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Invariant TI-subgroups and structure of finite groups

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Let G be a finite group and assume that a group of automorphisms A is acting on G such that A and G have coprime orders. Recall that a subgroup H of G is said to be a TI-subgroup if it has trivial intersection with its distinct conjugates in G. We study the solubility and other properties of G when we assume that certain invariant subgroups of G are TI-subgroups, precisely when all A-invariant subgroups, all non-nilpotent A-invariant subgroups, and all non-abelian A-invariant subgroups of G, respectively, are TI-subgroups.

1. Introduction

Let G be a finite group. Recall that a subgroup H of G is called a TI-subgroup (trivial intersection subgroup) of G if either $H \cap H^g = 1$ or H for every $g \in G$. Trivial examples of TI-subgroups are minimal subgroups and normal subgroups of every group, and likewise, every Frobenius complement of a Frobenius group. Much less obvious is the fact that a cyclic Sylow subgroup of a finite simple group is a TI-subgroup [2]. During the last few years, a topic of interest has been to investigate the influence of TI-subgroups on the structure of finite groups, and particularly, to classify the finite groups in which certain subgroups are assumed to be TI-subgroups. This study was started by Walls [8], who gave a complete classification of the finite groups all whose subgroups are TI-subgroups. Later, other authors have obtained some classifications, for instance, that of those groups in which every abelian subgroup is a TI-subgroup [6]. This result, however, is much deeper and its proof requires the Classification of the Finite Simple Groups.

In this note, we assume that a group of automorphisms A is acting on G and that A and G have coprime orders. Then we only consider the subgroups of G that are invariant under the action of A, so that our goal is to determine the structure of G or some properties of G, such as solubility, under the assumption that

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certain A-invariant subgroups of G are TI-subgroups. Our first result extends Wall's classification (Main Theorem of [8]) by giving a somewhat wider classification within the coprime action setting. We point out that, however, we need to appeal to a result based on the Classification, concretely, on a coprime action version of the Schimdt groups (minimal non-nilpotent groups), which was previously obtained by the authors (Theorem A of [1]).

Theorem A. *Suppose that a group A acts coprimely on a finite group G. If all A-invariant subgroups of G are* TI-*groups then either G is nilpotent or a Frobenius group whose kernel is elementary abelian and its complement is either a cyclic group or the direct product of Q_8 and a cyclic group of odd order.*

In [7], the solubility and other properties of those groups whose non-nilpotent subgroups are TI-subgroups were studied. We extend these results into the coprime action scenario in the following way.

Theorem B. *Suppose that a finite group A acts coprimely on a finite group G. If every non-nilpotent A-invariant subgroup of G is a* TI-*subgroup, then G is soluble and every non-nilpotent A-invariant subgroup is normal in G.*

Groups in which every abelian subgroup is a TI-subgroup are not necessarily soluble. In fact, there exist exactly three such non-soluble groups, which furthermore are simple: $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(9)$ (see the Main Theorem of [3]). We have not been able to obtain a classification for those groups acted on by an coprime automorphism group such that all their abelian invariant subgroups are TI-subgroups. But inspired by the results of [6], we address the opposite situation, that is, when all the non-abelian invariant subgroups of a group are TI-subgroups. For brevity, we will say that a subgroup H of a group G is an A-TI-subgroup when H is an A-invariant TI-subgroup of G.

Theorem C. *Suppose that a finite group A acts coprimely on a finite group G. If every non-abelian A-invariant subgroup of G is a* TI-*subgroup, then G is soluble. Furthermore:*

(a) *If G is nilpotent, then every A-*TI-*subgroup of G is normal.*
(b) *If G is non-nilpotent, then*
  (1) *$G = KM$ is a Frobenius group with a kernel K and a complement M, both A-invariant, where K is minimal A-invariant normal subgroup of G and M is either a cyclic group or a product of Q_8 with a cyclic group of odd order.*
  (2) *$G = PH$ with P an A-invariant normal Sylow p-subgroup of G and H an A-invariant abelian p-complement of G. Furthermore, $\mathbf{C}_P(H)$ is abelian and normal in G, and $P/\mathbf{C}_P(H)$ is elementary abelian.*

2. Preliminaries

Regarding coprime action, we refer to [4, Chapter 8] for a detailed presentation and basic properties. The following two lemmas, however, do not require the fact that the action is coprime, but we include this hypothesis in their statements since this is the framework in which we are working in.

Lemma 2.1. *Suppose that a finite group A acts coprimely on a finite group G. Let N be a non-trivial A-invariant normal subgroup of G. If H is an A-*TI-*subgroup of G with $N \leq H$, then $H \trianglelefteq G$.*

Proof. For every $g \in G$, we have $N^g \leq H^g$. Since $N \trianglelefteq G$, then $N \leq H \cap H^g$. As H is an A-TI subgroup, we get $H = H^g$, and hence $H \trianglelefteq G$. □

If a group A acts on a group G (not necessarily their orders are coprime), we recall the concept of A-composition series of G (see for instance 7.32 of [5]). An A-series of G is a series of G the terms of which are A-invariant, and an A-composition series of G is a proper A-series of G which has no proper refinement (as an A-series). Notice that if G is soluble, then every factor of an A-composition series of G is elementary abelian, because such factor does not have any non-trivial proper A-invariant normal subgroup. We need the following property regarding A-composition series and TI-subgroups.

Lemma 2.2. *Suppose that a finite group A acts coprimely on a finite soluble group G. Let H be an non-abelian (non-nilpotent) A-invariant subgroup of G. If there exists a A-composition series*

$$H := H_s \trianglelefteq H_{s-1} \trianglelefteq H_{s-1} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G,$$

such that H_i is a TI*-subgroup of G for every $1 \leq i \leq s$, then $H \trianglelefteq G$.*

Proof. We argue by induction on s. Since H is non-abelian (non-nilpotent) and $H < H_{s-1}$, we have that H_{s-1} is also non-abelian (non-nilpotent), and by induction $H_{s-1} \trianglelefteq G$. Note that each factor H_i/H_{i-1} has no proper and non-trivial normal A-invariant subgroup, and since G is soluble, it follows that H_i/H_{i-1} is elementary abelian for every i. Then, for every $g \in G$, we have $H^g \leq H^g_{s-1} = H_{s-1}$. As $H \trianglelefteq H_{s-1}$, then $H \trianglelefteq HH^g \leq H_{s-1}$ (HH^g is not necessarily A-invariant). If $H \neq H^g$, then $H \cap H^g = 1$ for H being a TI-subgroup. But then $H^g \cong HH^g/H$ is a subgroup of H_{s-1}/H, which is elementary abelian. This contradicts the fact that H is non-abelian (non-nilpotent). Therefore, $H = H^g$ for every $g \in G$, that is, H is normal in G. $\square$

Unlike the above two lemmas, the coprimeness of the action is essential for proving the next result. Besides, we want to remark that, in the statement of the following lemma (and likewise throughout the paper), by maximal A-invariant subgroup we mean a subgroup that is maximal among all proper A-invariant subgroups. Of course, if the action is non-trivial, then such a subgroup need not be a maximal subgroup.

Lemma 2.3. *Suppose that a finite group A acts coprimely on a finite group G. If every maximal A-invariant subgroup of G is normal, then G is nilpotent.*

Proof. For every prime p dividing $|G|$ we can choose an A-invariant Sylow p-subgroup P of G. If P is not normal in G, then we take a maximal A-invariant subgroup M of G such that $\mathbf{N}_G(P) \leq M$. By Sylow properties $\mathbf{N}_G(M) \leq M$, which certainly leads to a contradiction because $M \trianglelefteq G$ by the hypothesis. This proves that G is nilpotent. $\square$

As already pointed in the Introduction, for our proofs we use the following extension of Schmidt groups, whose proof relies on the Classification of the Finite Simple Groups.

Theorem 2.4. *Suppose that a finite group A acts coprimely on a finite group G. If every A-invariant proper subgroup of G is nilpotent, then G is soluble. Furthermore, if G is non-nilpotent, then $|G| = p^a q^b$ and G has an A-invariant normal Sylow p-subgroup.*

Proof. This is equivalent to Theorem B of [1]. $\square$

We can say more about the structure of those groups from the above result if, in addition, all of their invariant proper subgroups are abelian.

Theorem 2.5. *Suppose that a finite group A acts coprimely on a finite group G. If every A-invariant proper subgroup of G is abelian and G is non-nilpotent, then $G = PQ$, where P is an A-invariant normal Sylow p-subgroup, Q is an A-invariant Sylow q-subgroup (both abelian), such that $P/\mathbf{C}_P(Q)$ is elementary abelian.*

Proof. By applying Theorem 2.4, we only have to prove that $\Phi(P) \leq \mathbf{C}_P(Q)$. Since Q is not normal in G, we can choose a maximal A-invariant M such that $\mathbf{N}_G(Q) \leq M$. Then $M = Q(M \cap P)$ and since M is abelian we have

$$M \cap P \leq \mathbf{C}_P(Q) \leq \mathbf{N}_P(Q) = M \cap P.$$

As a consequence, $M = Q \times \mathbf{C}_P(Q)$. Now, we distinguish two possibilities. If $\Phi(P) \leq M$, then we trivially have the result. So by the maximality of M, we can assume that $G = \Phi(P)M$. This certainly implies that $\Phi(P)\mathbf{C}_P(Q) = P$ and this forces $\mathbf{C}_P(Q) = P$, that is, G is nilpotent, a contradiction. □

3. Main results

Theorem A. *Suppose that a group A acts coprimely on a group G. If all A-invariant subgroups of G are* TI-*subgroups, then either G is nilpotent or a Frobenius group whose kernel is elementary abelian and its complement is either a cyclic group or the direct product of Q_8 and a cyclic group of odd order.*

Proof. Assume that G is non-nilpotent, and take some A-invariant Sylow subgroup Q such that $\mathbf{N}_G(Q) < G$. Then we take a maximal A-invariant subgroup M containing $\mathbf{N}_G(Q)$, which obviously satisfies $\mathbf{N}_G(M) = M$. As M is a TI-subgroup, then G is a Frobenius group with complement M. Write $G = K \rtimes M$, where K is the kernel. First, we prove that K is minimal A-invariant normal subgroup of G. Assume that this is false and take N to be a minimal A-invariant normal subgroup of G, with $N \leq K$. Then $NM < G$ and this contradicts the maximality of M. Consequently, K is p-elementary abelian. Let M_q be an A-invariant Sylow q-subgroup of M, for some prime q. Then $K \rtimes M_q$ is also a Frobenius group with kernel K and complement M_q. Moreover, by Lemma 2.1, we have $K \rtimes M_q \trianglelefteq G$, so $M_q \trianglelefteq M$. This proves that M is nilpotent. Now, if $q \neq 2$, then M_q is cyclic by [4, 8.3.8 and 8.3.2], and if $q = 2$, then M is a generalized quaternion group by the same well-known result. Thus, if A does not act trivially on M_q, we get $|M_q| = 8$ according to the fact that the automorphism group of a generalized quaternion group of order larger than 8 is a 2-group (see for instance [4, Exercise 5.7]). So we only have to consider the case in which A acts trivially on M. In that case, for every non-trivial subgroup M_1 of M_q, we have $KM_1 \trianglelefteq G$ by Lemma 2.1, and this trivially implies that $M_1 \trianglelefteq M_q$. This means that M_q is a Dedekind group, so again $|M_q| = 8$ and $M_q \cong Q_8$. We conclude that the complements of G have the structure of the statement. □

Theorem B. *Suppose that a finite group A acts coprimely on a finite group G. If every non-nilpotent A-invariant subgroup of G is a* TI-*subgroup, then G is soluble and every non-nilpotent A-invariant subgroup is normal in G.*

Proof. We prove first that G is soluble by counterexample of minimal order. Thus, assume that G is a non-soluble group of minimal order satisfying that every non-nilpotent A-invariant subgroup is a TI-subgroup. We easily get that every non-nilpotent maximal A-invariant subgroup of G is soluble by minimality. First, notice that there exists a proper non-nilpotent A-invariant subgroup in G, otherwise Theorem 2.4 leads to the solubility of G. Thus, let M be a non-nilpotent maximal A-invariant subgroup of G. Note that $\mathbf{N}_G(M)$ is also A-invariant, so $M \trianglelefteq G$ or $\mathbf{N}_G(M) = M$. Suppose first that $M \trianglelefteq G$, and notice that G/M is A-irreducible, that is, G/M does not have any non-trivial and proper A-invariant subgroup. By [4, 8.2.3], it follows that G/M is a group of prime power order (in fact, it is elementary abelian) and hence it is soluble. As M is soluble, we conclude that G is soluble, a contradiction. Now we consider the case $M = \mathbf{N}_G(M)$. Since M is a TI-subgroup, we get that G is a Frobenius group with complement M by [4, 4.1.7]. Write $G = U \rtimes M$, where U is the Frobenius kernel, which is well-known that is nilpotent. Consequently, G is soluble too, a contradiction.

Next we prove that every non-nilpotent A-invariant subgroup is normal in G. Let H be a non-nilpotent A-invariant subgroup of G, which is a TI-subgroup by hypothesis. Assume that H is not normal in G. Let $N := \mathbf{N}_G(H) < G$. We claim that $H < N$. Assume on the contrary that $H = N$ and then G is a Frobenius group with complement H. Write $G = K \rtimes H$, where K is the kernel. As H is non-nilpotent, it does not have prime power order. Then we take H_p an A-invariant Sylow p-subgroup of H for each prime divisor p of $|H|$. It follows that KH_p is also a Frobenius group, so in particular is non-nilpotent. By hypothesis, KH_p is a TI-subgroup and $KH_p \trianglelefteq G$ by Lemma 2.2. Let $\overline{G} := G/K$. Then $\overline{H_p} \trianglelefteq \overline{G}$. By the arbitrariness of p, we conclude that $\overline{G} \cong H$ is nilpotent, a contradiction. Therefore $H < N$ as claimed.

Now we consider the following A-series

$$H := H_s \trianglelefteq N := H_{s-1} \trianglelefteq H_{s-2} \trianglelefteq \cdots \trianglelefteq H_m \trianglelefteq \cdots ,$$

where $H_i = \mathbf{N}_G(H_{i+1})$. If there is some i such that $H_i = G$, then we can refine this series to an A-composition series, from H to G. By Lemma 2.2, we obtain that H is normal in G and we are done. If such i does not exist, then there is some j such that $H_j = H_{j+1} < G$. Let $L := H_j$. By the same reason as above and using Lemma 2.2, we have $H \trianglelefteq L$, that is, $L = N$ and also $N = \mathbf{N}_G(N)$. Since H is non-nilpotent and $H < N$, then N is a non-nilpotent A-invariant subgroup of G, which moreover is a TI-subgroup by hypothesis. By [4, 4.1.7], G is a Frobenius group with complement N. Arguing as in the above paragraph, we easily obtain that N is nilpotent, and so is H. This contradiction finishes the proof. □

Theorem C. *Suppose that a finite group A acts coprimely on a finite group G. If every non-abelian A-invariant subgroup of G is a TI-subgroup, then G is soluble. Furthermore:*

(a) *If G is nilpotent, then every non-abelian A-TI-subgroup of G is normal.*
(b) *If G is non-nilpotent, then*
  (1) *$G = KM$ is a Frobenius group with a kernel K and a complement M, both A-invariant, where K is minimal A-invariant normal subgroup of G and M is either a cyclic group or a product of Q_8 with a cyclic group of odd order.*
  (2) *$G = PH$ with P an A-invariant normal Sylow p-subgroup of G and H an A-invariant abelian p-complement of G. Furthermore, $\mathbf{C}_P(H)$ is abelian and normal in G, and $P/\mathbf{C}_P(H)$ is elementary abelian.*

Proof. First we prove that G is soluble by induction on $|G|$. We can certainly assume, by Theorem 2.4, that G possesses a non-abelian maximal A-invariant subgroup, say M. Also, every non-abelian maximal A-invariant subgroup of G is soluble by the inductive hypothesis. Thus, if M is normal in G, then G/M is A-irreducible, so it is a group of prime power order. In particular, G/M is soluble and thus G is soluble, so we are finished. If M is not normal in G, by maximality we have $M = \mathbf{N}_G(M)$. Since M is a TI-subgroup of G, by [4, 4.1.7], we obtain that G is a Frobenius group with complement M. Write $G = K \rtimes M$, where K is the kernel. If M is a group of prime power order, since K is nilpotent, it follows that G is also soluble. So we can assume that M does not have prime power order. Since M is A-invariant, by [4, 8.2.3], we have that M has an A-invariant Sylow p-subgroup M_p of M for every prime divisor p of the order of M. Note that K is an A-invariant normal subgroup of G. Then KM_p is non-abelian and hence KM_p is a TI-subgroup. By Lemma 2.2, $KM_p \trianglelefteq G$. Let $\overline{G} := G/K$. By the arbitrariness of p, we conclude that $\overline{G}$ is nilpotent. Note that $\overline{G} \cong M$, so M is soluble too. Therefore G is soluble.

The rest of the proof is divided into two cases.

Case 1. G is nilpotent.

We prove that every non-abelian A-invariant subgroup of G is normal. Let M be a non-abelian A-invariant subgroup of G that is not normal in G and let $N := \mathbf{N}_G(M)$. We have the following A-invariant series

$$M = M_1 \trianglelefteq N := M_2 \trianglelefteq M_3 \trianglelefteq \cdots \trianglelefteq M_s \trianglelefteq \cdots,$$

where $M_{i+1} = \mathbf{N}_G(M_i)$. Since G is nilpotent, there is some i such that $M_i = G$. Then we can refine the series to be an A-composition series of G. By Lemma 2.2, we have that M is normal in G, a contradiction. This proves (a).

Case 2. G is non-nilpotent. We distinguish two subcases corresponding to two excluding assumptions.

Case 2.1. Assume that there is a non-abelian maximal A-invariant subgroup M of G which is not normal in G.

We certainly have $M = \mathbf{N}_G(M)$. As M is a TI-subgroup, then G is a Frobenius group with complement M. Write $G = K \rtimes M$, where K is the kernel. First, we prove that K is minimal A-invariant normal subgroup of G. Assume this is false and let p be a prime divisor of $|K|$. Since K is MA-invariant, by coprime action there is a MA-invariant Sylow p-subgroup K_p of K. It is trivial that $K_pM < G$ and is also non-abelian. But this contradicts the maximality of M. Hence K is a p-group for some prime p. By a similar argument, we can get that K is minimal A-invariant normal subgroup of G. Next we prove that M has prime power order. Let M_q be an A-invariant Sylow q-subgroup of M, for some prime q. Then $K \rtimes M_q$ is also a Frobenius group with kernel K and complement M_q. By the same reason as above, we have $K \rtimes M_q \trianglelefteq G$. If $q > 2$, then M_q is a cyclic group by [4, 8.3.8 and 8.3.2]. If $q = 2$, then M is a generalized quaternion group by [4, 8.3.8 and 8.3.2]. Now, if A does not act trivially on M_q, we get $|M_q| = 8$ according to the fact that the automorphism group of a generalized quaternion group of order larger than 8 is a 2-group (see for instance [4, Exercise 5.7]). So we only have to consider the case $M_q \leq \mathbf{C}_G(A)$. For every non-trivial subgroup M_1 of M_q, we have $KM_1 \trianglelefteq G$ by the same reason as above, and this trivially implies that $M_1 \trianglelefteq M_q$. This means that M_q is a Dedekind group, so again $|M_q| = 8$ and $M_q \cong Q_8$. By the same reason as above, we can get $KM_r \trianglelefteq G$ for every A-invariant Sylow r-subgroup of M. Therefore, we conclude that M is nilpotent and the 2-complement of M is cyclic. This is part (1) of the Theorem.

Case 2.2. Assume that every non-abelian maximal A-invariant subgroup of G is normal in G.

It may happen that every proper A-invariant subgroup of G is abelian. In that case, the structure of G is given by Theorem 2.5, that is, $G = PQ$ with P an A-invariant normal Sylow p-subgroup of G and Q an A-invariant Sylow q-subgroup of G satisfying all the conditions of case (2)(b). Henceforth, we assume that G has a non-abelian maximal A-invariant subgroup. We claim that G has at least one normal A-invariant Sylow subgroup. Suppose not and let us take an A-invariant Sylow p-subgroup P of G for every prime p dividing $|G|$. Then $\mathbf{N}_G(P)$ is A-invariant and we note that must be abelian, otherwise $\mathbf{N}_G(P)$ would be contained in some non-abelian maximal A-invariant subgroup, say M, which by our assumption is normal in G. However, the Frattini argument gives $G = M\mathbf{N}_G(P)$, so we get a contradiction. Now, the fact that all Sylow normalizers are abelian implies that G is abelian. This can be seen, for instance, by applying the well-known Burnside p-complement Theorem for every prime p dividing the order of G, since then G would be p-nilpotent for every p, and as a consequence, nilpotent. This contradiction proves the claim.

We construct the following subgroups. Let R be the (direct) product of all normal Sylow subgroups of G and let π be the set of primes corresponding to those Sylow subgroups. Of course, R is an A-invariant normal Hall subgroup of G, and we take T to be an A-invariant π-complement of G. For every prime $q \in \pi'$

and every A-invariant Sylow q-subgroup Q_1 of T, by the above argument, we know that $\mathbf{N}_G(Q_1)$ is abelian. In particular, $\mathbf{N}_T(Q_1)$ is also abelian, so we deduce that T is abelian.

It is clear that there exists a prime $p \in \pi$ and an A-invariant Sylow p-subgroup P of G such that $[P,T] \neq 1$, otherwise T would be central in G, which contradicts our assumption that G is non-nilpotent. Thus PT is non-abelian, and by Lemma 2.1, $PT \trianglelefteq G$. Now, let $q \in \pi'$ and choose an A-invariant Sylow q-subgroup Q of T. The Frattini argument gives $P\mathbf{N}_G(Q) = G$, where we know that $\mathbf{N}_G(Q)$ is abelian. Now, let M be a maximal A-invariant subgroup of G containing $\mathbf{N}_G(Q)$. Since M cannot be normal in G, by our assumption, M must be abelian. In particular $M = \mathbf{N}_G(Q)$. We conclude that G possesses abelian Hall p-complements. Now, for every prime $s \in \pi$, $s \neq p$, if we take S the Sylow s-subgroup of R, it follows that $[PT,S] \leq PT \cap S = 1$, so $[S,T] = 1$. This proves that

$$G = PT \times S_1 \times \ldots \times S_n,$$

where S_i for $i = 1,\ldots,n$ are the (A-invariant) Sylow subgroups of R distinct from P. Furthermore, S_i is abelian for every i (indeed $S_i \leq \mathbf{Z}(G)$ for every i). Now write $H = T \times S_1 \times \ldots \times S_n$, which is an A-invariant abelian p-complement of G. Therefore, the first assertion of (2) is proved.

Let $P_0 := \mathbf{C}_P(H) = \mathbf{C}_P(T)$. First, we prove that $M = P_0 \times H$. As H is abelian, then $H \leq \mathbf{N}_G(Q) = M$. Hence, as $G = PH$, then $M = (M \cap P)H$. Moreover, as M is abelian

$$M \cap P \leq P_0 \leq \mathbf{C}_P(Q) \leq \mathbf{N}_P(Q) = M \cap P,$$

so the equality $M = P_0 \times H$ holds.

Finally, we prove that $\Phi(P) \leq P_0$ and that $P_0 \trianglelefteq G$. We know that $\mathbf{C}_G(P_0) \geq M$ and by maximality of M we have $\mathbf{C}_G(P_0) = G$ or $\mathbf{C}_G(P_0) = M$. If $\mathbf{C}_G(P_0) = G$, then $P_0 \leq \mathbf{Z}(G)$ and thus $P_0 \trianglelefteq G$. Let $\overline{G} := G/P_0$. Assume that there is some non-trivial proper $A\overline{M}$-invariant subgroup $\overline{U}$ of $\overline{P}$. Then $\overline{MU}$ is also a proper A-invariant subgroup of $\overline{G}$ and thus $M < MU$ is a proper A-invariant subgroup of G, contradicting the maximality of M. As a result $A\overline{M}$ acts irreducibly on $\overline{P}$. So we get $\Phi(P) \leq P_0$, that is, P/P_0 is elementary abelian. Assume now that $\mathbf{C}_G(P_0) = M$ and we prove again that $\Phi(P) \leq P_0$. Otherwise, $M < \Phi(P)M$ is A-invariant. By the maximality of M, we have $G = \Phi(P)M$. Since $M = P_0 \times H$, we get $G = \Phi(P)P_0H$. Note that $\Phi(P)P_0 \leq P$ implies that $P = \Phi(P)P_0$, and then $P = P_0$, a contradiction. Hence $\Phi(P) \leq P_0$, as wanted. Also, this implies that $P_0 \trianglelefteq P$, and since H centralizes P_0, we conclude that $P_0 \trianglelefteq G$. Thus, the proof of (2) is finished. $\square$

Examples. We show that all cases in Theorem C are feasible. Every Dedekind group admitting a coprime automorphism group is an immediate example of case (a). For a non-trivial example of this case, let us take two copies, H_1 and H_2, of the quaternion group of order 8. Each of these has a coprime automorphism of order 3, say α and β respectively, and we suppose further that α is acting trivially on H_2 and that β is acting trivially on H_1. Then $A = \langle\alpha\rangle \times \langle\beta\rangle$ acts coprimely on $G = H_1 \times H_2$, and this group clearly satisfies the hypotheses of Theorem C. The non-abelian A-invariant (proper) subgroups of G are exactly H_i for $i = 1, 2$, and $H_1 \times \mathbf{Z}(H_2)$ and $\mathbf{Z}(H_1) \times H_2$, all of which are normal in G.

Let us consider $D_{2n} = \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1}\rangle$, the dihedral group of order $2n$, where n is an integer such that $\varphi(n)$ is divisible by some odd prime q (where φ denotes the Euler function). It is clear that D_{2n} is a Frobenius group which admits a coprime automorphism of order q that acts as an automorphism group of $\langle x\rangle$ and trivially on $\langle y\rangle$. Every proper invariant subgroup of D_{2n} is abelian, so this group satisfies the hypotheses of case (b)(1).

The group $G =$ SL(2, 3) with the trivial action is an example of group for case (b)(2), which in addition is not a Frobenius group. For an example of case (b)(2) with a non-trivial action, let $p \neq 2, 3$ be a prime, C_p the cyclic group of order p and consider the permutation wreath product $W = C_p \mathrm{Wr} S_3$, where $S_3 = HA$

is the symmetric group of degree 3, and H and A are cyclic groups of order 3 and 2, respectively. Let $P = C_p \times C_p \times C_p$ be the base group of W. Then A acts coprimely on $G = PH$. It is easily seen that every A-invariant proper subgroup of G is abelian, so in particular, G satisfies the hypotheses of Theorem C, and this is another example of case (2)(b), (which is not either a Frobenius group). Indeed, G verifies the hypotheses of Theorem 2.5.

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