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Dynamics of Newton method for symmetric quartic polynomial

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Abstract

We consider the Newton method on symmetric quartic polynomials. The parameter space is divided into different regions with different dynamical behaviours. In this paper, we study the dynamics for values of the parameter on the bifurcations curves that separate those regions, checking its dynamical behaviour due to the presence of multiple roots. We also consider a modified Newton method and we prove that this method is always conjugate to the rational map $\frac{m}{n}z^2$ when it is applied on polynomials with two roots of multiplicity m and n, respectively.

1 Introduction

Newton method is the best known algorithm for finding the roots of an equation $F(z) = 0$. The study of the global dynamics of this method goes back to Ernest Schröder and Artur Caley who considered the method applied to low degree polynomials as a rational map defined on the Riemann sphere. This global study provides important implications at computational level (see for instance [4]).

The dynamics of Newton method has been widely studied for low degree polynomials. Although there is not a general study on the dynamics of Newton method for high order polynomials, several results on concrete families of polynomials have been approached. In [1] and [2] we consider quartic and quintic polynomials, respectively.

In [1] we study the Newton method applied on symmetric quartic polynomials, because they frequently appear in the dynamical study of other families of iterative methods (see [3], for example). We select the family of symmetric quartic polynomials with two real parameters, thus the parameter space is $\mathbb{R}^2$. From the theoretical point of view these results are a first step for having a better understanding of Newton's method applied to quartic polynomials. In that paper we show the existence of bifurcations curves separating the parameter space into different regions where different dynamical behaviours are exhibited and we carry out the dynamical study of Newton method in each region.

In this paper we study the dynamics for values of the parameter on the bifurcations curves. The following section is devoted to recall the main results obtained in [1]. In Section 3, we study the dynamics of Newton method on the bifurcation curves, checking that its dynamical behaviour is more complicated due to the presence of multiple roots. In the last section we consider a modified Newton method in order to improve this study. Finally, we prove that this method is always conjugate to the rational map $\frac{m}{n}z^2$ when it is applied on polynomials with two roots of multiplicity m and n, respectively (Theorem 5).

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2 Previous results

In [1], we study some topological properties of the parameter and dynamical plane of Newton's method applied to the family of four degree symmetric polynomials:

$$p_{a,b} := p_{a,b}(z) = z^4 + az^3 + bz^2 + az + 1, \tag{1}$$

when a and b are real parameters. In that paper, we split the parameter plane into twelve regions in which the roots of the polynomials p, p' and p'' have simple zeroes. We only study the parameter plane for $a \geq 0$ because of the symmetry. We determine in which of those parameter regions it can be guaranteed that, except for a measure zero set (which is no relevant from the numerical point of view), any seed in dynamical plane converges to a root of p (Proposition 1 and Proposition 2). We also give numerical evidences that a more complicated and chaotic dynamics is possible in other regions.

The expression of the Newton's map applied to $p_{a,b}$ writes as

$$N := N_{a,b}(z) = z - \frac{p_{a,b}(z)}{p'_{a,b}(z)} = z - \frac{z^4 + az^3 + bz^2 + az + 1}{4z^3 + 3az^2 + 2bz + a}. \tag{2}$$

The critical points of N are the solutions of $N'(z) = 0$; that is, the roots of p and p''. For each root of p, $r_i(a,b) := r_i, i = 1, \cdots, 4$, we define its *basin of attraction*, $\mathcal{A}_{a,b}(r_i)$, as the set of points in the complex plane which tend to r_i under the Newton's map iteration. In general, $A_{a,b}(r_i)$ may have infinitely many connected components but only one of them, called *immediate basin of attraction of* r_i, contains $z = r_i$.

Primary bifurcations correspond to parameters (a, b) for which the roots of p, p' or p'' collide and we denote them as $L_i, i = 1, ..., 5$. The connected components of the complement of this set of parameters define regions in the parameter plane where the polynomials p, p', and p'' have simple roots. In each of those regions the Newton's methods $N_{a,b}$ need not be, in general, dynamically equivalent.

The roots of $p\left(x\right)$ are given by

$$x = \frac{y_\pm \pm \sqrt{y_\pm^2 - 4}}{2},$$

where

$$y_\pm = \frac{-a \pm \sqrt{a^2 - 4b + 8}}{2}.$$

So, the curves:

$$\begin{aligned} L_1 &: = \{(a,b) \in \mathbb{R}^2 \mid a^2 - 4b + 8 = 0\}, \\ L_2 &: = \{(a,b) \in \mathbb{R}^2 \mid b = 2a - 2\}, \\ L_3 &: = \{(a,b) \in \mathbb{R}^2 \mid b = -2a - 2\}, \end{aligned}$$

separate the plane into different regions, depending on the number of real and complex roots contained in each of them.

There are one specific choice of the parameters, $(a, b) = (4, 6)$, for which $p_{4,6}(x)$ has a unique root of multiplicity four. The bifurcations of the roots along the curves $L_1 \cup L_2 \cup L_3$ are of different nature. When the parameters $(a, b) \neq (4, 6)$ are in L_1 the polynomial $p_{a,b}$ exhibits two roots of multiplicity two. These two double roots are complex for $0 \leq a < 4$ and real for $a > 4$. When the parameters $(a, b) \neq (4, 6)$ are in $L_2 \cup L_3$ the polynomial $p_{a,b}$ has a double real root and two simple roots. The roots of $p'(x) = 4x^3 + 3ax^2 + 2bx + a$ are also studied. The number of real roots of $p'\left(x\right)$ gives us the number of vertical asymptotes of the Newton's operator. On the curve

$$L_4 := \{(a,b) \in \mathbb{R}^2 | 27a^4 + 108a^2 - 108a^2b - 9a^2b^2 + 32b^3 = 0\},$$

$p'\left(x\right)$ has two real roots, one simple and one double. This curve delimits the regions where $p'\left(x\right)$ has one or three real roots.

Finally, the roots of $p''(x)$, c_1 and c_2, collide on the curve

$$L_5 := \{(a,b) \in \mathbb{R}^2 \mid b = 3a^2/8\}.$$

The curves $L_1 \cup \cdots \cup L_5$ define the primary bifurcation parameters and the connected components of the complement in the parameter plane:

$$\{(a,b) \in \mathbb{R}^2 \, , \, a \geq 0\} \setminus \bigcup_{i=1}^{5} L_i,$$

denoted by $\mathcal{R}$, are formed by parameter values where the polynomials p, p' and p'' have a constant number of simple roots. The number of real roots of p, p' and p'' in each region is shown in the following table.

Regions	Roots of $p\left(x\right)$	Roots of $p'\left(x\right)$	Roots of $p''\left(x\right)$
$\mathcal{R}_1$	2	1	2
$\mathcal{R}_2$	0	1	0
$\mathcal{R}_3$	0	1	2
$\mathcal{R}_4$	0	3	2
$\mathcal{R}_5$	4	3	2
$\mathcal{R}_6$	2	3	2
$\mathcal{R}_7$	2	1	0
$\mathcal{R}_8$	0	1	2
$\mathcal{R}_9$	0	3	2
$\mathcal{R}_{10}$	4	3	2
$\mathcal{R}_{11}$	2	3	2
$\mathcal{R}_{12}$	0	1	0

Table 1. Regions where the zeroes of p, p' and p'' are simple.

In (Figure 1) we draw these bifurcations curves and the regions R_i for a, b real parameters.

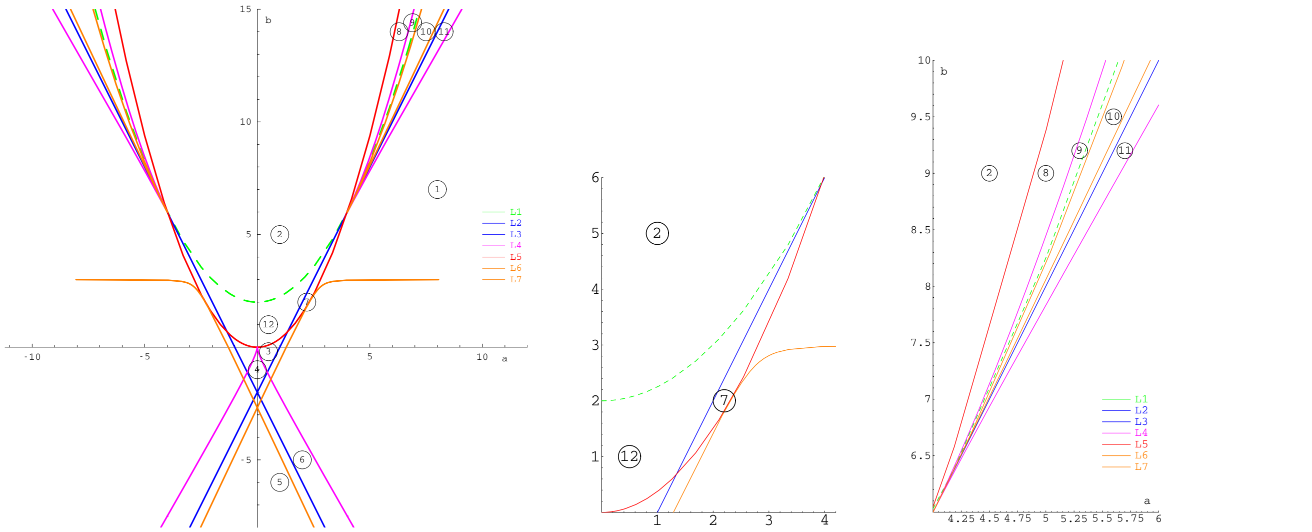

Figure 1: Regions in parameter plane bounded by the curves L_1, L_2, L_3, L_4 and L_5 and two zooms. The curves L_6 and L_7, corresponding to a collision of the roots of p and p''.

Afterwards, we study the dynamical behaviour of the Newton method in each of these regions obtaining the following results:

Firstly, we show that if the polynomial p has four different real roots all initial conditions converge to one of them:

Proposition 1 *If the polynomial $p_{a,b}$ has four different real roots then, except for a measure zero set, all initial conditions converge to one of them, that is:*

$$\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4).$$

If p has two real roots (regions $\mathcal{R}_1, \mathcal{R}_6, \mathcal{R}_7$ and $\mathcal{R}_{11}$), depending on the region under consideration, the dynamics of $N_{a,b}$ can be from *simple* (that is, the *free* critical points are captured and the Fatou set coincide with the union of the attracting basins of the zeroes of $p_{a,b}$) to *rich* (that is either one or both of the *free* critical points are allowable to do their own dynamics and the Fatou set is not reduced to points whose orbits converge to one of the roots of $p_{a,b}$). For example, all initial conditions converge to one of the roots of p if $(a, b) \in \mathcal{R}_7$:

Proposition 2 *Let $(a, b) \in \mathcal{R}_7$. Then, except for a measure zero set, all initial conditions converge to one of the roots of p, that is:*

$$\mathcal{F}(N_{a,b}) = A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4).$$

Nevertheless, in R_1 and R_6 the situation is quite different. Although the smallest free critical point is captured by one root, we still do not know in general the dynamics of the other critical point. In fact, numerical examples illustrate that, for some parameters, there are open sets of initial conditions (in the dynamical plane) where the orbit do not converge to any of the roots. Moreover, by means of the Implicit Function Theorem, those bad parameter values form an open set in parameter plane.

Proposition 3 *Let assume $(a, b) \in \mathcal{R}_1 \cup \mathcal{R}_6$. Then $c_1 \in A(r_2)$.*

In the case of region $\mathcal{R}_{11}$ there are parameter values for which none of the free critical points is captured by any of the attracting basins of the roots of p.

Finally, if p has not real roots, i.e. parameter values in $\mathcal{R}_i$ with $i \in \{2, 3, 4, 8, 9, 12\}$, the orbit of the two *free* critical points, c_1 and c_2 need not be captured by the roots of the polynomial p. So, the Fatou set will be larger than the union of the attracting basins of the roots of p.

Proposition 4 *There are parameters values $(a, b) \in \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_8, \mathcal{R}_{12}$ such that*

$$A(r_1) \cup A(r_2) \cup A(r_3) \cup A(r_4) \subsetneq \mathcal{F}(N_{a,b}).$$

Now, we analyze in this paper the dynamics of Newton method for those values of the parameters that give rise to multiple roots of p, p' or p''.

3 Dynamics of Newton method on the bifurcation curves

If we consider parameter values on the primary bifurcation curves, the Newton method depends only on one parameter and we observe that the double roots are not critical points. For $a = \pm 4, b = 6$ the polynomial has one root of multiplicity four and we do not consider them. Moreover, the Newton method has two free critical points, but one of them must be in the basin of attraction of the double root; so, there are only one parameter plane and we can carry out a dynamical study in the complex plane. Let us see what happens on every curve.

3.1 Dynamics on the curve L_1

For $(a, b) \in L_1$, that is, if $a^2 - 4b + 8 = 0$ the polynomial is $p(x) = \frac{1}{4}(2 + ax + 2x^2)^2$; so, it has two double roots $r_{1,2} = \frac{1}{4}(-a \pm \sqrt{-16 + a^2})$. These fixed points are always attractive being that $|N'(r_{1,2})| = \frac{1}{2}$; so, one of the critical points

$$c_{1,2} = \frac{1}{12}(-3a \pm \sqrt{3(-16 + a^2)}$$

must be in the basin of attraction of one of the roots. The parameter plane can be see in Figure 2.

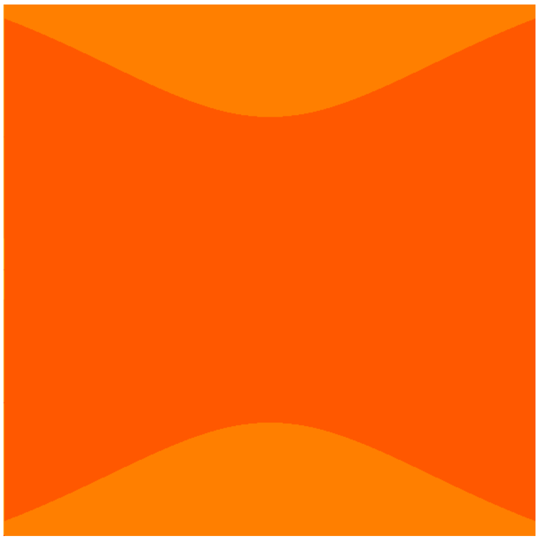

Figure 2: The parameter plane of $\mathcal{N}_a$ for complex values of a on the bifurcation curve L_1

3.2 Dynamics on the curves L_2 and L_3

These two curves correspond to quartic polynomials with one double root. For $(a, b) \in L_2$, that is, if $b = 2a - 2$ the polynomial is $p(x) = (1 + x)^2(2 + (a - 2)x + x^2)$, whose roots are $r = -1$ and $r_{1,2} = \frac{1}{2}(2 - a \pm \sqrt{-4a + a^2})$. As before, we can check that $x = -1$ is an attractive fixed point that becomes superattractive for $a = 4$. The fixed points $r_{1,2}$ are always superattractive. Moreover, there are two free critical points:

$$c_{1,2} = \frac{1}{12}(-3a \pm \sqrt{3(16 - 16a + 3a^2)}.$$

As $r = -1$ is an attractor, one of the critical points must be in his attraction basin and the other critical points makes its own dynamics.

Similarly, if $(a, b) \in L_3$ that is $b = -2a - 2$, then the polynomial is $p(x) = (-1+x)^2(2+(2+a)x+x^2)$ and the double root $r = 1$ is an attractor that becomes superattractor for $a = -4$. For any other value of a, there exist two free critical points

$$c_{1,2} = \frac{1}{12}(-3a \pm \sqrt{3(16 + 16a + 3a^2)}$$

and one of them must be in the attraction basin on $r = -1$. (See Figure 3).

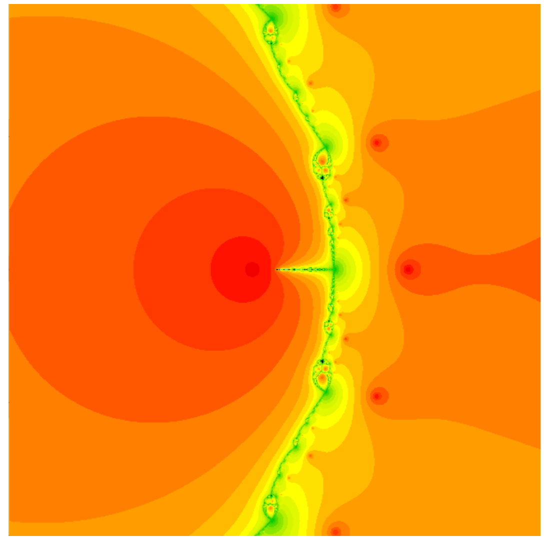

Figure 3: The parameter plane of $\mathcal{N}_a$ for complex values of a on the bifurcation curve L_2

3.3 Dynamics on the curve L_5

Now, we show the dynamics on the bifurcation curve L_5 defined by

$$L_5 = \{(a, b) \in \mathbb{R}^2 | b = 3a^2/8\}.$$

This curve consists of the set of parameter values for which the two simple critical points c_1 and c_2 collide in a double critical point located at $c_1 = c_2 = -a/4$. Under this assumption we can treat the Newton's map

$$\mathcal{N}_a := N_{a,b} \qquad \text{where} \qquad b = \frac{3a^2}{8}$$

as a one *complex* parameter family of rational maps in the Riemann sphere. In this case, fixed points are also critical points and there is a unique *free* critical point of multiplicity two located at $-a/4$.

$$\mathcal{N}_a(z) = z - \frac{p_{a,3a^2/8}(z)}{p'_{a,3a^2/8}(z)} = z - \frac{z^4 + a(z^3 + z) + \frac{3}{8}a^2z^2 + 1}{4z^3 + a(3z^2 + 1) + \frac{3}{4}a^2z},$$

In Figures 4 and 5 we plot the parameter plane of $\mathcal{N}_a$.

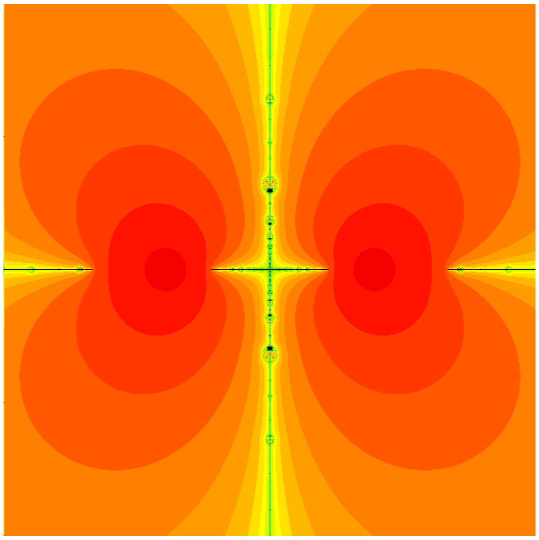

Figure 4: The parameter plane of $\mathcal{N}_a$ for complex values of a on the bifurcation curve L_5

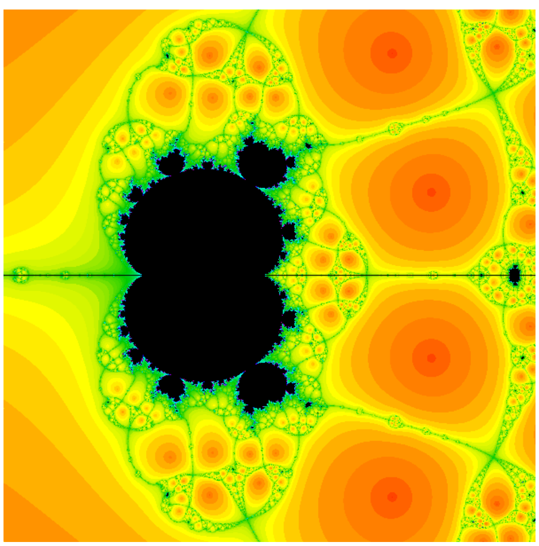

Figure 5: Zoom of $\mathcal{N}_a$

From the symmetries described in [1], we know that those small copies of $\mathcal{M}_3$ in the parameter plane of $\mathcal{N}_a$ are symmetrically located with respect the real (and complex) line. Moreover, it is possible to find numerically some real parameters $a \in \mathbb{R}$ such that the critical point $-a/4$ belongs a superattracting cycle of period $k \geq 2$.

4 The modified Newton method for multiple roots

As we have seen in the previous section, Newton's method is not a good method when we are working with a function that has multiple roots since they are eliminated in the expression of the rational operator. To avoid this, we can use the modified Newton method [5] :

$$(3) \qquad H(x) = x - \frac{f(x)\, f'(x)}{(f'(x))^2 - f(x)\, f''(x)}.$$

Let us apply this operator on the polynomial obtained on the curves L_1, L_2 and L_3.

For $(a,b) \in L_1$ the quartic polynomial becomes $p(x) = \frac{1}{4}\left(2 + ax + 2x^2\right)^2$ and the operator writes as

$$H_1(x) = -\frac{2\left(a + 8x + ax^2\right)}{-8 + a^2 + 4ax + 8x^2}$$

whose fixed points coincide with the roots of the polynomial. The derivative of this operator

$$H_1'(x) = \frac{4(a-16)\left(2 + ax + 2x^2\right)}{\left(-8 + a^2 + 4ax + 8x^2\right)^2}$$

gives also the roots of the polynomial as the only critical point. So, every initial condition goes to a root of the polynomial.

However, if $(a,b) \in L_2$ the quartic polynomial becomes $p(x) = (1+x)^2\left(1 + (a-2)x + x^2\right)$ and

$$H_2(x) = x - \frac{(1+x)\left(1 + (a-2)x + x^2\right)\left(a + (3a-4)x + 4x^2\right)}{q_4(x)}$$

where

$$\begin{aligned} q_4(x) &= (a-2)^2 + 2\left(-4 - a + a^2\right)x + \\ &+ \left(8 - 8a + 3a^2\right)x^2 + (-8 + 6a)\,x^3 + 4x^4. \end{aligned}$$

In this case, the roots are also fixed points of the operator and it is easy to check that they are also critical points. Nevertheless, the dynamics of this operator is more complicated because there appear three free critical points.

Similarly, for parameter values on L_3, the roots of the polynomial $p(x) = (x-1)^2\left(1 + (2+a)x + x^2\right)$ are fixed and critical points of the corresponding operator. As before, there appear three more free critical points and the dynamics of this operator becomes more complicated.

4.1 Modified Newton method for two multiple roots

From the previous section we observe that the modified Newton method works really well when we the polynomial has two double roots. As we prove in the following, this result is also true when the multiplicity of the roots is arbitrary:

Theorem 5 *Let $p(z)$ be a polynomial with two multiple roots, i.e. $p(z) = (z-a)^m (z-b)^n$. The operator of the modified Newton method $H(z)$ (Eq. (3)) is globally analytically conjugate to the quadratic polynomial $\frac{m}{n}z^2$.*

Proof We prove this result by considering the conjugacy map

$$h(z) = \frac{-z+b}{z-a}$$

with the following properties

$$h(a)=\infty, h(b)=0, h(\infty)=-1.$$

Then,

$$\left(h\circ H\circ h^{-1}\right)(z)=\frac{m}{n}z^2$$

Therefore, for polynomials with two multiple roots, the modified Newton operator is always conjugate to the rational map $\frac{m}{n}z^2$, satisfying the following properties:

(1) The dynamics of this operator gives the circle $S^1(z)=\{z\in C:|z|^2=\frac{n}{m}\}$ as the invariant Julia set.

(2) The Fatou set is defined by the two basins of attraction of the superattracting fixed points: 0 and ∞, that correspond to the roots a and b. □

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