Also, we show the relation between the solutions of the problems (6) and (7), as well as some properties of the problem (6).

Theorem 6. (Existence and uniqueness) *The initial value problem (6), with* $\gamma_0 \in \mathcal{A}_{r_0}$ *as initial condition, has a unique solution* $\gamma : \mathbb{S}^1 \times [0, T) \to M^3_w$ *given by*

$$\gamma(p,\, t) := exp\Big(\widetilde{\gamma}(p,\, \widetilde{t}(t)),\, (R(t) - r_0)\partial_r\Big), \tag{17}$$

where $\widetilde{\gamma}$ *is the unique solution for the initial value problem (7) with* $\widetilde{\gamma}_0$ *as initial condition,* $R(t)$ *is the unique solution for the ODE:*

$$\begin{cases} R'(t) &= -B(R(t)), \\ R(0) &= r_0, \end{cases} \tag{18}$$

and

$$\begin{aligned} \widetilde{t} : [0,\, T) &\longrightarrow [0,\, \widetilde{T}) \\ t &\longmapsto \widetilde{t}(t) := \int_0^t \Big(\frac{w(r_0)}{w(R(\beta))}\Big)^2 \, d\beta, \end{aligned} \tag{19}$$

with $\widetilde{T} \equiv \int_0^T \Big(\frac{w(r_0)}{w(R(\beta))}\Big)^2 \, d\beta$.

Prior to the proof, we need some preparatory lemmas.

Given $\gamma \in \mathcal{A}_r$ we shall denote by $\vec{k}_{S_r,\,\psi}$ the geodesic curvature vector with density of the curve $\widetilde{\gamma}$, considered as a curve in $(S_r,\, g_{S_r},\, \psi)$.

Lemma 1. *Given* $\gamma \in \mathcal{A}_r$*:*

$$\vec{H}_\xi = \vec{k}_{S_r,\,\psi} + \vec{B}(\gamma) \tag{20}$$

Proof. Let $\{\tau,\, \nu,\, \partial_r\}$ be an orthonormal frame over the curve with τ the unit tangent vector to the curve, ν the unit normal of the curve that is tangent to the geodesic sphere where the curve is contained, we note that $\{\tau,\, \nu\}$ is an orthonormal frame over the curve in the geodesic sphere $(S_r,\, g_{S_r})$. Now we may calculate the expression of the mean curvature vector with density:

$$\begin{aligned} \vec{H}_\xi &= \vec{H} - \big(\overline{\nabla}\xi\big)^\perp = \big\langle \overline{\nabla}_\tau \tau,\, \nu \big\rangle\, \nu + \big\langle \overline{\nabla}_\tau \tau,\, \partial_r \big\rangle\, \partial_r - \big\langle \overline{\nabla}\xi,\, \nu \big\rangle\, \nu - \big\langle \overline{\nabla}\xi,\, \partial_r \big\rangle\, \partial_r \\ &= \Big(\big\langle \overline{\nabla}_\tau \tau,\, \nu \big\rangle - \big\langle \overline{\nabla}\xi,\, \nu \big\rangle \Big)\nu + \Big(\big\langle \overline{\nabla}_\tau \tau,\, \partial_r \big\rangle - \big\langle \overline{\nabla}\xi,\, \partial_r \big\rangle \Big)\partial_r \\ &= \Big(\big\langle \nabla^{S_r}_\tau \tau + \alpha^{S_r}(\tau,\, \tau),\, \nu \big\rangle - \big\langle \overline{\nabla}(\psi \circ \sigma),\, \nu \big\rangle \Big)\nu \\ &\qquad + \Big(- \frac{\langle \tau,\, \tau \rangle}{w(r)} \big\langle \overline{\nabla} w,\, \partial_r \big\rangle - \big\langle \overline{\nabla}(\varphi \circ \pi),\, \partial_r \big\rangle \Big)\partial_r \\ &= \Big(\big\langle \nabla^{S_r}_\tau \tau,\, \nu \big\rangle - \big\langle \overline{\nabla}(\psi \circ \sigma),\, \nu \big\rangle \Big)\nu + \Big(- \frac{w'(r)}{w(r)} - \varphi'(r) \Big)\partial_r \\ &= \vec{k}_{S_r,\,\psi} + \vec{B}(\gamma), \end{aligned}$$

where $\overline{\nabla}$ is the covariant derivative of $(M^3_w,\, g_w)$. □

Lemma 2. *The function (19) is a diffeomorphism.*

Proof. This function

- is smooth for being composed of smooth functions,
- $\widetilde{t}(0) = 0$, $\widetilde{t}(T) = \widetilde{T}$,
- its derivative is strictly positive for all t:

$$\widetilde{t}'(t) = \Big(\frac{w(r_0)}{w(R(t))}\Big)^2 > 0 \text{ for all } t \in [0,\ T).$$

So by the inverse function theorem, it is a local diffeomorphism which, together the injectivity of the function, implies that it is a diffeomorphism. □

Lemma 3. *Let $\gamma \in \mathcal{A}_r$ and let $\hat{\gamma} := exp(\gamma,\ -(r - r_0)\partial_r)$ with $r_0 \in (0,\ \infty)$, then:*

$$\vec{k}_{\widetilde{\gamma},\, S_r,\, \psi} = \frac{w^2(r_0)}{w^2(r)}\, \vec{k}_{\hat{\gamma},\, S_{r_0},\, \psi}$$

as a vector field on $\mathbb{S}^2$.

Proof. We note that $g_{S_r} = \dfrac{w^2(r)}{w^2(r_0)} g_{S_{r_0}}$, that is, g_{S_r} and $g_{S_{r_0}}$ are conformally equivalent, then:

$$\begin{aligned}\vec{k}_{\widetilde{\gamma},\, S_r,\, \psi} &= \vec{k}_{\widetilde{\gamma},\, S_r} - \nabla^{S_r}(\psi \circ \sigma)^{\perp} = \frac{1}{\frac{w^2(r)}{w^2(r_0)}} \vec{k}_{\hat{\gamma},\, S_{r_0}} - \Bigg(\frac{1}{\frac{w^2(r)}{w^2(r_0)}} \nabla^{S_{r_0}}(\psi \circ \sigma)\Bigg)^{\perp} \\ &= \frac{w^2(r_0)}{w^2(r)} \Big(\vec{k}_{\hat{\gamma},\, S_{r_0}} - \nabla^{S_{r_0}}(\psi \circ \sigma)^{\perp}\Big) = \frac{w^2(r_0)}{w^2(r)} \vec{k}_{\hat{\gamma},\, S_{r_0},\, \psi}.\end{aligned}$$

□

Now, we are ready to give the proof of Theorem 6.

Proof. **(Theorem 6)** Problem (7) has a unique short-time solution by Theorem 1. Further, since B is locally Lipschitz, (18) also has a unique short-time solution. Then, since (19) is a diffeomorphism by Lemma 2, the expression for γ given in (17) is well-defined on $\mathbb{S}^1 \times [0,\ T)$. Let us remark that γ is smooth being composed of smooth functions.

We will now show that (17) is a solution to (6). We will denote by $\eta_{\widetilde{\gamma}(p,\, \widetilde{t}_1),\, \partial_r}$ the geodesic that starts at $\widetilde{\gamma}(p,\ \widetilde{t}_1)$ whose tangent vector at this point is ∂_r. Then:

$$\begin{aligned}\gamma(p,\ 0) &= exp\Big(\widetilde{\gamma}(p,\ 0),\ (R(0) - r_0)\partial_r\Big) = exp\Big(\widetilde{\gamma}(p,\ 0),\ 0\Big) \\ &= \widetilde{\gamma}(p,\ 0) = \widetilde{\gamma}_0(p) = \gamma_0(p),\ \forall\, p \in \mathbb{S}^1,\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t_1}\gamma(p,\,t) &= \frac{\partial}{\partial t}\Big|_{t_1} exp\Big(\widetilde{\gamma}(p,\,\widetilde{t}),\,(R(t)-r_0)\partial_r\Big)\\
&= \frac{\partial}{\partial t}\Big|_{t_1} exp\Big(\widetilde{\gamma}(p,\,\widetilde{t}_1),\,(R(t)-r_0)\partial_r\Big) + \frac{\partial}{\partial t}\Big|_{t_1} exp\Big(\widetilde{\gamma}(p,\,\widetilde{t}),\,(R(t_1)-r_0)\partial_r\Big)\\
&= \frac{\partial}{\partial t}\Big|_{t_1}\eta_{\widetilde{\gamma}(p,\,\widetilde{t}_1),\,\partial_r}\Big(R(t)-r_0\Big) + exp_\star\Big(\frac{\partial}{\partial t}\Big|_{t_1}\widetilde{\gamma}(p,\,\widetilde{t}),\,(R(t_1)-r_0)\partial_r\Big)\\
&= \eta'_{\widetilde{\gamma}(p,\,\widetilde{t}_1),\,\partial_r}\Big(R(t_1)-r_0\Big)R'(t_1) + exp_\star\Big(\frac{\partial\widetilde{t}}{\partial t}\frac{\partial}{\partial\widetilde{t}}\Big|_{\widetilde{t}_1}\widetilde{\gamma}(p,\,\widetilde{t}),\,(R(t_1)-r_0)\partial_r\Big)\\
&= R'(t_1)\partial_r|_{\gamma(p,\,t_1)} + exp_\star\Big(\Big(\frac{w(r_0)}{w(R(t_1))}\Big)^2\big(\overrightarrow{k}_{S_{r_0},\,\psi}\big)_{\widetilde{\gamma}(p,\,\widetilde{t}_1)},\,(R(t_1)-r_0)\partial_r\Big)\\
&= -B(R(t_1))\partial_r|_{\gamma(p,\,t_1)} + \Big(\frac{w(r_0)}{w(R(t_1))}\Big)^2 exp_\star\Big(\big(\overrightarrow{k}_{S_{r_0},\,\psi}\big)_{\widetilde{\gamma}(p,\,\widetilde{t}_1)},\,(R(t_1)-r_0)\partial_r\Big)\\
&= \vec{B}(\gamma(p,\,t_1)) + (\vec{k}_{S_{R(t_1)},\,\psi})_{\gamma(p,\,t_1)} = (\vec{H}_\xi)_{\gamma(p,\,t_1)},
\end{aligned}$$

where we have used Lemma 3 and also Lemma 1, in the last equality, given that from (17) we have that $\gamma(\cdot,\,t)\in\mathcal{A}_{R(t)}$ for all $t\in[0,\,T)$. Therefore (17) is a solution for problem (6).

To guarantee that (17) is the unique solution of the problem (6) we may use the existence and uniqueness theorem for the MCF in higher codimension. The statement of this theorem was presented as a special case of a theorem by R. Hamilton [12] in the survey about this topic, Part II Chap. 3 in [21], by K. Smoczyk (see Prop. 3.2 page 248 in [21]). Let us remind the link between the ξMCF and the MCF showed on (12) and (13). □

Corollary 1. *$Im\,\gamma(\cdot,\,t)\subset S_{R(t)}$, $\forall\, t\in[0,\,T)$.*

Proof. By Theorem 6 the solution is

$$\gamma(p,\,t) = exp\Big(\widetilde{\gamma}(p,\,\widetilde{t}(t)),\,(R(t)-r_0)\partial_r\Big),$$

this expression implies that $Im\,\gamma(\cdot,\,t)\subset S_{R(t)}$, $\forall\, t\in[0,\,T)$. □

Corollary 2. *Let $\gamma:\mathbb{S}^1\times[0,\,T)\to M^3_w$ be the maximal solution of the problem (6) with initial condition $\gamma_0\in\mathcal{A}_{r_0}$, then*

$$\widetilde{\gamma}(p,\,\widetilde{t}) := exp\Big(\gamma(p,\,t(\widetilde{t})),\,-(R(t(\widetilde{t}))-r_0)\partial_r\Big),$$

is the unique solution defined in $\mathbb{S}^1\times[0,\,\widetilde{T})$ for the problem (7) with initial condition $\widetilde{\gamma}_0$. The function $t:[0,\,\widetilde{T})\to[0,\,T)$ is the inverse function of (19).

Proof. From Theorem 6 and Lemma 2. □

Remark. *It is highly relevant that T could be the maximal time of the problem (6) and $\widetilde{T}$ could not be the maximal time of the problem (7).*

Theorem 7. (Preservation of the embedded property) *Let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M^3_w$ be a solution of the problem (6) such that γ_0 is an embedded curve then, $\gamma(\cdot,\, t)$ is an embedded curve for all $t \in [0,\, T)$.*

Proof. From Theorem 6 and Theorem 4. □

Theorem 8. (Comparison Principle) *Let $\gamma_1 : \mathbb{S}^1 \times [0,\, T_1) \to M^3_w$, $\gamma_2 : \mathbb{S}^1 \times [0,\, T_2) \to M^3_w$ be solutions of the initial value problem (6) with initial conditions $\gamma_1(\cdot,\, 0), \gamma_2(\cdot,\, 0) \in \mathcal{A}$, respectively. If $Im\, \gamma_1(\cdot,\, 0) \cap Im\, \gamma_2(\cdot,\, 0) = \emptyset$ then $Im\, \gamma_1(\cdot,\, t) \cap Im\, \gamma_2(\cdot,\, t) = \emptyset$ for all $t \in [0,\, \min\{T_1,\, T_2\})$.*

Proof. From Theorem 6 and Theorem 5. □

4 Bounded solution

4.1 Finite maximal time

In this section we analyse the different situations when the maximal time of the solution is finite, obtaining a proof for Theorem A.

Theorem 9. *Let $\gamma_0 \in \mathcal{A}$ and let $\gamma : \mathbb{S}^1 \times [0, T) \to M^3_w$ be the unique maximal solution for the initial value problem (6) with γ_0 as initial condition. If the maximal time T is finite and the solution is bounded, then the solution collapses to a point.*

Proof. We note that the continuous function $R(t)$ is either strictly decreasing, strictly increasing or constant. Thus, the $\lim_{t\to T} R(t)$ exists and it is finite. Now, we have two possibilities:

- $\lim_{t\to T} R(t) = R(T) > 0$: In this situation $\widetilde{T} < \infty$ and (17) only can have problems coming from $\widetilde{\gamma}(\cdot,\, \widetilde{t}(t))$. Then, the maximal time for (7) is finite, moreover it is exactly $\widetilde{T}$, and by Theorem 1 the flow $\widetilde{\gamma}$ collapses to a point. Therefore, the flow γ collapses to a point.

- $\lim_{t\to T} R(t) = R(T) = 0$: By Corollary 1 $Im\gamma(\cdot, t) \subset S_{R(t)}, \forall t \in [0, T)$ then, the flow collapses to the pole o.

□

Interestingly, we notice that in the second part of the proof, $\widetilde{T}$ may not be the maximal time of the flow $\widetilde{\gamma}$.

Now that we know that the solution collapses to a point, let's analyse the shape of the singularity.

Theorem 10. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M_w^3$ be the unique maximal solution for the initial value problem (6) with γ_0 as initial condition. If the maximal time T is finite and the solution is bounded, then:*

i) If $\lim_{t\to T} R(t) \neq 0$ the curve collapses to a spherical round point in the geodesic sphere $S_{R(T)}$.

ii) If $\lim_{t\to T} R(t) = 0$, that is, the curve collapses to the pole o of the manifold M_w^3, and there exists $\widetilde{\varphi} \in C^1([0,\, \infty))$ such that $\widetilde{\varphi}|_{(0,\,\infty)} = \varphi$ then, a blow-up centered at the pole o gives a limit flow by the ψMCF in $(S_{r_0},\, g_{S_{r_0}},\, \psi)$ that C^∞-subconverges, after a reparametrization of the curves, to a closed ψ-minimal curve.

Proof. By Theorem 9 the solution collapses to a point.

Case i) If $\lim_{t\to T} R(t) \neq 0$ then, the maximal time of the solution of the problem (7), with initial condition $\widetilde{\gamma}_0$, is finite due to the relation between the flows. This, together with Theorem 2, proves i). We note that, in this situation, the maximal time of (7) is $\widetilde{T}$.

Case ii) We need to study if $\widetilde{T}$ is finite or infinite. For this, we note that:

$$\widetilde{T} = \int_0^T \Big(\frac{w(r_0)}{w(R(t))}\Big)^2\, dt \sim \int_0^T \frac{1}{R(t)^2}\, dt \tag{21}$$

we shall check this fact:

$$\lim_{x\to 0^+} \frac{\dfrac{1}{w(x)^2}}{\dfrac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{w(x)^2} = \lim_{x\to 0^+} \frac{2x}{2w(x)w'(x)} = \lim_{x\to 0^+} \frac{x}{w(x)}$$
$$= \lim_{x\to 0^+} \frac{1}{w'(x)} = 1 \in (0,\, \infty),$$

where we have used $w(0) = 0$, $w'(0) = 1$ and L'Hôpital's rule twice.

Now, we are going to study the second integral of (21):

$$\int_0^T \frac{1}{R(t)^2}\, dt = \int_0^T \frac{R'(t)}{R'(t)R(t)^2}\, dt = \int_0^T \frac{R'(t)}{\Big(-\dfrac{w'(R(t))}{w(R(t))} - \varphi'(R(t))\Big)R(t)^2}\, dt$$
$$= \int_{R(0)}^{R(T)} \frac{1}{\Big(-\dfrac{w'(x)}{w(x)} - \varphi'(x)\Big)x^2}\, dx = \int_{r_0}^{0} \frac{1}{\Big(-\dfrac{w'(x)}{w(x)} - \varphi'(x)\Big)x^2}\, dx$$
$$= \int_0^{r_0} \frac{1}{\Big(\dfrac{w'(x)}{w(x)} + \varphi'(x)\Big)x^2}\, dx,$$

we remark that $R'(t) < 0$ for all $t \in [0, T)$, as otherwise the hypothesis $\lim_{t\to T} R(t) = 0$ is impossible. Going back to the integral:

$$\lim_{x\to 0^+} \frac{\dfrac{1}{\left(\dfrac{w'(x)}{w(x)} + \varphi'(x)\right)x^2}}{\dfrac{1}{x}} = \lim_{x\to 0^+} \frac{1}{\left(\dfrac{w'(x)}{w(x)} + \varphi'(x)\right)x} = \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{\dfrac{1}{w(x)}}$$

$$= \lim_{x\to 0^+} \frac{w(x)}{x} = \lim_{x\to 0^+} w'(x) = 1 \in (0, \infty),$$

we have used L'Hôpital's rule and also that φ has a C^1-extension to $[0, \infty)$, so

$$\int_0^{r_0} \frac{1}{\left(\dfrac{w'(x)}{w(x)} + \varphi'(x)\right)x^2}\, dx \sim \int_0^{r_0} \frac{1}{x}\, dx. \tag{22}$$

Therefore, from (21) and (22), $\widetilde{T} = \infty$ and if we perform a rescaling at the singularity we obtain exactly the flow $\widetilde{\gamma}$ as the rescaled flow of γ, so the rescaled flow subconverges, in the sense of Theorem 3, to a closed ψ-minimal curve in $(S_{r_0}, g_{S_{r_0}}, \psi)$ by this theorem. Note that the rescaled flow is given by

$$\exp\big(\gamma(\cdot, t), (r_0 - R(t))\partial_r\big),$$

and this flow is the flow $\widetilde{\gamma}$ by Corollary 2.

□

Remark. *The case ii) of the previous theorem is not true if the density φ does not have a C^0-extension to $[0, \infty)$. For example, if we consider $\varphi(r) := -\ln w(r) - \dfrac{1}{r}$ then:*

$$\widetilde{T} \sim \int_0^{r_0} \frac{1}{\left(\dfrac{w'(x)}{w(x)} + \varphi'(x)\right)x^2}\, dx = \int_0^{r_0} \frac{1}{\dfrac{1}{x^2}x^2}\, dx = r_0 < \infty$$

Therefore, if the solution collapses to the origin, the rescaled flow of γ faces two possibilities: either it converges to the curve $\widetilde{\gamma}(\cdot, \widetilde{T})$, if $\widetilde{T}$ is not the maximal time of $\widetilde{\gamma}$, or, if $\widetilde{T}$ is the maximal time of $\widetilde{\gamma}$, we need to make a new rescaling in the sphere and with this second rescaling we obtain that the curve converges to a round point by Theorem 2.

4.2 Infinite maximal time

In this section we study the situation in which the maximal time of the solution for the problem (6) is infinite, obtaining a proof for Theorem B. In this situation, there is a delicate scenario: when the solution collapses to the pole. This fact motivates splitting the study in two cases: when the solution collapses to the pole and when the solution is in a bounded region $0 < C_1 \leq R(t) \leq C_2$.

First, we are going to tackle the following question: Could the solution collapse to the pole in infinite time?

Proposition 1. *Let $\gamma_0 \in \mathcal{A}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M^3_w$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition. If this solution collapses to the pole at the maximal time and there exists $\widetilde{\varphi} \in C^1([0,\, \infty))$ such that $\widetilde{\varphi}|_{(0,\, \infty)} = \varphi$ then, the maximal time of the flow is finite.*

Proof. As $\lim_{t\to T} R(t) = 0$ then $\lim_{t\to T} w(R(t)) = 0$, $\lim_{t\to T} w'(R(t)) = 1$ and $\lim_{t\to T} \varphi'(R(t)) = \varphi'(0)$, the last equality relies on the hypothesis about φ. So $\lim_{t\to T} B(R(t)) = \infty$ and therefore there are $t^\star \in [0,\, T)$ and $C > 0$ such that

$$R'(t) \leq -C,\ \forall\, t \in [t^\star,\, T).$$

If we integrate this inequality, we obtain:

$$R(t_2) - R(t_1) \leq -C(t_2 - t_1),\ \forall\, t_2 \geq t_1 \geq t^\star,$$

so

$$C(t_2 - t_1) \leq R(t_2) + C(t_2 - t_1) \leq R(t_1) < \infty,\ \forall\, t_2 \geq t_1 \geq t^\star,$$

if we take $t_1 = t^\star$ and $t_2 \to T$:

$$C(T - t^\star) \leq R(t^\star) < \infty,$$

therefore $T < \infty$. □

Having this in mind, the answer to the question is: No if the density φ has a C^1-extension to $[0,\, \infty)$. However, if the density φ does not have a C^1-extension to $[0,\, \infty)$, the solution could collapse to the pole in infinite time. For example, let $(\mathbb{R}^3,\, g_{\mathbb{R}^3},\, \xi)$ be the ambient manifold with $\varphi(r) = -\ln(r) + \dfrac{1}{2}r^2$. In this situation, the derivative of $R(t)$ satisfies:

$$R'(t) = -\frac{1}{R(t)} - \varphi'(R(t)) = -\frac{1}{R(t)} + \frac{1}{R(t)} - R(t) = -R(t)$$
$$\Rightarrow R(t) = r_0 e^{-t}.$$

Then, if the flow collapses to the pole, the maximal time of the solution is infinite.

Now, we continue studying the other case. We need some preparatory lemmas. Let s be the arc length parameter of $\gamma(\cdot,\,t)$ and let $\widetilde{s}$ be the arc length parameter of $\widetilde{\gamma}(\cdot,\,\widetilde{t})$.

Lemma 4.

- $\partial_s = \dfrac{w(r_0)}{w(R(t))}\partial_{\widetilde{s}}$,
- $\partial_s^n\Big(k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\Big) = \Big(\dfrac{w(r_0)}{w(R(t))}\Big)^{n+1}\partial_{\widetilde{s}}^n\Big(k_{\widetilde{\gamma}(\cdot,\,\widetilde{t}),\,S_{r_0},\,\psi}\Big)$.

Proof. The relation between ds and $d\widetilde{s}$ is

$$ds = |\partial_\alpha\gamma(\alpha,\,t)|_{g_{S_{R(t)}}}d\alpha = \frac{w(R(t))}{w(r_0)}|\partial_\alpha\gamma(\alpha,\,t)|_{g_{S_{r_0}}}d\alpha = \frac{w(R(t))}{w(r_0)}d\widetilde{s}.$$

Therefore

$$\partial_s = \frac{w(r_0)}{w(R(t))}\partial_{\widetilde{s}}.$$

On the other hand, the relation between the mean curvature vectors with density is given from Lemma 3 by

$$\vec{k}_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi} = \frac{w^2(r_0)}{w^2(R(t))}\vec{k}_{\widetilde{\gamma}(\cdot,\,\widetilde{t}),\,S_{r_0},\,\psi}$$

and $g_{S_{R(t)}} = \dfrac{w(R(t))^2}{w(r_0)^2}g_{S_{r_0}}$, then

$$k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi} = \frac{w(r_0)}{w(R(t))}k_{\widetilde{\gamma}(\cdot,\,\widetilde{t}),\,S_{r_0},\,\psi}.$$

So we obtain that

$$\partial_s^n\Big(k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\Big) = \Big(\frac{w(r_0)}{w(R(t))}\Big)^{n+1}\partial_{\widetilde{s}}^n\Big(k_{\widetilde{\gamma}(\cdot,\,\widetilde{t}),\,S_{r_0},\,\psi}\Big).$$

□

Also, we need an expression that links $\partial_s^n\big(k_{\gamma(\cdot,\,t),\,S_{R(t)}}\big)$ with $\partial_s^n\big(k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\big)$. We may borrow the expression given in (51) of [15]:

$$\begin{aligned}\partial_s^n\big(k_{\gamma(\cdot,\,t),\,S_{R(t)}}\big) &= \partial_s^n\big(k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\big) + \nabla^{n+1}\psi(\partial_s,\,\cdots,\,\partial_s,\,\nu)\\ &\quad+\sum_{n,\,1}^{1,\,n-1} c_{i,\,J,\,K}\,k^i_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\,\partial_s^J\big(k_{\gamma(\cdot,\,t),\,S_{R(t)},\,\psi}\big)\,C(\nabla^K\psi),\end{aligned} \tag{23}$$

we denote by ∇ the covariant derivative in $S_{R(t)}$, which is independent of t. $J = (j_1, \cdots, j_q), 0 < j_1 \le j_2 \le \cdots \le j_q$ is an ordered multi-index and we denote by $|J| := j_1 + \cdots + j_q$, $d(J) := q$, $o(J) := j_q$, $\partial_s^J x := \partial_s^{j_1} x \cdots \partial_s^{j_q} x$, $\nabla^J x := \nabla^{j_1} x \otimes \cdots \otimes \nabla^{j_q} x$. About the summation notation, we only need to know that given $\sum_{m,\, r}^{s,\, t}$ then $i + |J| + d(J) + |K| = m + r$ and we consider that if $|J| = 0$ then $\partial_s^J \big(k_{\gamma(\cdot,\, t),\, S_{R(t)},\, \psi}\big)$ do not appear. Also, $0 \le d(J) \le [(m + r - s)/2]$, $|K| \ge s$, $o(J) \le t$, $1 \le d(K) \le m + 1$. Finally, by $C(\nabla^K \psi)$ we denote $\nabla^K \psi$ acting on $|K|$ copies of ∂_s or/and ν.

We shall denote by $\overline{\nabla}$ the covariant derivative of the Riemannian manifold $(M_w^3,\, g_w)$.

Lemma 5. *Given $\gamma \in \mathcal{A}_r$*

$$\left\{ \begin{array}{lcll} \overline{\nabla}_\tau \tau & = & k\nu & -\dfrac{w'}{w}\partial_r \\ \overline{\nabla}_\tau \nu & = & -k\tau & \\ \overline{\nabla}_\tau \partial_r & = & \dfrac{w'}{w}\tau & \end{array} \right. \tag{24}$$

where $\{\tau,\, \nu,\, \partial_r\}$ is an orthonormal frame over the curve with τ, the unit tangent vector to γ, ν, unit normal to γ and tangent to the geodesic sphere S_r where the curve is contained, and with k, the geodesic curvature of the curve γ as curve of the sphere $(S_r,\, g_r)$.

Proof. See Chap. 7, Prop. 35 in [18]:

$$\begin{aligned} \overline{\nabla}_\tau \tau &= \big\langle \overline{\nabla}_\tau \tau,\, \nu \big\rangle \nu + \big\langle \overline{\nabla}_\tau \tau,\, \partial_r \big\rangle \partial_r = \big\langle \nabla_\tau^{S_r} \tau,\, \nu \big\rangle \nu + \left\langle -\frac{\langle \tau,\, \tau \rangle}{w} \overline{\nabla} w,\, \partial_r \right\rangle \partial_r \\ &= k\nu - \frac{w'}{w}\partial_r, \\ \overline{\nabla}_\tau \nu &= \big\langle \overline{\nabla}_\tau \nu,\, \tau \big\rangle \tau + \big\langle \overline{\nabla}_\tau \nu,\, \partial_r \big\rangle \partial_r = -\big\langle \nu,\, \overline{\nabla}_\tau \tau \big\rangle \tau + \left\langle -\frac{\langle \tau,\, \nu \rangle}{w} \overline{\nabla} w,\, \partial_r \right\rangle \partial_r \\ &= -k\tau, \\ \overline{\nabla}_\tau \partial_r &= \frac{\partial_r(w)}{w}\tau = \frac{w'}{w}\tau. \end{aligned}$$

□

Theorem 11. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M_w^3$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition. If the solution exists for all t, that is, $T = \infty$, and $0 < C_1 \le R(t) \le C_2$ with C_1, C_2 some constants for all $t \in [0, \infty)$, then the flow C^∞-subconverges, after a reparametrization of the curves $\gamma(\cdot,\, t)$, to a closed ψ-minimal spherical curve contained in the B-minimal geodesic sphere $S_{\lim_{t\to\infty} R(t)}$.*

Proof. If $0 < C_1 \leq R(t) \leq C_2$ for all $t \in [0,\, \infty)$ and the maximal time of the solution of (6) is infinite then, $\widetilde{T} = \infty$:
The $\lim_{t\to\infty} R(t) = R_\infty \in \mathbb{R} \Rightarrow \lim_{t\to\infty} w(R(t)) = w(R_\infty) \in \mathbb{R}$, so $\forall\, \epsilon > 0\, \exists\, t_\epsilon > 0$ such that $|w(R(t)) - w(R_\infty)| < \epsilon$ for all $t \geq t_\epsilon$. Therefore

$$\begin{aligned}\widetilde{T} &= \int_0^\infty \Big(\frac{w(r_0)}{w(R(t))}\Big)^2 dt \\ &\geq \int_0^{t_\epsilon} \Big(\frac{w(r_0)}{w(R(t))}\Big)^2 dt + \int_{t_\epsilon}^\infty \Big(\frac{w(r_0)}{w(R_\infty)+\epsilon}\Big)^2 dt = \infty \Rightarrow \widetilde{T} = \infty.\end{aligned}$$

As $\widetilde{t} : [0, \infty) \longrightarrow [0, \infty)$ is a diffeomorphism by Lemma 2, we obtain that the behaviour of the flows (6) and (7) is the same.

From Step 4 on page 23 of [15], Lemma 4 and the hypothesis about $R(t)$, we obtain that

$$\partial_s^n \Big(k_{\gamma(\cdot,\, t),\, S_{R(t)},\, \psi}\Big) \text{ converges uniformly to zero when } t \to \infty \\ \text{for every } n \in \mathbb{N}. \tag{25}$$

On the other hand,

$$\begin{aligned}\partial_s \gamma &= \tau, \\ \partial_s^2 \gamma &= \overline{\nabla}_\tau \tau = k\nu - \frac{w'}{w}\partial_r, \\ \partial_s^3 \gamma &= \overline{\nabla}_\tau \Big(k\nu - \frac{w'}{w}\partial_r\Big) = \partial_s k\, \nu + k\overline{\nabla}_\tau \nu - \partial_s\Big(\frac{w'}{w}\Big)\partial_r - \frac{w'}{w}\overline{\nabla}_\tau \partial_r \\ &= \partial_s k\, \nu - k^2\tau - (\frac{w'}{w})^2\tau = \Big(-k^2 - (\frac{w'}{w})^2\Big)\tau + \partial_s k\, \nu, \\ \partial_s^4 \gamma &= \overline{\nabla}_\tau \Big[\Big(-k^2 - (\frac{w'}{w})^2\Big)\tau + \partial_s k\, \nu\Big] \\ &= -2k\partial_s k\, \tau + \Big(-k^2 - (\frac{w'}{w})^2\Big)\overline{\nabla}_\tau \tau + \partial_s^2 k\, \nu + \partial_s k\, \overline{\nabla}_\tau \nu \\ &= -2k\partial_s k\, \tau + \Big(-k^2 - (\frac{w'}{w})^2\Big)\Big(k\nu - \frac{w'}{w}\partial_r\Big) + \partial_s^2 k\, \nu - k\partial_s k\, \tau \\ &= -3k\partial_s k\, \tau + \Big(-k^3 - k(\frac{w'}{w})^2 + \partial_s^2 k\Big)\nu + \Big(k^2\frac{w'}{w} + (\frac{w'}{w})^3\Big)\partial_r,\end{aligned}$$

so that we can obtain an expression for $\partial_s^n \gamma$ of the form:

$$\begin{aligned}\partial_s^n \gamma = f_n(\frac{w'}{w},\, k,\, \partial_s k, \cdots,\, \partial_s^{n-3} k)\tau + g_n(\frac{w'}{w},\, k,\, \partial_s k, \cdots,\, \partial_s^{n-2} k)\nu \\ + h_n(\frac{w'}{w},\, k,\, \partial_s k, \cdots,\, \partial_s^{n-4} k)\partial_r,\, n \geq 1,\end{aligned} \tag{26}$$

where f_n, g_n and h_n are polynomials in $\frac{w'}{w}$, k, $\partial_s k, \cdots,\, \partial_s^j k$ with $j \leq n-2$ where all monomials have degree $n-1$, which is obtained counting $\partial_s^i k$ as

$i+1$. Here, the map γ had the form $\gamma = \gamma(s_t,\, t)$ where s_t is the arc-length parameter of $\gamma(\cdot,\, t)$. Now, let L_t be the length without density of $\gamma(\cdot,\, t)$, we consider the change of parameter:

$$\begin{array}{ccc} [0,\, L_t] & \longrightarrow & [0,\, 1] \\ s_t & \longmapsto & \alpha := \dfrac{s_t}{L_t} \end{array}$$

and we denote by $\hat{\gamma}$ the reparametrization of the curves with the parameter α, that is $\hat{\gamma}(\alpha,\, t) = \gamma(s_t(\alpha),\, t)$. We notice that

$$\frac{\partial s_t}{\partial \alpha} = L_t \quad \text{and} \quad \frac{\partial^n s_t}{\partial \alpha} = 0,\, \forall\, n \geq 2.$$

Then we have that:

$$\partial^n_\alpha\, \hat{\gamma} = L^n_t\, \partial^n_{s_t} \gamma,\, \forall\, n = 1,\, 2,\, \cdots$$

and therefore:

$$|\partial^n_\alpha\, \hat{\gamma}| = |L^n_t\, \partial^n_{s_t} \gamma| = L^n_t |\partial^n_{s_t} \gamma| = L^n_t \sqrt{f_n^2 + g_n^2 + h_n^2},\, \forall\, n = 1,\, 2,\, \cdots \qquad (27)$$

We may also obtain the following bound for the length of the curves:

$$e^{\xi(\gamma_t)} = e^{\varphi\circ\pi(\gamma_t) + \psi\circ\sigma(\gamma_t)} = e^{\varphi\circ\pi(\gamma_t)} e^{\psi\circ\sigma(\gamma_t)} \geq \min_{r\in[C_1,\, C_2]} e^{\varphi(r)} \min_{p\in\mathbb{S}^2} e^{\psi(p)},$$

we denote by D the last expression, then:

$$\begin{aligned} L_t &= \int_{\mathbb{S}^1} ds_t = \int_{\mathbb{S}^1} \frac{e^{\xi(\gamma_t)}}{e^{\xi(\gamma_t)}} ds_t \leq \frac{1}{D} \int_{\mathbb{S}^1} e^{\xi(\gamma_t)} ds_t \\ &= \frac{1}{D} L_\xi(\gamma_t) \leq \frac{1}{D} L_\xi(\gamma_0), \end{aligned} \qquad (28)$$

as $L_\xi(\gamma_t)$ decreases throughout the flow, so L_t is bounded independently of t.

On the other hand, the hypothesis $C_1 \leq R(t) \leq C_2$ for all $t \in [0,\, \infty)$ implies that $\dfrac{w'}{w}$ and w are bounded. This fact, together with (25) and (23), imply that

$$\sqrt{f_n^2 + g_n^2 + h_n^2} \text{ is bounded independently of t.} \qquad (29)$$

Therefore, from (27), (28) and (29):

$$|\partial^n_\alpha\, \hat{\gamma}(\alpha,\, t)| \leq C_n,\, \forall\, (\alpha,\, t) \in [0,\, 1] \times [0,\, \infty),\, \forall\, n = 1,\, 2,\, \cdots \qquad (30)$$

with C_n independent of $(\alpha,\, t)$. The case $n = 0$ is immediate from the hypothesis about $R(t)$. We might use the Arzelà-Ascoli theorem to conclude that there is a family $\{\hat{\gamma}(\cdot,\, t_m)\}_{m\in\mathbb{N}}$, $t_m \to \infty$, such that C^∞-converges to a limit curve $\hat{\gamma}_\infty$ which is closed and regular. To obtain this result, we use a diagonal type argument.

The limit curve is regular because of Lemma 8 of [15]. This implies that, in our situation, $L_t \geq c$ for all $t \in [0, \infty)$ for some constant c, so $|\partial_\alpha \hat{\gamma}| = L_t \geq c$ and the limit curve $\hat{\gamma}_\infty$ is regular.

We note that the geodesic sphere whose radius is $R_\infty = \lim_{t\to\infty} R(t)$ is B-minimal and from (25) $k_{\gamma(\cdot,\, t),\, S_{R(t)},\, \psi}$ converges uniformly to zero when $t \to \infty$, then $\hat{\gamma}_\infty$ is a ψ-minimal curve contained in the B-minimal geodesic sphere S_{R_∞}. □

We note that Proposition 1 together with Theorem 11 give us a proof for Theorem B.

5 Unbounded solution

In this section we analyze the situation in which the solution of (6) is not bounded, obtaining a proof for Theorem C. Further, the maximal time of the solution of (6) can be finite or infinite.

Theorem 12. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to M^3_w$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition. If the solution is not bounded, $\lim_{t\to T} R(t) = \infty$, then necessarily $B(r) < 0$ for all $r \in [r_0,\, \infty)$. Further:*

a) If M^3_w is parabolic and $\liminf_{r\to\infty} B(r)$ is finite then, the flow topologically subconverges to $\gamma_\infty : \mathbb{S}^1 \to [0,\, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty,\, \chi(p))$ where $\chi : \mathbb{S}^1 \to S_{r_0}$ is a smooth embedded closed ψ-minimal curve in $(S_{r_0},\, g_{S_{r_0}},\, \psi)$.

b) If M^3_w is hyperbolic and $\limsup_{r\to\infty} B(r) \neq 0$ then, the flow either

- *topologically converges to $\gamma_\infty : \mathbb{S}^1 \to [0, \infty]\times\mathbb{S}^2, p \mapsto (\infty, \widetilde{\gamma}(p, \widetilde{T}))$, which is a curve contained in $\mathbb{S}_\infty \equiv \{\infty\} \times \mathbb{S}^2 \subset [0,\, \infty] \times \mathbb{S}^2$ the infinite radius sphere,*
- *or topologically converges to a point p_∞ in $\mathbb{S}_\infty \equiv \{\infty\} \times \mathbb{S}^2 \subset [0,\, \infty] \times \mathbb{S}^2$ the infinite radius sphere.*

Proof. As $\lim_{t\to T} R(t) = \infty$, $R'(t) > 0$ for all $t \in [0, T)$. Otherwise, there is $t^\star \in [0,\, T)$ such that $S_{R(t^\star)}$ is a B-minimal sphere and the solution $\gamma(\cdot,\, t)$ is contained in $\mathbb{B}_{R(t^\star)}$ for all $t \in [0,\, T)$, therefore $\lim_{t\to T} R(t) \leq R(t^\star)$. As a consequence, we obtain that $B(R) < 0$ for all $R \in [r_0, \infty)$, due to $R'(t) = -B(R(t))$.

The hypothesis of the case a) about the function B implies that there exists a constant $C > 0$ such that $-C \leq B(r) < 0$ for all $r \in [r_0, \infty)$. Besides, the hypothesis of the case b) about the function B implies that there exists a constant $C>0$ such that $B(r)\le -C<0$ for all $r\in[r_0,\infty)$. As a summary:

$$\begin{array}{lll} \text{In the case a)} & -\dfrac{1}{B(r)} \geq \dfrac{1}{C}, & \text{for all } r \in [r_0,\ \infty). \\ \text{In the case b)} & \dfrac{1}{C} \geq -\dfrac{1}{B(r)}, & \text{for all } r \in [r_0,\ \infty). \end{array}$$

On the other hand, we could obtain the following expression for $\widetilde{T}$:

$$\begin{aligned} \widetilde{T} &= \int_0^T \Big(\frac{w(r_0)}{w(R(t))}\Big)^2\, dt = \int_0^T \frac{Area(S_{r_0})}{Area(S_{R(t)})}\, dt \\ &= Area(S_{r_0}) \int_0^T \frac{R'(t)}{R'(t)\ Area(S_{R(t)})}\, dt \\ &= -Area(S_{r_0}) \int_0^T \frac{R'(t)}{B(R(t))\ Area(S_{R(t)})}\, dt \\ &= -Area(S_{r_0}) \int_{r_0}^\infty \frac{1}{B(r)\ Area(S_r)}\, dr, \end{aligned} \tag{31}$$

where we used that $R:[0,\,T)\to[r_0,\,\infty)$ defines a diffeomorphism on its image.

Case a) M^3_w is parabolic and $-\dfrac{1}{B(r)} \geq \dfrac{1}{C}$ for all $r\in[r_0,\ \infty)$. Then

$$\begin{aligned} \widetilde{T} &= -Area(S_{r_0}) \int_{r_0}^\infty \frac{1}{B(r)\ Area(S_r)}\, dr \\ &\geq \frac{Area(S_{r_0})}{C} \int_{r_0}^\infty \frac{1}{Area(S_r)}\, dr = \infty, \end{aligned}$$

where the last equality is true because the Riemannian manifold M^3_w is parabolic (see Prop. 3.1 in [11]).
As we have that $\widetilde{t}:[0,\,T)\longrightarrow[0,\infty)$ is a diffeomorphism, we conclude that the behaviour of the flow γ in $t=T$ is the behaviour of the flow $\widetilde{\gamma}$ in infinite time. The behaviour of the flow $\widetilde{\gamma}$, with infinite maximal time, is given by Theorem 3.

Case b) M^3_w is hyperbolic and $\dfrac{1}{C} \geq -\dfrac{1}{B(r)}$ for all $r\in[r_0,\ \infty)$. Then

$$\begin{aligned} \widetilde{T} &= -Area(S_{r_0}) \int_{r_0}^\infty \frac{1}{B(r)\ Area(S_r)}\, dr \\ &\leq \frac{Area(S_{r_0})}{C} \int_{r_0}^\infty \frac{1}{Area(S_r)}\, dr < \infty, \end{aligned}$$

where we note that the last equality is true because the Riemannian manifold M_w^3 is hyperbolic (see Prop. 3.1 in [11]).
As we have that $\widetilde{t} : [0,\, T) \longrightarrow [0, \widetilde{T})$ is a diffeomorphism with $\widetilde{T} < \infty$, we get that the behaviour of the flow γ in $t = T$ is the behaviour of the flow $\widetilde{\gamma}$ in finite time. This flow in $\widetilde{t} = \widetilde{T}$ has two options: either the flow is defined, $\widetilde{\gamma}(\cdot, \widetilde{T})$ is a smooth curve, or the flow collapses to a point by Theorem 1. We remark that in the first situation $\widetilde{T}$ is not the maximal time of the flow (7).

□

We note that Theorem 12 is essentially Theorem C.

Remark. *The case a) is not true if we eliminate the condition on the function B. We may find situations where M_w^3 is parabolic, $\liminf_{r\to\infty} B(r) = -\infty$ and $\widetilde{T} < \infty$. For example, let $(M_w^3,\, g_w,\, \xi)$ be a smooth rotationally symmetric space such that $w|_{[C,\,\infty)}(r) = \sqrt{r}$ with $C > 0$ and $\varphi(r) = -\frac{1}{2}\ln(r) - \frac{r^2}{2}$. Then, M_w^3 is parabolic,*

$$\int_C^\infty \frac{dr}{Area(S_r)} = \int_C^\infty \frac{dr}{4\pi w(r)^2} = \int_C^\infty \frac{dr}{4\pi r} = \infty,$$

and

$$B(r) = \frac{w'}{w} + \varphi'(r) = \frac{1}{2r} - \frac{1}{2r} - r = -r < 0, \textit{ for all } r \in (C,\, \infty),$$

so

$$\liminf_{r\to\infty} B(r) = -\infty.$$

Given $\gamma_0 \in \mathcal{A}_{r_0}$, such that $r_0 > C$, as initial condition then, the system for R is as follows:

$$\begin{cases} R'(t) &= R(t), \\ R(0) &= r_0, \end{cases}$$

thus

$$R(t) = r_0 e^t. \tag{32}$$

On the other hand, if we assume that $\psi \equiv 0$, we note that the problem (7) is the curve shortening flow with $\widetilde{\gamma}_0$ as initial condition. Then, it is known that we can calculate the maximal time of the flow $\widetilde{\gamma}$ from the variation formula of enclosed area by the curve $\widetilde{\gamma}(\cdot,\, \widetilde{t})$. We shall denote by Ω_0 the region enclosed by the curve $\widetilde{\gamma}_0$ in the sphere S_{r_0} such that $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} \leq 1/2$. We *also take the inward-pointing normal $\widetilde{\nu}$ to $\partial\Omega_0$. We shall denote by $\Omega_{\widetilde{t}}$ the region enclosed by the curve $\widetilde{\gamma}(\cdot,\, \widetilde{t})$ in the sphere S_{r_0} where $\widetilde{\nu}|_{\widetilde{\gamma}(\cdot,\, \widetilde{t})}$ points inwards to the set $\Omega_{\widetilde{t}}$. This variation formula is given by*

$$\frac{\partial}{\partial \widetilde{t}} Area(\Omega_{\widetilde{t}}) = -\int_{\mathbb{S}^1} k_{\widetilde{\gamma}(\cdot,\, \widetilde{t}),\, S_{r_0}}\, d\widetilde{s} = -2\pi + \int_{\Omega_{\widetilde{t}}} K_{S_{r_0}}\, da_{S_{r_0}}$$
$$= -2\pi + \frac{1}{w(r_0)^2} Area(\Omega_{\widetilde{t}}),$$

where the first equality is well-known and we have used the Gauss-Bonnet theorem in the second equality. We obtain the following expression for the enclosed area:

$$Area(\Omega_{\widetilde{t}}) = 2\pi w(r_0)^2 - (2\pi w(r_0)^2 - Area(\Omega_0))e^{\widetilde{t}/w(r_0)^2}. \tag{33}$$

Then, if we assume that the area enclosed by the curve $\widetilde{\gamma}_0$ is $Area(S_{r_0})/2$, we obtain from (33) that $Area(\Omega_{\widetilde{t}}) = Area(S_{r_0})/2$ for all $\widetilde{t} \in [0,\, \widetilde{T}_{max})$. So, the flow $\widetilde{\gamma}$ do not collapses to a point. Therefore, by Theorem 1, Theorem 6 and (32) we conclude that $\widetilde{T}_{max} = \infty$, $T = \infty$ and $\gamma(\cdot,\, t)$ is not bounded. Bearing all this in mind, the value of $\widetilde{T}$ is

$$\widetilde{T} = \int_0^\infty \Big(\frac{w(r_0)}{w(R(t))}\Big)^2\, dt = \int_0^\infty \frac{r_0}{r_0 e^t}\, dt < \infty.$$

Therefore, the flow topologically converges to $\gamma_\infty : \mathbb{S}^1 \to [0,\, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty,\, \widetilde{\gamma}(p,\, \widetilde{T}))$.

Remark. *In the case b) the situation is analogous: it is not true if we eliminate the condition on the function B. We may find situations where M^3_w is hyperbolic, $\limsup_{r\to\infty} B(r) = 0$ and $\widetilde{T} = \infty$. For example, let $(\mathbb{R}^3,\, g_{\mathbb{R}^3},\, \xi)$ be the 3-dimensional Euclidean space with a density ξ such that $\varphi(r) = -2\ln(r)$, we note that $w(r) = r$. Then, $\mathbb{R}^3$ is a hyperbolic manifold and*

$$B(r) = \frac{1}{r} + \varphi'(r) = -\frac{1}{r} < 0, \textit{ for all } r \in (0,\, \infty),$$

so

$$\limsup_{r\to\infty} B(r) = 0.$$

Given $\gamma_0 \in \mathcal{A}_{r_0}$ as initial condition, we have that $R(t)$ satisfies the system

$$\begin{cases} R'(t) &= \dfrac{1}{R(t)}, \\ R(0) &= r_0, \end{cases}$$

so

$$R(t) = \sqrt{r_0^2 + 2t}. \tag{34}$$

On the other hand, as in the last remark, if we assume that $\psi \equiv 0$ and that the area enclosed by the curve $\widetilde{\gamma}_0$ is $Area(S_{r_0})/2$, then we can use (33), Theorem 1, Theorem 6 and (34) to conclude that $\widetilde{T}_{max} = \infty$, $T = \infty$ and the solution $\gamma(\cdot,\, t)$ is not bounded.
Now, we can calculate the value of $\widetilde{T}$:

$$\widetilde{T} = \int_0^\infty \Big(\frac{r_0}{R(t)}\Big)^2 \, dt = \int_0^\infty \frac{r_0^2}{r_0^2 + 2t} \, dt = \frac{r_0^2}{2} \ln(r_0^2 + 2t)\big|_0^\infty = \infty.$$

Then, the flow topologically subconverges to $\gamma_\infty : \mathbb{S}^1 \to [0,\, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty,\, \chi(p))$ with $\chi : \mathbb{S}^1 \to S_{r_0}$ a smooth embedded ψ-minimal curve in $(S_{r_0},\, g_{S_{r_0}},\, \psi)$.

In the following result, we provide an equivalence to the hypothesis about the function B in the cases a) and b) of the last theorem.

Let $(\mathbb{R}^2,\, g_w,\, \varphi)$ be the Riemannian manifold with density where $g_w := dr^2 + w^2(r) g_{\mathbb{S}^1}$ and $\varphi = \varphi(r)$ then, $B(r)$ is the geodesic curvature with density of the C_r circle centered at the origin whose radius is r respect to the normal field $-\partial_r$.

Proposition 2. *Define*

$$\begin{aligned} L : (0, \infty) &\longrightarrow \mathbb{R} \\ r &\longmapsto L(r) := \ln\big(Length_\varphi(C_r)\big), \end{aligned}$$

where $Length_\varphi(C_r)$ is the length with density of the circle C_r with respect to the manifold $(\mathbb{R}^2,\, g_w,\, \varphi)$. If we are in the situation of the last theorem, then:

a) The function $L|_{[r_0,\, \infty)}$ is Lipschitz if and only if $\liminf_{r\to\infty} B(r)$ is finite.

b) The function $\Big(L|_{[r_0,\, \infty)}\Big)^{-1}$ is Lipschitz if and only if $\limsup_{r\to\infty} B(r) \neq 0$.

Proof. We note that $L|_{[r_0,\, \infty)}$ is a C^1-function.

Case a) If $L|_{[r_0,\, \infty)}$ is Lipschitz then its derivative is bounded in $[r_0,\, \infty)$, that is, there exists $C > 0$ such that $|L|'_{[r_0,\, \infty)}(r)| \le C$ for all $r \in [r_0,\, \infty)$. As $L|'_{[r_0,\, \infty)}(r) = B(r)$ we obtain that $\liminf_{r\to\infty} B(r)$ is finite.

Conversely, if we assume that $\liminf_{r\to\infty} B(r)$ is finite as $B(r) < 0$ for all $r \in [r_0,\, \infty)$ then there exists $C > 0$ such that $|B(r)| < C$ for all $r \in [r_0, \infty)$ and $L|'_{[r_0,\, \infty)}(r) = B(r)$ so $|L|'_{[r_0,\, \infty)}(r)| < C$ for all $r \in [r_0,\, \infty)$. Therefore, $L|_{[r_0,\, \infty)}$ is Lipschitz.

Case b) We have that

$$L|'_{[r_0,\,\infty)}(r) = B(r) < 0, \text{ for all } r \in [r_0,\,\infty),$$

so $L|_{[r_0,\,\infty)}$ has an inverse C^1-function by the inverse function theorem, moreover

$$\left(L|^{-1}_{[r_0,\,\infty)}\right)'(s) = \frac{1}{L|'_{[r_0,\,\infty)}(r(s))} = \frac{1}{B(r(s))} \text{ for all } s \in (\liminf_{r\to\infty} L(r),\ L(r_0)]$$

where the link between r and s is $s = L(r)$. Also we note that $L|_{[r_0,\,\infty)}$ is a diffeomorphism. From this last equality we obtain the result, the proof is analogous to the case a).

□

6 Gaussian density

In this section we aim to study a particular case where b) of Theorem 12 applies. We shall assume that $\psi \equiv 0$, in order to use the variation formula for the area enclosed by the curve, which ultimately provides us a feasible way to explicitly perform the calculus.

Let $(\mathbb{R}^3,\ g_{\mathbb{R}^3},\ \xi)$ be the 3-dimensional Euclidean space with a density ξ such that $\psi(r) \equiv 0$ and $\varphi(r) = -\frac{1}{2}\mu^2 r^2$, that is, the radial part of the density is the Gaussian density. In this situation, given $\gamma_0 \in \mathcal{A}_{r_0}$ as initial condition of the problem (6), the system of ODE for $R(t)$ is:

$$\begin{cases} R'(t) &= -\dfrac{1}{R(t)} + \mu^2 R(t), \\ R(0) &= r_0. \end{cases} \tag{35}$$

Thus, R is:

$$R(t) = \frac{1}{\mu}\sqrt{1 + (\mu^2 r_0^2 - 1)e^{2\mu^2 t}}. \tag{36}$$

Proposition 3. *The link between the time parameters is given by*

$$\begin{aligned} &\widetilde{t} : [0,\ T) \longrightarrow [0,\ \widetilde{T}) \\ &\qquad t \longmapsto \widetilde{t}(t) = \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2 e^{2\mu^2 t}}{1 + (\mu^2 r_0^2 - 1)e^{2\mu^2 t}}\Big), \\ &t : [0,\ \widetilde{T}) \longrightarrow [0,\ T) \\ &\qquad \widetilde{t} \longmapsto t(\widetilde{t}) = \frac{1}{2\mu^2} \ln\Big(\frac{e^{2\widetilde{t}/r_0^2}}{\mu^2 r_0^2 - (\mu^2 r_0^2 - 1)e^{2\widetilde{t}/r_0^2}}\Big). \end{aligned}$$

Proof. From (19) the relation between times is given by

$$\begin{aligned}
\widetilde{t}(t_1) &= \int_0^{t_1} \Big(\frac{w(r_0)}{w(R(t))}\Big)^2\, dt = \int_0^{t_1} \Big(\frac{r_0}{R(t)}\Big)^2\, dt \\
&= r_0^2 \int_0^{t_1} \frac{\mu^2}{1+(\mu^2 r_0^2-1)e^{2\mu^2 t}}\, dt = r_0^2 \int_1^{e^{2\mu^2 t_1}} \frac{\mu^2}{1+(\mu^2 r_0^2-1)x}\, \frac{1}{2\mu^2}\frac{dx}{x} \\
&= \frac{r_0^2}{2}\int_1^{e^{2\mu^2 t_1}} \Big(\frac{1}{x} - \frac{(\mu^2 r_0^2-1)}{1+(\mu^2 r_0^2-1)x}\Big)\, dx = \frac{r_0^2}{2}\Big(\ln(x) - \ln\big(1+(\mu^2 r_0^2-1)x\big)\Big|_1^{e^{2\mu^2 t_1}} \\
&= \frac{r_0^2}{2} \ln\Big(\frac{x}{1+(\mu^2 r_0^2-1)x}\Big)\Big|_1^{e^{2\mu^2 t_1}} \\
&= \frac{r_0^2}{2}\Big(\ln\Big(\frac{e^{2\mu^2 t_1}}{1+(\mu^2 r_0^2-1)e^{2\mu^2 t_1}}\Big) - \ln\Big(\frac{1}{1+\mu^2 r_0^2 - 1}\Big)\Big) \\
&= \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2 e^{2\mu^2 t_1}}{1+(\mu^2 r_0^2-1)e^{2\mu^2 t_1}}\Big),
\end{aligned}$$

where we have used the change of parameter $t = \dfrac{1}{2\mu^2}\ln x$, $dt = \dfrac{1}{2\mu^2}\dfrac{dx}{x}$. Now, we move on calculating the inverse function:

$$\begin{aligned}
\widetilde{t} = \frac{r_0^2}{2}\ln\Big(\frac{\mu^2 r_0^2 e^{2\mu^2 t}}{1+(\mu^2 r_0^2-1)e^{2\mu^2 t}}\Big) \quad &\Leftrightarrow \quad e^{2\widetilde{t}/r_0^2} = \frac{\mu^2 r_0^2 e^{2\mu^2 t}}{1+(\mu^2 r_0^2-1)e^{2\mu^2 t}} \quad \Leftrightarrow \\
(1+(\mu^2 r_0^2-1)e^{2\mu^2 t})e^{2\widetilde{t}/r_0^2} = \mu^2 r_0^2 e^{2\mu^2 t} \quad &\Leftrightarrow \quad e^{2\widetilde{t}/r_0^2} = \Big(\mu^2 r_0^2 - (\mu^2 r_0^2 - 1)e^{2\widetilde{t}/r_0^2}\Big) e^{2\mu^2 t} \quad \Leftrightarrow \\
\frac{e^{2\widetilde{t}/r_0^2}}{\mu^2 r_0^2 - (\mu^2 r_0^2-1)e^{2\widetilde{t}/r_0^2}} = e^{2\mu^2 t} \quad &\Leftrightarrow \quad t = \frac{1}{2\mu^2}\ln\Big(\frac{e^{2\widetilde{t}/r_0^2}}{\mu^2 r_0^2 - (\mu^2 r_0^2-1)e^{2\widetilde{t}/r_0^2}}\Big).
\end{aligned}$$

□

Given $\gamma_0 \in \mathcal{A}_{r_0}$ we shall denote by Ω_0 to the region enclosed by the curve $\widetilde{\gamma}_0$ in the sphere S_{r_0} such that $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} \leq 1/2$. We also take the inward-pointing normal $\widetilde{\nu}$ to $\partial\Omega_0$.

Theorem 13. *Let $\gamma_0 \in \mathcal{A}_{r_0}$ and let $\gamma : \mathbb{S}^1 \times [0,\, T) \to \mathbb{R}^3$ be the unique maximal solution of the initial value problem (6) with γ_0 as initial condition, then:*

i) If $r_0 > \dfrac{1}{\mu}$:

- *If $\dfrac{1}{2}\Big(1 - \Big(\dfrac{\mu^2 r_0^2 - 1}{\mu^2 r_0^2}\Big)^{1/2}\Big) < \dfrac{Area(\Omega_0)}{Area(S_{r_0})} \leq 1/2$ the flow topologically converges to $\gamma_\infty : \mathbb{S}^1 \to [0,\, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty,\, \widetilde{\gamma}(p,\, \widetilde{T}))$.*

- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} = \dfrac{1}{2}\Big(1-\Big(\dfrac{\mu^2 r_0^2 - 1}{\mu^2 r_0^2}\Big)^{1/2}\Big)$ *the flow topologically converges to a point* $p_\infty \in \mathbb{S}_\infty \equiv \{\infty\} \times \mathbb{S}^2 \subset [0,\,\infty] \times \mathbb{S}^2$ *in the infinite radius sphere.*
- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} < \dfrac{1}{2}\Big(1-\Big(\dfrac{\mu^2 r_0^2 - 1}{\mu^2 r_0^2}\Big)^{1/2}\Big)$ *the flow collapses to a spherical round point in the Euclidean space* $\mathbb{R}^3$.

ii) If $r_0 = \dfrac{1}{\mu}$:

- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} = 1/2$ *the flow* C^∞*-subconverges, after a reparametrization of the curves* $\gamma(\cdot,\,t)$*, to a closed geodesic in* $(S_{r_0},\,g_{r_0})$.
- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} < 1/2$ *the flow collapses to a round point in* (S_{r_0}, g_{r_0}).

iii) If $r_0 < \dfrac{1}{\mu}$:

- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} = 1/2$ *the flow collapses to the coordinate origin in the Euclidean space* $\mathbb{R}^3$*. A blow-up centered at the origin coordinate gives a limit flow by the curve shortening problem in* $(S_{r_0},\,g_{S_{r_0}})$ *that* C^∞*-subconverges, after a reparametrization of the curves, to a closed geodesic.*
- *If* $\dfrac{Area(\Omega_0)}{Area(S_{r_0})} < 1/2$ *the flow collapses to a spherical round point in* $\mathbb{R}^3 - \{0\}$.

Proof.

Case i) We note that the problem (7) is the curve shortening problem. It is known that we can calculate the maximal time of the flow $\widetilde{\gamma}$ from the variation formula of enclosed area by the curve $\widetilde{\gamma}(\cdot,\,\widetilde{t})$. This formula is given by

$$\begin{aligned}\frac{\partial}{\partial \widetilde{t}} Area(\Omega_{\widetilde{t}}) &= -\int_{\mathbb{S}^1} k_{\widetilde{\gamma}(\cdot,\,\widetilde{t}),\,S_{r_0}}\, d\widetilde{s} = -2\pi + \int_{\Omega_{\widetilde{t}}} K_{S_{r_0}}\, da_{S_{r_0}} \\ &= -2\pi + \frac{1}{r_0^2} Area(\Omega_{\widetilde{t}}),\end{aligned}$$

where the first equality is well-known and we have used the Gauss-Bonnet theorem in the second equality, so we obtain the following expression for the enclosed area:

$$Area(\Omega_{\widetilde{t}}) = 2\pi r_0^2 - (2\pi r_0^2 - Area(\Omega_0))e^{\widetilde{t}/r_0^2}. \tag{37}$$

So, if $Area(\Omega_0) = 2\pi r_0^2$ the maximal time of the solution $\widetilde{\gamma}$ is infinite. Let us remind that if the maximal time is finite, the curve collapses to a point and, if $Area(\Omega_0) < 2\pi r_0^2$, the maximal time is finite and it is given by

$$\widetilde{T}_{\max} = r_0^2 \ln\Big(\frac{2\pi r_0^2}{2\pi r_0^2 - Area(\Omega_0)}\Big). \tag{38}$$

From the hypothesis $r_0 > 1/\mu$ and Proposition 3 we obtain that the function $\widetilde{t}(t)$ is defined on $[0, \infty)$ and that $\lim_{t\to\infty} \widetilde{t}(t) = \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)$. Then

- If $\widetilde{T}_{\max} < \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)$, the flow γ collapses to a point in the Euclidean space $\mathbb{R}^3$, because there is $t^\star \in [0, \infty)$ such that $\widetilde{t}(t^\star) = \widetilde{T}_{\max}$. Then, by the relation between the flows, $t^\star = T_{\max} < \infty$.
- If $\widetilde{T}_{\max} = \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)$, the flow γ topologically converges to a point p_∞ in $\mathbb{S}_\infty$ the infinite radius sphere.
- If $\widetilde{T}_{\max} > \frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)$, the flow topologically converges to $\gamma_\infty : \mathbb{S}^1 \to [0, \infty] \times \mathbb{S}^2$, $p \mapsto (\infty, \widetilde{\gamma}(p, \widetilde{T}))$, a curve contained in $\mathbb{S}_\infty$ the infinite radius sphere.

We can translate these inequalities in the following sense:

$$\begin{array}{lll}
\lim_{t\to\infty} \widetilde{t}(t) & < \ (=)\ (>)\ \widetilde{T}_{\max} & \Leftrightarrow \\
\frac{r_0^2}{2} \ln\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big) & < \ (=)\ (>)\ r_0^2 \ln\Big(\frac{2\pi r_0^2}{2\pi r_0^2 - Area(\Omega_0)}\Big) & \Leftrightarrow \\
\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)^{1/2} & < \ (=)\ (>)\ \frac{2\pi r_0^2}{2\pi r_0^2 - Area(\Omega_0)} & \Leftrightarrow \\
\Big(\frac{\mu^2 r_0^2}{\mu^2 r_0^2 - 1}\Big)^{1/2} & < \ (=)\ (>)\ \dfrac{\frac{1}{2}}{\frac{1}{2} - \frac{Area(\Omega_0)}{Area(S_{r_0})}} & \Leftrightarrow \\
\dfrac{Area(\Omega_0)}{Area(S_{r_0})} & > \ (=)\ (<)\ \frac{1}{2}\Big(1 - \Big(\frac{\mu^2 r_0^2 - 1}{\mu^2 r_0^2}\Big)^{1/2}\Big). &
\end{array}$$

At this point, if we write the previous classification in these terms, we obtain the statement of the theorem.

Case ii) In this situation, $R(t)$ is constant for all t then, this case is the classic curve shortening problem [7, 8, 10, 9].

Case iii) We note that if $r_0 < \dfrac{1}{\mu}$ then, T is the maximal time of the solution and it is less than or equal to $\dfrac{1}{2\mu^2}\ln\Big(\dfrac{1}{1-\mu^2 r_0^2}\Big)$. This time is the first time such that $R(\dfrac{1}{2\mu^2}\ln\Big(\dfrac{1}{1-\mu^2 r_0^2}\Big)) = 0$, so the maximal time is finite. Then from Theorem 9, the curve collapses to a point; moreover, from Theorem 10 we know the nature of the singularities. We also notice that in this situation:

$$\text{the flow collapses to the pole o} \Leftrightarrow T = \frac{1}{2\mu^2}\ln\Big(\frac{1}{1-\mu^2 r_0^2}\Big) \Leftrightarrow \widetilde{T} = \infty,$$

and by the variation formula we obtain that

$$\widetilde{T} = \infty \Leftrightarrow \frac{Area(\Omega_0)}{Area(S_{r_0})} = 1/2$$

□

Acknowledgements

The author thanks Dr. Vicent Gimeno and Dr. Vicente Palmer for their support and useful comments throughout our fruitful seminars that encouraged the writing of the present manuscript.

The author also thanks the referee for the detailed comments and valuable suggestions that greatly helped to improve the quality of this manuscript.

References

[1] Angenent, S. Parabolic equations for curves on surfaces. i. curves with p-integrable curvature. *Annals of Mathematics (2) 132*, 3 (1990), 451–483.

[2] Angenent, S. Parabolic equations for curves on surfaces. ii. intersections, blow-up and generalized solutions. *Annals of Mathematics (2) 133*, 1 (January 1991), 171–215.

[3] Borisenko, A., and Miquel, V. Gaussian mean curvature flow. *Journal of Evolution Equations 10* (2010), 415–423.

[4] Castro, I., Lerma, A. M., and Miquel, V. Evolution by mean curvature flow of lagrangian spherical surfaces in complex euclidean plane. *Journal of Mathematical Analysis and Applications 462* (2018), 637–647.

[5] Chen, J., and Li, J. Mean curvature flow of surface in 4-manifolds. *Advances in Mathematics 163*, 2 (November 2001), 287–309.

[6] Chou, K.-S., and Zhu, X.-P. *The Curve Shortening Problem.* Chapman and Hall/CRC, Boca Raton, FL, March 2001.

[7] Gage, M., and Hamilton, R. The heat equation shrinking convex plane curves. *Journal Differential Geometry 23* (1986), 69–96.

[8] Gage, M. E. Curve shortening makes convex curves circular. *Inventiones mathematicae 76* (1984), 357–364.

[9] Gage, M. E. Curve shortening on surfaces. *Annales Scientifiques de l'École Normale Supérieure 23*, 2 (1990), 229–256.

[10] Grayson, M. Shortening embedded curves. *Annals of Mathematics 129* (1989), 71–111.

[11] Grigor'yan, A. Analytic and geometric background of recurrence and non-explosion of the brownian motion on riemannian manifolds. *Bull. Amer. Math. Soc. (N.S.) 36* (1999), 135–249.

[12] Hamilton, R. Three-manifolds with positive ricci curvature. *Journal Differential Geometry 17*, 2 (1982), 255–306.

[13] Huisken, G. Asymptotic behavior for singularities of the mean curvature flow. *Journal Differential Geometry 31*, 1 (1990), 285 – 299.

[14] Ma, L., and Chen, D. Curve shortening in a riemannian manifold. *Annali di Matematica Pura ed Applicata 186*, 4 (October 2007), 663–684.

[15] Miquel, V., and Viñado-Lereu, F. The curve shortening problem associated to a density. *Calculus of Variations and Partial Differential Equations 55:61*, 3 (June 2016), 1 – 30.

[16] Miquel, V., and Viñado-Lereu, F. Type i singularities in the curve shortening flow associated to a density. *The Journal of Geometric Analysis 28*, 3 (July 2018), 2361–2394.

[17] Oaks, J. A. Singularities and self intersections of curve evolving on surfaces. *Indiana University Mathematics Journal 43*, 3 (1994), 959–981.

[18] O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity.* 1983.

[19] Schnürer, O. C., and Smoczyk, K. Evolution of hypersurfaces in central force fields. *Journal für die reine und angewandte Mathematik 550* (2002), 77–95.

[20] SMOCZYK, K. Symmetric hypersurfaces in riemannian manifolds contracting to lie-groups by their mean curvature. *Calculus of Variations and Partial Differential Equations 4*, 2 (February 1996), 155 – 170.

[21] SMOCZYK, K. Mean curvature flow in higher codimension: Introduction and survey. In *Global Differential Geometry.*, M. S. Christian Bär, Joachim Lohkamp, Ed., vol. 17 of *Springer Proceedings in Mathematics*. Springer, Berlin, Heidelberg, 2012, pp. 231–274.

[22] STOKER, J. *Differential Geometry*. Wiley-Interscience, 1969.

[23] WANG, M.-T. Mean curvature flow of surfaces in einstein four-manifolds. *Journal of Differential Geometry 57*, 2 (2001), 301–338.

[24] ZHOU, H. Curve shortening flows in warped product manifolds. *Proceedings of the American Mathematical Society 145*, 10 (October 2017), 4503–4516.

[25] ZHU, X.-P. Asymptotic behavior of anisotropic curve flows. *Journal Differential Geometry 48* (1998), 225–274.

This project was developed at the Department of Mathematics, School of Technology and Experimental Sciences, Universitat Jaume I, Av. de Vicente Sos Baynat, s/n, E-12071 Castelló de la Plana, Spain
fvinado@uji.es