

NON-POSITIVE AND NEGATIVE AT INFINITY DIVISORIAL VALUATIONS OF HIRZEBRUCH SURFACES

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ABSTRACT. We consider rational surfaces Z defined by divisorial valuations ν of Hirzebruch surfaces. We introduce concepts of non-positivity and negativity at infinity for these valuations and prove that these concepts admit nice local and global equivalent conditions. In particular we prove that, when ν is non-positive at infinity, the extremal rays of the cone of curves of Z can be explicitly given.

1. INTRODUCTION

Valuations were used in the middle of the past century to treat resolution of singularities of algebraic varieties. They are essentially local objects which could be used to prove local uniformization [30]. However, in the last years, they have been used to study global properties. The best known situation corresponds with valuations of the field of rational functions of \mathbb{P}^2 centered at the local ring $\mathcal{O}_{\mathbb{P}^2,p}$, \mathbb{P}^2 being the projective plane over an algebraically closed field k and p a closed point in \mathbb{P}^2 . These valuations were classified by Spivakovsky [29] (see also [12, 19]). This classification has five types and works for valuations of the fraction field of any two-dimensional regular local ring R centered at R .

Divisorial and irrational valuations are two of these types and, for them and suitable divisors, it can be defined Seshadri-like constants [7], that is, objects which basically contain the same information for valuations as Seshadri constants for points. Recall that Seshadri constants were used by Demailly [10] for studying the Fujita conjecture. Even in the simplest case, where the local ring is $\mathcal{O}_{\mathbb{P}^2,p}$ and the divisor is a line, Seshadri-like constants are very difficult to compute. They allow us to establish the concept of minimal valuation and propose a conjecture which implies the Nagata conjecture and is implied by the Greuel-Lossen-Shustin conjecture (see [17] and also [11], where the mentioned Seshadri-like constant is denoted by $\hat{\mu}(\nu)$).

Recently, in [16], it was considered a class \mathcal{N} of divisorial valuations ν (of the field of rational functions of \mathbb{P}^2 centered at $\mathcal{O}_{\mathbb{P}^2,p}$) which have a similar behaviour as that of curves with only one place at infinity [1]. They were named non-positive at infinity because they satisfy $\nu(f) \leq 0$ for every $f \in k[x, y] \setminus \{0\}$, $\{x, y\}$ being affine coordinates in the chart of points which are not in the line at infinity (which contains p). When $\nu(f) < 0$, they are called negative at infinity. These valuations are centered at infinity because the point p is in the line at infinity [13]. Recently, this last class of valuations has been studied and used in different contexts [4, 13, 14, 22, 26]. There is a natural bijection between the set of divisorial valuations ν centered at $\mathcal{O}_{\mathbb{P}^2,p}$ and that of finite simple sequences of point blowups starting with the blowup at p , and each valuation ν determines a rational projective surface X . In [16] we proved that the fact that ν belongs to \mathcal{N} is equivalent to that the cone of curves $NE(X)$ is regular, and we gave a

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simple characterization of this fact. Even more, we are able to compute the Seshadri-like constant with respect to a line divisor for valuations in \mathcal{N} and valuations in this set are used to provide evidences in the direction of the Nagata conjecture [17].

Since the projective plane and the Hirzebruch surfaces provide the classical minimal models for rational surfaces, we consider divisorial valuations of the field of rational functions of \mathbb{F}_δ centered at $\mathcal{O}_{\mathbb{F}_\delta, p}$ (called here divisorial valuations of \mathbb{F}_δ), where \mathbb{F}_δ is any Hirzebruch surface and $p \in \mathbb{F}_\delta$ is a closed point. Our objective is to find suitable affine charts on a Hirzebruch surface such that, as in the case of the class \mathcal{N} , valuations ν which are non-positive (or negative) on non-zero regular functions on these charts give rise to surfaces (obtained by the sequence of blowups given by ν which starts at the Hirzebruch surface) with nice geometrical global properties. Notice that, as algebraic objects, valuations of \mathbb{F}_δ do not differ from valuations centered at regular closed points of other birationally equivalent surfaces; however we desire to relate the mentioned valuations with global geometric aspects of Hirzebruch surfaces. We will show that, in this case, there exist two natural charts “at infinity”. On the one hand that given by points which are neither in the fiber F_1 that contains p nor in the special section M_0 of \mathbb{F}_δ , and, on the other hand, by points which are neither in F_1 nor in a particular uniquely defined section $M_1 \neq M_0$. In this paper, we will divide the divisorial valuations of \mathbb{F}_δ in two classes, special and non-special, according to the chart at infinity to be used for introducing the concepts of non-positive and negative at infinity divisorial valuation of \mathbb{F}_δ . That is, over each point $p \in \mathbb{F}_\delta$, we will consider special or non-special divisorial valuations which will determine the chart to be used to define non-positivity or negativity at infinity. We will give several characterizations of those concepts, including one which is very easy to check from the dual graph of the valuation ν (that involves only topological information) and the images by ν of (the germs at p of) the fiber and sections before introduced (see Theorems 3.6 (c) and 4.8 (c)), and, in the case of negative at infinity valuations, the Iitaka dimension of certain divisor (Theorems 3.9 (b) and 4.12 (b)).

Each divisorial valuation ν of \mathbb{F}_δ defines in a unique way a rational surface Z obtained from the simple sequence of point blowups given by ν . A remarkable property of non-positive at infinity valuations is that they determine (in fact, are equivalent to) the surfaces Z as above such that its cone of curves $NE(Z)$ is finite polyhedral and generated either by the classes of the strict transforms of the fiber F_1 , the special section M_0 and the exceptional divisors (special valuations) or by the mentioned generators plus the class of the section M_1 (non-special valuations). Since the Hirzebruch surface \mathbb{F}_1 can be obtained by blowing-up a point in \mathbb{P}^2 , our results recover those in [16] concerning the characterization of non-positive and negative at infinity divisorial valuations of \mathbb{P}^2 , and provide a very simple characterization of the rational surfaces (obtained from a classical minimal model by a finite simple sequence of point blowups) whose cone of curves has the above mentioned generators.

Some complementary results on the effective monoid of surfaces given by blowing-up some very concrete configurations of infinitely near points over Hirzebruch surfaces can be found in [8].

For surfaces defined by non-positive at infinity valuations of \mathbb{P}^2 , we are able to decide whether their Cox rings are finitely generated [16], and, as mentioned, for these same valuations, we know how to compute their Seshadri-like constants and also to explicitly obtain the Newton-Okounkov bodies [23, 27, 6] of flags defined by them [18]. In a forthcoming paper, we will prove that similar properties can be deduced when considering non-positive at infinity valuations of \mathbb{F}_δ .

Section 2 of the paper reviews some known material which will be used later. Special and non-special divisorial valuations of \mathbb{F}_δ are introduced in Definition 3.1. Section

3 studies the special ones and characterizes its non-positivity (respectively, negativity) at infinity in Theorem 3.6 (respectively, Theorem 3.9). The non-special divisorial valuations are considered in Section 4, being Theorems 4.8 and 4.12 the main results.

2. PRELIMINARIES

Given a surface Z_0 , a (finite) simple sequence of blowups starting at Z_0 is a sequence

$$\pi : Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \rightarrow \dots \rightarrow Z_1 \xrightarrow{\pi_1} Z_0, \quad (2.1)$$

of blowups $\pi_i : Z_i \rightarrow Z_{i-1}$, $1 \leq i \leq n$, centered at closed points $p_i \in Z_{i-1}$, such that $p_1 = p \in Z_0$ and each p_i , $2 \leq i \leq n$, belongs to the exceptional divisor created by π_{i-1} .

In this paper, we will study some global properties concerning rational surfaces obtained from simple sequences π as above where Z_0 is a Hirzebruch surface. We start by recalling some basic facts about these surfaces (see [20, 2, 28, 25] for additional information).

2.1. Hirzebruch surfaces. Let k be an algebraically closed field and $\mathbb{P}^1 = \mathbb{P}_k^1$ the projective line over k . Let δ be a non-negative integer and consider the δ th *Hirzebruch surface* $\mathbb{F}_\delta := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$. Let F and M be two prime divisors such that $F^2 = 0$, $F \cdot M = 1$, $M^2 = \delta$ and the classes $[F]$ and $[M]$ generate the Picard group $\text{Pic}(\mathbb{F}_\delta)$. If $\delta > 0$, we denote by M_0 the $(-\delta)$ -curve of \mathbb{F}_δ which we call *special section*. Otherwise ($\delta = 0$), M_0 is a linearly equivalent to M divisor on \mathbb{F}_δ . If a divisor D is linearly equivalent to $aF + bM$, we will say that it has *degree* (a, b) . Notice that any irreducible curve $C \neq F, M_0$ on \mathbb{F}_δ has degree (a, b) , where $a \geq 0$ and $b > 0$ [20, V, Proposition 2.20].

Later we will use coordinates. From this point of view Hirzebruch surfaces \mathbb{F}_δ can be obtained as the quotient of the product of punctured affine planes over k ,

$$(\mathbb{A}^2 \setminus \{(0, 0)\}) \times (\mathbb{A}^2 \setminus \{(0, 0)\}),$$

by an action of the product, $k^* \times k^*$, of multiplicative groups of the field k (see [28, §2.2]). For each $(\lambda, \mu) \in k^* \times k^*$, the action goes as follows:

$$\begin{aligned} (\lambda, 1) : (X_0, X_1; Y_0, Y_1) &\rightarrow (\lambda X_0, \lambda X_1; Y_0, \lambda^{-\delta} Y_1) \\ (1, \mu) : (X_0, X_1; Y_0, Y_1) &\rightarrow (X_0, X_1; \mu Y_0, \mu Y_1). \end{aligned} \quad (2.2)$$

\mathbb{F}_δ is equipped with a projection morphism $pr : \mathbb{F}_\delta \rightarrow \mathbb{P}^1$ that, in terms of coordinates, is the projection onto the first factor. The classes $[M_0], [M]$ and $[F]$ are the classes of the curves with equations $Y_1 = 0, Y_0 = 0$ and that of any fiber of pr , respectively. Notice that \mathbb{F}_δ is covered by four affine open sets $U_{ij} := \mathbb{F}_\delta \setminus \mathbf{V}(X_i Y_j)$, $0 \leq i, j \leq 1$.

By convention, if $\delta \geq 1$, a point of \mathbb{F}_δ will be called *special* or *general* depending on whether it belongs or not to the special section M_0 . We warn the reader not to get confused with special and non-special valuations (see Definition 3.1), which are different concepts.

2.2. Divisorial valuations. A *valuation* of a field K is a surjective map

$$\nu : K \setminus \{0\} \rightarrow G,$$

where G is a totally ordered commutative group (the value group of ν), such that, for $f, g \in K \setminus \{0\}$, satisfies:

$$\nu(f + g) \geq \min\{\nu(f), \nu(g)\} \text{ and also } \nu(fg) = \nu(f) + \nu(g).$$

The ring $R_\nu = \{f \in K \setminus \{0\} \mid \nu(f) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν . It is a local ring whose maximal ideal is $\mathfrak{m}_\nu = \{f \in K \setminus \{0\} \mid \nu(f) > 0\} \cup \{0\}$. When K is the fraction field of a local regular ring (R, \mathfrak{m}) and $R \cap \mathfrak{m}_\nu = \mathfrak{m}$, one says that ν is *centered* at R . When $\dim R = 2$, valuations centered at R (R/\mathfrak{m} is algebraically closed)

are in one-to-one correspondence with (non-necessarily finite) simple sequences of point blowups starting at $\text{Spec}R$. Divisorial valuations are those corresponding with finite simple sequences [31, 29].

In this paper p will be a closed point in \mathbb{F}_δ . Denote by K the field of rational functions of \mathbb{F}_δ . Then to give a sequence as (2.1) where $Z_0 = \mathbb{F}_\delta$ is equivalent to give a divisorial valuation ν of K centered at $R = \mathcal{O}_{\mathbb{F}_\delta, p}$. The valuation ν is defined by the last exceptional divisor E in the sequence $\pi : Z \rightarrow Z_0$ and, frequently and by simplicity, we will say that ν is a valuation of \mathbb{F}_δ . The map π_1 is the blowup of Z_0 at $p = p_1$ and π_{i+1} , $1 \leq i \leq n-1$, the blowup of Z_i at the unique point p_{i+1} of the exceptional divisor defined by π_i , E_i , such that ν is centered at the local ring $\mathcal{O}_{Z_i, p_{i+1}}$. Write $\mathcal{C}_\nu := \{p_i\}_{i=1}^n$ the sequence (or configuration) of infinitely near points defined above; p_i is said to be *proximate* to p_j , denoted by $p_i \rightarrow p_j$, whenever $i > j$ and p_i belongs either to E_j or to the strict transform of E_j on Z_{i-1} . A point p_i is *satellite* whenever there exists $j < i-1$ satisfying $p_i \rightarrow p_j$; otherwise, it is called *free*.

As in the case of germs of plane curves [3], plane divisorial valuations admit sets of invariants that help to study them. For a valuation ν as above, we will use its sequence of maximal contact values $\{\bar{\beta}_j\}_{j=0}^{g+1}$ [9, (1.5.3)]. The dual graph Ω of ν is a labelled tree, where each vertex represents an exceptional divisor appearing in the sequence of blowups (2.1) and two vertices are joined whenever their corresponding divisors meet. Each vertex is labelled with the number of blowups needed to create the corresponding divisor. Set $E_{l_{g+1}} = E_n$ the last exceptional divisor and consider the set $\{E_{l_j}\}_{0 \leq j \leq g}$ of divisors different from $E_{l_{g+1}}$, ordered according their labels, whose corresponding vertices in Ω have degree 1. Denote by φ_i , $1 \leq i \leq n$, an analytically irreducible germ of curve at p whose strict transform on Z_i is transversal to E_i at a non-singular point of the exceptional locus. Then each value $\bar{\beta}_j$, $0 \leq j \leq g+1$, can be defined as $\bar{\beta}_j := \nu(\varphi_{l_j})$. The set $\{\bar{\beta}_j\}_{j=0}^g$ generates the *semigroup of values* of ν , $S(\nu) = \nu(R \setminus \{0\})$, [29, Remark 6.1], and the sequence of maximal contact values has an extra value $\bar{\beta}_{g+1}$ which coincides with the inverse of the volume of ν , $[\text{vol}(\nu)]^{-1}$. Indeed, by definition,

$$\text{vol}(\nu) = \lim_{\alpha \rightarrow \infty} \frac{\dim_k(R/\mathcal{P}_\alpha)}{\alpha^2/2},$$

where $\mathcal{P}_\alpha = \{f \in R \mid \nu(f) \geq \alpha\} \cup \{0\}$; taking into account that the above dimensions depend only on local data, the equality $\bar{\beta}_{g+1} = [\text{vol}(\nu)]^{-1}$ follows as in [18, Remark 2.3].

Along the paper, for any curve C on \mathbb{F}_δ , φ_C denotes its germ at p and $(\varphi_i, \varphi_C)_p$ equals 0 (respectively, the intersection multiplicity at p of the germs φ_i and φ_C) if C does not pass through p (respectively, otherwise). Finally, $\text{mult}_{p_j}(\varphi_i)$ (respectively, $\text{mult}_{p_j}(\varphi_C)$), $1 \leq i, j \leq n$, means multiplicity of the strict transform of φ_i (respectively, φ_C) at p_j . We will use frequently, without any mention, the so-called Noether's formula for valuations, that we recall here for the convenience of the reader and whose proof can be found in [5, Theorem 8.1.6].

Lemma 2.1. *Let ν be a divisorial valuation of K centered at R , with associated configuration $\mathcal{C}_\nu := \{p_i\}_{i=1}^n$, and let C be a curve on \mathbb{F}_δ . Then*

$$\nu(\varphi_C) = (\varphi_n, \varphi_C)_p = \sum_{j=1}^n \text{mult}_{p_j}(\varphi_n) \cdot \text{mult}_{p_j}(\varphi_C).$$

2.3. Non-positive and negative at infinity valuations. A divisorial valuation of the field of rational functions of \mathbb{P}^2 centered at $\mathcal{O}_{\mathbb{P}^2, p}$, $p \in \mathbb{P}^2$ being closed point, (or, simply, a divisorial valuation of \mathbb{P}^2), is called *non-positive at infinity* when $\nu(h) \leq 0$ for all $h \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$, L being a line (the line at infinity) containing p . In case it satisfies $\nu(h) < 0$ for every non-constant function $h \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$, ν is named *negative*

at infinity. As we mentioned in the introduction, non-positive and negative at infinity divisorial valuations of \mathbb{P}^2 have nice global properties involving the surfaces they define.

Afterwards we will introduce the concepts of non-positivity and negativity at infinity for valuations of the field of rational functions of \mathbb{F}_δ centered at $\mathcal{O}_{\mathbb{F}_\delta, p}$, p being a point in \mathbb{F}_δ . As we will see, our definition will depend on the point p and a chart of the Hirzebruch surface which does not contain p . The goal of the paper is to show that the rational surfaces given by these valuations are easy to characterize and also enjoy nice global properties. We focus on the cone of curves and positivity properties of divisors.

3. THE SIGN AT INFINITY OF SPECIAL VALUATIONS

We start this section by partitioning the set of divisorial valuations of Hirzebruch surfaces in two subsets because our main results have to do with the concept of “sign at infinity” of valuations which depends on the considered subset. First we need to fix some notations.

Let \mathbb{F}_δ be a Hirzebruch surface and $p \in \mathbb{F}_\delta$ a point on it. Consider a divisorial valuation ν of the field of rational functions of \mathbb{F}_δ centered at $\mathcal{O}_{\mathbb{F}_\delta, p}$ and their associated configuration $\mathcal{C}_\nu = \{p_i\}_{i=1}^n$, and a composition of blowups $\pi : Z \rightarrow Z_0 = \mathbb{F}_\delta$ as in (2.1). Set E_i , $1 \leq i \leq n$, the exceptional divisor produced after blowing-up p_i , and denote also by E_i (respectively, E_i^*) its strict (respectively, total) transform on the surface Z . We will write \tilde{C} and C^* the strict and total transforms on Z of other divisors C . Occasionally, the above transforms could be on some surface Z_i appearing in (2.1).

Definition 3.1. A divisorial valuation ν as before is called to be *special* (with respect to \mathbb{F}_δ and p) when one of the following conditions holds:

- (1) $\delta = 0$.
- (2) $\delta > 0$ and p is a special point.
- (3) $\delta > 0$, p is a general point and there is no integral curve in the complete linear system $|M|$ whose strict transform on Z has negative self-intersection.

The remaining valuations will be called *non-special*.

Remark 3.2. Looking at the local equations of the linear system $|M|$ and taking into account their evolution by blowups, it is not difficult to show that the above condition (3) holds if and only if either p_2 belongs to strict transform of the fiber of pr passing through p on Z_1 , or there does not exist $j \geq \delta + 1$ such that the points p_i , $1 \leq i \leq j$, of \mathcal{C}_ν are free.

Throughout this section ν will be a *special* divisorial valuation of \mathbb{F}_δ . In addition, F_1 will denote the fiber passing through p , and M_0 will denote either the special section (in case $\delta \geq 1$), or the section of degree $(0, 1)$ passing through p (otherwise).

Denote by $\text{Pic}(Z)$ the Picard group of Z and \cdot the intersection pairing associated to $\text{Pic}(Z)$. Consider the linear space $\text{Pic}_{\mathbb{Q}}(Z) = \text{Pic}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ and, by abuse of notation, \cdot will denote the corresponding bilinear pairing. Recall that the convex cone of $\text{Pic}_{\mathbb{Q}}(Z)$ generated by the classes of effective divisors (respectively, nef divisors) is called the *cone of curves* (respectively, *nef cone*) and denoted by $NE(Z)$ (respectively, $P(Z)$). Notice that $P(Z)$ is the dual cone of $NE(Z)$. In the following we will denote by $\overline{NE}(Z)$ the closure of $NE(Z)$ in the usual topology.

By [24, Lemma 1.22], the classes $[\tilde{F}_1]$ and $[\tilde{M}_0]$ span extremal rays of both cones $NE(Z)$ and $\overline{NE}(Z)$. For our purposes, it will be useful to consider the strongly convex cone of $\text{Pic}_{\mathbb{Q}}(Z)$, $S_1(Z)$, generated by the set of classes $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$, and also its dual cone

$$S_1^\vee(Z) := \{[C] \in \text{Pic}_{\mathbb{Q}}(Z) \mid [C] \cdot [D] \geq 0 \text{ for all } [D] \in S_1(Z)\}.$$

Our next result provides generators for $S_1^\vee(Z)$.

Proposition 3.3. *The dual cone $S_1^\vee(Z)$ is generated by $[F^*], [M^*]$ and the classes $\{[\Lambda_i]\}_{i=1}^n$ of the divisors*

$$\Lambda_i := a_i F^* + b_i M^* - \sum_{j=1}^i \text{mult}_{p_j}(\varphi_i) E_j^*, \quad (3.1)$$

where $a_i := (\varphi_i, \varphi_{M_0})_p$ and $b_i := (\varphi_i, \varphi_{F_1})_p$.

Proof. It is enough to prove that $\{[F^*], [M^*]\} \cup \{[\Lambda_i]\}_{i=1}^n$ is the dual basis of $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$ with respect to the intersection product.

Let $p_{i_{F_1}}$ be the last point in the configuration \mathcal{C}_ν of the valuation ν giving rise to Z through which the strict transform of F_1 passes. Also, if p belongs to M_0 , we define i_{M_0} such that $p_{i_{M_0}}$ is the last point of \mathcal{C}_ν through which the strict transform of M_0 passes; otherwise we define $i_{M_0} := 0$. Taking into account that φ_i is analytically irreducible, the proximity equalities [5, Theorem 3.5.3] show that $\Lambda_i \cdot E_j = \delta_{ij}$, where δ_{ij} denotes the Kronecker's delta. Also, for each $i \in \{1, 2, \dots, n\}$, it holds

$$\Lambda_i \cdot \tilde{F}_1 = b_i - \sum_{j=1}^{\min\{i, i_{F_1}\}} \text{mult}_{p_j}(\varphi_i) = 0,$$

and

$$\Lambda_i \cdot \tilde{M}_0 = a_i - \sum_{j=1}^{\min\{i, i_{M_0}\}} \text{mult}_{p_j}(\varphi_i) = 0,$$

where the summations with upper index equal to 0 are defined to be 0. Finally notice that $F^* \cdot \tilde{F}_1 = 0$, $F^* \cdot \tilde{M}_0 = 1$, $M^* \cdot \tilde{F}_1 = 1$, $M^* \cdot \tilde{M}_0 = 0$ and $F^* \cdot E_i = M^* \cdot E_i = 0$ for all $i = 1, 2, \dots, n$. This concludes the proof. \square

Recall that we are considering a special divisorial valuation ν of \mathbb{F}_δ and the surface Z that ν defines. The divisors Λ_i defined in (3.1) will be useful in this section because, as we are going to prove, they satisfy nice properties.

Lemma 3.4. *Let ν be a special divisorial valuation of \mathbb{F}_δ . Then, with the above notation, it holds that $\Lambda_1^2 \geq 0$, and the inequality $\Lambda_i^2 \geq 0$ for some index $i \in \{2, 3, \dots, n\}$ implies:*

- (a) $\Lambda_i^2 > 0$, whenever p_i is a satellite point of the configuration \mathcal{C}_ν .
- (b) $\Lambda_{i-1}^2 \geq 0$ and in case $\Lambda_{i-1}^2 = 0$, the point p_i is satellite and the point p_{i-1} is free.

Proof. The self-intersection of the divisor Λ_1 satisfies $\Lambda_1^2 = 1 + \delta$ when p_1 is a special point and also when $\delta = 0$. Otherwise, $\Lambda_1^2 = \delta - 1$.

For proving the remaining statements, we can assume, without loss of generality, that $i = n \geq 2$.

We are going to prove the result when p_1 is a special point. Otherwise, the proof is the same after setting $\delta = 0$ or $a_n = 0$.

We start with the proof of (a) for which we will use some properties of the set of maximal contact values of ν , $\{\bar{\beta}_j\}_{j=0}^{g+1}$, (see [29] and [9, Section 1.5]). We divide this proof in two cases.

Case 1(a): $g > 1$. Reasoning by contradiction and taking into account that the point p_n is satellite, we get that

$$0 = \Lambda_n^2 = 2a_n b_n + \delta b_n^2 - e_{g-1} \bar{\beta}_g = e_{g-1} \left[\frac{2a_n b_n + \delta b_n^2}{e_{g-1}} - \bar{\beta}_g \right],$$

where $e_{g-1} := \gcd(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_{g-1})$. Since both a_n and b_n are either a multiple of $\bar{\beta}_0$ or $\bar{\beta}_1$, the first addend in the brackets is a multiple of e_{g-1} , which gives a contradiction because $\gcd(e_{g-1}, \bar{\beta}_g) = 1$.

Case 2(a): $g = 1$. We distinguish three sub-cases: *The values a_n and b_n are divisible by $\bar{\beta}_0$.* Then $e_{g-1} = e_0 = \bar{\beta}_0$ and the proof follows as above. *The value a_n satisfies $a_n = \bar{\beta}_1$.* Then $\Lambda_n^2 = \bar{\beta}_0(2\bar{\beta}_1 + \bar{\beta}_0\delta - \bar{\beta}_1) > 0$. *Otherwise.* Then $\Lambda_n^2 = \bar{\beta}_1(2\bar{\beta}_0 + \bar{\beta}_1\delta - \bar{\beta}_0) > 0$, which concludes the proof of (a).

Now we prove (b). Again we can suppose that $i = n$. We also assume that the point p_n is satellite, otherwise $\Lambda_{n-1}^2 > 0$ by Noether's formula. Denote by $\hat{\nu}$ the divisorial valuation defined by the divisor E_{n-1} . Let $\{\hat{\beta}_j\}_{j=0}^{\hat{g}+1}$ be the sequence of maximal contact values of $\hat{\nu}$, set $\hat{e}_{g-1} := \gcd(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_{g-1})$ and $e := \hat{e}_{g-1}/e_{g-1}$. Consider two cases with two sub-cases.

Case 1(b): $g = \hat{g}$. Assume first that $g > 1$. From the following equality, which is proved in [16, Lemma 2],

$$|\hat{\beta}_g - e\bar{\beta}_g| = \frac{1}{e_{g-1}}, \quad (3.2)$$

one can deduce that

$$-\frac{e_{g-1}\hat{\beta}_g}{e} \geq -\frac{1}{e} - e_{g-1}\bar{\beta}_g. \quad (3.3)$$

In this case both valuations ν and $\hat{\nu}$ are defined by satellite points, therefore $a_{n-1} = ea_n, b_{n-1} = eb_n, \bar{\beta}_{g+1} = e_{g-1}\bar{\beta}_g$ and $\hat{\beta}_{g+1} = \hat{e}_{g-1}\hat{\beta}_g$. As a consequence

$$\begin{aligned} \Lambda_{n-1}^2 &= e^2 \left[2a_nb_n + \delta b_n^2 - \frac{e_{g-1}\hat{\beta}_g}{e} \right] \geq e^2 \left[2a_nb_n + \delta b_n^2 - \frac{1}{e} - e_{g-1}\bar{\beta}_g \right] \\ &= e^2 \left[\Lambda_n^2 - \frac{1}{e} \right] > 0, \end{aligned}$$

where the first inequality is deduced from (3.3) and the last one holds since $\Lambda_n^2 > e_{g-1} > 1/e$.

To conclude the proof in this case, it remains to study what happens when $g = 1$. We consider the same subcases as above: The values a_n and b_n are both divisible by $\bar{\beta}_0$, then the fact $\Lambda_{n-1}^2 > 0$ can be proved as before. The value a_n equals $\bar{\beta}_1$, then

$$\Lambda_{n-1}^2 = 2\hat{\beta}_0\hat{\beta}_1 + \delta\hat{\beta}_0^2 - \hat{\beta}_2 = \hat{\beta}_0(2\hat{\beta}_1 + \delta\hat{\beta}_0 - \hat{\beta}_1) = \hat{\beta}_0(\hat{\beta}_1 + \delta\hat{\beta}_0) > 0.$$

Otherwise, then $\Lambda_{n-1}^2 = \hat{\beta}_1(\hat{\beta}_0 + \delta\hat{\beta}_1) > 0$.

Case 2(b): $\hat{g} = g - 1$. When $g > 1$, it holds

$$\hat{\beta}_{\hat{g}+1} = \frac{\bar{\beta}_{g+1} + 2}{4}$$

and thus

$$\Lambda_{n-1}^2 = \frac{1}{4} (2a_nb_n + \delta b_n^2 - \bar{\beta}_{g+1} - 2) = \frac{1}{4} \Lambda_n^2 - \frac{1}{2} \geq 0,$$

because $\Lambda_n^2 \geq 2$.

Finally we must assume that $g = 1$ and, as above, when the values a_n and b_n are divisible by $e_0 = 2$, $\Lambda_{n-1}^2 \geq 0$. When $a_n = \bar{\beta}_1$, $\Lambda_{n-1}^2 = \hat{\beta}_1 + \delta \geq 0$, and otherwise,

$$\Lambda_{n-1}^2 = 2\hat{\beta}_1 + \delta\hat{\beta}_1^2 - \hat{\beta}_1 \geq 0,$$

which concludes the proof. \square

Next we introduce the concepts of non-positivity and negativity at infinity for special divisorial valuations. Afterwards a similar concept will be given for non-special valuations. We consider a Hirzebruch surface \mathbb{F}_δ , a closed point p in \mathbb{F}_δ and denote by R the local ring $\mathcal{O}_{\mathbb{F}_\delta, p}$.

Definition 3.5. Let ν be a special divisorial valuation of the field of rational functions of \mathbb{F}_δ centered at R . The valuation ν is called *non-positive* (respectively, *negative*) at infinity whenever $\nu(h) \leq 0$ (respectively, $\nu(h) < 0$) for all $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$, $h \notin k$).

We devote the remaining of this section to give two results, Theorems 3.6 and 3.9, which give several equivalent conditions to the fact that a special divisorial valuation of a Hirzebruch surface is non-positive or negative at infinity. We will use the divisor Λ_n and the values a_n and b_n , defined in Proposition 3.3.

Theorem 3.6. *Let ν be a special divisorial valuation of the field of rational functions of \mathbb{F}_δ centered at R . Set Z the surface that ν defines. Consider the divisor Λ_n given in (3.1) and the inverse of the volume of ν , $[\text{vol}(\nu)]^{-1}$. Then the following conditions are equivalent:*

- (a) *The valuation ν is non-positive at infinity.*
- (b) *The divisor Λ_n is nef.*
- (c) *The inequality $2a_n b_n + b_n^2 \delta \geq [\text{vol}(\nu)]^{-1}$ holds.*
- (d) *The cone of curves $NE(Z)$ is generated by the classes of the strict transforms on Z of the fiber passing through p , the special section and the irreducible exceptional divisors associated with the map π given by ν .*

Proof. Our first step is to prove the equivalence between (a) and (b), and we start by proving that (b) implies (a). We assume firstly here that $\delta > 0$ and $p = p_1$ is a special point. Without loss of generality, suppose that the special point p has coordinates $(1 : 0; 1, 0)$. The point p belongs to the fiber F_1 whose equation is $X_1 = 0$, and the special section M_0 is defined by the equation $Y_1 = 0$. Set U_{00} the affine open set of \mathbb{F}_δ given by $X_0 \neq 0$ and $Y_0 \neq 0$, whose associated affine coordinates are $\{u, v\} = \left\{ \frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0} \right\}$. Consider also the affine open set of \mathbb{F}_δ , U_{11} , defined by $X_1 \neq 0$ and $Y_1 \neq 0$, with coordinates $\{x, y\} = \left\{ \frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1} \right\}$. It holds that $p \in U_{00}$ and F_1 and M_0 have local equations $u = 0$ and $v = 0$, respectively. Denote by \mathcal{P} the set of non-constant functions in $\mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them. In terms of the coordinates $\{u, v\}$, $f \in \mathcal{P}$ can be expressed as

$$f(x, y) = f(1/u, 1/u^\delta v) = \frac{h_f(u, v)}{u^{\deg_1(h_f) + \delta \deg_2(h_f)} v^{\deg_2(h_f)}}, \quad (3.4)$$

where $h_f(u, v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{00})$. The bi-homogeneous polynomial

$$X_0^{\deg_1(h_f)} Y_0^{\deg_2(h_f)} \cdot h_f \left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0} \right)$$

defines a curve C_f on the surface \mathbb{F}_δ of degree $(\deg_1(h_f), \deg_2(h_f))$ and, if F' and M' are the fiber and the section on \mathbb{F}_δ with equations $X_0 = 0$ and $Y_0 = 0$, it holds that the map $f \rightarrow C_f$ defines a one-to-one correspondence between \mathcal{P} and the set of the curves on \mathbb{F}_δ containing no curve in $\{F_1, F', M_0, M'\}$ as a component. Now, the condition Λ_n nef and (3.4) show that

$$\begin{aligned} 0 \leq \Lambda_n \cdot C_f &= \Lambda_n \cdot \left[\deg_1(h_f) F^* + \deg_2(h_f) M^* - \sum_{i=1}^n \text{mult}_{p_i}(h_f) E_i^* \right] \\ &= - [-(\deg_1(h_f) + \deg_2(h_f) \delta) \nu(u) - \deg_2(h_f) \nu(v) + \nu(h_f)] = -\nu(f). \end{aligned}$$

So, to finish the proof of (a) in this case (p is a special point), it only remains to assume that either x or y or both are factors of f . Then the proof follows from the existence of non-negative integers α, β with $\alpha + \beta \neq 0$ and $f_1 \in \mathcal{P}$ such that

$$\nu(f) = \nu(x^\alpha y^\beta f_1) = -(\alpha + \beta\delta)\nu(u) - \beta\nu(v) + \nu(f_1) \leq 0.$$

If $\delta = 0$ the proof is analogous, and the non-positivity of ν for the case when p is a general point can be proved in a similar way after assuming that p has coordinates $(0 : 1; 0, 1)$ and considering local coordinates $\{u, v\} = \left\{ \frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1} \right\}$ in the affine open set U_{11} and $\{x, y\} = \left\{ \frac{X_1}{X_0}, \frac{Y_0}{X_0^\delta Y_1} \right\}$ in U_{01} .

Now we are going to prove that (a) implies (b). Assume by contradiction that the divisor Λ_n is not nef and, therefore, that there exists an effective divisor C such that $\Lambda_n \cdot C < 0$. This implies that, with the above notation, if p is a special point –or $p \in \mathbb{F}_{0^-}$, (respectively, p is a general point), then there exists $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (respectively, $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{01})$) such that $-\nu(f) = \Lambda_n \cdot C < 0$, a contradiction.

The fact that (b) implies (c) follows easily from previous computations given in the proof of Lemma 3.4.

Let us prove that (d) can be deduced from (c). Fix any ample divisor H on the surface Z and consider the set

$$A(Z) := \{[D] \in \text{Pic}_{\mathbb{Q}}(Z) \mid [D]^2 \geq 0 \text{ and } [H] \cdot [D] \geq 0\}.$$

Recall that the above defined cone $S_1(Z)$ is generated by the classes $[\tilde{F}_1]$, $[\tilde{M}_0]$ and $[E_i]$, $1 \leq i \leq n$, and we are going to prove that

$$\overline{NE}(Z) = S_1(Z) + S_1^\vee(Z) = NE(Z) \quad (3.5)$$

and

$$S_1^\vee(Z) \subseteq A(Z) \subseteq S_1(Z) \quad (3.6)$$

hold, which shows (d). Our Hypothesis (c) means that $\Lambda_n^2 \geq 0$ and by Lemma 3.4, one has that $\Lambda_i^2 \geq 0$, $1 \leq i \leq n-1$. Proposition 3.3 proves the first inclusion in (3.6) and the last one follows from the first one and the equality $A(Z)^\vee = A(Z)$ (that holds taking into account the Hodge index theorem [20, Theorem 1.9]). It remains to prove the chain of equalities (3.5). For a start, notice that $A(Z) \subseteq \overline{NE}(Z)$ by [24, Lemma 1.20]. Thus $S_1^\vee(Z) \subseteq \overline{NE}(Z)$. Now, if $[C]$ is the class of an irreducible curve on Z and it is not one of the given generators of $S_1(Z)$, then $[C] \in S_1^\vee(Z)$ because otherwise $[C] \cdot [D] < 0$ for $[D] \in S_1(Z)$ and C and D would have a common component. Therefore we have proved (3.5) with inclusions \supseteq instead of equalities. Taking topological closures we deduce that (3.5) holds.

Finally (d) implies (b) by Proposition 3.3, which concludes the proof. \square

Remark 3.7. The Hirzebruch surface \mathbb{F}_1 can be regarded as the projective space \mathbb{P}^2 with a point blown-up. If one considers the line at infinity L in \mathbb{P}^2 and one regards \mathbb{F}_1 as the blowup of \mathbb{P}^2 at a point of L , then it is not difficult to deduce that $M_0 = E_1$ and F is a general divisor in $|\tilde{L}|$. As a consequence, Theorem 3.6 allows us to provide equivalent conditions to the non-positivity of a valuation of \mathbb{P}^2 (see Section 2.3). In fact, our Theorem 3.6 recovers Theorem 1 in [16] considering $\delta = 1$ and p_1 a special point, and the above proof is an adaptation and extension to our more general situation of that of [16, Theorem 1].

Remark 3.8. Assume that Z is a surface as in Theorem 3.6 defined by a non-positive at infinity special valuation. Then, on the one hand, Lemma 3.4 and Theorem 3.6 prove that the divisorial valuation ν_i defined by any exceptional divisor E_i given by (2.1) is also special and non-positive at infinity. On the other hand, every divisor Λ_i , $1 \leq i \leq n$,

is effective. Indeed, under these conditions, the expression of Λ_i , in the basis of strict transforms $\{\tilde{F}_1, \tilde{M}_0\} \cup \{E_j\}_{j=1}^n$, is

$$\Lambda_i = (\Lambda_i \cdot M^*)\tilde{F}_1 + (\Lambda_i \cdot F^*)\tilde{M}_0 + \sum_{j=1}^n (\Lambda_i \cdot \Lambda_j)E_j,$$

which is effective because Λ_i is nef.

We conclude this section by stating a characterization result for negative at infinity special valuations of Hirzebruch surfaces. Arguing as in Remark 3.7, one can see that our result also proves [16, Theorem 2].

Theorem 3.9. *Let ν (respectively, Z, Λ_n) a divisorial valuation (respectively, a surface, a divisor on Z) as in Theorem 3.6. Then the following conditions are equivalent:*

- (a) *The valuation ν is negative at infinity.*
- (b) *It holds that either $2a_n b_n + b_n^2 \delta > [\text{vol}(\nu)]^{-1}$, or $2a_n b_n + b_n^2 \delta = [\text{vol}(\nu)]^{-1}$ and the Iitaka dimension of the divisor Λ_n vanishes.*
- (c) *The inequality $\Lambda_n \cdot \tilde{C} > 0$ holds for the strict transform on Z , \tilde{C} , of any curve C on \mathbb{F}_δ , $C \neq F_1, M_0$.*

Proof. For a start, we recall that the Iitaka dimension [21] of a divisor D on Z is the maximum of the projective dimensions of the closures of the images of the rational maps defined by the complete linear systems $|nD|$, when n runs over those positive integers m such that $H^0(Z, \mathcal{O}_Z(mD)) \neq 0$.

We assume that p_1 is a special point. The other cases can be proved similarly. Assume also, without loss of generality, that p_1 has coordinates $(1 : 0; 1, 0)$ and consider the same notations as in the proof of Theorem 3.6.

We start by proving by contradiction that (b) can be deduced from (a). Hence, assume that (a) holds but (b) is false (what means, taking into account Theorem 3.6, that $\Lambda_n^2 = 0$ and $\dim |m\Lambda_n| > 0$ for m large enough). Therefore, there exists $f \in \mathcal{P}$ such that the class of $m\Lambda_n - \tilde{C}_f$ is effective for m large enough. This implies that

$$0 \leq \Lambda_n \cdot (m\Lambda_n - \tilde{C}_f) = m\Lambda_n^2 - \Lambda_n \cdot \tilde{C}_f = -\Lambda_n \cdot \tilde{C}_f.$$

Hence $0 = \Lambda_n \cdot \tilde{C}_f = -\nu(f)$ because Λ_n is nef (by Theorem 3.6), and this fact contradicts (a).

To prove that (b) implies (c), we reason again by contradiction and consider C an integral curve on \mathbb{F}_δ different from F_1 and M_0 , and such that $\Lambda_n \cdot \tilde{C} \leq 0$. In fact $\Lambda_n \cdot \tilde{C} = 0$ because, by Theorem 3.6, Λ_n is nef. Let \mathcal{F} be the face of the cone of curves of Z spanned by the classes $[\tilde{F}_1], [\tilde{M}_0], [E_1], \dots, [E_{n-1}]$, that is, $\mathcal{F} = [\Lambda_n]^\perp \cap NE(Z)$. It is clear that $[\tilde{C}] \in \mathcal{F}$ and, since the extremal rays of $NE(Z)$ are generated by classes of irreducible curves with negative self-intersection, $\tilde{C}^2 = 0$. \tilde{C} is nef, so $[\tilde{C}]^\perp \cap NE(Z)$ is a face of $NE(Z)$ which contains $[\tilde{C}]$ and, then, it must coincide with $[\Lambda_n]^\perp \cap NE(Z)$. Indeed, this is a consequence of the fact that, in suitable coordinates, $A(Z)$ is the projective cone over an Euclidean ball B (by the Hodge index theorem [20, Theorem 1.9]) and B is strictly convex. Then, \tilde{C} is linearly equivalent to a multiple of Λ_n and, by Remark 3.8, we get a contradiction.

To finish, the fact that (c) implies (a) can be proved as in Theorem 3.6 when proving that (b) implies (a). □

Remark 3.10. The concepts of non-positivity (and negativity) at infinity of valuations of \mathbb{P}^2 and \mathbb{F}_δ are different. For instance, by [16, Theorem 1] there is no non-positive at infinity divisorial valuation of \mathbb{P}^2 with maximal contact values 3, 11 and 122. However,

Theorem 3.6 proves the existence of non-positive at infinity valuations of \mathbb{F}_δ with those maximal contact values, when $\delta \geq 2$.

4. THE SIGN AT INFINITY OF NON-SPECIAL VALUATIONS

This section gives results that characterize the non-positivity and negativity at infinity of non-special divisorial valuations (see Definition 4.7 therein) of a Hirzebruch surface \mathbb{F}_δ , $\delta > 0$.

Notice that, when considering non-special valuations, there exists a unique irreducible section that is linearly equivalent to M and whose strict transform on Z has negative self-intersection. We will denote this section by M_1 . Notice that its class gives an extremal ray of the cone $NE(Z)$.

For reaching our objectives, we need to describe the dual cone of the strongly convex cone of $\text{Pic}_\mathbb{Q}(Z)$, $S_2(Z)$, generated by the classes $[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1]$ and $[E_i]$, $1 \leq i \leq n$. Our first result is a lemma which we will use in the forthcoming Proposition 4.2 (that gives generators for the mentioned dual cone).

Lemma 4.1. *The class of the strict transform of M_1 on Z , $[\tilde{M}_1]$, can be written as*

$$[\tilde{M}_1] = \delta[\tilde{F}_1] + [\tilde{M}_0] + (\delta - 1)[E_1] + (\delta - 2)[E_2] + \dots \\ + [E_{\delta-1}] + d_{\delta+1}[E_{\delta+1}] + \dots + d_n[E_n],$$

where $d_i \in \mathbb{Z}$ and $d_i \leq -1$ for all $i = \delta + 1, \delta + 2, \dots, n$.

Proof. It is clear that we can write $[\tilde{M}_1]$ as

$$[\tilde{M}_1] = d_{01}[\tilde{F}_1] + d_{02}[\tilde{M}_0] + d_1[E_1] + \dots + d_n[E_n],$$

for some values $d_{01}, d_{02}, d_i \in \mathbb{Z}$, $1 \leq i \leq n$. Now, using the equalities

$$[\tilde{F}_1] = [F^*] - [E_1^*], \quad [\tilde{M}_0] = \delta[F^*] + [M^*] \quad \text{and} \quad [E_i] = [E_i^*] - \sum_{p_j \rightarrow p_i} [E_j^*],$$

we can compare the above expression with the equality $[\tilde{M}_1] = [M^*] - \sum_{j=1}^{i_{M_1}} [E_j^*]$, where i_{M_1} is the index of the last point of the configuration of infinitely near points given by ν , \mathcal{C}_ν , through which \tilde{M}_1 goes. This gives rise to a system of linear equations in the variables $d_{01}, d_{02}, \{d_i\}_{i=1}^n$, whose first equations are

$$d_{01} - \delta d_{02} = 0, \quad d_{02} = 1, \quad d_1 - d_{01} = -1, \quad d_2 - d_1 = -1, \dots, \quad d_{\delta-1} - d_{\delta-2} = -1, \\ d_\delta - d_{\delta-1} = -1.$$

These equations determine the values of d_{01}, d_{02}, d_i for $i \in \{1, 2, \dots, \delta\}$, that coincide with those given in the statement. The fact that $d_i \leq -1$ for $i \geq \delta + 1$ follows from considering the remaining equations and recalling that non-free points can only appear when $j > \delta + 1$. \square

Proposition 4.2. *Let Z be the surface given by a non-special divisorial valuation and let $S_2(Z)$ be the cone of $\text{Pic}_\mathbb{Q}(Z)$ defined before Lemma 4.1. Then the dual cone $S_2^\vee(Z)$ is generated by the following classes of divisors: $[F^*], [M^*], \{[\Theta_i]\}_{i=1}^\delta, \{[\Delta_i]\}_{i=\delta+1}^n, \{[\Gamma_i]\}_{i=\delta+1}^n$ and $\{[\Upsilon_{ik}]\}_{i=\delta+1, k=1, \dots, \delta-1}^n$, where*

$$\Theta_i := b_i M^* - \sum_{j=1}^i \text{mult}_{p_j}(\varphi_i) E_j^*, \\ \Delta_i := (-\delta b_i + c_i) F^* + b_i M^* - \sum_{j=1}^i \text{mult}_{p_j}(\varphi_i) E_j^*,$$

$$\Gamma_i := c_i M^* - \sum_{j=1}^i (\delta \operatorname{mult}_{p_j}(\varphi_i)) E_j^*, \text{ and}$$

$$\Upsilon_{ik} := (c_i - kb_i) M^* - \sum_{j=1}^k (c_i - kb_i) E_j^* - \sum_{j=k+1}^i ((\delta - k) \operatorname{mult}_{p_j}(\varphi_i)) E_j^*,$$

and where

$$b_i := (\varphi_{F_1}, \varphi_i)_p, 1 \leq i \leq n, \text{ and } c_i := (\varphi_{M_1}, \varphi_i)_p, \delta + 1 \leq i \leq n.$$

Proof. The cone $S_2(Z)$ is the intersection of the half-spaces

$$H_\tau := \{x \in \operatorname{Pic}_{\mathbb{Q}}(Z) \mid u_\tau \cdot x \geq 0\},$$

where τ varies in the set of faces of $S_2(Z)$ of codimension one and $u_\tau \in S_2^\vee(Z)$ is such that $\tau = S_2(Z) \cap u_\tau^\perp$; then the vectors u_τ generate $S_2^\vee(Z)$ (see [15, Section 1.2]). This shows that it suffices to consider every $(n+1)$ -dimensional linear subspace H generated by elements of $S_2(Z)$ and check whether H^\perp is generated by an element of $S_2^\vee(Z)$. We will see that these generators will be those in the statement.

Denote by $\langle S \rangle$ the linear subspace generated by a set $S \subseteq \operatorname{Pic}_{\mathbb{Q}}(Z)$. Then

$$\langle \{\tilde{F}_1\} \cup \{[E_i]_{i=1}^n\}^\perp \rangle = \langle [F^*] \rangle, \quad \langle \{\tilde{M}_0\} \cup \{[E_i]_{i=1}^n\}^\perp \rangle = \langle [M^*] \rangle,$$

and $[F^*], [M^*] \in S_2^\vee(Z)$. Moreover $\langle \{\tilde{M}_1\} \cup \{[E_i]_{i=1}^n\}^\perp \rangle$ is not generated by an element in $S_2^\vee(Z)$.

We have studied spaces H whose generators contain all the classes $[E_i]$. Now we will treat the cases where a class $[E_i]$, $1 \leq i \leq n$, is not considered. Let us start with the linear space $\langle \{\tilde{F}_1, \tilde{M}_0\} \cup \{[E_j]_{1 \leq j \leq n, j \neq i}\} \rangle$, set

$$[D_i] = d_{i01}[F^*] + d_{i02}[M^*] + d_{i1}[E_1^*] + \cdots + d_{in}[E_n^*] \in \operatorname{Pic}_{\mathbb{Q}}(Z)$$

with arbitrary coefficients and impose the conditions:

$$[D_i] \cdot [\tilde{F}_1] = 0, [D_i] \cdot [\tilde{M}_0] = 0, [D_i] \cdot [E_j] = 0, [D_i] \cdot [\tilde{M}_1] \geq 0 \text{ and } [D_i] \cdot [E_i] \geq 0.$$

Then we obtain the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0, \quad d_{i01} + \delta d_{i02} - \delta d_{i02} = 0, \quad -d_{ij} + \sum_{p_s \rightarrow p_j} d_{is} = 0,$$

$$d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} \geq 0 \text{ and } -d_{ii} + \sum_{p_s \rightarrow p_i} d_{is} \geq 0,$$

where i_{M_1} is the index defined as in the proof of Lemma 4.1. Solving it, we obtain $d_{ii} = -1$; $d_{ij} = \sum_{p_s \rightarrow p_j} d_{is}$, $1 \leq j \leq i-1$; $d_{ij} = 0$, $i+1 \leq j \leq n$; $d_{i01} = 0$; and $d_{i02} = -d_{i1}$. This proves that $d_{ij} = -\operatorname{mult}_{p_j}(\varphi_i)$ holds and also

$$\delta \operatorname{mult}_{p_1}(\varphi_i) - \sum_{j=1}^{\min\{i, i_{M_1}\}} \operatorname{mult}_{p_j}(\varphi_i) \geq 0$$

by our first inequality, which shows that the classes $\{[\Theta_i]_{1 \leq i \leq \delta}\}$ in the statement give generators of the dual cone $S_2^\vee(Z)$.

Reasoning as above for the subspace $\langle \{\tilde{F}_1, \tilde{M}_1\} \cup \{[E_j]_{1 \leq j \leq n, j \neq i}\} \rangle$ and with the same notation, we get the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0, \quad d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} = 0, \quad -d_{ij} + \sum_{p_s \rightarrow p_j} d_{is} = 0,$$

$$d_{i01} + \delta d_{i02} - \delta d_{i02} \geq 0 \text{ and } -d_{ii} + \sum_{p_s \rightarrow p_i} d_{is} \geq 0.$$

Here, the equality $d_{ij} = -\text{mult}_{p_j}(\varphi_i)$ is again true and the first inequality means that

$$\sum_{j=1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_j}(\varphi_i) - \delta \text{mult}_{p_1}(\varphi_i) \geq 0$$

must hold. As a consequence, we have proved that the classes $\{[\Delta_i]\}_{\delta+1 \leq i \leq n}$ in the statement give extremal rays of $S_2^\vee(Z)$.

Repeating the procedure with $\langle\langle [\tilde{M}_0], [\tilde{M}_1] \rangle\rangle \cup \langle\langle [E_j] \rangle\rangle_{1 \leq j \leq n, j \neq i}$, the obtained system is

$$\begin{aligned} d_{i01} + \delta d_{i02} - \delta d_{i02} = 0, \quad d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} = 0, \quad -d_{ij} + \sum_{p_s \rightarrow p_j} d_{is} = 0, \\ d_{i02} + d_{i1} \geq 0 \quad \text{and} \quad -d_{ii} + \sum_{p_s \rightarrow p_i} d_{is} \geq 0. \end{aligned}$$

This proves, on the one hand, that $d_{ii} = -1$; $d_{ij} = \sum_{p_s \rightarrow p_j} d_{is}$, $1 \leq j \leq i-1$; $d_{ij} = 0$, $i+1 \leq j \leq n$; $d_{i01} = 0$; and $d_{i02} = (1/\delta) \sum_{j=1}^{\min\{i, i_{M_1}\}} -d_{ij}$. On the other hand, reasoning as above, $d_{ij} = \delta \text{mult}_{p_j}(\varphi_i)$ and then

$$\sum_{j=1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_j}(\varphi_i) - \delta \text{mult}_{p_1}(\varphi_i) \geq 0,$$

which shows that the set of classes $\{[\Gamma_i]\}_{\delta+1 \leq i \leq n}$ gives generators of $S_2^\vee(Z)$.

It only remains to consider those subspaces $\langle\langle [\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1] \rangle\rangle \cup \langle\langle [E_j] \rangle\rangle_{j \in \{1, 2, \dots, n\} \setminus \{k, i\}}$ attached to pairs of indices k, i , $1 \leq k < i \leq n$. Lemma 4.1 proves the $(n+1)$ -dimensionality of these subspaces. Our computations depend on two indices i and k . So, we will write

$$[D_{ik}] = d_{ik01}[F^*] + d_{ik02}[M^*] + d_{ik1}[E_1^*] + d_{ik2}[E_2^*] + \cdots + d_{ikn}[E_n^*].$$

We must impose the following conditions:

$$\begin{aligned} [D_{ik}] \cdot [\tilde{F}_1] = 0, \quad [D_{ik}] \cdot [\tilde{M}_0] = 0, \quad [D_{ik}] \cdot [\tilde{M}_1] = 0, \quad [D_{ik}] \cdot [E_j] = 0, \quad [D_{ik}] \cdot [E_k] \geq 0 \\ \text{and} \quad [D_{ik}] \cdot [E_i] \geq 0, \end{aligned}$$

which give the equivalent system

$$\begin{aligned} d_{ik02} + d_{ik1} = 0, \quad d_{ik01} = 0, \quad d_{ik01} + \delta d_{ik02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ikj} = 0, \\ -d_{ikj} + \sum_{p_s \rightarrow p_j} d_{iks} = 0, \quad -d_{ikk} + \sum_{p_s \rightarrow p_k} d_{iks} \geq 0 \quad \text{and} \quad -d_{iki} + \sum_{p_s \rightarrow p_i} d_{iks} \geq 0. \end{aligned}$$

To solve it we can assume that the inequalities are strict because, otherwise, we would obtain that $[D_{ik}]$ either vanishes or it gives the class $[\Theta_\delta]$. Indeed, if both inequalities are equalities, then $[D_{ik}] = 0$. Otherwise, taking into account that the first $\delta+1$ points in \mathcal{C}_ν are free, by considering the third equality and $\delta+1 \leq i_{M_1}$, it holds that one of the indices i or k equals δ . This shows that we obtain $[\Theta_\delta]$ as a solution.

The solutions of the system satisfy that $d_{iki} = -1$; $-d_{ikk} > -\sum_{p_s \rightarrow p_k} d_{iks}$; $d_{ikj} = \sum_{p_s \rightarrow p_j} d_{iks}$, $1 \leq j \neq k \leq i-1$; $d_{ikj} = 0$, $i+1 \leq j \leq n$; $d_{ik01} = 0$; $d_{ik02} = -d_{ik1}$; and it must hold that

$$-\delta d_{ik1} = - \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ikj}, \quad (4.1)$$

by the third equation. Note that, for $k+1 \leq j \leq i$, $d_{ikj} = -\text{mult}_{p_j}(\varphi_i)$ up to a positive factor, and also that $-d_{ikk} > -\sum_{p_s \rightarrow p_k} d_{iks} \geq 0$.

The indices i and k must satisfy that $1 \leq k \leq \delta - 1$ and $\delta + 1 \leq i \leq n$. Indeed, with respect to k and reasoning by contradiction, suppose that $k \geq \delta$. By hypothesis, $k < i$, $\delta + 1 \leq i_{M_1}$, and $d_{ikj} = d_{ik\delta}$ for $1 \leq j \leq \delta - 1$, because the first $\delta + 1$ points in \mathcal{C}_ν are free, then

$$-\sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ikj} = -\delta d_{ik1} - \sum_{j=\delta+1}^{\min\{i, i_{M_1}\}} d_{ikj},$$

where $-\sum_{j=\delta+1}^{\min\{i, i_{M_1}\}} d_{ikj} > 0$, which does not hold by (4.1). Notice that this equality is true by our imposed equalities. With respect to the index i , again reasoning by contradiction, suppose that $i \leq \delta$. As $1 \leq k \leq \delta - 1$, Equality (4.1) is equivalent to

$$-(\delta - k)d_{ikk} = -\sum_{j=k+1}^{\min\{i, i_{M_1}\}} d_{ikj}, \quad (4.2)$$

because $d_{ikj} = d_{ikk}$ for $1 \leq j \leq k$. This implies that $-(\delta - k)d_{ikk} = -(i - k)d_{ikk+1}$, which is a contradiction since $-d_{ikk} > -d_{ikk+1}$.

Notice that (4.1) also gives us the value of d_{ikk} , which can be obtained from the following chain of equalities:

$$d_{ikk} = \delta d_{ik1} - \sum_{j=1, j \neq k}^{\min\{i, i_{M_1}\}} d_{ikj} = \delta d_{ikk} - (k-1)d_{ikk} - \sum_{j=k+1}^{\min\{i, i_{M_1}\}} d_{ikj}.$$

Thus, if we take $d_{ikj} = -(\delta - k)\text{mult}_{p_j}(\varphi_i)$, $k+1 \leq j \leq i$, one gets that

$$d_{ik1} = \dots = d_{ikk} = \frac{-(\delta - k) \sum_{j=k+1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_j}(\varphi_i)}{(\delta - k)} = -\sum_{j=k+1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_j}(\varphi_i),$$

and the coefficient of $[M^*]$ is $d_{ik02} = -d_{ik1}$.

As a result, we have that $[D_{ik}] = [\Upsilon_{ik}]$, where

$$\begin{aligned} \Upsilon_{ik} := & \left(\sum_{j=k+1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_j}(\varphi_i) \right) M^* - \sum_{j=1}^k \left(\sum_{s=k+1}^{\min\{i, i_{M_1}\}} \text{mult}_{p_s}(\varphi_i) \right) E_j^* \\ & - \sum_{j=k+1}^i ((\delta - k)\text{mult}_{p_j}(\varphi_i)) E_j^*, \end{aligned}$$

where $\delta + 1 \leq i \leq n$ and $1 \leq k \leq \delta - 1$. This finishes the proof. \square

Remark 4.3. From the above proof, it can be deduced that, when considering the surface \mathbb{F}_1 and a non-special valuation ν , no class $[\Upsilon_{ik}]$ appears as a generator of $S_2^\vee(Z)$.

We are interested in determining conditions under which the generators of the cone $NE(Z)$ of the surfaces Z given by non-special valuations are known. The divisors introduced in Proposition 4.2 will be important for this purpose. The next lemma states some of their properties.

Lemma 4.4. *Let Z (respectively, ν) be a rational surface (respectively, valuation) as in Proposition 4.2. Consider the set of divisors there defined. Then $\Delta_{\delta+1}^2 > 0$, $\Gamma_{\delta+1}^2 > 0$ and $\Upsilon_{\delta+1k}^2 > 0$ for all $k \in \{1, 2, \dots, \delta - 1\}$. In addition, for any index $i \in \{\delta + 2, \delta + 3, \dots, n\}$ such that $\Delta_i^2 \geq 0$ (respectively, $\Gamma_i^2 \geq 0$, $\Upsilon_{ik}^2 \geq 0$), the following properties are satisfied:*

- (a) If p_i is a satellite point of the configuration \mathcal{C}_ν , that ν defines, it holds $\Delta_i^2 > 0$ (respectively, $\Gamma_i^2 > 0$, $\Upsilon_{ik}^2 > 0$).
- (b) $\Delta_{i-1}^2 \geq 0$ (respectively, $\Gamma_{i-1}^2 \geq 0, \Upsilon_{i-1k}^2 \geq 0$) and, moreover, if $\Delta_{i-1}^2 = 0$ (respectively, $\Gamma_{i-1}^2 = 0, \Upsilon_{i-1k}^2 = 0$) then p_i is a satellite point and p_{i-1} is free.

Proof. To prove our first assertion, it suffices to notice that the following three equalities hold:

$$\begin{aligned} \Delta_{\delta+1}^2 &= 2 + \delta - (\delta + 1) = 1 > 0, \\ \Gamma_{\delta+1}^2 &= \delta(\delta + 1)^2 - \delta^2(\delta + 1) = \delta(\delta + 1)(\delta + 1 - \delta) = \delta(\delta + 1) > 0, \\ \Upsilon_{\delta+1k}^2 &= \delta(\delta + 1 - k)^2 - k(\delta + 1 - k)^2 - (\delta + 1 - k)(\delta - k)^2 \\ &= (\delta - k)(\delta + 1 - k)^2 - (\delta - k)^2(\delta + 1 - k) \\ &= (\delta - k)(\delta + 1 - k)[\delta - k + 1 - (\delta - k)] \\ &= (\delta - k)(\delta + 1 - k) > 0. \end{aligned}$$

Finally we remark that (a) and (b) can be proved reasoning as in the proof of Lemma 3.4. Indeed, recalling that $g + 2$ is the cardinality of the set of maximal contact values of ν , the case $g = 1$ follows as in that proof, and, when $g > 1$, with notations as in that lemma and in Proposition 4.2, it suffices to consider the following equalities and to reason again as we did in the mentioned Lemma 3.4.

$$\begin{aligned} \Delta_n^2 &= 2(-\delta b_n + c_n)b_n + b_n^2\delta - \bar{\beta}_{g+1} = 2b_n c_n - \delta b_n^2 - \bar{\beta}_{g+1}, \\ &= e_{g-1} \left[\frac{2b_n c_n - \delta b_n^2}{e_{g-1}} - \bar{\beta}_g \right], \\ \Gamma_n^2 &= c_n^2 \delta - \delta^2 \bar{\beta}_{g+1}, \quad \text{and} \\ \Upsilon_{nk}^2 &= (c_n - kb_n)^2 \delta - k(c_n - kb_n)^2 - (\delta - k)^2 \sum_{j=k+1}^n \text{mult}_{p_j}^2(\varphi_n) \\ &= (\delta - k)[(c_n - kb_n)^2 - (\delta - k)(\bar{\beta}_{g+1} - kb_n^2)] \\ &= (\delta - k)[c_n^2 - 2kc_n b_n + \delta kb_n^2 - (\delta - k)\bar{\beta}_{g+1}] \\ &= (\delta - k)[c_n^2 - k(2c_n b_n - \delta b_n^2) - (\delta - k)\bar{\beta}_{g+1}]. \end{aligned}$$

□

Remark 4.5. Lemma 4.4 allows us to get numerical conditions which imply the non-negativity of the self-intersection of the divisors, appearing in Proposition 4.2, whose classes generate the above defined dual cone $S_2^\vee(Z)$. Let us show which are those numerical conditions.

It is clear that the divisors Θ_i , $1 \leq i \leq \delta$, satisfy $\Theta_i^2 = \delta - i \geq 0$ because each p_j , $1 \leq j \leq i$, is a free point.

Now, $2b_n c_n - \delta b_n^2 \geq [\text{vol}(\nu)]^{-1}$ implies that, for all $i \in \{1, 2, \dots, n\}$, $\Delta_i^2 \geq 0$, which is equivalent to the fact that $2b_i c_i - \delta b_i^2 \geq [\text{vol}(\nu_i)]^{-1}$, where ν_i is the divisorial valuation defined as in Remark 3.8. In a similar way, it holds that if $c_n^2 \geq \delta[\text{vol}(\nu)]^{-1}$, then $\Gamma_i^2 \geq 0$ or, equivalently, $c_i^2 \geq \delta[\text{vol}(\nu_i)]^{-1}$ for all $i \in \{1, 2, \dots, n\}$.

Finally, for each integer k , $1 \leq k \leq \delta - 1$, if one assumes

$$c_n^2 - k(2c_n b_n - \delta b_n^2) \geq (\delta - k)[\text{vol}(\nu)]^{-1},$$

one can deduce that, for all $i \in \{1, 2, \dots, n\}$, $\Upsilon_{ik}^2 \geq 0$ or, equivalently, $c_i^2 - k(2c_i b_i - \delta b_i^2) \geq (\delta - k)[\text{vol}(\nu_i)]^{-1}$.

Before giving our main result in this section, we need to state a last lemma.

Lemma 4.6. *Let ν be a non-special divisorial valuation of a Hirzebruch surface and Z the surface that it defines. Consider the divisors Δ_i, Γ_i and Υ_{ik} , $\delta + 1 \leq i \leq n$; $1 \leq k \leq \delta - 1$, given in Proposition 4.2. Then, for each index i , $\Delta_i^2 \geq 0$ implies $\Gamma_i^2 \geq 0$ and $\Upsilon_{ik}^2 \geq 0$ for all $k \in \{1, 2, \dots, \delta - 1\}$.*

Proof. Our proof follows from the following two properties:

Property 1: If the self-intersections of the divisors Δ_i and $\Upsilon_{i\delta-1}$ are non-negative, then the same property holds for the divisors Γ_i and Υ_{ik} , $1 \leq k \leq \delta - 1$.

Property 2: If the self-intersection of the divisor Δ_i is non-negative, so is the self-intersection of $\Upsilon_{i\delta-1}$.

For proving Property 1, our hypothesis are, by Remark 4.5,

$$[\text{vol}(\nu_i)]^{-1} \leq 2c_i b_i - \delta b_i^2 \quad \text{and} \quad (4.3)$$

$$(\delta - 1)(2c_i b_i - \delta b_i^2 - [\text{vol}(\nu_i)]^{-1}) \leq c_i^2 - \delta [\text{vol}(\nu_i)]^{-1}. \quad (4.4)$$

The inequality in (4.4) and the following one

$$[\text{vol}(\nu_i)]^{-1} \leq c_i^2 - (\delta - 1)(2c_i b_i - \delta b_i^2)$$

are equivalent. From the last inequality and the one in (4.3), we get that $c_i^2 \geq \delta [\text{vol}(\nu_i)]^{-1}$ and then $\Gamma_i^2 \geq 0$. Finally, $\Upsilon_{ik}^2 \geq 0$, $1 \leq k \leq \delta - 1$, if and only if the inequality

$$k(2c_i b_i - \delta b_i^2 - [\text{vol}(\nu_i)]^{-1}) \leq c_i^2 - \delta [\text{vol}(\nu_i)]^{-1}$$

holds, fact that follows straightforwardly from (4.3) and (4.4).

To conclude we prove Property 2. It suffices to check that the following inequalities

$$[\text{vol}(\nu_i)]^{-1} \leq 2c_i b_i - \delta b_i^2 < c_i^2 - (\delta - 1)(2c_i b_i - \delta b_i^2) \quad (4.5)$$

are true. In fact, the first inequality comes from our hypothesis $\Delta_i^2 \geq 0$ and the inequality given by the first and the last sides in (4.5) allows us to show $\Upsilon_{i\delta-1}^2 \geq 0$. To prove the second inequality in (4.5), set $c_i = x$ and $b_i = b$ for simplicity. We are considering non-special valuations, which means that $x > \delta b$. In our new notation we want to prove that

$$2bx - \delta b^2 < x^2 - (\delta - 1)(2bx - \delta b^2).$$

This inequality is equivalent to

$$0 < x^2 - (2b\delta)x + \delta^2 b^2,$$

and it holds for all $x \neq \delta b$ since the point $(\delta b, 0)$ is the vertex of the parabola given by the right-hand side of the inequality. \square

Theorem 3.6 considered special valuations of Hirzebruch surfaces. There we gave equivalent conditions to the non-positivity at infinity of valuations of that type. Our next result gives the corresponding conditions for non-special valuations. In fact, it gives an easy to check numerical and local condition, and two global properties concerning the surfaces that these valuations define. Before stating our result, we introduce the concepts of non-positive, and negative, at infinity, non-special valuation.

Definition 4.7. Let ν be a non-special divisorial valuation of a Hirzebruch surface \mathbb{F}_δ and keep the above notation. The valuation ν is called to be *non-positive* (respectively, *negative*) at infinity if $\nu(h) \leq 0$ (respectively, $\nu(h) < 0$) for all $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$) such that $h \notin k$.

Theorem 4.8. *Let ν be a non-special divisorial valuation of the field of rational functions of \mathbb{F}_δ centered at $R := \mathcal{O}_{\mathbb{F}_\delta, p}$ and $C_\nu = \{p_i\}_{i=1}^n$ the configuration of infinitely near points defined by ν . Let Z be the surface that ν defines and consider the divisor Δ_n on Z defined in Proposition 4.2. Then the following conditions are equivalent:*

- (a) *The valuation ν is non-positive at infinity.*
- (b) *The divisor Δ_n is nef.*
- (c) *It holds the following inequality $2c_n b_n - \delta b_n^2 \geq [\text{vol}(\nu)]^{-1}$.*
- (d) *The cone of curves of Z is generated by $[\tilde{F}_1], [M_0], [M_1], [E_1], [E_2], \dots, [E_n]$.*

Proof. Our proof uses a close reasoning to that of Theorem 3.6. Keeping the notation as in that theorem, we are going to give a sketch of the proof emphasizing only the main differences.

To prove that (a) can be deduced from (b), we can suppose that p is a general point of \mathbb{F}_δ with coordinates $(0 : 1; 0, 1)$. Consider local coordinates $\{x, y\} = \left\{ \frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0} \right\}$ in the affine open set U_{00} and $\{u, v\} = \left\{ \frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1} \right\}$ in U_{11} . Notice that, with our notation, F_1 and M_1 are defined by the equations $X_0 = 0$ and $Y_0 = 0$, $p \in U_{11}$, and F_1 and M_1 have local equations $u = 0$ and $v = 0$, respectively.

If now \mathcal{S} denotes the set of non-constant polynomials in $\mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them, $f \in \mathcal{S}$ satisfies

$$f(x, y) = f(1/u, u^\delta/v) = \frac{h_f(u, v)}{u^{\deg_1(h_f)} v^{\deg_2(h_f)}}, \quad (4.6)$$

where $h_f(u, v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$.

The bi-homogeneous polynomial $X_1^{\deg_1(h_f) + \delta \deg_2(h_f)} Y_1^{\deg_2(h_f)} h_f\left(\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right)$ defines a curve C_f on \mathbb{F}_δ of degree $(\deg_1(h_f), \deg_2(h_f))$ and $f \mapsto C_f$ is a one-to-one correspondence between \mathcal{S} and the set of curves on \mathbb{F}_δ containing no curve F_1, F', M_0, M_1 as a component, where F' and M_0 are defined by the equations $X_0 = 0$ and $Y_0 = 0$. Then $\Delta_n \cdot C_f = -\nu(f)$ and by (b), $-\nu(f) \geq 0$. The case when $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ and x or y or both are factors of f follows as in Theorem 3.6 and (a) is proved.

A proof of the fact that (a) implies (b), (b) implies (c) and (d) implies (b) can be done as in Theorem 3.6.

To see that (c) implies (d), it suffices to notice that, by Lemmas 4.4 and 4.6,

$$S_2^\vee(Z) \subseteq \{[D] \in \text{Pic}_\mathbb{Q}(Z) \mid [D]^2 \geq 0 \text{ and } [H] \cdot [D] \geq 0\} =: A(Z),$$

where $S_2^\vee(Z)$ is the dual cone defined in Proposition 4.2 and H an ample divisor on Z . Finally, the fact

$$S_2^\vee(Z) \subseteq A(Z) \subseteq (S_2^\vee(Z))^\vee = S_2(Z)$$

and a reasoning as in Theorem 3.6 completes our proof. \square

An immediate consequence of the above result is the following one.

Corollary 4.9. *Let ν be a non-positive at infinity non-special divisorial valuation of \mathbb{F}_δ . Consider the divisorial valuations ν_i defined by the divisors E_i associated to the simple sequence of point blowups that ν defines. Then the valuations ν_i , $\delta + 1 \leq i \leq n - 1$, are non-positive at infinity.*

Remark 4.10. Let Z be a surface as in Theorem 4.8 defined by a non-positive at infinity non-special valuation. Then all the divisors Θ_i , $i = 1, 2, \dots, \delta$; Δ_i, Γ_i and Υ_{ik} , $i = \delta + 1, \delta + 2, \dots, n$ and $k = 1, 2, \dots, \delta - 1$, are effective. Indeed, under this assumption,

all the divisors $\Theta_i, \Delta_i, \Gamma_i$ and Υ_{ik} can be expressed as

$$\begin{aligned}\Theta_i &= (\Theta_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^n (\Theta_i \cdot \Delta_j)E_j, \\ \Delta_i &= (\Delta_i \cdot M_0^*)\tilde{F}_1 + (\Delta_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^n (\Delta_i \cdot \Delta_j)E_j, \\ \Gamma_i &= (\Gamma_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^n (\Gamma_i \cdot \Delta_j)E_j, \\ \Upsilon_{ik} &= (\Upsilon_{ik} \cdot F^*)\tilde{M}_1 + \sum_{j=1}^n (\Upsilon_{ik} \cdot \Delta_j)E_j,\end{aligned}$$

which are effective divisors since $\Theta_i, \Delta_i, \Gamma_i$ and Υ_{ik} are nef divisors.

Example 4.11. Let ν be a non-special divisorial valuation of the Hirzebruch surface \mathbb{F}_2 whose sequence of maximal contact values is $\{15, 51, 262, 786\}$. Set $\mathcal{C}_\nu = \{p_i\}_{i=1}^{12}$ the configuration of infinitely near points of ν . Let F_1 be the fiber of \mathbb{F}_2 that goes through p_1 , M_0 the special section and M_1 the section that is linearly equivalent to M and passes through p_1, p_2 and p_3 . Then $b_{12} = 15, c_{12} = 45$ and $[\text{vol}(\nu)]^{-1} = 786$, and so, Theorem 4.8 (c) is satisfied. Therefore, the cone of curves of the surface Z defined by ν is generated by $\{[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1]\} \cup \{E_i\}_{i=1}^{12}$ and the divisors $\Delta_i, 1 \leq i \leq 12$, defined in Proposition 4.2 are nef.

We finish this paper with a result that gives two equivalent properties to the fact of that a non-special valuation is negative at infinite. It can be proved as we did in Theorem 3.9.

Theorem 4.12. *Keeping the same assumptions and notations as in Theorem 4.8. Then the following conditions are equivalent:*

- (a) *The valuation ν is negative at infinity.*
- (b) *It holds that either $2c_n b_n - b_n^2 \delta > [\text{vol}(\nu)]^{-1}$, or $2c_n b_n - b_n^2 \delta = [\text{vol}(\nu)]^{-1}$ and the Iitaka dimension of the divisor Δ_n vanishes.*
- (c) *The inequality $\Delta_n \cdot \tilde{C} > 0$ is satisfied for the strict transform on Z , \tilde{C} , of any curve C on \mathbb{F}_δ , $C \neq F_1, M_1$.*

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REFERENCES

- [1] S. S. Abhyankar and T. T. Moh. Newton-Puiseux expansion and generalized Tschirnhausen transformation. I, II. *J. Reine Angew. Math.*, 260:47–83; *ibid.* 261 (1973), 29–54, 1973.
- [2] A. Beauville. *Complex Algebraic Surfaces*, volume 34 of *London Math. Soc. Student Texts*. Cambridge University Press, Cambridge, second edition, 1996.
- [3] A. Campillo. *Algebroid curves in positive characteristic*, volume 613 of *Lect. Notes Math.* Springer, Berlin, 1980.
- [4] A. Campillo, O. Piltant, and A. Reguera. Curves and divisors on surfaces associated to plane curves with one place at infinity. *Proc. London Math. Soc.*, 84:559–580, 2002.
- [5] E. Casas-Alvero. *Singularities of plane curves*, volume 276 of *London Math. Soc. Lect. Notes Ser.* Cambridge Univ. Press, 2000.
- [6] C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, and C. Shramov. Newton-Okounkov bodies sprouting on the valuative tree. *Rend. Circ. Mat. Palermo (2)*, 66:161–194, 2017.
- [7] S. D. Cutkosky, L. Ein, and R. Lazarsfeld. Positivity and complexity of ideal sheaves. *Math. Ann.*, 321(2):213–234, 2001.

- [8] B. L. de la Rosa-Navarro, J. B. Frías-Medina, and M. Lahyane. Rational surfaces with finitely generated Cox rings and very high Picard numbers. *RACSAM*, 111:297–306, 2017.
- [9] F. Delgado, C. Galindo, and A. Núñez. Saturation for valuations on two-dimensional regular local rings. *Math. Z.*, 234:519–550, 2000.
- [10] J. P. Demailly. Singular Hermitian metrics on positive line bundles. In *Complex algebraic varieties (Bayreuth, 1990)*, volume 1507 of *Lect. Notes Math.*, pages 87–104. Springer, Berlin, 1992.
- [11] M. Dumnicki, B. Harbourne, A. Küronya, J. Roé, and T. Szemberg. Very general monomial valuations of \mathbb{P}^2 and a Nagata type conjecture. *Comm. Anal. Geom.*, 25:125–161, 2017.
- [12] C. Favre and M. Jonsson. *The valuative tree*, volume 1853 of *Lect. Notes Math.* Springer-Verlag, 2004.
- [13] C. Favre and M. Jonsson. Eigenvaluations. *Ann. Sci. Éc. Norm. Sup.*, 40:309–349, 2007.
- [14] C. Favre and M. Jonsson. Dynamical compactifications of \mathbb{C}^2 . *Ann. Math.*, 173:211–248, 2011.
- [15] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton Univ. Press, Princeton, NJ, 1993.
- [16] C. Galindo and F. Monserrat. The cone of curves and the Cox ring of rational surfaces given by divisorial valuations. *Adv. Math.*, 290:1040–1061, 2016.
- [17] C. Galindo, F. Monserrat, and J. Moyano-Fernández. Minimal plane valuations. *J. Alg. Geom.*, 27:751–783, 2018.
- [18] C. Galindo, F. Monserrat, J. Moyano-Fernández, and M. Nickel. Newton-Okounkov bodies of exceptional curve valuations. *ArxivMath*, 1705.03948, 2017.
- [19] S. Greco and K. Kiyek. General elements of complete ideals and valuations centered at a two-dimensional regular local ring. In *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, pages 381–455. Springer, Berlin, 2004.
- [20] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.
- [21] S. Itaka. On D -dimensions of algebraic varieties. *J. Math. Soc. Japan*, 23:356–373, 1971.
- [22] M. Jonsson. *Dynamics on Berkovich spaces in low dimensions*. Springer, 2015.
- [23] K. Kaveh and A. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. Math.*, 176:925–978, 2012.
- [24] J. Kollar and S. Mori. *Birational Geometry of Algebraic Varieties*, volume 134 of *Cambridge Tracts Math.* Cambridge University Press, Cambridge, 1998.
- [25] T. K. Moe. *Cuspidal curves on Hirzebruch surfaces*. PhD thesis, Department of Mathematics, University of Oslo, 2013. <https://www.duo.uio.no/handle/10852/37197>.
- [26] P. Mondal. How to determine the sign of a valuation on $\mathbb{C}[x, y]$. *Michigan Math. J.*, 66:227–244, 2017.
- [27] A. Okounkov. Why would multiplicities be log-concave? In *The orbit method in geometry and physics (Marseille, 2000)*, volume 213 of *Progr. Math.*, pages 329–347. Birkhäuser Boston, Boston, MA, 2003.
- [28] M. Reid. Chapters on algebraic surfaces. In *Complex algebraic geometry (Park city, UT, 1993)*, volume 3 of *IAS/Park City Math. Ser.*, pages 3–159. Amer. Math. Soc., Providence, RI, 1997.
- [29] M. Spivakovsky. Valuations in function fields of surfaces. *Amer. J. Math.*, 112:107–156, 1990.
- [30] B. Teissier. Valuations, deformations, and toric geometry. In *Valuation theory and its applications, II (Saskatoon, SK, 1999)*, volume 33 of *Fields Inst. Commun.*, pages 361–459. Amer. Math. Soc., Providence, RI.
- [31] O. Zariski and P. Samuel. *Commutative Algebra II. Vol. II*. Graduate Texts in Mathematics, Vol. 29, 1975.

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