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Minimal Controllable Set for Takagi-Sugeno Fuzzy Systems with disturbances

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Abstract: In this paper, an approximation of the smallest set (in a given norm) to which the state of a system can be steered (and kept inside once reached) is provided, when bounded disturbances are present. A non-LMI geometric polytope-manipulation approach is pursued over a Takagi-Sugeno model in order to solve this disturbance-rejection setup. Once this set is available, the techniques in this paper compute, too, explicit controllers driving initial conditions to such set, as well as controllers that make this set invariant. This work generalises the earlier particular undisturbed case (i.e., just stabilization) developed in Ariño et al. (2017b).

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1. INTRODUCTION

Takagi-Sugeno fuzzy systems (also known as quasi-LPV ones) are good option to systematically obtain linear parameter-varying embeddings of nonlinear systems via the sector-nonlinearity modelling approach (Tanaka et al., 2001), when the state does not leave a given *modelling region*; note, however, that TS models of a given nonlinear system are not unique and some may give better performance than other ones (Robles et al. (2019) pursues finding the optimal-performance TS model); polynomial-fuzzy models of nonlinear systems may also be obtained (Sala and Ariño, 2009), amenable to sum-of-squares tools, but they will not be pursued in this work.

Controller design for them can be tackled with tools derived from linear approaches, and may lie in two fields:

(1) Linear matrix inequalities (ellipsoid manipulation),
(2) Polytopic manipulation approaches (multiparametric linear and quadratic programming).

The first "LMI" approach was pursued in Boyd et al. (1994); Tanaka et al. (2001) and other seminal works in the 1990s; many extensions and refinements appeared, both in the fuzzy control community (Sala et al., 2005; Guerra et al., 2015) and in the LPV gain-scheduling one (Scherer, 2001; Wu and Dong, 2006; Mohammadpour and Scherer, 2012; Sala, 2019). Nowadays, it is the mainstream approach to Takagi-Sugeno/LPV control. Regarding the disturbance-rejection issues to be discussed in this paper, invariant sets in in TS/polynomial systems under finite-time integral disturbance bounds appear in Sala and Pitarch (2016), under an LMI/SOS formulation.

The second "polytopic" approach was introduced in linear robust and predictive control setups by Kerrigan (2000); Kvasnica et al. (2004), and our team was responsible to extend it to gain-scheduled (membership-function dependent) control laws: stabilisation was addressed in Ariño et al. (2017b), and predictive control was addressed in Ariño et al. (2017a). The first theoretical advantage of the geometric polytopic approach is that it can incorporate saturation in control action and the definition of the modelling region out of which the TS model is invalid, and such sets may even be non-symmetric. The second and most important advantage is that it can be proved asymptotically exact, i.e., the domain of attraction of a given system ("maximal" contractive set) can be obtained with progressively better accuracy as some Polya-related parameters increase. Given the conservatism in most LMI Lyapunov function choices and controller structure, as well as in the handling of saturation, basically the polytopic approach can theoretically beat any LMI formulation of the problem, even with multi-step or membership-dependent Lyapunov functions, under some assumptions. Despite its theoretical elegance, the most important drawback of the polytopic framework is that it scales poorly to systems with high order or a high number of inputs.

This work pursues finding the "smallest" region into which a closed-loop disturbance-rejection controller can keep the state into (for any membership shape), i.e., changing the "maximal" invariant set goal in prior works to a sort of "minimal" one [1]. Actually, the procedure does not assume any specific controller; on the contrary, the controller minimising the size of such zone is sought, too. In a later stage, a controller steering the state to that small region around the origin is also constructed.

The structure of this paper is as follows: next section discusses preliminary ideas, notation and problem statement. Section 3 discusses modifications to the key one-step-set

[1] The concept of minimal invariant set is difficult to define; we will actually pursue a norm-minimisation goal, as discussed in later sections.

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concept under disturbances in a TS setup. Section 4 uses such one-step set to propose an algorithm to compute a minimum-norm feasible set for our disturbance rejection problem. Section 5 provides the controller that, under zero initial condition, keeps the state inside the previously obtained set, and Section 6 builds up the controller that drives the state to such minimum-norm region around the origin. A numerical example is presented in Section 7 and a conclusion section closes this paper.

2. PRELIMINARIES

This section will discuss Takagi-Sugeno systems with disturbances and the method to compute invariant and contractive sets.

2.1 Takagi-Sugeno Fuzzy Model

Consider a discrete-time nonlinear system:

$$x_{k+1} = f(x_k, u_k, v_k) \tag{1}$$

such that f has continuous partial derivatives, where $x_k \in \mathbb{R}^n$ represents the state vector, $u_k \in \mathbb{R}^m$ stands for the control actions and $v_k \in \mathbb{W} \in \mathbb{R}^s$ are the disturbances at time instant k.

It is well known that, under mild $\mathcal{C}^1$ assumptions, such system can be expressed (*locally* in a compact region $\mathbb{X}$ of the state-space Tanaka et al. (2001), denoted as modelling region), as a TS fuzzy system with r rules or local models:

$$x_{k+1} = \tilde{f}(\mu(x_k), x_k, u_k, v_k) = \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k + E_i v) \tag{2}$$

where A_i, B_i, E_i are the so-called consequent model matrices and $\mu_i(x_k)$ represents membership functions such that the vector of membership functions $\mu(x_k)$ belongs to the $(r-1)$-dimensional standard simplex $\Delta \subset \mathbb{R}^r$, defined as:

$$\Delta := \{\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r \mid \sum_{i=1}^{r} \mu_i = 1, \quad \mu_i \geq 0 \;\; i : 1 \ldots r\} \tag{3}$$

Given an arbitrary set Ω, notation $\lambda\Omega$ will denote the linear scaling of the set Ω by $\lambda \geq 0$. If Ω is defined as $\Omega := \{x \in \mathbb{R}^n : M(x) \leq 0\}$, for an arbitrary vector of constraint functions $M(\cdot)$, the scaled set is $\lambda\Omega := \{x : M(\lambda^{-1}x) \leq 0\}$.

Definition 1. (Kerrigan (2000)). A set $\Omega \subset \mathbb{X}$ is *control λ-contractive* (given $0 \leq \lambda \leq 1$) for the system (1) if and only if, for any x in Ω there exists an admissible input such that the successor state lies in $\lambda\Omega$, i.e., if $x \in \Omega \Rightarrow \exists u \in \mathbb{U} : f(x, u, v) \in \lambda\Omega \; \forall v \in \mathbb{W}$. A λ-contractive set in Ω is *maximal* if any other λ-contractive set in Ω is a subset of the said maximal one.

Definition 2. Given an arbitrary target set Ω, the one-step set $\mathcal{Q}(\Omega)$ is the set of states x in $\mathbb{X}$ from which the next state of system (1) can be driven to Ω with an admissible $u \in \mathbb{U}$ for all possible disturbance values, i.e.,

$$\mathcal{Q}(\Omega) := \{x \in \mathbb{X} | \exists u \in \mathbb{U} : f(x, u, v) \in \Omega \; \forall v \in \mathbb{W}\}$$

Efficient computational characterisation of the one-step set $\mathcal{Q}$ in Definition 2 can only be easily carried out for special cases of f; for instance, the linear case (Kerrigan, 2000). For the TS systems (2), an approximated (asymptotically exact) solution can be obtain using the Polya's theorem (Sala and Ariño, 2007). This algorithm is presented in Ariño et al. (2017b), used in disturbance-free stabilisation problems.

Given any expression $\tilde{f}(\mu, x, u, v)$ being $\tilde{f}$ an homogeneous polynomial in μ of degree d_f, we will denote by

$$Po(\tilde{f}, d) := \left(\sum_{s=1}^{r} \mu_s\right)^{d-d_f} \cdot \tilde{f}$$

referred to as Polya expansion of $\tilde{f}$, which is, again, an homogeneous polynomial of degree d in μ. The coefficients of such polynomial will be denoted as $Po_i^{[\tilde{f},d]}$, which are themselves functions of (x, u, v), for $1 \leq i \leq N_d$, $N_d = \begin{pmatrix} d+r-1 \\ d \end{pmatrix}$.

For instance, if $\tilde{f} = \sum_{i=1}^{2} \mu_i A_i x$, then $Po(\tilde{f}, 3) = \mu_1^3 Po_1^{[\tilde{f},d]} + \mu_1^2\mu_2 Po_2^{[\tilde{f},d]} + \mu_1\mu_2^2 Po_3^{[\tilde{f},d]} + \mu_2^3 Po_4^{[\tilde{f},d]}$, with $Po_1^{[\tilde{f},d]} = A_1 x$, $Po_2^{[\tilde{f},d]} = (2A_1 + A_2)x$, $Po_3^{[\tilde{f},d]} = (A_1 + 2A_2)x$, $Po_4^{[\tilde{f},d]} = A_2 x$. For details on Polya-related manipulations, see, for instance, Sala and Ariño (2007); Ariño et al. (2017b). Note: the fact that $Po_i^{[\tilde{f},d]}$ are linear functions of state and input when $\tilde{f}$ comes from (2) is important, and will be later exploited.

Polya's Theorem says that if $\tilde{f}(\mu, x, u, v) > 0$, there exists a large enough d such that $Po_i^{[\tilde{f},d]}(x, u, v) \geq 0$ for all i.

In this paper, we will use controllers wich are a polynomial in the membership functions, being its coefficients considered as decision variables. If we denote as $\bar{u}$ the set of such decision variables, the control law will be represented by $u = g(\mu, \bar{u})$, being g assumed an homogeneous polynomial of degree c in the memberships. The closed-loop equations will be:

$$x_+ = \tilde{f}(\mu, x, g(\mu, \bar{u}), v) \tag{4}$$

In this way, the number of inputs has been expanded, so the initial fuzzy control design problem (finding $u(\mu, x)$) can now be stated as finding a "non-fuzzy" control law $\bar{u}(x)$ because the membership dependence has been explicity fixed to be in the form of the said homogeneous polynomial. So if an unique control action is considered we would have a non-fuzzy control ($c = 0$), the standard PCD-like control would amount to $c = 1$ but more complex parameterisations can be considered for any arbitrarily large c, subject to computational resources availability.

For instance, in a system (2), we could set up a control $u = g(\mu, \bar{u}) = \sum_{i=1}^{r} \mu_i \bar{u}_i$ so we can write (4) as $x_+ = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(A_i x + B_i \bar{u}_j)$. Setting $\bar{u}_j = -K_j x$ would conform the standard PDC controller in, say, Tanaka et al. (2001); in Ariño et al. (2017b) a non-PDC piecewise affine structure $\bar{u}_j = -K_{jl}x + m_l$ was proposed, where l denoted a particular polyhedron in a partition of the state space resulting from optimisation. Such structure will be used, too, in this work.

Regarding input saturation, it can be easily proven that $u = g(\mu, \bar{u}) \in \mathbb{U}$ if each coefficient in $\bar{u}$ is in $\mathbb{U}$; abusing the notation, this will be expressed as the constraint $\bar{u} \in \mathbb{U}$.

Problem statement. The paper Ariño et al. (2017b) presents a polytopic inner *approximation* of the one step set denoted as $\widetilde{\mathcal{Q}}_d^c$ where parameter c defines the controller *complexity* and d defines the required Polya expansion degree. However, the above-cited paper does not consider disturbances in its developments, so incorporating them is the objective of this work.

Formally, we seek to obtain the minimum scaling γ such that there exists a λ-contractive set $\mathcal{C}^\lambda$ with $\|x\|_p \leq \gamma$ for all $x \in \mathcal{C}^\lambda$, being the chosen norm $p = 1$ or $p = \infty$. Actually, minor modifications will allow to find the minimum γ such that $\mathcal{C}^\lambda \in \gamma P$ being P any user-defined compact polyhedron.

3. ONE-STEP SET UNDER BOUNDED DISTURBANCES

Based in the disturbance-free ideas in Ariño et al. (2017b), if Ω is a polyhedral set, i.e. $\Omega := \{x \in \mathbb{X} : R_\Omega x \leq l_\Omega\}$, then, for a system (2), an inner approximation[2] to the ideal one-step set $\mathcal{Q}(\Omega)$ in Definition 2 is:

$$\widetilde{\mathcal{Q}}_{c,d}^\lambda(\Omega) := \Big\{x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U} : Po_i^{[L,d]}(x, \bar{u}, v) \leq 0 \quad \forall v \in \mathbb{W}\ \forall i \Big\} \quad (5)$$

where $L(\mu, x, \bar{u}, v) = R_\Omega \tilde{f}(\mu, x, g(\mu, \bar{u}), v) - l_\Omega$ and trivial manipulations are assumed in (5) to express L as an homogeneous polynomial of degree $c + 1$ in μ.

Note that the approximation of the one step set is done for all possible values of the membership functions, i.e., it is a so-called *shape-independent* approach.

Note that $\widetilde{\mathcal{Q}}_{c,d}^\lambda(\Omega)$ will get larger if either c or d are increased, asymptotically approaching the ideal one-step set with large values of complexity of the control law and Polya expansions, as discussed in footnote 2 and references therein.

Multi-parametric linear programming. Note that $Po_i^{[L,d]}$ is a linear function of $(x, \bar{u}, v)$, to be denoted by

$$Po_i^{[L,d]} = G_i \begin{bmatrix} x \\ \bar{u} \end{bmatrix} + H_i v$$

hence, evaluating (5) is needed only in the vertices of $\mathbb{W}$, i.e., $vert(\mathbb{W}) = \{w_1, \ldots, w_T\}$. Thus, we can express (5) as a finite set of linear constraints, so $\widetilde{\mathcal{Q}}_{c,d}^\lambda(\Omega)$ is a polyhedron:

$$\widetilde{\mathcal{Q}}_{c,d}^\lambda(\Omega) := \Big\{x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U} : G_i \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \leq -H_i w_j \quad \forall i, j \Big\} \quad (6)$$

Actually, $\widetilde{\mathcal{Q}}_{c,d}^\lambda(\Omega)$ is the projection over $\mathbb{X}$ of a polyhedron in the product state-input space, details omitted for brevity Ariño et al. (2017b).

The Algorithm 1 presented at Ariño et al. (2017b), repeated here for convenience, is the base of the sets computed in this work. As the one step set is restricted to polytopic sets, the computational geometry tools in the MPT toolbox, see Kvasnica et al. (2004), allow implementing the above algorithm to find the λ-contractive set, $\mathcal{C}_\infty^\lambda$ in a few lines of MATLAB code.

Algorithm 1 Computation of the maximal λ-contractive set $\mathcal{C}_\infty^\lambda(\Omega)$

Inputs: c, d, Ω, λ, $\mathbb{X}$, $\mathbb{U}$.

(1) Make $i = 0$, $\mathcal{C}_0^\lambda = \Omega$
(2) Repeat :
  (a) i=i+1
  (b) $\mathcal{C}_i^\lambda = \widetilde{\mathcal{Q}}_d^c\left(\lambda \mathcal{C}_{i-1}^\lambda\right) \cap \Omega$
  Until $\mathcal{C}_i^\lambda = \mathcal{C}_{i-1}^\lambda$;
(3) Set $\mathcal{C}_\infty^\lambda = \mathcal{C}_i^\lambda$; END.

Note that, in a disturbance-free setting, if $\mathcal{C}^\lambda$ is control λ-contractive, then any scaling $\gamma\mathcal{C}^\lambda$, with $\gamma \leq 1$ is λ-contractive, too, for a TS system (Ariño et al., 2017b, Proposition 2). However, that does not hold in disturbed systems as we are considering here: indeed, reducing the size of the states makes the disturbance vector comparatively "larger" with respect to the state so contraction rate gets slower. This requires some algorithmic changes to Ariño et al. (2017b), which will be discussed in next sections of this manuscript.

4. MINIMAL-NORM DISTURBANCE REJECTION SET

Continuing with the above discussion motivating this section, it is easy to understand that there is a minimum region inside which there does *not* exists any λ-contractive set under disturbances; indeed $x = 0$ will forcefully entail $x_+ \neq 0$ for $v \neq 0$, as the "equilibrium" control $u = 0$ cannot reject unmeasurable disturbances at the same time instant.

As stated in the introduction, the goal of this paper is driving the state to a small invariant (contractive with $\lambda = 1$) set around the origin. In order to compare the different λ-contractive sets the p-norm of a compact set Ω is defined as:

$$\|\Omega\|_p = \max_{x \in \Omega} \|x\|_p \quad (7)$$

In this work, the results obtained with $\|x\|_\infty = \max_i(x_i)$, $\|x\|_1 = \sum_i |x_i|$ will be compared in the example section.

The bisection algorithm below can solve this problem, given a termination tolerance ϵ, contraction rate λ and complexity parameters c (controller) and d (Polya expansion).

[2] Except in the case $\tilde{f}$ is of degree 1 in μ, $\mathcal{Q}(\Omega)$ involves checking if a homogeneous polynomial in the memberships fulfills some inequalities; this can be reduced to copositive programming which is a computationally hard problem that can be approximated by Polya relaxations: exact computation of $\mathcal{Q}(\Omega)$ is out of reach but a progressively tighter sequence of approximations can be easily built, see Ariño et al. (2017b) for details.

Algorithm 2 Computation of minimum λ-contractive set $\mathcal{C}_m^\lambda$

Inputs: c, d, λ, $\mathbb{X}$, $\mathbb{U}$.

(1) Compute the set $\mathcal{C}_m^\lambda := \mathcal{C}_\infty^\lambda(\mathbb{X})$ with Algorithm 1. If such set is empty then END [disturbances are too large and no contractive set is found in the whole modelling region].
(2) Define Ω as the polytope $V_1 := \{x \in \mathbb{X} : \|x\|_p \leq 1\}$ and $\gamma := \|\mathcal{C}_m^\lambda\|_p$, with $p = 1$ or $p = \infty$.
(3) Define $\gamma_2 = \gamma$ and $\gamma_1 = 0$.
(4) Let $\gamma = \frac{\gamma_2+\gamma_1}{2}$. Compute $\mathcal{C}_\infty^\lambda(\gamma\Omega)$ with Algorithm 1.
(5) if $\mathcal{C}_\infty^\lambda(\gamma\Omega)$ is empty, let $\gamma_1 = \gamma$, else let $\mathcal{C}_m^\lambda = \mathcal{C}_\infty^\lambda(\gamma\Omega)$ and $\gamma_2 = \gamma$.
(6) if $\gamma_2 - \gamma_1 > \epsilon$ then go to step 4 else END

The basic idea of the above algorithm is that small sets Ω must render Algorithm 1 infeasible, as earlier discussed and, on the other hand, if a disturbance is very large (and, say, the system is unstable or control saturation is tight) there may not exist λ-contractive sets in the modelling region $\mathbb{X}$. This justifies the initial algorithm step, which starts the iteration with the maximal contractive set in the whole modelling region $\mathbb{X}$. If feasible, the bisection steps try to shrink the set in which λ-contractive sets are found until smaller trials render infeasible.

Thus, the algorithm obtains (within a tolerance ϵ) the minimum scaling $\gamma^* = \|\mathcal{C}_m^\lambda\|_p$ such that there exist a contractive set $\mathcal{C}$ with $\|\mathcal{C}\|_p \leq \gamma^*$, as pursued in our problem statement. Indeed, $\|\mathcal{C}_m^\lambda,\|_p \leq \gamma^*$ ensues from the fact that $\mathcal{C}_m^\lambda \subseteq \gamma^* V_1$, being V_1 the unit-norm level set defined in step 2 of Algorithm 2. Note that, due to the scaling property, any $\mathcal{C}_m^\lambda$ from any iteration was a polyhedral Lyapunov level set in Ariño et al. (2017b), but Lyapunov level sets are not meaningful in disturbance rejection in a strict interpretation (of course, bounded-real lemma and like results (Boyd et al., 1994) induce quadratic forms proving attenuation, passivity, invariance, etc.; the sets $\mathcal{C}_m^\lambda$ are the polyhedral analogue in polytopic-manipulation approaches).

Remark: In the problem statement our goal was an approximation to the minimal disturbance-rejection set. Indeed, approximations come from three sources:

(1) The needed Polya relaxations.
(2) The need of $\lambda < 1$ and not $\lambda \leq 1$ to ensure that states outside the finally obtained set $\mathcal{C}_m^\lambda$ can be driven to it in finite time.
(3) The fact that algorithm 1 computes a *maximal* contractive set inside $\gamma^* V_1$: even if the maximal set is empty for $\gamma < \gamma^*$, so we can assert that there does not exist any λ-contractive set whose norm is strictly lower than γ^*, there may be another contractive set inside $\gamma^* V_1$ contained in $\mathcal{C}_m^\lambda$.

5. CONTROLLER COMPUTATION IN THE MINIMAL DISTURBANCE REJECTION SET

Once $\mathcal{C}_m^\lambda$ is obtained by Algorithm 2, a procedure to compute a controller law is needed; the control law should keep next state inside $\mathcal{C}_m^\lambda$. This is due to the fact that the referred algorithm output (a polyhedral set) only ensures the existence of that control action (indeed, $\mathcal{C}_m^\lambda$ with $\lambda \leq 1$ is contractive and, henceforth, invariant); however, the algorithm does not explicitly compute a control law $u(x)$ to close the loop with.

In order to obtain this controller two possibilities are considered: The first one is the on-line computation of the control action solving a linear programming problem at each sample time. The second one is an off-line optimization. In this case a explicit solution of the optimization problem is obtained an the controller consists of a piecewise polynomial function.

5.1 On-line Controller

In on-line operation, state and membership values are known at the time of computing the control action, so the model $x_{k+1} = A(\mu(x_k))x_k + B(\mu(x_k))u_k + E(\mu(x_k))v_k$, affine in the control action u_k, renders:

$$x_{k+1} = M_k + N_k u_k + R_k v_k \tag{8}$$

with

$$M_k := A(\mu(x_k))x_k,\ N_k := B(\mu(x_k)),\ R_k := E(\mu(x_k))$$

and M_k, N_k and R_k are matrices known at time k once x_k has been measured. A reasonable course of action would be proposing a cost index depending only on the current control action u_k, choosing a suitable one in the convex set $U_{\lambda\mathcal{C}_m^\lambda}(x_k, \mu(x_k)) = \{u \in \mathbb{U} \,|\, \forall v \in \mathbb{W}\ M_k + N_k u + R_k v \in \lambda\mathcal{C}_m^\lambda\}$; dependence on x_k is implicit in matrices M_k, N_k, and R_k. In this way, there would be no need to actually build up an explicit "fuzzy" controller , as u_k can be directly optimised in a way akin to, say, predictive control (whose fuzzy extension appears in Ariño et al. (2017a), using the same class of polyhedron manipulation techniques under discussion here).

If $\mathbb{W}$ is a polyhedron, then the set $U_{\lambda\mathcal{C}_m^\lambda}(x_k, \mu(x_k))$ is a polyhedron, too (elementary convexity argumentations easily conclude that only the vertices of Ω need to be considered in the definition of $U_{\lambda\mathcal{C}_m^\lambda}(x_k, \mu(x_k))$ so it is characterised by a finite number of linear inequalities (depending on the currently measured state and memberships). Optimisation of the 1 or ∞ norms of u can be done with plain linear programming, and optimisation of the 2 norm of u with quadratic programming; this is analogue to (Ariño et al., 2017b, Sect. 6.1) so, for brevity, the reader is referred there for details.

5.2 off-line Controller

Multiparametric linear or quadratic programming can be used to obtain a partition of $\mathcal{C}_m^\lambda$ so that an explicit piecewise-affine fuzzy control law is computed in the same way as (Ariño et al., 2017b, Sect. 6.2); the reader is, again, referred to such work for further details.

6. CONTROLLER COMPUTATION OUTSIDE THE MINIMAL DISTURBANCE REJECTION SET

Section 5 discussed how to keep the state inside the invariant set $\mathcal{C}_m^\lambda$ if initial conditions were already in it. Usually, most disturbance-rejection problems assume zero initial conditions (for instance, most $\mathcal{H}_\infty$ ones). In our case, the following approaches can provide controllers for states outside $\mathcal{C}_m^\lambda$ that drive the system's trajectory to it in finite time.

The basic idea roots on the fact that disturbances are progressively less significant in relative terms as the state gets larger. Indeed, if $\tilde{f}(\mu, x, g(\mu, \bar{u}), v) \in \Omega$ for all $\mu \in \Delta$, for all $v \in \mathbb{W}$ then, linearity in $(x, \bar{u}, v)$ of the closed-loop TS expression in (4) entails that $\tilde{f}(\mu, \beta x, g(\mu, \beta\bar{u}), v) \in \beta\Omega$ for all $v \in \beta\mathbb{W}$, given $\beta \geq 1$, thus, evidently, $\tilde{f}(\mu, \beta x, g(\mu, \beta\bar{u}), v) \in \beta\Omega$ for all $v \in \mathbb{W}$, as $\mathbb{W} \in \beta\mathbb{W}$. The key for this fact to be useful is $\beta\bar{u}$ being a valid (non-saturating) control action, $\beta\bar{u} \in \mathbb{U}$.

In the rest of this section, we will assume that there is a non-saturating control action making $\mathcal{C}_m^\lambda$ λ-contractive, i.e., the set $\mathcal{C}_m^\lambda$ is a consequence of "uncertainty" and not of "limits in the control action". For instance, $x_{k+1} = x_k + u_k + d_k$ with $|d_k| \leq 0.1$ can be kept inside $\mathcal{C}_m^\lambda = [-0.1, +0.1]$ with $|u_k| \leq 0.1$: if the control saturation limit is larger, the said assumption will be fulfilled.

6.1 On-line controller

Given x, once memberships are measured, we can write (8). The control goal will be minimising β such that $x_{k+1} \in \beta\mathcal{C}_m^\lambda$. Of course, $\beta = \lambda$ can be guaranteed for $x \in \mathcal{C}_m^\lambda$. If the above non-saturating assumption holds, for sure there will be an scaling $\kappa\mathcal{C}_m^\lambda$, $\kappa > 1$ in which this problem will be feasible. Determining the largest domain of attraction of $\mathcal{C}_m^\lambda$ (i.e., largest set of initial conditions that can be steered to $\mathcal{C}_m^\lambda$) is left for further research.

6.2 Off-line controller

The explicit shape-independent solution in Section 5.2 can be used, too, to steer states outside $\mathcal{C}_m^\lambda$ to it. Indeed, let us first determine the minimum β such that $x \in \beta\mathcal{C}_m^\lambda$. Then, the control action $\bar{u}(x) = \beta\bar{u}(\beta^{-1}x)$ ensures that $x_+ \in \lambda\beta\mathcal{C}_m^\lambda$ if it is non-saturating, i.e., $\bar{u}(x) \in \mathbb{U}$.

7. EXAMPLE

Consider a Takagi-Sugeno system

$$x_{k+1} = \sum_{i=1}^{2} \mu_i (A_i x_k + B_i u_k + E_i v_k)$$

with vertex model matrices:

$$A_1 = \begin{pmatrix} 0.95 & 0.3 \\ 0.7 & 1.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0.1 & 0.7 \\ 0.2 & 0.4 \end{pmatrix} \tag{9}$$

$$B_1 = \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0.1 \\ 2 \end{pmatrix} \tag{10}$$

$$E_1 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \qquad E_2 = \begin{pmatrix} -0.1 \\ 1 \end{pmatrix} \tag{11}$$

The system will be subject to the following constraints in inputs and states:

$$-10 \leq u_k \leq 10, \quad -10 \leq x_k \leq 10 \tag{12}$$

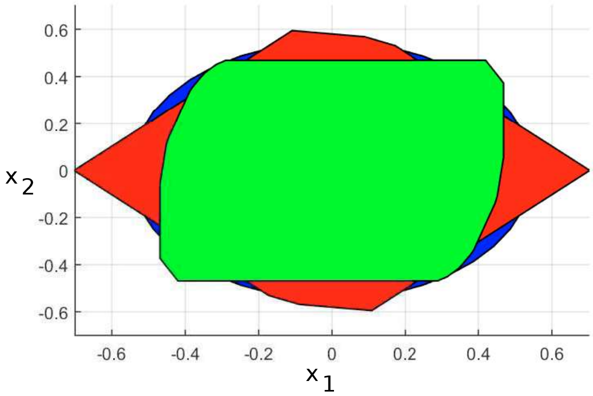

Fig. 1. Minimal λ-contractive sets: green ∞-norm, red 1-norm, blue 2-norm

Additionally, we will assume that the disturbance is bounded by $-0.3 \leq v_k \leq 0.3$. The membership functions [3] are:

$$\mu_1(x) = (10 - (1\ 0)x)/20, \quad \mu_2(x) = 1 - \mu_1(x) \tag{13}$$

Following Algorithm 2, the minimal λ-contractive set is obtained, with $\lambda = 0.99$, $c = 1$ and $d = 8$, for both 1-norm and infinity norm optimization. Additionally, a 30 vertex polygonal approximation of a circle was also used, with trivial modifications to the algorithms, left to the reader, to approximate the 2-norm minimisation disturbance-rejection problem.

The result of the three executions of the algorithm are shown in Figure 1. Note that the resulting sets depend on the chosen norm, i.e., there does not seem to exist a minimal invariant set contained in all of them (indeed, the intersection of them can be proven *not* invariant).

For the ∞-norm case the explicit controller outlined in Section 5.2 has been computed. The explicit controller is formed by a partition of 141 sets; the regions conforming the partition of the green set in Figure 1 and a simulation (black line) are shown in Figure 2. The simulation was carried out assuming uniformly distributed random disturbances in the constraint interval $[-0.3, 0.3]$.

Finally the controllers are tested outside the minimal λ-contractive set, following the ideas in Section 6. Figure 3 shows the trajectories of the system with both approaches. The on-line controller is shown in blue line and the trajectory of off-line controller is shown in black.

8. CONCLUSION

This paper has presented a geometric approach to drive the system state to the "smallest" possible invariant set (in certain norm) around the origin, generalising earlier disturbance-free stabilization approaches based on polyhedron manipulations. Two control laws are proposed: one for the "reaching" phase if initial conditions are outside such small invariant set, and another one for the stationary

[3] The given memberships are provided for simulation purposes only, as results here are shape-independent , i.e., they will work for any pair of membership functions as long as they are positive and add up to one.

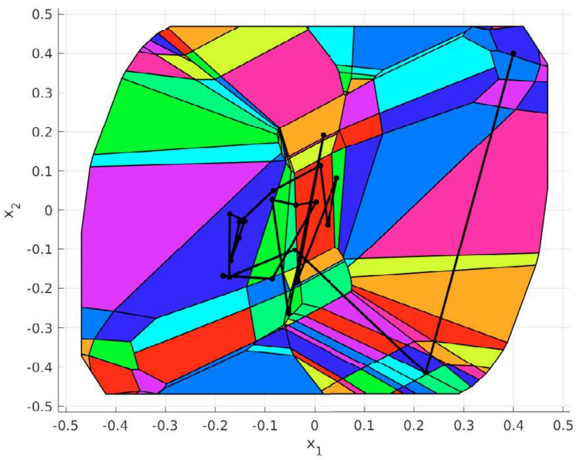

Fig. 2. Explicit Controller partition and state trajectory

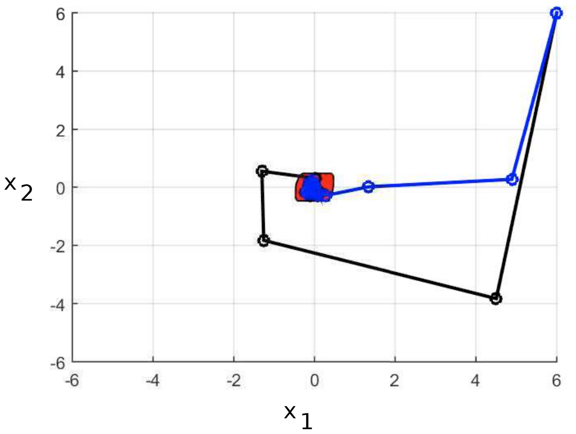

Fig. 3. Trajectories of the state with on-line and off-line controllers

situation in which the trajectories are kept under control inside the referred invariant set.

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