Dynamics of subfamilies of Ostrowski - Chun methods

B. Campos and P. Vindel

Instituto de Matemáticas y Aplicaciones de Castellón,
Universitat Jaume I, Spain
campos@uji.es, vindel@uji.es

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Abstract

In this paper, we classify the fixed and critical points of the bi-parametric family of Ostrowski-Chun methods applied on quadratic polynomials. We obtain the values of the parameters that reduce the number of free critical points.

We select the one-parametric subfamilies with only one free critical point and we carry out a dynamical study of these subfamilies. For each subfamily we study the parameter plane in order to consider which numerical methods are more appropriate for solving nonlinear equations.

1 Introduction

Computers are a universal tool for solving a broad class of problems by implementing numerical methods. This is the reason why they have become indispensable for much current scientific research. For the case of the search of the roots of an equation, iterative methods are used. The iterative methods require at least one initial estimate for the location of the root being sought. If this initial estimate is "close enough" to a root then, in general, the procedures converge to this root under standard conditions. The problem is, of course, how to obtain such "good" initial estimate. The study of the dynamics of iterative methods is a useful way to see what initial conditions are most appropriate for a proposed problem.

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, gives rise to rational functions whose iterates define a discrete dynamical system. In general, the study of this dynamics is very complicated and it is not well-known. The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative process considered is Newton's method. The study of the dynamics of other iterative methods used for solving nonlinear equations have been widely developed in recent years (see, for example, [1], [2], [3],[4], [5], [6], [7], and references therein).

In recent studies, many authors ([8], [9], [10], [11], [12], [13], [14], [15], for example) have found interesting dynamical behaviours, including periodicity and others anomalies. One of the main interests in these papers has been the study of the parameter spaces associated to families of iterative methods, which allow us to distinguish members with better behaviour.

In this paper we study a bi-parametric family with order of convergence four. The bi-parametric families appear with a certain frequency in the design of new numerical methods since they allow to adjust the values of the parameters in order to achieve a higher degree of convergence, see for example [16], [17], [18], [19], [20], [21], [22], [23], or even for building methods with memory (see [24], for example). Some of these families have very high order of convergence ([25], [26]) that complicates their dynamical study.

A dynamical study of an iterative method includes the calculus of the fixed and the critical points of the system. The solutions of the initial equation are fixed points of the iterative system. The numerical method fails if an initial condition goes to a fixed point different from the solution of the equation or to another strange attractor. Then, it is interesting to analyze the stability of the strange fixed points. Critical points also play an important role because every attractor attracts a critical point (see [27], [28]); then, the iteration of the critical points detects the existence of strange attractors. Parameter planes are built by iterating the critical points and give us information about the values of the parameters corresponding the iterative methods with bad behaviour.

In this paper, we consider the bi-parametric family defined by a generalization of the Ostrowski and the Chun methods introduced by A. Cordero et al. ([8], [9]) applied on quadratic polynomials. We make an exhaustive study of the fixed and critical points in terms of the parameters. First, in Section 2, we consider that both parameters are real and we study those bifurcation curves where the character and the number of fixed and critical points change.

From this study we select, in Section 3, the one-parameter subfamilies defined by the bifurcation curves previously obtained proving that the number of free critical points decreases on these curves. We select the five subfamilies for which there is only one free critical point in order to carry out a dynamical study of such methods; the fact of having only one free critical point allows us to draw the corresponding parameter plane.

These subfamilies are considered in Section 4. As there is only one parameter we consider it as a complex parameter and provide a dynamical study on the complex plane. Observing the parameter planes obtained, we can see that two of these subfamilies show very good behaviour, the black areas of no convergence of the critical point to the roots are very small. We make a detailed study analyzing the stability of the strange fixed points in order to locate these black areas. We also show dynamical planes corresponding to different values of the parameter in order to visualize the basins of attraction of the roots and the strange attractors.

2 The bi-parametric family

In [8] the authors design new parametric families of iterative methods for nonlinear equations by using Ostrowski and Chun methods, whose iterative schemes are:

$$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)} \quad \text{and} \quad x_{k+1} = y_k - \frac{f(x_k) + 2f(y_k)}{f(x_k)} \frac{f(y_k)}{f'(x_k)}$$

where $y_k = x_k - \frac{f(x_k)}{f'(x_k)}$ is the step of Newton's method. They design a new family as a generalization of these two schemes:

$$y_k = x_k - \alpha \frac{f(x_k)}{f'(x_k)}; \quad x_{k+1} = y_k - \left(\frac{f(x_k)}{a_1 f(x_k) + a_2 f(y_k)} + \frac{b_1 f(x_k) + b_2 f(y_k)}{f(x_k)} \right) \frac{f(y_k)}{f'(x_k)}.$$

They show that the order of convergence of this new family is at least four if $\alpha = 1,\ a_2 = a_1^2(b_2 - 2)$ and $b_1 = 1 - \frac{1}{a_1}$, with $a_1 \neq 0$. With these values of the parameters, in [9] the authors apply this family of methods on the quadratic polynomials $p(z) = z^2 + c$ in order to study its behavior. It is known (see, for example, [29, 30]) that the roots of a polynomial can be transformed by the conjugacy map $h(z)$ with no qualitative changes on the dynamics of the family of polynomials, where

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}.$$

This map, known as Möbius transformation, has the following properties:

$$(i) \;\; h(i\sqrt{c}) = 0, \quad (ii) \;\; h(-i\sqrt{c}) = \infty, \quad (iii) \;\; h(\infty) = 1.$$

By applying this conjugacy map, the operator of this class of numerical methods is conjugated to the rational function

$$O_{a_1,b_2}(z) = \frac{-z^4 \left((z+1)^2 \left(z^2 + 4z + 5\right) - a_1 \left(b_2^2 - b_2 \left(z^3 + 4z^2 + 5z + 4\right) + 2(z+1)^2 (z+2) \right) \right)}{z^4 \left(a_1 (b_2 - 2)^2 - 5 \right) + z^3 \left(-5a_1 (b_2 - 2) - 14\right) - 2z^2 \left(2a_1 (b_2 - 2) + 7\right) - z \left(a_1 (b_2 - 2) + 6\right) - 1}.$$

For our study, we rename the parameters in order to simplify the calculations:

$$a = a_1 (b_2 - 2) \;\text{ and }\; b = b_2$$

obtaining the following operator

$$O_{a,b}(z) = z^4 \frac{-5 - 2a + ab + (-14 - 5a) z + (-14 - 4a) z^2 + (-6 - a) z^3 - z^4}{-1 + (-6 - a) z + (-14 - 4a) z^2 + (-14 - 5a) z^3 + (-5 + a(-2 + b)) z^4}. \tag{1}$$

We start our analysis by considering a and b as real parameters and studying the change in the number and the nature of the fixed and critical points of the associated operator. This study allows us to find interesting relations between these parameters that lead to one-parametric subfamilies with only one free critical point. We continue our work making a more exhaustive study of the dynamics of these subfamilies in the complex plane.

2.1 Fixed points

The first step of the dynamical study of the operator $O_{a,b}(z)$ as a function of the parameters a and b consists of calculating its fixed and critical points. As we will see, the number and the stability of the fixed and critical points depend on these parameters.

Let us recall that a point z_0 is a *fixed point* of $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if $F(z_0) = z_0$. The *basin of attraction* of an attracting fixed point z_0 consists of the set of points that accumulate on z_0 under iteration of F. A point z_0 is called a *critical point* of F if $F'(z_0) = 0$. Moreover, any rational map of degree d has $d+1$ fixed points (including multiplicity) and $2d-2$ critical points (with multiplicity) (see [29], for example). In our case, there are 9 fixed and 14 critical points.

As fixed points satisfy $O_{a,b}(z) = z$, we obtain the points $z = 0$, $z = 1$, $z = \infty$ and the roots of the sixth degree polynomial

$$p_6(z) = 1 + (7+a)z + (21+5a)z^2 + (30 + a(8+b))z^3 + (21+5a)z^4 + (7+a)z^5 + z^6. \tag{2}$$

The points $z = 0$ and $z = \infty$ correspond to the roots of the quadratic polynomial $p(z) = z^2 + c$ and have their own basin of attraction for all values of the parameters. The points $z = 1$ and the roots of the sixth degree polynomial (2) are called *strange fixed points*, since they do not correspond to any root of the initial polynomial. Strange fixed points can be attractors; so, a key part in the dynamical study of an iterative method is the search of their basins of attraction, because in these regions the method fails.

Let us solve $p_6(z) = 0$ in order to obtain the other 6 strange fixed points. As $p_6(z)$ is a symmetric polynomial, we make the change of variables

$$z + \frac{1}{z} = x$$

that transforms equation $p_6(z) = 0$ into the third order equation

$$16 + 6a + ab + (18+5a)x + (7+a)x^2 + x^3 = 0, \tag{3}$$

whose solutions are

$$\begin{aligned}
x_1 &= \frac{-1}{3}(7+a) + \frac{1}{3}\left(\sqrt[3]{f} + \sqrt[3]{g}\right)\\
x_2 &= \frac{-1}{3}(7+a) - \frac{1}{6}\left(\sqrt[3]{f} + \sqrt[3]{g}\right) - \frac{\sqrt{3}}{6}i\left(\sqrt[3]{f} - \sqrt[3]{g}\right)\\
x_3 &= \frac{-1}{3}(7+a) - \frac{1}{6}\left(\sqrt[3]{f} + \sqrt[3]{g}\right) + \frac{\sqrt{3}}{6}i\left(\sqrt[3]{f} - \sqrt[3]{g}\right)
\end{aligned}$$

where

$$\begin{aligned}
f &= \frac{1}{2}(16 + 21a + 3a^2 - 2a^3 - 27ab - \sqrt{\Delta_1})\\
g &= \frac{1}{2}(16 + 21a + 3a^2 - 2a^3 - 27ab + \sqrt{\Delta_1})
\end{aligned}$$

and

$$\Delta_1 = 4\left(5 + a - a^2\right)^3 + \left(16 + 21a + 3a^2 - 2a^3 - 27ab\right)^2.$$

As parameters a and b are real, if $\Delta_1 \geq 0$ the three roots x_1, x_2, x_3 are real while we obtain one real and two complex roots for $\Delta_1 < 0$.

Undoing the previous change of variables, the 6 roots of $p_6(z)$ are given by:

$$z = \frac{x_i \pm \sqrt{x_i^2 - 4}}{2}, \quad i = 1, 2, 3. \tag{4}$$

Therefore, the curves $\Delta_1 = 0$ and $x_i = \pm 2$ are bifurcation curves that separate the $(a,b)-$plane into regions with different type of roots.

The explicit formula for the bifurcation curve $\Delta_1 = 0$ is

$$b = \frac{-(a+1)\left(2a^2 - 5a - 16\right) \pm 2\sqrt{\left(a^2 - a - 5\right)^3}}{27a}.$$

For solving $x_i = 2$, we substitute this value in equation (3), obtaining

$$16 + 6a + ab + 2(18 + 5a) + 4(7 + a) + 8 = 0$$

that leads to the bifurcation curve

$$b = \frac{-88 - 20a}{a}.$$

Following the same reasoning for condition $x_i = -2$, we obtain $-54ab = 0$, that leads to the bifurcation curves $a = 0$ and $b = 0$.

Let us remark that for $x = 2$ all the fixed points coincide with the fixed point $z = 1$. If $x = -2$ all the fixed points coincide with $z = -1$ which is a preimage of $z = 1$ since $O_{a,b}(-1) = 1$.

The real or complex character of the strange fixed points for each region of the (a, b)-plane separated by these bifurcations curves can be seen in the Figure 1. We use different colours in order to distinguish these regions and we indicate the number of real (R) and complex (C) strange fixed points in each region.

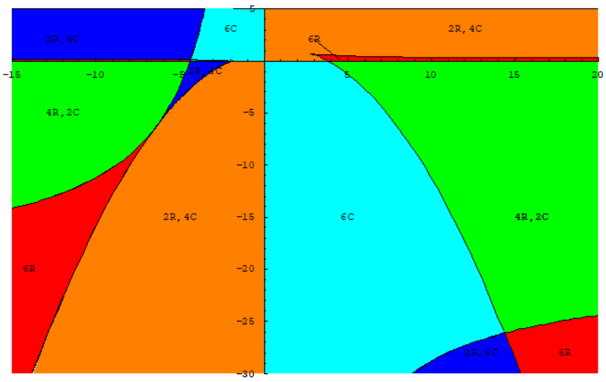


Figure 1: Regions in (a, b)-plane where the real and complex character of strange fixed points is indicated.

In Figure 2 we show a zoom of the previous figure for small negative values of the parameter b.

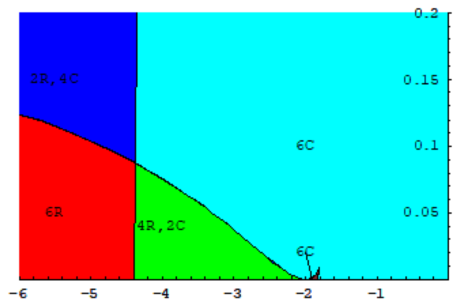


Figure 2: A zoom of Figure 1.

Then, for given values of the parameters a and b, we know the number of real and complex strange fixed points of the corresponding dynamical system. If we focus on a real study of the dynamical system, those values of the parameters corresponding to regions where all the strange fixed points are complex will lead to a more stable real dynamics.

2.2 Critical points

As commented above, every attractor has a critical point in its basin of attraction; then, the iteration of the critical points tells us the existence of strange attractors. Parameter planes are built by iterating the critical points, colouring in black the regions where the critical point goes to an attractor different from a solution of the problem.

Critical points satisfy $O'_{a,b}(z) = 0$. The derivative of our operator is given by

$$O'_{a,b}(z) = -\frac{z^3(1+z)^4 p_4(z)}{\left(1+(6+a)z+2(7+2a)z^2+(14+5a)z^3+(5-a(b-2))z^4\right)^2} \qquad (5)$$

where $p_4(z)$ is the symmetric fourth degree polynomial

$$\begin{aligned} p_4(z) &= (-20-8a+4ab)+\left(-80-44a-6a^2+2ab+3a^2b\right)z+\left(-120-72a-12a^2-4ab-4a^2b\right)z^2+ \\ &\quad \left(-80-44a-6a^2+2ab+3a^2b\right)z^3+(-20-8a+4ab)z^4. \end{aligned}$$

Then, from (5) we obtain that the fixed points $z=0$ and $z=\infty$ are also critical points of degree 3 and consequently, these fixed points (that are associated to the roots of the quadratic family) are always superattracting points and have their own basin of attraction for any values of the parameters. As we have seen above, the point $z=-1$ is a preimage of $z=1$.

We obtain other four critical points given by the roots of $p_4(z)$. As $p_4(z)$ is a symmetric polynomial, we use again the change of variables

$$z+\frac{1}{z}=x$$

that transforms equation $p_4(z)=0$ into the two order equation

$$(-80-56a-12a^2-12ab-4a^2b)+\left(-80-44a-6a^2+2ab+3a^2b\right)x+(-20-8a+4ab)x^2=0,$$

whose solutions are

$$x_\pm = \frac{80-3(b-2)a^2-2(b-22)a \pm a\sqrt{\Delta_3}}{8(-5+a(b-2))},$$

where

$$\Delta_3 = (b-2)(-72+9a^2(b-2)+196b+a(76b-72)).$$

Then, these four critical points are given by

$$c = \frac{x_\pm \pm \sqrt{x_\pm^2-4}}{2}.$$

The curves $\Delta_3=0$ and $x_\pm=\pm 2$ are bifurcation curves that separate the $(a,b)-$plane into regions with different type of critical points.

From equation $\Delta_3=0$ we obtain the bifurcation curves:

$$\begin{aligned} b &= 2, \\ b &= \frac{18(a+2)^2}{196+76a+9a^2}. \end{aligned}$$

From equations $x_\pm=\pm 2$, we obtain the bifurcation curves:

$$\begin{aligned} a &= -4, \\ b &= \frac{5+2a}{a}, \\ b &= \frac{4(10+3a)}{a}, \\ a &= 0, \\ b &= 0. \end{aligned}$$

In Figure 3 we can observe the different regions of the $(a,b)-$plane defined by these bifurcation curves. We use different colours in order to distinguish these regions and we indicate the number of real (R) and complex (C) free critical points in each region.

Figure 4 gives a zoom of the previous one for small positive values of the parameter b.

Let us remark that, for given values of the parameters a and b, we know the number of real and complex free critical points of the corresponding dynamical system. If we focus on a real study of the system, those values of the parameters corresponding to regions where all free critical points are complex implies that there is no other stable behaviour than the basins of attraction of the roots of the quadratic polynomial.

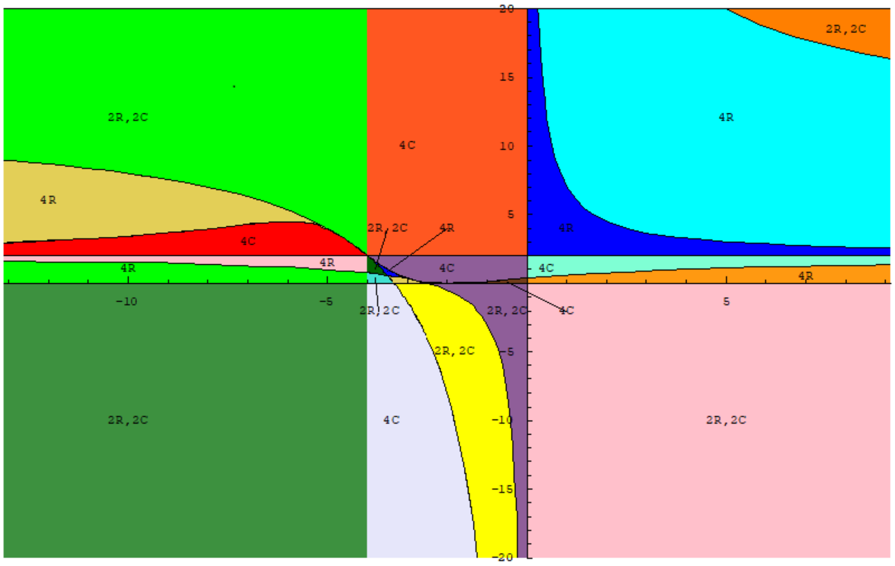


Figure 3: Regions in (a, b)-plane where the real and complex character of free critical points is indicated.

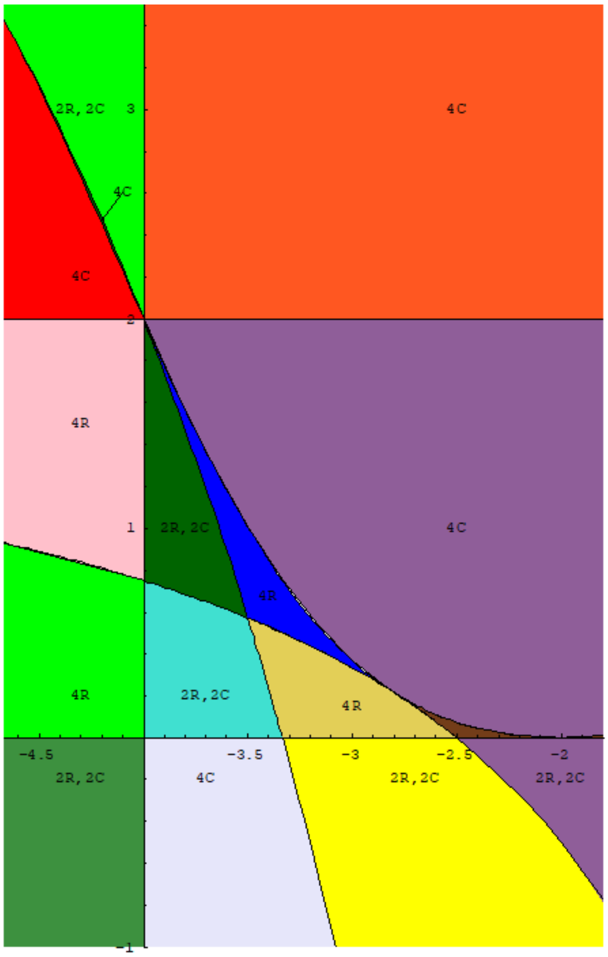


Figure 4: A zoom of Figure 3.

3 One-parameter subfamilies with one free critical point

Let us see that the number of free critical points decrease on the bifurcation curves obtained above.

- On the bifurcation curve $b = \frac{18(a+2)^2}{196+76a+9a^2}$, we have that

$$x_\pm = \frac{(28+16a+3a^2)}{(-14-5a)},$$

then

$$c_1 = c_3 = \frac{-(28+16a+3a^2) + \sqrt{3a(2+a)(4+a)(14+3a)}}{2(14+5a)}$$

and

$$c_2 = c_4 = \frac{-(28+16a+3a^2) - \sqrt{3a(2+a)(4+a)(14+3a)}}{2(14+5a)}.$$

As $c_1 c_2 = 1$, there is only one free critical point, since both points have the same dynamical behavior. It is easy to show that if two critical points of this family satisfy $c_i = \frac{1}{c_j}$ the dynamics of the system is equivalent (see [31]) and there are only one free critical point.

For $-\frac{14}{3} < a < -4$ or $-2 < a < 0$ we have that $c_i \in \mathbb{C}$; for other values of a we have that $c_i \in \mathbb{R}$.

- For $a = -4$, we have that

$$p_4(z) = -4(z-1)^2(4b-3+(-6-2b)z+(4b-3)z^2.$$

Then, the critical points are $z = 1$ (corresponding to a fixed point) and

$$c_\pm = \frac{3+b \pm \sqrt{15b(2-b)}}{4b-3}.$$

As $c_+c_- = 1$, there is only one free critical point.

Moreover, for $b < 0$ or $2 < b$ we have that $c_\pm \in \mathbb{C}$; for other values of b we have that $c_\pm \in \mathbb{R}$.

- On the bifurcation curve $b = \frac{5+2a}{a}$, the four order coefficient of the polynomial $p_4(z)$ vanishes and its degree is reduced to three. In this case, the free critical points are the solutions of the equation

$$(5a+14)\,z^3 + \left(4a^2+20a+28\right)z^2 + (5a+14)z = 0,$$

one of them is $z = 0$ (which is a fixed point) and the other two are

$$c_\pm = \frac{-2\left(a^2+5a+7\right) \pm \sqrt{a\,(2a+5)\,(2a+7)\,(a+4)}}{5a+14}.$$

It can be checked that $c_+c_- = 1$; then, for these values of the parameters, there is only one free critical point. We can observe that $c_\pm \in \mathbb{C}$ for $-4 < a < -7/2$ and $-5/2 < a < 0$; for other values of a we have that $c_\pm \in \mathbb{R}$.

- On the curve $b = \frac{4(10+3a)}{a}$ we have that

$$p_4(z) = 10(z-1)^2(14+4a+(28+18a+3a^2)z+(14+4a)z^2).$$

Then, the critical points are $z = 1$ (corresponding to a fixed point) and

$$c_\pm = \frac{-(28+18a+3a^2) \pm \sqrt{a\,(4+a)\,(10+3a)\,(14+31)}}{4(2a+7)}.$$

As $c_+c_- = 1$, there is only one free critical point.

Moreover, $c_\pm \in \mathbb{C}$ for $b < -14/3$ or $b > -10/3$ and $c_\pm \in \mathbb{R}$ for other values of b.

- For $b=0$, we have that
$$x_+ = \frac{20+14a+3a^2}{-10-4a} \quad \text{and } x_- = -2,$$
then
$$c_{1,2} = \frac{-(20+14a+3a^2)+\sqrt{3a(2+a)(4+a)(10+3a)}}{4(5+2a)}$$
and
$$c_3 = c_4 = -1.$$
The point $z=-1$ is a preimage of the strange fixed point $z=1$. As $c_1c_2=1$, there is one free critical point.

  Moreover, $-4 < a < -\frac{10}{3}$ or $-2 < a < 0$ we have that $c_{1,2} \in \mathbb{C}$; for other values of a we have that $c_{1,2} \in \mathbb{R}$.

- The cases $a=0$ and $b=2$ come together. As $a = a_1(b-2)$ and $a_1 \neq 0$, we have that $a=0$ if and only if $b=2$. In this case the operator of the corresponding numerical method is
$$O_{0,2}(z) = z^4 \frac{z^2+4z+5}{5z^2+4z+1}$$
and is derivative is:
$$O'_{0,2}(z) = 20z^3 \frac{(1+z)^4}{(5z^2+4z+1)^2}.$$
The point $z=-1$ is a preimage of strange fixed point $z=1$, which is repelling since $O'_{0,2}(1) = \frac{16}{5} > 1$. Then, there are not other critical points than $z=0$ and $z=\infty$.

We summarize the previous calculus in the following propositions.

Proposition 3.1. *For $a=0$ and $b=2$ the operator $O_{a,b}(z)$ has not free critical points.*

Proposition 3.2. *The operator $O_{a,b}(z)$ has only one free critical point for the following values of the parameters:*

$$\begin{aligned}
b &= 0,\\
b &= \frac{5+2a}{a},\\
b &= \frac{18(a+2)^2}{196+76a+9a^2},\\
a &= -4,\\
b &= \frac{4(10+3a)}{a}.
\end{aligned}$$

Some dynamical planes corresponding to values of $b=0$ and $b=2$ have been obtained in [9].

Let us remark the results obtained in the previous propositions. Since a critical point is needed in the basin of attraction of any attractor, Proposition 3.1 shows that all the dynamical planes corresponding to the values of the parameters $a=0$ and $b=2$ are divided into two basins of attraction associated to $z=0$ and $z=\infty$. In this case, the resulting scheme corresponds to the Chun's method.

On the other hand, the values of the parameters stated in Proposition 3.2 are those for which this bi-parametric family has only one free critical point. In these cases the associated methods can show a more stable dynamical behaviour since, at most, there is one strange attractor; that is, the dynamical planes are divided into, at most, three different basins of attraction. Note that for $b=0$ the resulting scheme corresponds to the King's family.

For any values of the parameters outside the lines described in Propositions 3.1 and 3.2, the bi-parametric family has two free critical points. In this case, the system can present two different strange attractors.

In the following section we carry out a dynamical study of the subfamilies defined by Proposition 3.2.

4 Dynamical study of the one-parametric subfamilies

In the following, we consider the families associated to the values obtained in Proposition 3.2 and call them $S1$, $S2$, $S3$, $S4$ and $S5$. As they are one-parametric families, from now on, we consider the parameter as complex and we introduce a new notation for the corresponding operators. For each subfamily, we draw the parameter plane associated to one critical point. As we have seen, the critical points are related and there is one free critical point.

Parameter planes are drawn by iterating the critical point and studying its asymptotic behaviour. A grid of 1501×1501 points is considered. The critical point is iterated up to 100 times; if before reaching 100 iterations a point w is close enough to $z = 0$ or $z = \infty$ ($|w| < 10^{-4}$ or $|w| > 10^{4}$), then we conclude that the critical orbit converges to one of the roots of the polynomial and plot the parameter using a scaling of red colours depending on the number of iterates taken before escaping. If the critical orbit has not escaped to $z = 0$ or $z = \infty$ in less than 100 iterates, then this value of the parameter is painted in black. Black parameters are, precisely, those parameters for which the critical orbit can accumulate on an strange attractor. Hence, black parameters are not good for the stability of the numerical method. Dynamical planes are obtained with the same number of points, the same number of iterations and stopping criterium.

We first recall some definitions of complex dynamics, see [30] and [28] for a more complete study of this subject. The book [29] presents a broad dynamical study of rational functions.

Given a rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere, the set of its iterates can be considered as a discrete dynamical system. The *orbit* $\mathcal{O}(z)$ of a point $w \in \widehat{\mathbb{C}}$ is given by the subsequent iterates of w under $R(z)$, i.e.

$$\mathcal{O}(z) = \{z,\, R\left(w\right),\, R^2\left(w\right), \ldots, R^n(z), \ldots\}.$$

A point $z_0 \in \widehat{\mathbb{C}}$ is called *fixed* if $R\left(z_0\right) = z_0$. A point z_0 is called *periodic* of period $p > 1$ if $R^p\left(z_0\right) = z_0$ and $R^k\left(z_0\right) \neq z_0$, $k < p$; the orbit of z_0 is called a periodic orbit of period p. A point z_0 is *pre-periodic* if it is not periodic but it is eventually mapped under iteration of $R(z)$ to a periodic one.

Fixed points are classified depending on their multiplier $\lambda = R'(z_0)$. A fixed point z_0 is called:

- *attractor* if $|\lambda| < 1$ and *superattractor* if $\lambda = 0$;
- *repulsor* if $|\lambda| > 1$;
- *indifferent* or neutral if $|\lambda| = 1$.

The same classification can be used for periodic points of any given period p since they are fixed points of the map $R^p(z)$. The multiplier λ of a fixed point z_0 determines the possible dynamics which might take place in a small neighborhood of it (see [28]).

The *basin of attraction* $\mathcal{A}(z_0)$ of an attracting point z_0 consists of the set of points $z \in \widehat{\mathbb{C}}$ that accumulate on z_0 under iteration of $R(z)$, i.e.

$$\mathcal{A}\left(z_0\right) = \{z \in \widehat{\mathbb{C}} \;:\; R^n\left(z\right) \to z_0 \;\text{ when }\; n \to \infty\}.$$

The dynamics of $R(z)$ provides a totally invariant partition of the Riemann sphere. The *Fatou set*, $\mathcal{F}\left(R\right)$, of a rational map $R(z)$ consists of the points $z \in \widehat{\mathbb{C}}$ such that the family of iterates $\{R(z), R^2(z), \ldots, R^n(z), \ldots\}$ is normal in some open neighborhood U of z. Its complement, the *Julia set* $\mathcal{J}(R)$, consists of the points where the dynamics of $R(z)$ is chaotic. The Fatou set is open and the Julia set is closed. The connected components of the Fatou set are called *Fatou components* and are mapped among themselves under iteration.

The *critical points* of a rational map $R(z)$ are defined as the $z \in \widehat{\mathbb{C}}$ where $R(z)$ fails to be injective in any neighborhood of z or, equivalently, the $z \in \widehat{\mathbb{C}}$ such that $R'(z) = 0$ (see [29], for example). Moreover, all periodic Fatou components are related to critical points being that the basins of attraction of attracting and rationally indifferent points contain, at least, a critical point (see, for example, [28]). Then, if there is a free critical point, there can be values of the parameter for which the free critical point is in the basin of attraction of an attractor different from $z = 0$ or $z = \infty$. These values of the parameter are depicted in black in the parameter plane.

The simplest example of complex dynamics is given by the dynamical system that appears by iterating Newton's method on polynomials of degree two. In this case, the Fatou set consists of two basins of attraction of the superattracting fixed points, corresponding to the roots of the polynomial, while the Julia set consists of a straight line which separates these basins of attraction. Nevertheless, this behavior becomes more complicated in the case of other families of iterative methods, as those cited in the introduction.

4.1 The subfamily $S1$

The one-parametric family $S1$ is obtained for $b = 0$ and corresponds to the King's method, whose dynamics is briefly studied in [7]. The associated operator is:

$$OS1_a(z) = z^4 \frac{5 + 2a + (4 + a)z + z^2}{1 + (4 + a)z + (5 + 2a)z^2}.$$

The fixed points are obtained by solving $OS1_a(z) = z$ or by making $b = 0$ in the expressions obtained in Section 2.1. The points $z = 0$ and $z = \infty$ are associated to the roots of the quadratic polynomial $p(z) = z^2 + c$. The strange fixed points are $z = 1$ and the solution of the fourth degree equation

$$1 + (5 + a)z + (10 + 3a)z^2 + (5 + a)z^3 + z^4 = 0 \tag{6}$$

The change of variables

$$z + \frac{1}{z} = x$$

transforms equation (6) into the second order equation

$$y^2 + (5 + a)y + (8 + 3a) = 0,$$

whose solutions are

$$y_\pm = \frac{-5 - a \pm \sqrt{a^2 - 2a - 7}}{2}.$$

The four solutions of the equation (6) are

$$ex_i = \frac{y_\pm \pm \sqrt{y_\pm^2 - 4}}{2}$$

for $i = 1, ..., 4$.

In this case, the critical points are $z = 0$, $z = \infty$ and

$$c_\pm = \frac{-20 - 14a - 3a^2 \pm \sqrt{3a(2 + a)(4 + a)(10 + 3a)}}{4(5 + 2a)}.$$

As we have seen, $c_+c_- = 1$, then we have only one free critical point.

Now, we consider the parameter a complex; then, we write $a = \alpha + \beta i$. The parameter plane (α, β) for the critical point c_+ is shown in Figure 5. Black regions correspond to values of the parameter where the critical point c_+ does not converge to a root ($z = 0$ or $z = \infty$).

We study the stability of a fixed point p by analyzing the regions of the complex plane where $|OS1'_a(p)| < 1$. In Figure 6, we show the curves where $z = 1$ and the strange fixed points are indifferent. The orange circle corresponds to $|OS1'_a(1)| = 1$. If $|OS1'_a(ex_1)| = |OS1'_a(ex_2)| = 1$ we obtain the cyan curve.

For values of the parameter on the two red cardioids, ex_3 and ex_4 are indifferent points. A zoom of the Figure 6 is given in Figure 7.

The stability of the fixed point $z = 1$ is studied in the following proposition.

Proposition 4.1. *The strange fixed point $z = 1$ satisfies one of the following statements.*

1. *If $|a + \frac{226}{55}| < \frac{16}{55}$, then $z = 1$ is an attractor; moreover, it is a superattractor for $a = -4$.*
2. *If $|a + \frac{226}{55}| = \frac{16}{55}$, then $z = 1$ is an indifferent fixed point.*
3. *For any other value of the complex parameter a, the point $z = 1$ is a repulsor.*

Proof. The point $z = 1$ is attracting for values of the parameter such that

$$|OS1'_a(1)| < 1.$$

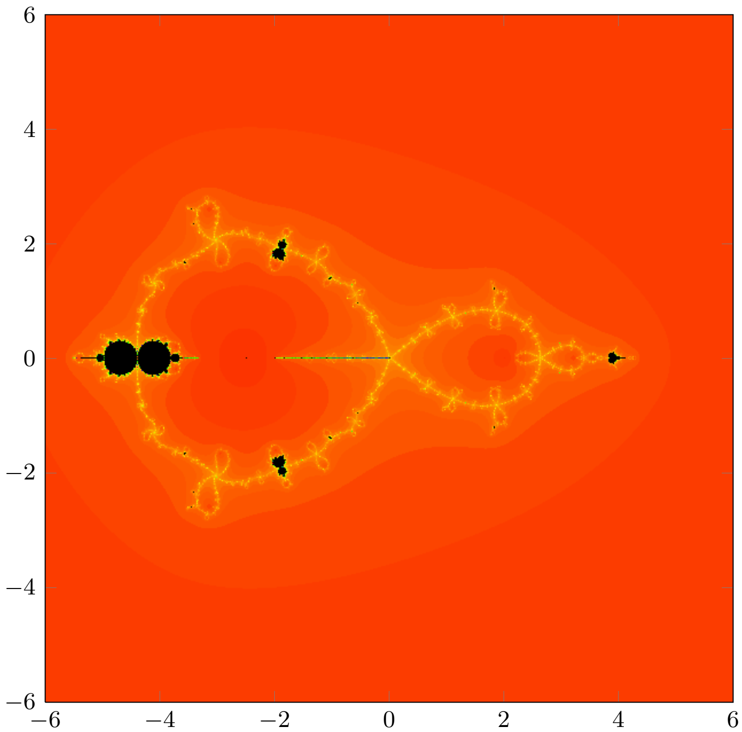


Figure 5: Parameter plane of the subfamily $S1$.

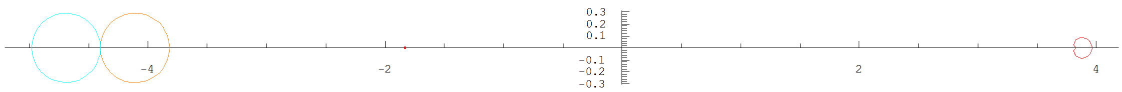
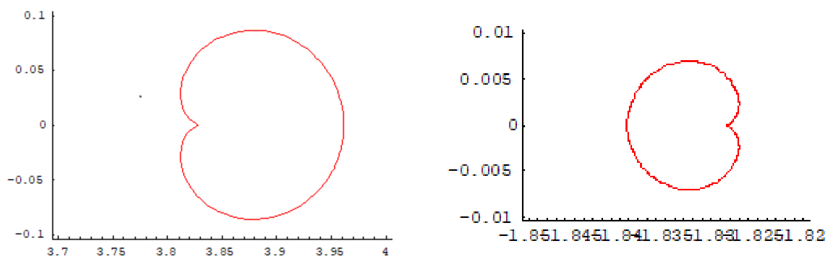


Figure 6: Curves in the parameter plane where the strange fixed points are indifferent.



(a) (b)

Figure 7: A zoom of the two red cardioids

We consider equation $|OS1'_a(1)| = 1$, then

$$\begin{aligned} \left|OS1'_{\alpha+\beta i}(1)\right| &= 1 \to \left|\frac{8(4+\alpha+\beta i)}{10+3(\alpha+\beta i)}\right| = 1 \to 64\,|4+\alpha+\beta i|^2 = |10+3(\alpha+\beta i)|^2 \to \\ 64\,((4+\alpha)^2+\beta^2) &= (10+3\alpha)^2+9\beta^2 \to (\alpha+\frac{226}{55})^2+\beta^2 = (\frac{16}{55})^2 \end{aligned}$$

which defines a circle in the $(\alpha,\beta)-$plane.

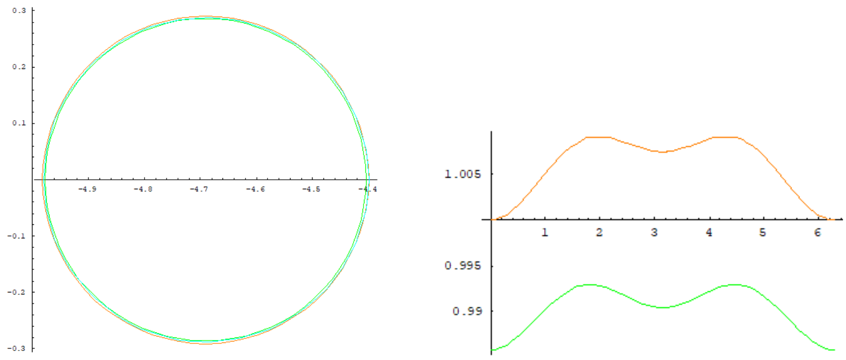
As $a=-4$ is inside this circle, $\left|OS1'_{\alpha+\beta i}(1)\right| < 1$ for values of $a=\alpha+\beta i$ inside this circle, the strange fixed point $z=1$ is an attractor inside this circle, it is a repulsor outside this circle and it is indifferent along its boundary. □

Now, we study the stability of the strange fixed points ex_1 and ex_2.

Proposition 4.2. *Let D_1 be the region of the complex plane where the fixed points ex_1 and ex_2 are attractors. Then, D_1 satisfies the following statements.*

1. *The boundary of D_1 is delimited by the circles $c_1 : \left|a + \frac{258}{55}\right| = \frac{63}{220}$ and $c_2 : \left|a + \frac{258}{55}\right| = \frac{16}{55}$.*
2. *The region D_1 corresponds to the disk containing c_1.*

Proof. The cyan curve in Figure 6 corresponds to the values of the parameter for which ex_1 and ex_1 are indifferent points; then, this curve is the boundary of D_1. In Figure 8(a) we can observe that this curve is enclosed between the circles $c_1 : \left(\alpha + \frac{258}{55}\right)^2 + \beta^2 = \left(\frac{63}{220}\right)^2$ and $c_2 : \left(\alpha + \frac{258}{55}\right)^2 + \beta^2 = \left(\frac{16}{55}\right)^2$. The stability of ex_1 and ex_2 is checked in Figure 8(b), where we can observe that $|OS1'_a(ex_i)| > 1$ when $a \in c_2$ and $|OS1'_a(ex_i)| < 1$ when $a \in c_1$, for $i = 1, 2$.

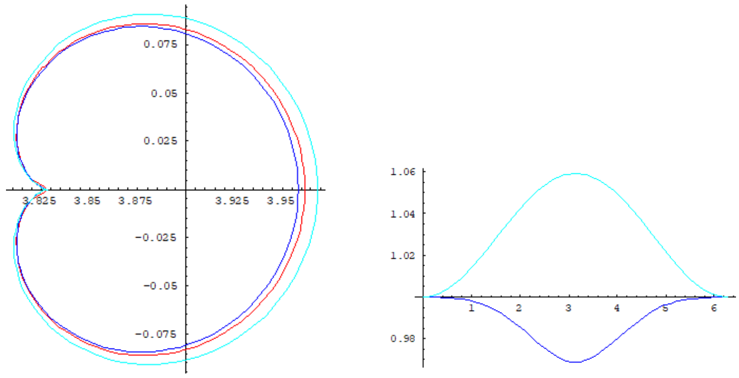


(a) ∂D_1 delimited by c_1 and c_2 (b) Value of $|OS1'_a(ex_{1,2})|$ on c_1 and c_2

Figure 8: Stability of $ex_1(a)$ and $ex_2(a)$.

Then, the points ex_1 and ex_2 are attractors in the region bounded by the curve $|OS1'_a(ex_1)| = |OS1'_a(ex_2)| = 1$ containing the circle c_1, and they are repulsors outside this region. $\square$

Similarly, we study the stability of the strange fixed points ex_3 and ex_4.



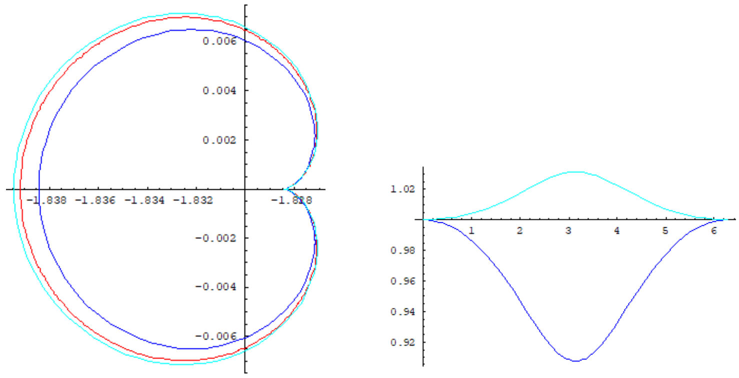
(a) ∂D_2 delimited by C_1 and C_2 (b) Value of $|OS1'_a(ex_{3,4})|$ on C_1 and C_2

Figure 9: Stability of $ex_3(a)$ and $ex_4(a)$.

Proposition 4.3. *Let D be the region of the complex plane where the fixed points ex_3 and ex_4 are attractors. Then, $D = D_2 \cup D_3$ with D_2 and D_3 satisfying the following statements.*

1. *The boundary of D_2 is contained in the region delimited by the cardioids*

$$C_1 : a = 1 + 2\sqrt{2} + \frac{1}{4}\frac{13}{100} - \frac{1}{2}\frac{13}{100}\cos t + \frac{1}{4}\frac{13}{100}\cos(2t) + \imath\left(\frac{1}{2}\frac{13}{100}\sin t - \frac{1}{4}\frac{13}{100}\sin(2t)\right), \quad 0 \le t \le 2\pi$$



(a) ∂D_3 delimited by C_3 and C_4

(b) Value of $|OS1'_a(ex_{3,4})|$ on C_3 and C_4

Figure 10: Stability of $ex_3(a)$ and $ex_4(a)$.

and

$$C_2 : a = 1 + 2\sqrt{2} + \frac{1}{4}\frac{14}{100} - \frac{1}{2}\frac{14}{100}\cos t + \frac{1}{4}\frac{14}{100}\cos(2t) + \imath\left(\frac{1}{2}\frac{14}{100}\sin t - \frac{1}{4}\frac{14}{100}\sin(2t)\right), \quad 0 \le t \le 2\pi$$

2. *The boundary of D_3 is contained in the region delimited by the cardioids*

$$C_3 : a = 1 - 2\sqrt{2} - \frac{1}{4}\frac{1}{100} + \frac{1}{2}\frac{1}{100}\cos t - \frac{1}{4}\frac{1}{100}\cos(2t) + \imath\left(\frac{1}{2}\frac{1}{100}\sin t - \frac{1}{4}\frac{1}{100}\sin(2t)\right), \quad 0 \le t \le 2\pi$$

and

$$C_4 : a = 1 - 2\sqrt{2} - \frac{1}{4}\frac{11}{1000} + \frac{1}{2}\frac{11}{1000}\cos t - \frac{1}{4}\frac{11}{1000}\cos(2t) + \imath\left(\frac{1}{2}\frac{11}{1000}\sin t - \frac{1}{4}\frac{11}{1000}\sin(2t)\right), \quad 0 \le t \le 2\pi.$$

3. *D_2 corresponds to the region containing the cardioid C_1 and D_3 corresponds to the region containing the cardioid C_3.*

Proof. The red curves in Figure 6 correspond to the values of the parameter for which ex_3 and ex_4 are indifferent points; then, these curves form the boundary of D. Let us write it as $\partial D_2 \cup \partial D_3$.

In Figure 9(a) we observe that ∂D_2 is enclosed between the cardioids C_1 and C_2. We can observe in Figure 9(b) that $|OS_1a'(ex_i)| < 1$ when $a \in C_1$ and $|OS1'_a(ex_i)| > 1$ when $a \in C_2$, for $i = 3, 4$.

In Figure 10(a) we observe that ∂D_3 is enclosed between the cardioids C_3 and C_4. We can observe in Figure 10(b) that $|OS1'_a(ex_i)| < 1$ when $a \in C_3$ and $|OS1'_a(ex_i)| > 1$ when $a \in C_4$, for $i = 3, 4$.

Then, the points ex_3 and ex_4 are attractors in the region bounded by the curves defined by $|OS1'_a(ex_3)| = |OS1'_a(ex_4)| = 1$ containing the cardioids C_1 and C_3 and they are repulsors outside these regions. □

Other black regions in the parameter plane correspond to the presence of attracting periodic orbits.

In the following figures we show dynamical planes for different values of the parameter a where the previous results can be observed. In these figures, green regions correspond to the basin of attraction of $z = 0$, red regions correspond to the basin of attraction of $z = \infty$ and black regions correspond to the basin of attraction of other attractors.

In Figure 11 we show the dynamical plane for $a = -4.25$. In this case, black regions correspond to the basin of attraction of the point $z = 1$.

In Figure 12 we show the dynamical plane corresponding to $a = -4.8 + 0.03i$. In this figure, black regions correspond to the basins of attraction of the fixed points $ex_1 = 1.90798 - 0.0443604i$ and $ex_2 = 0.523831 + 0.012179i$.

In Figure 13 we show the dynamical plane corresponding to $a = -1.83 + 0i$. In this figure, black regions correspond to the basins of attraction of the fixed points $ex_3 = -0.8160 + 0.5779i$ and $ex_4 = -0.8160 - 0.5779i$.

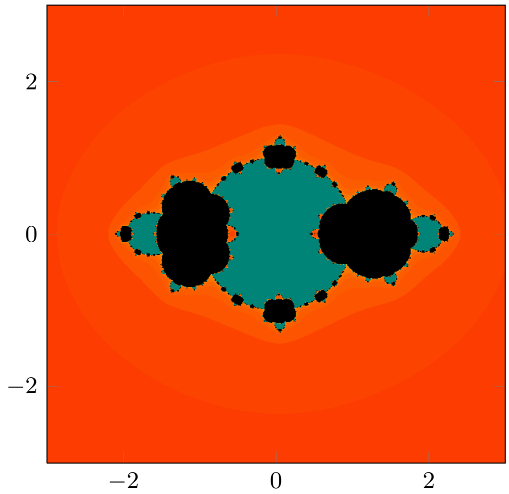


Figure 11: Dynamical plane for the family $S1$ and $a = -4.25$.

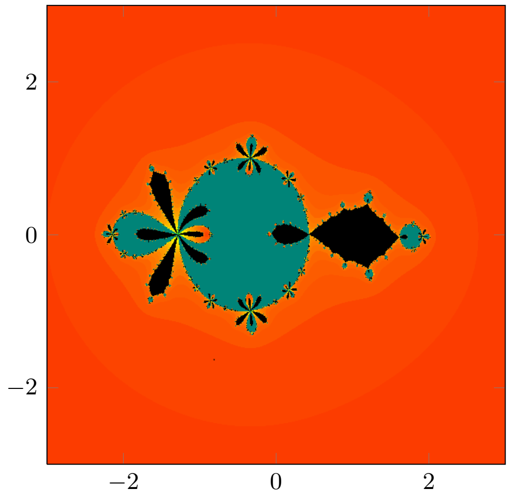


Figure 12: Dynamical plane for the family $S1$ and $a = -4.8 + 0.03i$.

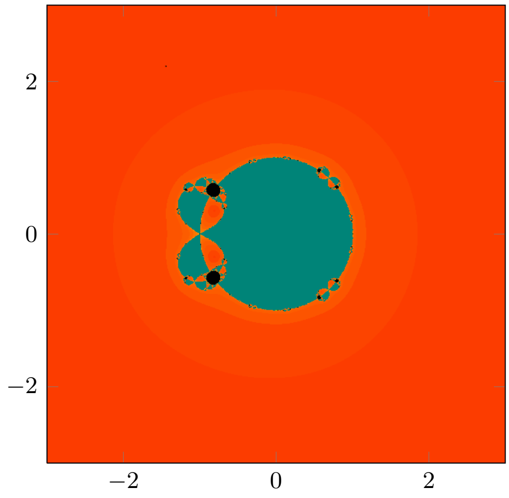


Figure 13: Dynamical plane for the family $S1$ and $a = -1.83$.

4.2 The subfamily $S2$

The subfamily $S2$ is obtained for $b = \frac{5+2a}{a}$. The corresponding operator is:

$$OS2_a(z) = z^5 \frac{14+5a+2(7+2a)z+(6+a)z^2+z^3}{1+(6+a)z+2(7+2a)z^2+(14+5a)z^3} \tag{7}$$

and its derivative is:

$$OS2'_a(z) = 5z^4(1+z)^4 \frac{14+5a+\left(28+20a+4a^{\ 2}\right)z+(14+5a)z^2}{\left(1+(6+a)z+2(7+2a)z^2+(14+5a)z^3\right)^2}.$$

The strange fixed points for this subfamily are $z = 1$ and other six points ex_i obtained by making $b = \frac{5+2a}{a}$ in the expressions of Section 2.1. The critical points are:

$$c_{\pm} = \frac{-2\left(7+5a+a^2\right) \pm \sqrt{a(a+4)(5+2a)(7+2a)}}{14+5a}.$$

The parameter plane for this family is shown in Figure 14. Let us note that this subfamily, in contrast to the previous one (Figure 5), presents good behaviour for real positive values of the parameter. Moreover, the corresponding operator (7) indicates that its order of convergence is 5 for quadratic polynomials.

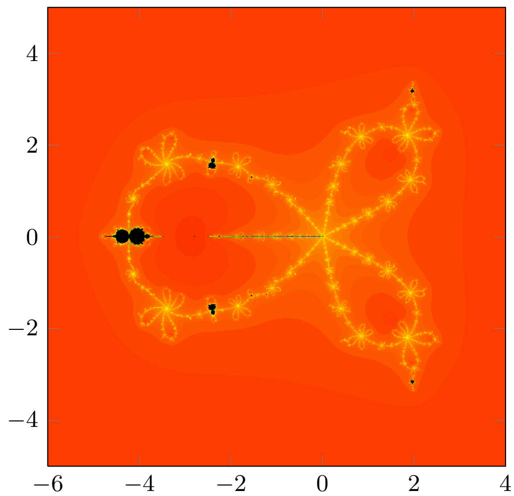


Figure 14: Parameter plane of the subfamily $S2$.

The stability of the fixed point $z = 1$ is studied in the following proposition.

Proposition 4.4. *The strange fixed point $z = 1$ satisfies one of the following statements.*

1. *If $|a + \frac{1873}{462}| < \frac{40}{231}$, then $z = 1$ is an attractor; moreover, it is a superattractor if $a = -4$.*
2. *If $|a + \frac{1873}{462}| = \frac{40}{231}$, then $z = 1$ is an indifferent fixed point.*
3. *For any other value of the complex parameter a, the point $z = 1$ is a repulsor.*

The proof is similar to the proof of Proposition 4.1.

In Figure 15 we observe a dynamical plane where the black region corresponds to the basin of attraction of $z = 1$. In the parameter plane given in Figure 14, we also observe a black disk at left of the attraction disk of $z = 1$. This disk corresponds to the values of the parameter where the strange fixed points ex_1 and ex_2 are attracting.

Proposition 4.5. *Let D_1 be the region of the complex plane where the strange fixed points ex_1 and ex_2 are attractors. Then,*

1. *∂D_1 is delimited by the circles $c_1 : \left|a + \frac{2021}{462}\right| = \frac{33.5}{231}$ and $c_2 : \left|a + \frac{2021}{462}\right| = \frac{34.5}{231}$.*
2. *The region D_1 corresponds to the disk containing c_1.*

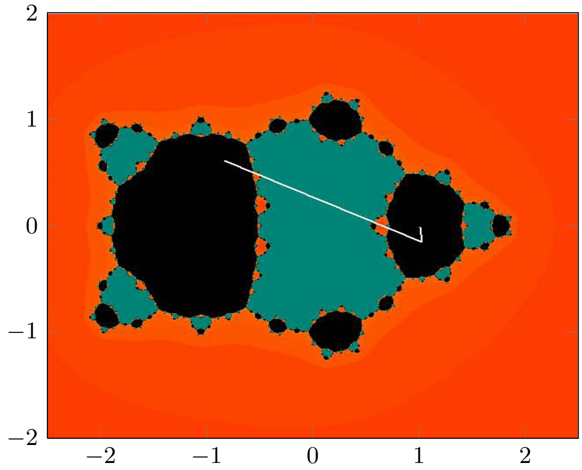


Figure 15: Dynamical plane for $a = -4$.

The proof is similar to the proof of Proposition 4.2.

Moreover, as $ex_1(a) = ex_2(a) = 1$ for $a = -\frac{93}{22}$, we have that $a = -\frac{93}{22}$ is a bifurcation point where the fixed points $z = 1, ex_1(a)$ and $ex_2(a)$ change their stability.

This bifurcation can be observed comparing Figure 15 and Figure 16. In Figure 15, corresponding to $a = -4$, the point $z = 1$ is an attractor and black regions correspond to the basin of attraction of $z = 1$. In Figure 16 corresponding to the dynamical plane obtained for $a = -4.4$, the point $z = 1$ has become repelling while the two points $ex_1(a)$ and $ex_2(a)$ have become attracting; in this case, black regions are the basins of attraction of $ex_1(a)$ and $ex_2(a)$.

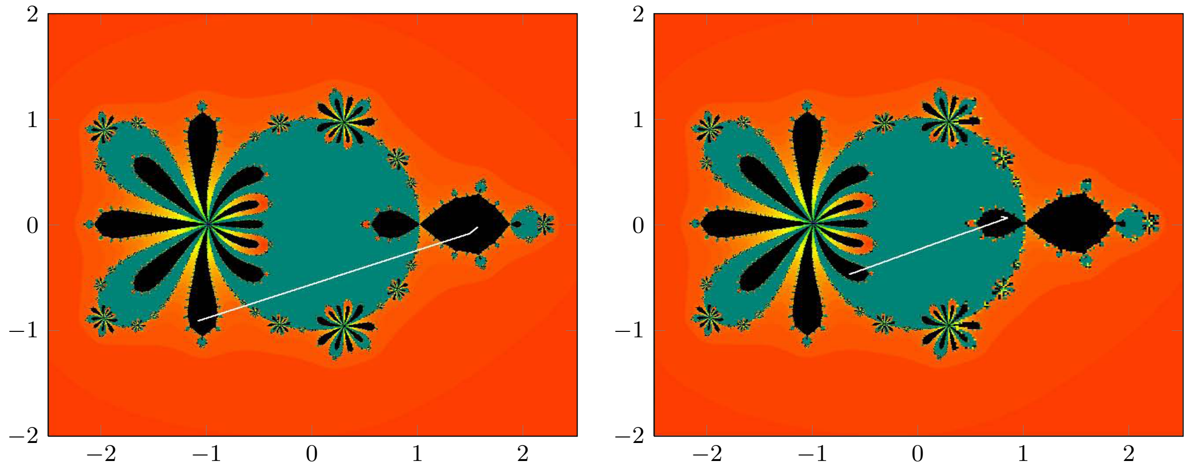


Figure 16: Dynamical plane for $a = -4.4$.

We have observed numerically that the two symmetric black regions in Figure 14 located around the points $a = -2.42 \pm 1.53i$ correspond to the values of the parameter where there are one attracting 2-periodic orbit that bifurcates to two attracting 2-periodic orbits when the parameter moves to the black regions located around the points $a = -2.40 \pm 1.65i$. Moreover, the two small regions containing the points $a = 1.94 \pm 3.14i$ correspond to values of the parameter for which the strange fixed points ex_3 and ex_4 are attractors.

4.3 The subfamily $S3$

This subfamily correspond to the value $b = \frac{18(a+2)^2}{196+76a+9a^2}$. The corresponding operator is:

$$OS3_a(z) = z^4 \frac{5(14+5a)^2 + \left(196+76a+9a^2\right)\left((14+5a)z + 2(7+2a)z^2 + (6+a)z^3 + z^4\right)}{\left(1+(6+a)z+2(7+2a)z^2+(14+5a)z^3\right)\left(196+76a+9a^2\right)+5(14+5a)^2 z^4}.$$

The derivative of this operator is

$$OS3'_a(z) = 20z^3(1+z)^4 \frac{\left(196+76a+9a^2\right)\left(14+5a+\left(28+16a+3a^2\right)z+(14+5a)z^2\right)^2}{\left(\left(1+(6+a)z+2(7+2a)z^2+(14+5a)z^3\right)\left(196+76a+9a^2\right)+5(14+5a)^2 z^4\right)^2}.$$

The strange fixed points are $z=1$ and other six points ex_i defined by the expressions of Section 2.1 for this value of b.

The critical points are:

$$c_{\pm} = \frac{-\left(28+16a+3a^2\right) \pm \sqrt{3a(2+a)(4+a)(14+3a)}}{2(14+5a)}.$$

The parameter plane is shown in Figure 17.

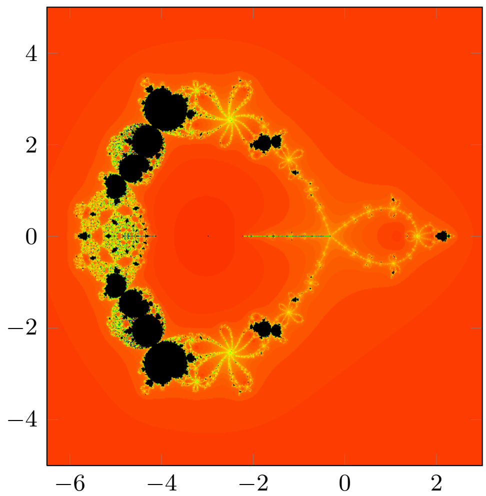


Figure 17: Parameter plane of the subfamily $S3$.

The study of the stability of the strange fixed point $z=1$ is carried out in the following proposition.

Proposition 4.6. *Let us write the parameter a as $\alpha + i\beta$ and let us define the functions:*

$$\vartheta_1(\alpha) = \sqrt{\frac{-243644 - 156204\alpha - 18711\alpha^2 - 128\sqrt{5(-1302256 - 624816\alpha - 74439\alpha^2)}}{18711}}$$

$$\vartheta_2(\alpha) = \sqrt{\frac{-243644 - 156204\alpha - 18711\alpha^2 + 128\sqrt{5(-1302256 - 624816\alpha - 74439\alpha^2)}}{18711}}$$

Then, the strange fixed point $z=1$ satisfies one of the following statements.

1. *The point $z=1$ is attractor if*

$$\frac{4(-8678 - 21\sqrt{1155})}{8271} < \alpha < \frac{4(-8678 + 21\sqrt{1155})}{8271} \quad \textit{and}$$

$$-\vartheta_2(\alpha) < \beta < -\vartheta_1(\alpha) \quad \textit{or} \quad \vartheta_1(\alpha) < \beta < \vartheta_2(\alpha).$$

   Moreover, $z=1$ is superattractor for $a = \frac{2}{9}(-19 \pm 4\sqrt{5}i)$.

2. $z = 1$ is an indifferent fixed point for

$$\frac{4(-8678 - 21\sqrt{1155})}{8271} < \alpha < \frac{4(-8678 + 21\sqrt{1155})}{8271} \quad \textit{and}$$

$$\beta = \pm\vartheta_1(\alpha) \quad \textit{or} \quad \beta = \pm\vartheta_2(\alpha).$$

3. The point $z = 1$ is repulsor for any other value of the complex parameter a.

Proof. The point $z = 1$ is an attractor for values of the parameter such that

$$|O'_a(1)| = \left|\frac{16(196 + 76a + 9a^2)}{5(14 + 3a)^2}\right| < 1$$

The fixed point is indifferent if $\left|\frac{16(196+76a+9a^2)}{5(14+3a)^2}\right| = 1$. So, by considering $a = \alpha + i\beta$ as a complex parameter we obtain

$$(8874096 + 6803552\alpha + 2117224\alpha^2 + 312408\alpha^3 + 18711\alpha^4 + (487288 + 312408\alpha + 37422\alpha^2)\beta^2 + 18711\beta^4 = 0$$

that implies the statements of the proposition. These curves can be seen in Figure 18. They are the boundaries of the two black disks in the parameter plane (see Figure 17). In particular, for $a = \frac{2}{9}(-19 \pm 4\sqrt{5}i)$ we have

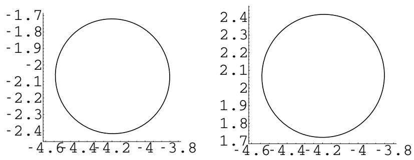


Figure 18: Curves in the parameter plane where $z = 1$ is indifferent.

that $|O'_a(1)| = 0$, then $z = 1$ is a superattractor for this value of the parameter. $\square$

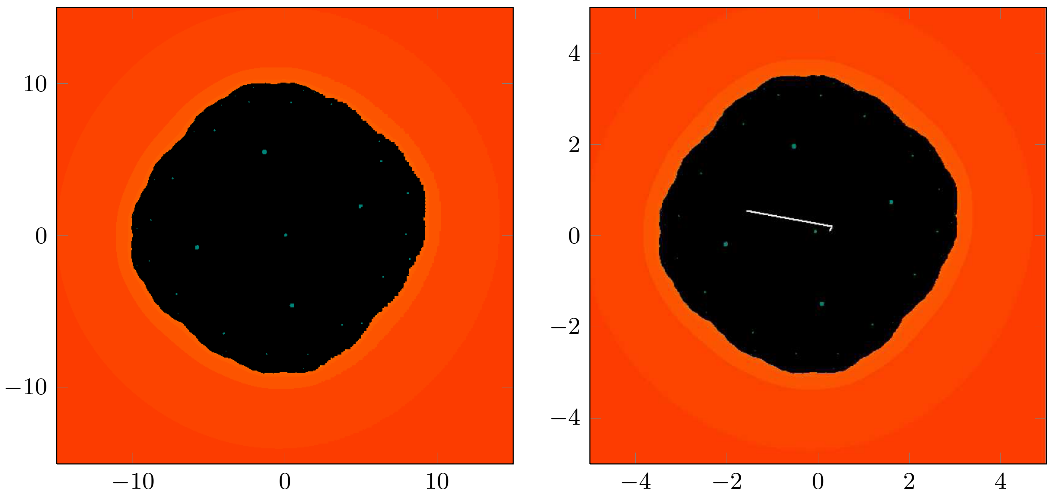


Figure 19: Dynamical plane of the family $S3$ for $a = -4.2 - 2i$.

In Figure 19 we observe a dynamical plane for $a = -4.2 - 2i$, where the black zone corresponds to the basin of attraction of $z = 1$.

Numerically, we have observed that the cardioids located on the right of the region of attraction of $z = 1$ correspond to values of the parameter where two of the strange fixed points are attracting, as it can be observed in the dynamical plane of Figure 20. As can be seen in Figure 21, another two strange fixed points are attracting for values of the parameter in the small black region that is observed at the right in the parameter plane.

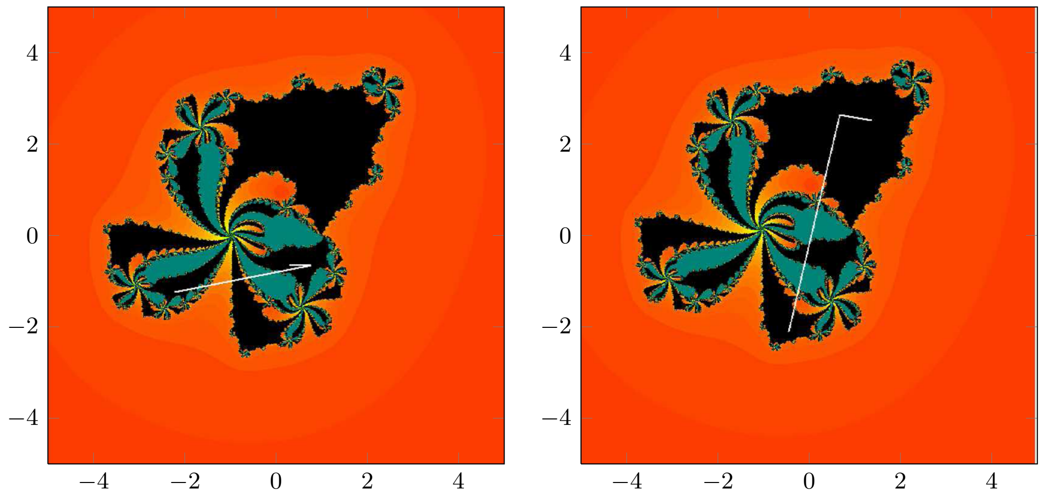


Figure 20: Dynamical plane of the family $S3$ for $a = -4 - 3i$.

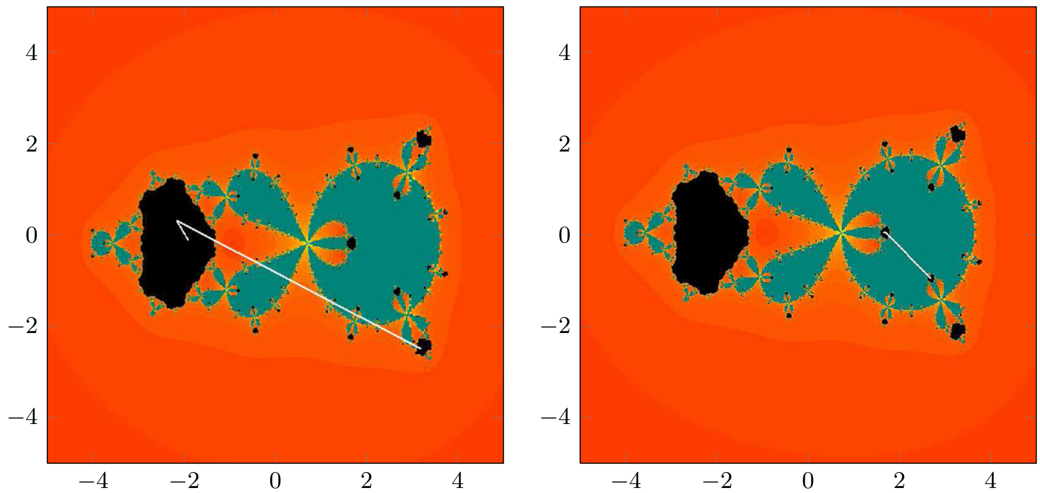


Figure 21: Dynamical plane of the family $S3$ for $a = 2.5$.

If we move to the left of the region of attraction of $z = 1$ in the parameter plane, we find two small cardioids that correspond to values of the parameter for which there is an attracting 2-periodic orbit. In Figure 22 we show a dynamical plane where the black zones correspond to the basin of attraction of a periodic orbit of period 2, coming from the bifurcation of $z = 1$.

At the left of these two cardioids in the parameter plane, there are two smaller ones that correspond to values of the parameter for which two attracting 2-periodic orbits appear. In Figure 23 we can observe a dynamical plane where the black zones correspond to the basin of attraction of these periodic orbits.

4.4 The subfamily $S4$

This subfamily is obtained for $a = -4$. The corresponding operator is:

$$OS4_b(z) = z^4 \frac{-3 + 4b - 6z - 2z^2 + 2z^3 + z^4}{1 + 2z - 2z^2 - 6z^3 + (-3 + 4b) z^4}.$$

The strange fixed points for this subfamily are $z = 1$ and other six points ex_i defined by the expressions of Section 2.1 for this value of a. The derivative of the operator is

$$OS4'_b(z) = 4z^3 (z-1)^2 (z+1)^4 \frac{4b - 3 - (2b + 6) z + (4b - 3) z^2}{\left(1 + 2z - 2z^2 - 6z^3 + (-3 + 4b) z^4\right)^2}.$$

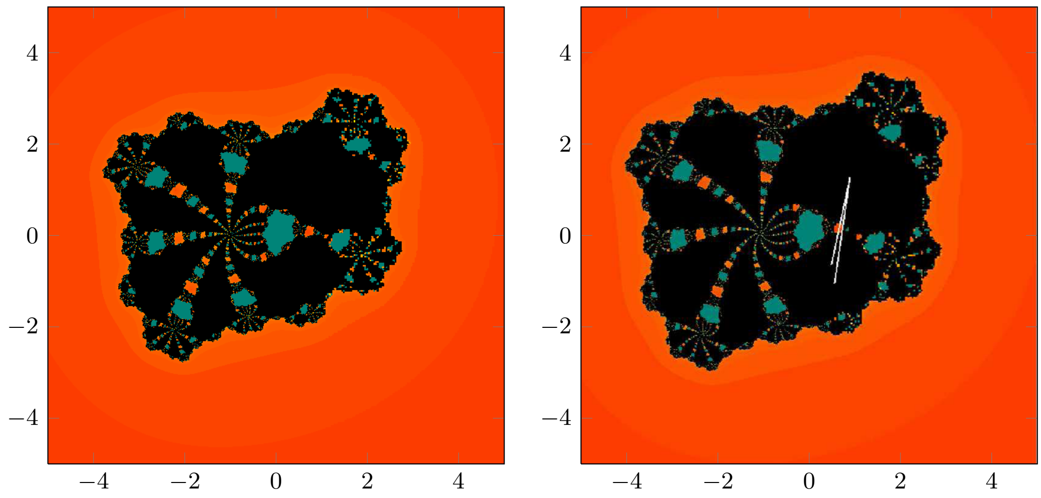


Figure 22: Dynamical plane of the family $S3$ for $a = -4.5 - 1.4i$.

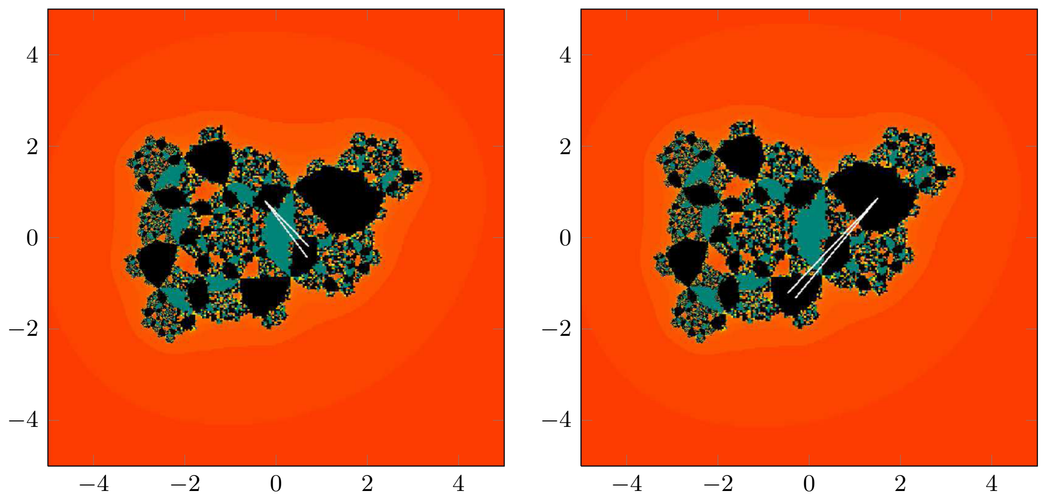


Figure 23: Dynamical plane of the family $S3$ for $a = -5 - i$.

This derivative vanishes for $z = 1$; then, the following result for the stability of the strange fixed point $z = 1$ holds.

Proposition 4.7. *The point $z = 1$ is a superattractor for any value of the parameter b.*

Moreover, as $OS4_b(-1) = 1$, the point $z = -1$ is a pre-periodic point of $z = 1$. Then, we have the critical points:

$$c_{\pm} = \frac{3 + b \pm \sqrt{15b\,(2-b)}}{-3+4b}.$$

The parameter plane of this family is shown in Figure 24.

The black shapeless zones correspond to values of the parameter where the free critical point is in the basin of attraction of $z = 1$. The black zones with cardioid shape correspond to values of the parameter where the free critical point goes to other strange attractors. For example, we have observed numerically that for values of the parameter in the cardioid on the left, two strange fixed points are attracting (see Figure 25); for values of the parameter belonging to the big cardioid on the right there is an attracting periodic orbit of period two (see Figure 26).

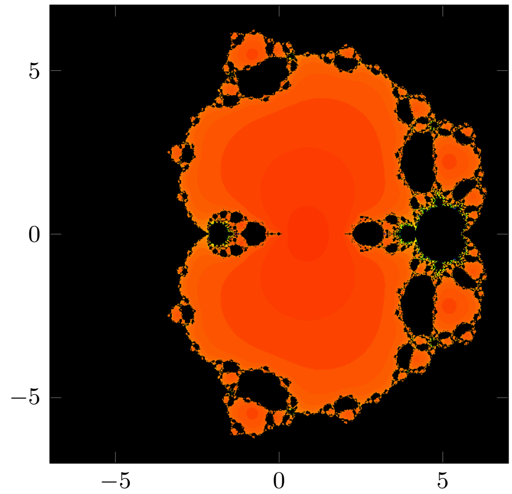


Figure 24: Parameter plane of the subfamily $S4$ for complex b.

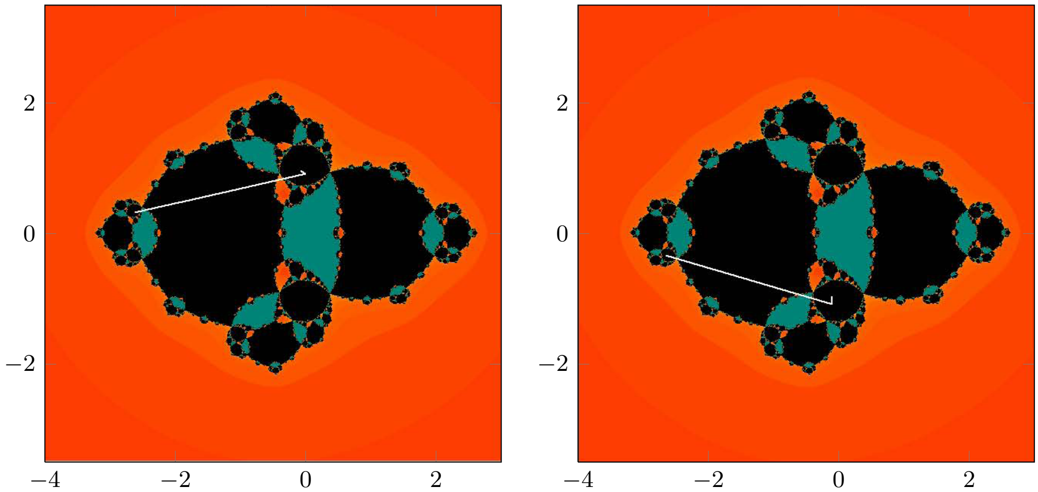


Figure 25: Dynamical plane of the family $S4$ for $b = -1.9$.

4.5 The subfamily $S5$

The subfamily $S5$ is obtained for $b = \frac{4(10+3a)}{a}$. The associated operator is:

$$OS5_a(z) = -z^4 \frac{5(7+2a) + (21+5a)z + (7+a)z^2 + z^3}{1 + (7+a)z + (21+5a)z^2 + 5(7+2a)z^3}.$$

The strange fixed points for this subfamily are $z = 1$ and other six points ex_i obtained by making $b = \frac{4(10+3a)}{a}$ in the expressions of Section 2.1.

The derivative of the operator is

$$OS5'_a(z) = -10z^3(z+1)^4 \frac{2(7+2a) + \left(28+18a+3a^2\right)z + 2(7+2a)z^2}{\left(1 + (7+a)z + (21+5a)z^2 + 5(7+2a)z^3\right)^2}.$$

In this case, $OS5_a(-1) = 1$ and $OS5_a(1) = -1$ and $OS5'_a(-1) = 0$; then, the following result holds.

Proposition 4.8. *The double period orbit $\{-1, 1\}$ is attracting for any value of the parameter a.*

The critical points are:

$$c_\pm = \frac{-\left(28+18a+3a^2\right) \pm \sqrt{a(a+4)(10+3a)(14+3a)}}{4(7+2a)}.$$

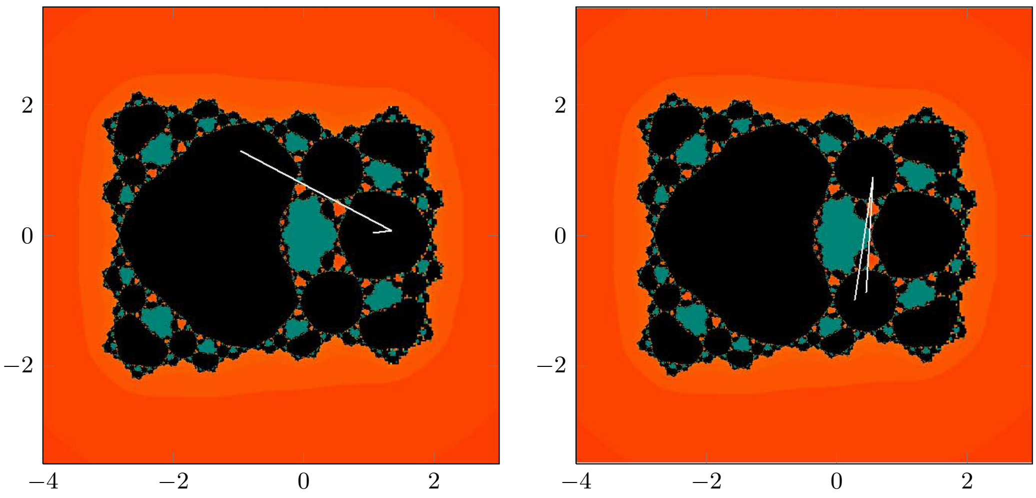


Figure 26: Dynamical plane of the family $S4$ for $b = 5$.

The parameter plane for $S5$ is given in Figure 27. For values of the parameter in the bigger black areas the free critical point is in the basin of attraction of the double period orbit $\{-1, 1\}$. The basin of attraction of $\{-1, 1\}$ can be observed in the dynamical planes given in Figure 28.

In the other black regions of the parameter plane, cardioid shaped, there exist other attractors. For example, in Figure 29 we observe two 2-period orbits in addition to $\{-1, 1\}$.

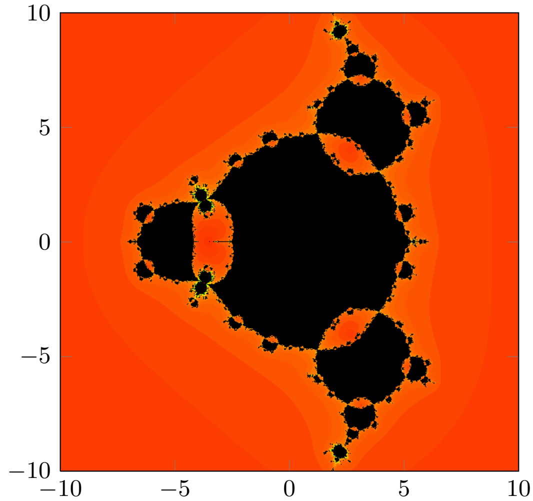


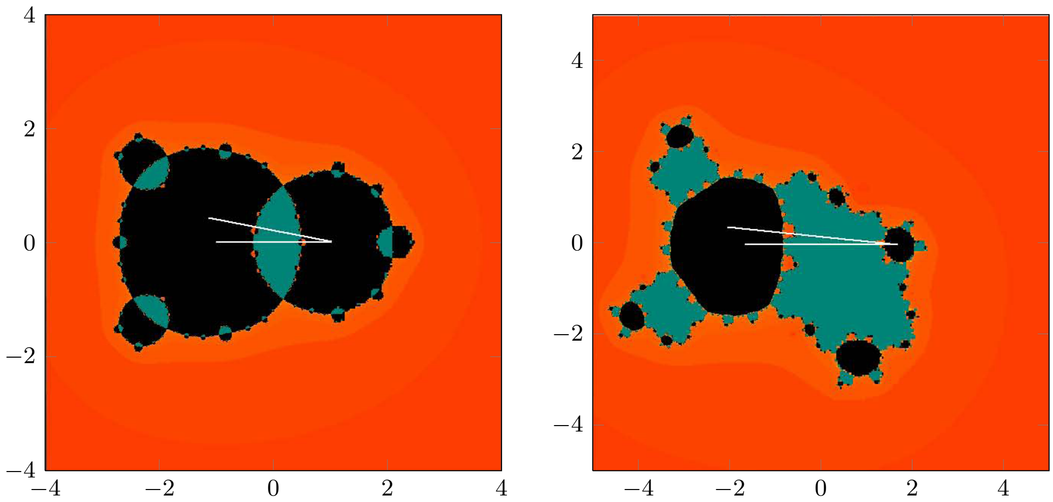
Figure 27: Parameter plane of the subfamily $S5$ for $a = \alpha + \beta i$.

5 Final remarks

Summarizing the results obtained in the previous section, we can see that two of these subfamilies, $S1$ (the King's family) and $S2$, show very good behaviour, they correspond to $b = 0$ and $b = \frac{5+2a}{a}$, respectively. Moreover, the subfamily $S2$ has order of convergence 5 for quadratic polynomials. For these families, the values of the parameter where there are strange attractors are located in very small regions.

The family $S3$, obtained for $b = \frac{18(a+2)^2}{196+76a+9a^2}$, presents zones of non-convergence greater than the previous two families.

For the family $S4$, obtained for $a = -4$, the fixed point $z = 1$ is a superattractor for every value of the parameter b; this fact implies that for all members of this family there are always initial conditions that converge



(a) $a = -4.67$ (b) $a = -3.7 + 0.5i$

Figure 28: Dynamical planes of $S5$ for values of the parameter where the black regions correspond to the basin of attraction of $\{-1, 1\}$.

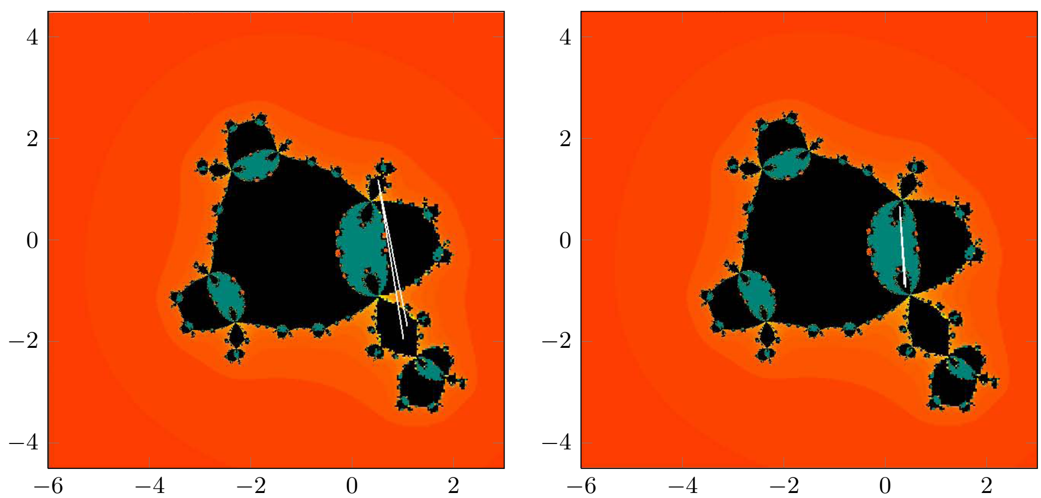


Figure 29: Dynamical planes of $S5$ for $a = -3.9 + 2.1i$.

to $z = 1$. The same occurs for the family $S5$, corresponding to $b = \frac{4(10+3a)}{a}$, since there is an attracting periodic orbit, $\{-1, 1\}$, for every value of the parameter.

On the other hand, given any of these families, the dynamical planes provided for different values of the parameters, show the initial conditions that lead to a wrong result of the corresponding numerical method. As we have pointed out, these initial conditions are located in the black regions of the dynamical planes, while green and red regions correspond to initial conditions leading to one of the solutions searched.

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