# NONLINEAR DIAMETER PRESERVING MAPS ON FUNCTION SPACES

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ABSTRACT. In this paper we study nonlinear diameter preserving mappings defined between function spaces and obtain generalizations of, basically, all known results concerning diameter preservers. In particular, we give a complete description for algebras of continuously differentiable functions, (little) Lipschitz algebras and dense function spaces.

## 1. Introduction

There has been always considerable interest in characterizing maps between spaces of functions that preserve a certain property or family of functions. There is a vast history in such problems when the map is assumed to be linear, the so-called *linear preserver problems*. Here we can include linear maps which are norm preserving (Banach-Stone type theorems), disjointness preserving, diameter preserving, spectrum preserving, etc. (see, e.g., [5], [11], [22], [18], [15]).

More recently, however, there has been an increasing interest in not considering linearity a priori. Thus, in [17], K. Jarosz proved that a not necessarily linear isometry between spaces of continuous functions is a composition operator in modulus. Subsequent papers have dealt with finding nonlinear conditions for isometries to be composition or weighted composition operators (see, e.g., [26]). Other authors, however, have focused primarily on nonlinear maps which preserve some property or subset of the spectrum of the functions, known as spectral preserver problems (see, e.g., [23], [25], [13]).

Following this trend of interest in nonlinear mappings, in this paper we focus on (not necessarily linear) diameter preserving maps defined between function algebras, that is, maps which preserve the diameter of the range of the functions. Besides the sup-norm, the diameter of the range is another possibility to measure the functions and even has been proved to be more appropriate with regard to isometries in certain contexts than the sup-norm (see [22, p. 57]). In its linear version, these maps, which are indeed linear isometries with respect to the diameter (semi)norm, were introduced by Győry and Molnár ([12]) and have been studied intensively by many authors since then (see e.g.,

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[7], [24], [6], [9], [2], [1], [3], and [4]). More recently, in [14], the authors studied linear diameter preserving maps between function spaces and extended the previous results. They also gave an example showing that their assumptions cannot be removed. Basically these papers show that, in a wide range of contexts, linear diameter preserving mappings can be written as the sum of a weighted composition operator and a linear functional.

Without assuming linearity, we provide a representation for diameter preserving mappings defined between two function algebras, $A_1$ and $A_2$, as the sum of an element of $A_2$, a weighted composition operator and a functional on $A_1$. Similar questions have been addressed in [16] for dense function spaces with different techniques. Here we study nonlinear diameter preserving mappings between function spaces and obtain generalizations of, basically, all known results concerning linear and nonlinear diameter preserving mappings. In particular, a complete description of such maps is given if we consider algebras of continuously differentiable functions, (little) Lipschitz algebras and dense function spaces.

## 2. Preliminaries

For a compact Hausdorff space $X$, the algebra of all continuous scalar-valued functions on $X$ is denoted by $C(X)$. For any $f \in C(X)$, diam$(f)$ is the diameter of the range of $f$. A linear subspace (resp. subalgebra) $A$ of $C(X)$ is called a *function space* (resp. *function algebra*) if $A$ contains the constant functions and separates the points of $X$.

Let $A$ be a linear subspace of $C(X)$. A nonempty subset $E$ of $X$ is called a *boundary* for $A$ if each function in $A$ attains its maximum modulus within $E$. The *Choquet boundary* for $A$, $Ch(A)$, consists of all $x \in X$ such that the evaluation functional $\delta_x$ at $x$ is an extreme point of the closed unit ball of the dual space of $(A, \|.\|_\infty)$ and is a boundary for $A$. Clearly, $Ch(A) = Ch(\overline{A})$, where $\overline{A}$ is the closure of $A$ in $C(X)$ endowed with the supremum norm $\|.\|_\infty$. For any two points $x, x' \in Ch(A)$, we shall write $\delta_{x,x'} := \delta_x - \delta_{x'}$.

Let $X$ be a compact Hausdorff space, and let $C$ denote the space of all the constant functions. For a function space $A$ on a compact Hausdorff space $X$, by $A_d$ we mean the quotient space $A/C$ endowed with the diameter norm, $\|\pi(f)\|_d :=$diam$(f)$ for all $f \in A$, where $\pi$ is the natural quotient map $\pi : A \to A/C$, and $(A_d^*, \|\cdot\|_d^*)$ is its dual space. Moreover, we denote the closed unit ball of the dual space $A_d^*$ by $\mathcal{B}_{A_d^*}$ and the set of its extreme points by $ext(\mathcal{B}_{A_d^*})$. Since $A_d$ is isomorphic to a quotient of $A$, then $A_d^*$ is isomorphic to a subspace of $A^* = (A, \|.\|_\infty)^*$. In fact, $A_d^* = \{\mu \in A^* : \mu(X) = 0\}$.

Given two function spaces $A$ and $B$, a map (not assumed to be linear) $T : A \to B$ is called *diameter preserving* if diam$(f - g) =$diam$(Tf - Tg)$ for all $f, g \in A$.

Let $A$ be a function space on a compact Hausdorff space $X$. By an argument similar to the proof of [10, Theorem 1], we can deduce that $ext(\mathcal{B}_{A_d^*})$ is a nonempty subset of $\{\alpha\delta_{x,x'} : x, x' \in Ch(A), x \neq$

$x', \alpha \in \mathbb{T}\}$, where $\mathbb{T}$ is the unit sphere of $\mathbb{C}$. Then the set

$$dch(A) := \{\{x_1, x_2\} : \delta_{x_1,x_2} \in ext(\mathcal{B}_{A_d^*})\},$$

introduced without any study in [3], is a nonempty set. Furthermore, $\{x_1, x_2\} \in dch(A)$ if and only if $(x_1, x_2)$ belongs to the Choquet boundary of the linear subspace

$$A - A := \{h \in C(X \times X) : h(x, y) = h_1(x) - h_1(y), h_1 \in A\}$$

of $C(X \times X)$ ([14, Proposition 2.2]). Moreover, we can define

$$\widetilde{X} := \{x \in Ch(A) : \delta_{x,x'} \in ext(\mathcal{B}_{A_d^*}) \,\text{for some}\, x' \in Ch(A)\}.$$

Let us note that $\widetilde{X} = \{x \in Ch(A) : \{x, x'\} \in dch(A) \,\text{for some}\, x' \in Ch(A)\}$.

Finally, let us remark that for a function space $A$ on a compact Hausdorff space $X$, the set $\{\delta_x : x \in X\}$ is not necessarily linearly independent in $(A, \|\cdot\|_\infty)^*$, but it is true for function algebras. To see this, assume that $x_1, ..., x_n$ are distinct points in $X$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$ such that $\sum_{i=1}^n \alpha_i \delta_{x_i} = 0$ on $A$. Then by applying this equation to a function $f_i \in A$, $i \in \{1, ..., n\}$, with $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for all $j \in \{1, ..., n\} \setminus \{i\}$, we get easily every $\alpha_i = 0$. Let us note that, the set $\{\delta_x : x \in X\}$ is linearly independent in $(A, \|\cdot\|_\infty)^*$ if and only if for any distinct points $x_1, ..., x_n \in X$ there exist functions $f_1, ..., f_n \in A$ as above. We also refer to [14] for more details and to [20] for any concept related to function algebras.

## 3. Diameter preserving mappings

The following is our main result, which can be considered the non-linear version of [14, Theorem 3.1].

**Theorem 3.1.** *Let $A_i$, $i = 1, 2$, be function spaces on compact spaces $X_i$ such that the set $\{\delta_x : x \in X_i\}$ is linearly independent in $(A_i, \|.\|_\infty)^*$ and $dch(A_1) = \{\{x, x'\} : x, x' \in \widetilde{X_1}, x \neq x'\}$. A (not necessarily linear) surjection $T : A_1 \to A_2$ is a diameter preserving map if and only if there exist a homeomorphism $\psi : \widetilde{X_2} \to \widetilde{X_1}$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : A_1 \to \mathbb{C}$ such that either*

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in \widetilde{X_2}), \text{ or}$$

$$Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in \widetilde{X_2}).$$

*Proof.* Assume that the surjection $T : A_1 \to A_2$ has the mentioned representation with $\psi$, $\lambda$, $L$ as in the statement above. For $i = 1, 2$, we recall that

$$A_i - A_i = \{h \in C(X_i \times X_i) : h(x, y) = h_1(x) - h_1(y), h_1 \in A_i\}.$$

Let $f, g \in A_1$. Since $Ch(A_i - A_i)$ is a boundary for $A_i - A_i$, then there exist $\{x, x'\} \in dch(A_1)$ and $\{y, y'\} \in dch(A_2)$ such that $\mathrm{diam}(f-g) = |(f-g)(x) - (f-g)(x')|$ and $\mathrm{diam}(Tf - Tg) = |(Tf - Tg)(y) - (Tf - Tg)(y')|$. Hence from the representation of $T$ we conclude that

$$\begin{aligned}\mathrm{diam}(f-g) &= |(f-g)(x) - (f-g)(x')| \\ &= |(Tf - Tg)(\psi^{-1}(x)) - (Tf - Tg)(\psi^{-1}(x'))| \\ &\leq \mathrm{diam}(Tf - Tg) = |(Tf - Tg)(y) - (Tf - Tg)(y')| \\ &= |(f-g)(\psi(y)) - (f-g)(\psi(y'))| \leq \mathrm{diam}(f-g),\end{aligned}$$

which shows that $\mathrm{diam}(f-g) = \mathrm{diam}(Tf - Tg)$.

We next prove the converse through several steps. Let us first introduce the following map

$$\begin{cases} T_d : A_{1d} \to A_{2d} \\ T_d(\pi(f)) = \pi(Tf), \end{cases}$$

which is a surjective isometry under the diameter norm, i.e., $\|T_d(\pi(f)) - T_d(\pi(g))\|_d = \|\pi(f) - \pi(g)\|_d$ for all $f, g \in A_1$, since $\mathrm{diam}(f-g) = \mathrm{diam}(Tf - Tg)$ for all $f, g \in A_1$. Based on $T_d$, we can define the map $\widetilde{T}_d := T_d - T_d(\pi(0))$.

**Step 1.** *$\widetilde{T}_d : A_{1d} \to A_{2d}$ is a surjective real-linear isometry with respect to the diameter norm.*

By the Mazur-Ulam theorem ([21]), we can deduce that $\widetilde{T}_d = T_d - T_d(\pi(0))$ is a surjective real-linear isometry with respect to the diameter norm.

In the sequel we assume that the spaces $A_1$ and $A_2$ consist of complex-valued functions. The argument for real-valued case is similar and even simpler.

**Step 2.** *There exists a surjective real-linear isometry $T_* : A_{2d}^* \to A_{1d}^*$ such that*

$$\mathrm{Re}(\mathrm{T}_*\Lambda)(\pi(\mathrm{f})) = \mathrm{Re}\Lambda(\widetilde{\mathrm{T}}_{\mathrm{d}}(\pi(\mathrm{f}))) \quad (\mathrm{f} \in \mathrm{A}_1, \Lambda \in \mathrm{A}_{2\mathrm{d}}^*).$$

If $\widetilde{T}_d$ is a complex-linear, we let $T_*$ be the adjoint operator $\widetilde{T}_d^*$ of $\widetilde{T}_d$, from $A_{2d}^*$ onto $A_{1d}^*$, which is a complex-linear isometry such that $(\widetilde{T}_d^*\Lambda)(\pi(f)) = \Lambda(\widetilde{T}_d(\pi(f)))$ for all $f \in A_1$ and $\Lambda \in A_{2d}^*$. Now suppose that $\widetilde{T}_d$ is just real-linear. According to [19, Lemma 3.1], there exists a real-linear isometry $T_*$ from $A_{2d}^*$ onto $A_{1d}^*$ satisfying the mentioned property.

**Step 3.** *Let $\{y, y'\} \in dch(A_2)$. Then either $T_*(i\delta_{y,y'}) = iT_*(\delta_{y,y'})$ or $T_*(i\delta_{y,y'}) = -iT_*(\delta_{y,y'})$.*

Since $T_* : A_{2d}^* \to A_{1d}^*$ is a real-linear bijective isometry, we infer that $T_*(ext(\mathcal{B}_{A_{2d}^*})) = ext(\mathcal{B}_{A_{1d}^*})$. Then there exist $\alpha, \alpha' \in \mathbb{T}$ and $\{x, x'\}, \{x_1, x_1'\} \in dch(A_1)$ such that $T_*(\delta_{y,y'}) = \alpha\delta_{x,x'}$ and $T_*(i\delta_{y,y'}) = \alpha'\delta_{x_1,x_1'}$. Let us assume, with no loss of generality, that the order of appearance of the common points of $Ch(A_1)$ in $\alpha\delta_{x,x'}$ and $\alpha'\delta_{x_1,x_1'}$ (if there exist) is the same. It is apparent that $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{y,y'} \in ext(\mathcal{B}_{A_{2d}^*})$ and so $T_*\left(\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{y,y'}\right) \in ext(\mathcal{B}_{A_{1d}^*})$. Hence $T_*\left(\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{y,y'}\right) = \alpha''\delta_{x_2,x_2'}$ for some $\alpha'' \in \mathbb{T}$ and $\{x_2, x_2'\} \in dch(A_1)$. On the other hand, from the real-linearity of $T_*$,

$$T_*\left(\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\delta_{y,y'}\right) = \frac{1}{\sqrt{2}}T_*(\delta_{y,y'}) + \frac{1}{\sqrt{2}}T_*(i\delta_{y,y'}).$$

Hence $\alpha''\delta_{x_2,x_2'} = \frac{1}{\sqrt{2}}\alpha\delta_{x,x'} + \frac{1}{\sqrt{2}}\alpha'\delta_{x_1,x_1'}$. Now since the set $\{\delta_x : x \in X_1\}$ is linearly independent in $(A_1, \|.\|_\infty)^*$, it is not difficult to conclude that $\{x, x'\} = \{x_1, x_1'\} = \{x_2, x_2'\}$. Moreover we have either $\alpha'' - \frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\alpha' = 0$ or $\alpha'' + \frac{1}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\alpha' = 0$. Setting $\alpha = a + ib$ and $\alpha' = a' + ib'$, where $a, a', b, b' \in \mathbb{R}$, we have

$$\left(\frac{a}{\sqrt{2}} + \frac{a'}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{2}} + \frac{b'}{\sqrt{2}}\right)^2 = 1,$$

which yields $aa' + bb' = 0$. Thus, $\mathrm{Re}(\alpha'\bar{\alpha}) = 0$ and so $\alpha'\bar{\alpha} = i$ or $\alpha'\bar{\alpha} = -i$, since $\alpha\alpha' \in \mathbb{T}$. Therefore, we have either $T_*(i\delta_{y,y'}) = iT_*(\delta_{y,y'})$, or $T_*(i\delta_{y,y'}) = -iT_*(\delta_{y,y'})$.

**Step 4.** *$\Phi : dch(A_1) \to dch(A_2)$ defined by $\Phi\{x_1, x_2\} := supp(T_*^{-1}(\delta_{x_1,x_2}))$ is a bijective map.*

We first note that $\Phi$ is well-defined since $T_*^{-1}(ext(\mathcal{B}_{A_{1d}^*})) = ext(\mathcal{B}_{A_{2d}^*})$. If $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are distinct elements of $dch(A_1)$ such that $\Phi\{x_1, x_2\} = \Phi\{x_3, x_4\}$, then there exist $\{y_1, y_2\} \in dch(A_2)$, $\beta_1, \beta_2 \in \mathbb{T}$ such that $T_*^{-1}(\delta_{x_1,x_2}) = \beta_1\delta_{y_1,y_2}$ and $T_*^{-1}(\delta_{x_3,x_4}) = \beta_2\delta_{y_1,y_2}$. Thus, $\delta_{x_1,x_2} = T_*(\beta_1\delta_{y_1,y_2})$ and $\delta_{x_3,x_4} = T_*(\beta_2\delta_{y_1,y_2})$. Now applying Step 3, and from the real-linearity of $T_*$, it follows that either $\delta_{x_1,x_2} = \beta_1 T_*(\delta_{y_1,y_2})$ or $\delta_{x_1,x_2} = \bar{\beta}_1 T_*(\delta_{y_1,y_2})$, and also either $\delta_{x_3,x_4} = \beta_2 T_*(\delta_{y_1,y_2})$ or $\delta_{x_3,x_4} = \bar{\beta}_2 T_*(\delta_{y_1,y_2})$. Then, $\alpha\delta_{x_1,x_2} + \alpha'\delta_{x_3,x_4} = 0$ for some $\alpha, \alpha' \in \mathbb{T}$, which contradicts the fact that the set $\{\delta_x : x \in X_1\}$ is linearly independent in $(A_1, \|.\|_\infty)^*$. Therefore, $\Phi$ is injective. The surjectivity of $\Phi$ is easily obtained since $T_*(ext(\mathcal{B}_{A_{2d}^*})) = ext(\mathcal{B}_{A_{1d}^*})$.

**Step 5.** *There exists an injective map $\varphi : \widetilde{X_1} \to Ch(A_2)$ and a scalar $\beta \in \mathbb{T}$ such that for each $x, x' \in \widetilde{X_1}$, $T_*^{-1}(\delta_{x,x'}) = \beta\delta_{\varphi(x),\varphi(x')}$.*

If $card(\widetilde{X_1}) = 2$, then there is only one point in $dch(A_1)$ and the claim is proved easily. So let us suppose that $card(\widetilde{X_1}) > 2$. In this case we divide the proof into three parts as follows:

(i) Let $x \in \widetilde{X_1}$. We shall next show that for each pair of different points $x_1, x_2 \in Ch(A_1)$ distinct from $x$ and such that $\{x_1, x\}, \{x, x_2\} \in dch(A_1)$, which exists because $dch(A_1) = \{\{x, x'\} : x, x' \in \widetilde{X_1}, x \neq x'\}$, we have $card(\Phi\{x_1, x\} \cap \Phi\{x_2, x\}) = 1$.

It is apparent, due to the injectivity of $\Phi$, that $card(\Phi\{x_1, x\} \cap \Phi\{x_2, x\}) \neq 2$. Therefore, let us suppose that $card(\Phi\{x_1, x\} \cap \Phi\{x_2, x\}) = 0$. By the assumption, $T_*^{-1}(\delta_{x_1,x_2}) = T_*^{-1}(\delta_{x_1,x}) + T_*^{-1}(\delta_{x,x_2}) \in T_*^{-1}(ext(\mathcal{B}_{A_{1d}^*})) = ext(\mathcal{B}_{A_{2d}^*})$. Then, since $ext(\mathcal{B}_{A_{2d}^*})$ is included in the set $\{\alpha\delta_{y,y'} : y, y' \in Ch(A_2), y \neq y', \alpha \in \mathbb{T}\}$, it follows that there exists a nonzero linear combination in $\{\delta_y : y \in X_2\}$, a contradiction. This argument yields that $card(\Phi\{x_1, x\} \cap \Phi\{x_2, x\}) = 1$.

We denote the unique point in the intersection $\Phi\{x_1, x\} \cap \Phi\{x_2, x\}$ by $\varphi(x)$. Next, it is shown that $\varphi(x)$ does not depend on the choice of the points $x_1, x_2$.

(ii) We prove that $\varphi(x) \in \Phi\{x, x'\}$ for each $x' \in Ch(A_1)$ such that $\{x, x'\} \in dch(A_1)$.

Contrary to what we claim, assume that there exists $x_3 \in Ch(A_1) \setminus \{x, x_1, x_2\}$ such that $\varphi(x)$ does not belong to $\Phi\{x, x_3\}$. Reasoning as in the above paragraph and since the set $\{\delta_y : y \in X_2\}$ is linearly independent in $(A_2, \|.\|_\infty)^*$, we can conclude that there exist distinct points $y_1, y_2 \in Ch(A_2) \setminus \{\varphi(x)\}$ with $\Phi\{x_1, x\} = \{\varphi(x), y_1\}$, $\Phi\{x_2, x\} = \{\varphi(x), y_2\}$ and $\Phi\{x_3, x\} = \{y_1, y_2\}$. Furthermore,

$$T_*^{-1}(\delta_{x_1} - \delta_{x_2}) = T_*^{-1}(\delta_{x_1,x}) + T_*^{-1}(\delta_{x,x_2}) = \beta\delta_{y_1,y_2},$$

for some $\beta \in \mathbb{T}$. Then $\Phi\{x_1, x_2\} = \{y_1, y_2\}$, which contradicts the injectivity of $\Phi$.

From the above argument we can define $\varphi : \widetilde{X_1} \to Ch(A_2)$ as the map assigning to each $x \in \widetilde{X_1}$ the unique point $\varphi(x)$. Furthermore, taking into account the injectivity of $T_*^{-1}$, it is easily seen that $\varphi$ is an injective map.

(iii) There is a unique scalar $\beta \in \mathbb{T}$ such that for each $x, x' \in \widetilde{X_1}$, $T_*^{-1}(\delta_{x,x'}) = \beta\delta_{\varphi(x),\varphi(x')}$.

Assume that $x, x', x''$ are distinct points in $\widetilde{X_1}$. From the equation $T_*^{-1}(ext(\mathcal{B}_{A_{1d}^*})) = ext(\mathcal{B}_{A_{2d}^*})$ and the above discussion, there exist scalars $\beta, \beta', \beta'' \in \mathbb{T}$ such that $T_*^{-1}(\delta_{x,x'}) = \beta\delta_{\varphi(x),\varphi(x')}$, $T_*^{-1}(\delta_{x,x''}) = \beta'\delta_{\varphi(x),\varphi(x'')}$, and $T_*^{-1}(\delta_{x'',x'}) = \beta''\delta_{\varphi(x''),\varphi(x')}$. Then since $T_*^{-1}(\delta_{x'',x'}) = T_*^{-1}(\delta_{x,x'}) - T_*^{-1}(\delta_{x,x''})$, we have

$$\beta''\delta_{\varphi(x''),\varphi(x')} = \beta\delta_{\varphi(x),\varphi(x')} - \beta'\delta_{\varphi(x),\varphi(x'')}.$$

Taking into account that the set $\{\delta_y : y \in X_2\}$ is linearly independent, we may choose $f \in A_2$ such that $f(\varphi(x)) = 1$ and $f(\varphi(x')) = f(\varphi(x'')) = 0$. Next, from evaluating the latter equation at the function $f$, it follows that $\beta = \beta' = \beta''$. Now, we can get (iii) immediately.

It is not difficult to see that $\varphi(\widetilde{X_1}) = \widetilde{X_2}$. Now let $\psi := \varphi^{-1} : \widetilde{X_2} \to \widetilde{X_1}$, which is a bijective map such that $T_*(\beta\delta_{y,y'}) = \delta_{\psi(y),\psi(y')}$ for all $y, y' \in \widetilde{X_2}$.

**Step 6.** *Let $y \in \widetilde{X_2}$. Then either $T_*(i\delta_{y,y'}) = iT_*(\delta_{y,y'})$ for all $y' \in \widetilde{X_2}$, or $T_*(i\delta_{y,y'}) = -iT_*(\delta_{y,y'})$ for all $y' \in \widetilde{X_2}$.*

Take $y' \in \widetilde{X_2}$ such that $T_*(i\delta_{y,y'}) = iT_*(\delta_{y,y'})$. Clearly, we may assume that $y' \neq y$. We claim that for a given $y'' \in \widetilde{X_2}$, $T_*(i\delta_{y,y''}) = iT_*(\delta_{y,y''})$. Suppose, contrary to what we claim, that

$T_*(i\delta_{y,y''}) \neq iT_*(\delta_{y,y''})$. Hence $T_*(i\delta_{y,y''}) = -iT_*(\delta_{y,y''})$, by Step 3, and thus

$$T_*(i\delta_{y'',y'}) = T_*(i\delta_{y,y'} - i\delta_{y,y''}) = iT_*(\delta_{y,y'}) + iT_*(\delta_{y,y''}) = iT_*(\delta_{y,y'} + \delta_{y,y''}).$$

On the other hand, again from Step 3, $T_*(i\delta_{y'',y'}) = \pm iT_*(\delta_{y'',y'})$ and from the injectivity of $T_*$, we conclude that either

$$\delta_{y'',y'} = \delta_{y,y'} + \delta_{y,y''}, \, or$$

$$-\delta_{y'',y'} = \delta_{y,y'} + \delta_{y,y''}.$$

Hence, $\delta_{y''} = \delta_y$, or $\delta_{y'} = \delta_y$, which is impossible since $A_2$ separates the points of $X_2$. The second case can be treated similarly.

**Step 7.** *There exist $\lambda \in \mathbb{T}$ and a functional $L$ on $A_1$ such that either we have*

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in \widetilde{X_2}), \text{ or}$$

$$Tf(y) = T0(y) + \lambda\overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in \widetilde{X_2}).$$

Given $y, y' \in \widetilde{X_2}$, from Step 5, we have $T_*(\beta\delta_{y,y'}) = \delta_{\psi(y),\psi(y')}$. Setting $\beta = a + ib$ $(a, b \in \mathbb{R})$, from the real-linearity of $T_*$ we get

$$\delta_{\psi(y),\psi(y')} = T_*(\beta\delta_{y,y'}) = aT_*(\delta_{y,y'}) + bT_*(i\delta_{y,y'}).$$

Hence, according to the previous step, one of the following two cases holds:

$$\begin{cases} \beta T_*(\delta_{y,y'}) = \delta_{\psi(y),\psi(y')} & (y, y' \in \widetilde{X_2}), \text{ or,} \\ \overline{\beta} T_*(\delta_{y,y'}) = \delta_{\psi(y),\psi(y')} & (y, y' \in \widetilde{X_2}). \end{cases}$$

Assume that we are in the first case. If $f \in A_1$, $y, y' \in \widetilde{X_2}$, then from Step 2 we deduce that

$$\begin{aligned} \mathrm{Re}(Tf(y) - Tf(y') - T0(y) + T0(y')) &= \mathrm{Re}(\delta_{y,y'}\widetilde{T}_d(\pi(f))) = \mathrm{Re}(T_*\delta_{y,y'}(\pi(f))) \\ &= \mathrm{Re}(\overline{\beta}\delta_{\psi(y),\psi(y')}(\pi(f))) \\ &= \mathrm{Re}(\overline{\beta}(f(\psi(y)) - f(\psi(y')))), \end{aligned}$$

moreover,

$$\begin{aligned} \mathrm{Im}(Tf(y) - Tf(y') - T0(y) + T0(y')) &= -\mathrm{Re}(i(Tf(y) - Tf(y') - T0(y) + T0(y'))) \\ &= -\mathrm{Re}(i\delta_{y,y'}\widetilde{T}_d(\pi(f))) = -\mathrm{Re}(iT_*\delta_{y,y'}(\pi(f))) \\ &= -\mathrm{Re}(i\overline{\beta}\delta_{\psi(y),\psi(y')}(\pi(f))) = -\mathrm{Re}(i\overline{\beta}(f(\psi(y)) - f(\psi(y')))) \\ &= \mathrm{Im}(\overline{\beta}(f(\psi(y)) - f(\psi(y')))), \end{aligned}$$

and consequently,

$$Tf(y) - Tf(y') - T0(y) + T0(y') = \overline{\beta}(f(\psi(y)) - f(\psi(y'))).$$

Similarly, in the second case, if $f \in A_1$, $y, y' \in \widetilde{X_2}$, then we get

$$\begin{aligned}\mathrm{Re}(Tf(y) - Tf(y') - T0(y) + T0(y')) &= \mathrm{Re}(\delta_{y,y'}\widetilde{T}_d(\pi(f))) = \mathrm{Re}(T_*\delta_{y,y'}(\pi(f))) \\ &= \mathrm{Re}(\beta\delta_{\psi(y),\psi(y')}(\pi(f))) \\ &= \mathrm{Re}(\beta(f(\psi(y)) - f(\psi(y')))).\end{aligned}$$

Furthermore, from Steps 2 and 6, we infer that

$$\begin{aligned}\mathrm{Im}(Tf(y) - Tf(y') - T0(y) + T0(y')) &= -\mathrm{Re}(i(Tf(y) - Tf(y') - T0(y) + T0(y'))) \\ &= -\mathrm{Re}(i\delta_{y,y'}\widetilde{T}_d(\pi(f))) = -\mathrm{Re}(T_*(i\delta_{y,y'})(\pi(f))) \\ &= -\mathrm{Re}(-iT_*\delta_{y,y'}(\pi(f))) \\ &= -\mathrm{Re}(-i\beta\delta_{\psi(y),\psi(y')}(\pi(f))) = -\mathrm{Re}(-i\beta(f(\psi(y)) - f(\psi(y')))) \\ &= -\mathrm{Im}(\beta(f(\psi(y)) - f(\psi(y')))),\end{aligned}$$

and so

$$Tf(y) - Tf(y') - T0(y) + T0(y') = \overline{\beta}(\overline{f(\psi(y))} - \overline{f(\psi(y'))}).$$

Setting $\lambda := \bar{\beta}$ we get $\lambda \in \mathbb{T}$ and one of the following cases holds:

$$Tf(y) - Tf(y') - T0(y) + T0(y') = \lambda(f(\psi(y)) - f(\psi(y'))) \quad (f \in A_1, y, y' \in \widetilde{X_2}), \text{ or}$$

$$Tf(y) - Tf(y') - T0(y) + T0(y') = \lambda(\overline{f(\psi(y))} - \overline{f(\psi(y'))}) \quad (f \in A_1, y, y' \in \widetilde{X_2}).$$

Next, define a functional $L : A_1 \to \mathbb{C}$ as follows

$$L(f) = Tf(y') - T0(y') - \lambda f(\psi(y'))^\star \quad (f \in A_1),$$

where $y'$ is an arbitrary element in $\widetilde{X_2}$ and $f(\psi(y'))^\star = f(\psi(y'))$ in the first case, and $f(\psi(y'))^\star = \overline{f(\psi(y'))}$ in the second case. From the recent relations, it is easily seen that $L$ is well-defined. Finally, we derive that either

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in \widetilde{X_2}), \text{ or}$$

$$Tf(y) = T0(y) + \lambda\overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in \widetilde{X_2}).$$

**Step 8.** *$\psi$ is a homeomorphism.*

Let $y_0 \in \widetilde{X_2}$ and $(y_i)_i$ be a net in $\widetilde{X_2}$ convergent to $y_0$. Since $X_1$ is compact, passing to a subnet we can assume, without loss of generality, that there exists $x_0 \in X_1$ such that $(\psi(y_i))_i \to x_0$. It is enough to show that $x_0 = \psi(y_0)$. Contrary to what we claim, let us suppose that $x_0 \neq \psi(y_0)$. Then there is a neighborhood $U$ of $\psi(y_0)$ such that $x_0 \in X_1 \setminus U$. Take $u \in C(X_1)$ with $0 \le u \le 1$ on $X_1$, $u(\psi(y_0)) = 1$ and $u = 0$ on $X_1 \setminus U$. Since $\psi(y_0) \in Ch(A_1)$, from [14, Lemma 2.3] it follows that $\sup\{\mathrm{Re}h(\psi(y_0)) : h \in \overline{A_1},\ \mathrm{Re}h \le u\} = 1$. Hence, it is not difficult to see that $\sup\{\mathrm{Re}h(\psi(y_0)) : h \in A_1,\ \mathrm{Re}h \le u\} = 1$. Then there exists $h \in A_1$ with $\mathrm{Re}h \le u$ and $\mathrm{Re}h(\psi(y_0)) > \frac{3}{4}$. We can consider $i_0$ such that, for all $i \ge i_0$, $|Th(y_i) - T0(y_i) + T0(y_0) - Th(y_0)| < \frac{1}{4}$. On the other hand, since

$$\begin{aligned}\lim_{i\to\infty} |Th(y_i) - T0(y_i) - Th(y_0) + T0(y_0)| &= \lim_{i\to\infty} |h(\psi(y_i)) - h(\psi(y_0))| \\ &= |h(x_0) - h(\psi(y_0))| \\ &\ge \mathrm{Re}h(\psi(y_0)) - \mathrm{Re}h(x_0) \ge \frac{3}{4},\end{aligned}$$

then, for a sufficiently large index $i$, $|Th(y_i) - T0(y_i) - Th(y_0) + T0(y_0)| \ge \frac{1}{4}$ and this contradicts the continuity of the function $Th - T0$. This argument implies that $x_0 = \psi(y_0)$ and so $\psi$ is continuous.

Similarly, we prove that $\varphi$ $(\psi^{-1})$ is continuous. Suppose, on the contrary, that $(x_i)_i$ is a net in $\widetilde{X_1}$ convergent to $x_0 \in \widetilde{X_1}$ such that $(\varphi(x_i))_i$ converges to $y_0$ in $X_2$ and $y_0 \ne \varphi(x_0)$. Let $V$ be a neighborhood of $\varphi(x_0)$ with $y_0 \in X_2 \setminus V$. As above, since $\varphi(x_0) \in Ch(A_2)$, we can choose $k \in A_2$ with $\mathrm{Re}k(\varphi(x_0)) > \frac{3}{4}$ and $\mathrm{Re}k \le 0$ on $X_2 \setminus V$. Then taking $h' \in A_1$ such that $Th' = k + T0$, from

$$\begin{aligned}\lim_{i} |h'(x_i) - h'(x_0)| = \lim_{i} |k(y_0) - k(\varphi(x_0))| &= |k(\varphi(x_i)) - k(\varphi(x_0))| \\ \ge \mathrm{Re}k(\varphi(y_0)) - \mathrm{Re}k(y_0) &\ge \frac{3}{4},\end{aligned}$$

we get a contradiction with the continuity of $h'$. Therefore, $\psi$ is a homeomorphism. □

**Remark 3.2.** (1) It is worth mentioning that in the above result, according to [14, Remark 3.2 (2)], the assumptions that the evaluation functionals are linearly independent and $dch(A_1) = \{\{x, x'\} : x, x' \in \widetilde{X_1}, x \ne x'\}$ cannot be removed.

(2) Note that the existence of the homeomorphism $\psi : \widetilde{X_2} \to \widetilde{X_1}$ implies that $A_2$ also satisfies the condition that $dch(A_2) = \{\{x, x'\} : x, x' \in \widetilde{X_2}, x \ne x'\}$. In other words, $ext(\mathcal{B}_{A^*_{2d}}) = \{\alpha\delta_{x,x'} : x, x' \in \widetilde{X_2}, x \ne x', \alpha \in \mathbb{T}\}$.

(3) It is worth pointing out that if $T$ is assumed to be complex-linear (resp. real-linear) in Theorem 3.1, then $L$ is complex-linear (resp. real-linear) too and we have either

$$Tf(y) = \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in \widetilde{X_2}), \text{ or}$$

$$Tf(y) = \lambda \overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in \widetilde{X_2}).$$

We also note that in the complex-linear case, just the first situation happens. So, Theorem 3.1 is a generalization of most of the known results in the category of linear diameter preservers.

(4) We would like to remark that our Theorem 3.1 includes all results in [16] with different techniques. Indeed, the main result of [16], which is a particular case of Theorem 3.1, is as follows. Let $A_1$ be a dense function space in $C(X_1)$ and $A_2$ be a function space such that the evaluation functionals are linearly independent. If $T : A_1 \to A_2$ is a diameter preserving, then there exist a subset $X_0$ of $X_2$, a continuous bijection $\psi : X_0 \to X_1$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : A_1 \to \mathbb{C}$ such that either

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in X_0), \text{ or}$$

$$Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in X_0).$$

It should be noted, according to Theorem 3.1, that the set $X_0$ is explicitly declared as the non-empty set $\widetilde{X_2}$, and the bijection $\psi$ is not only continuous but it is also a homeomorphism. Meantime, according to our result, the assumption of the regularity of $A_1$ is redundant in [16, Corollary 3.3] even in the lack of the complex-linearity of $T$.

Next we provide some interesting consequences of the main theorem. Let us first recall that a function algebra $A$ on $X$ is a *Banach function algebra* on $X$ if it is a Banach algebra with respect to a certain norm. For a Banach function algebra $A$ and $f \in A$, we denote the maximal ideal space of $A$ and the Gelfand transform of $f$ by $M_A$ and $\hat{f}$, respectively. The following result is a generalization of [8, Theorem 3.1].

**Corollary 3.3.** *(1) Let $A_i$, $i = 1, 2$, be complex function algebras on compact spaces $X_i$ such that $dch(A_i) = \{\{x_1, x_2\} : x_1, x_2 \in Ch(A_i), x_1 \neq x_2\}$. If $T : A_1 \to A_2$ is a (not necessarily linear) diameter preserving surjection, then there are a homeomorphism $\psi : Ch(A_2) \to Ch(A_1)$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : A_1 \to \mathbb{C}$ such that either $Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f)$ for all $f \in A_1$ and $y \in Ch(A_2)$, or $Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f)$ for all $f \in A_1$ and $y \in Ch(A_2)$.*

*(2) If $A_1$ and $A_2$ are Banach function algebras in (1), then $\psi$ can be extended to a homeomorphism $\tilde{\psi}$ from $M_{A_2}$ onto $M_{A_1}$ such that we have either $\hat{Tf} = \hat{T0} + \lambda \hat{f} \circ \tilde{\psi} + L(f)$ for all $f \in A_1$ on $M_{A_2}$, or $\hat{Tf} = \hat{T0} + \lambda \overline{\hat{f} \circ \tilde{\psi}} + L(f)$ for all $f \in A_1$ on $M_{A_2}$.*

*Proof.* (1) Clearly, $\widetilde{X_i} = Ch(A_i)$ $(i = 1, 2)$, and the result is a straightforward consequence of Theorem 3.1.

(2) By (1), there are a homeomorphism from $Ch(A_2)$ onto $Ch(A_1)$, a scalar $\lambda \in \mathbb{T}$, and a functional $L : A_1 \to \mathbb{C}$ such that one of the following two cases will happen:

$$\text{Case(i).} \quad Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in A_1, y \in Ch(A_2)),$$

$$\text{Case(ii).} \quad Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f) \quad (f \in A_1, y \in Ch(A_2)).$$

Since $Ch(A_2)$ is a boundary for $A_2$, then for each $f \in A_1$ there is a unique element $g_f \in A_2$ such that on $Ch(A_2)$ we have $g_f = f \circ \psi$ and $g_f = \overline{f \circ \psi}$, respectively, in Case (i) and Case (ii). Hence by this argument, we can define a map $S : A_1 \to A_2$ by $Sf = g_f$, and it is easily seen that $S$ is a real-algebra isomorphism. Moreover, from the representation of $T$ and the fact that the Choquet boundary is indeed a boundary, it follows that $Tf = T0 + \lambda Sf + L(f)$ for all $f \in A_1$. Now $\psi$ can be extended to a function $\tilde{\psi} : M_{A_2} \longrightarrow M_{A_1}$ in this way that for every $y \in M_{A_2}$, $\tilde{\psi}(y)$ is a member in $M_{A_1}$ defined by

$$\text{Case(i).} \quad \tilde{\psi}(y)(f) = \hat{Sf}(y) \quad (f \in A_1),$$

$$\text{Case(ii).} \quad \tilde{\psi}(y)(f) = \overline{\hat{Sf}(y)} \quad (f \in A_1).$$

It it is easily proved that $\tilde{\psi}$ is a homeomorphism and for each $f \in A_1$ and $y \in M_{A_2}$ we have either

$$\hat{Tf}(y) = y(Tf) = y(T0 + \lambda Sf + L(f)) = y(T0) + \lambda y(Sf) + L(f) = \hat{T0}(y) + \lambda \hat{f}(\tilde{\psi}(y)) + L(f),$$

or

$$\hat{Tf}(y) = y(Tf) = y(T0 + \lambda Sf + L(f)) = y(T0) + \lambda y(Sf) + L(f) = \hat{T0}(y) + \lambda \overline{\hat{f}(\tilde{\psi}(y))} + L(f).$$

□

For $n \in \mathbb{N} \cup \{\infty\}$, let $C^{(n)}(I)$ denote the function algebra of all $n$-times continuously differentiable functions on the interval $I = [0, 1]$. It is interesting to note that the function algebra $C^{\infty}(I)$ is not a Banach function algebra. In what follows we describe nonlinear diameter preserving maps between these function algebras.

**Corollary 3.4.** *Let $m, n \in \mathbb{N} \cup \{\infty\}$ and let $T : C^{(n)}(I) \longrightarrow C^{(m)}(I)$ be a (not necessarily linear) diameter preserving surjection. Then $m = n$ and there exist a $C^{(n)}$-diffeomorphism $\psi : I \longrightarrow I$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : C^{(n)}(I) \to \mathbb{C}$ such that either*

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in C^{(n)}(I), y \in I), \text{ or}$$

$$Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f) \quad (f \in C^{(n)}(I), y \in I).$$

*Proof.* Since $C^{(n)}(I)$ is uniformly dense in $C(I)$, then $dch(C^{(n)}(I)) = \{\{x, x'\} : x, x' \in I, x \neq x'\}$ by [8, Proposition 3.3]. Then, according to Theorem 3.1 (or Corollary 3.3), there exist a homeomorphism $\psi : I \longrightarrow I$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : C^{(n)}(I) \to \mathbb{C}$ such that either

$$Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f) \quad (f \in C^{(n)}(I), y \in I), \text{ or}$$

$$Tf(y) = T0(y) + \lambda \overline{f(\psi(y))} + L(f) \quad (f \in C^{(n)}(I), y \in I).$$

We now show that $\psi \in C^{(m)}(I)$. Let $f_0 \in C^{(n)}(I)$ be defined by $f_0(x) = x$ for all $x \in I$. Then we have

$$Tf_0(y) = T0(y) + \lambda\psi(y) + L(f_0) \quad (y \in I).$$

Consequently, $\psi = \overline{\lambda}(Tf_0 - T0 - L(f_0))$, which obviously belongs to $C^{(m)}(I)$. Analogously, it can be shown that $\varphi(= \psi^{-1}) \in C^{(n)}(I)$. To see this, choose $f_1 \in C^{(n)}(I)$ such that $Tf_1 - T0$ is the identity function on $I$. Hence, from the representation of $T$, it follows that

$$\varphi(x) = \lambda f_1(x) + L(f_1) \quad (x \in I), \text{ or}$$

$$\varphi(x) = \lambda\overline{f_1(x)} + L(f_1) \quad (x \in I),$$

which yields that $\varphi \in C^{(n)}(I)$.

We next prove that $m = n$. Assume that $f \in C^{(n)}(I)$. Thus, $g = f \circ \psi^{-1} \in C^{(n)}(I)$, and so

$$Tg(y) = T0(y) + \lambda f(y) + L(g) \quad (y \in I), \text{ or}$$

$$Tg(y) = T0(y) + \lambda\overline{f(y)} + L(g) \quad (y \in I).$$

Consequently, $f = \overline{\lambda}(Tg - T0 - L(g))$, or $f = \lambda(\overline{Tg - T0 - L(g)})$, which shows that $f \in C^{(m)}(I)$. Hence, $C^{(n)}(I) \subseteq C^{(m)}(I)$, and then $n \geq m$. Similarly, if $k \in C^{(m)}(I)$, then $g = k \circ \psi \in C^{(m)}(I)$, and so there exists a function $h \in C^{(n)}(I)$ such that $Th = k \circ \psi + T0$. Now from the representation of $T$, it follows that $k = \lambda h + L(h)$, or $k = \lambda\overline{h} + L(h)$, and thus $k \in C^{(n)}(I)$. Whence $C^{(m)}(I) \subseteq C^{(n)}(I)$, and then $m \geq n$. Therefore, $m = n$. ☐

Given a compact metric space $(X, d)$, the Lipschitz algebra Lip(X), is the space of all scalar-valued functions $f$ on $X$ such that $\mathcal{L}(f) < \infty$, where $\mathcal{L}(f) = \sup\limits_{x \neq x'} \frac{|f(x) - f(x')|}{d(x, x')}$ is the Lipschitz constant of $f$.

**Corollary 3.5.** *Let $X$ and $Y$ be compact metric spaces and let $T : \mathrm{Lip(X)} \longrightarrow \mathrm{Lip(Y)}$ be a (not necessarily linear) diameter preserving surjection. Then there exist a bi-Lipschitz homeomorphism $\psi : Y \to X$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : \mathrm{Lip(X)} \to \mathbb{C}$ such that either $Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f)$ for all $f \in \mathrm{Lip(X)}$ and $y \in Y$, or $Tf(y) = T0(y) + \lambda\overline{f(\psi(y))} + L(f)$ for all $y \in Y$ and $f \in \mathrm{Lip(X)}$.*

*Proof.* Because of the density of Lipschitz algebras in the space of all continuous functions, from [8, Proposition 3.3] and Theorem 3.1 (or Corollary 3.3), we deduce that there exist a homeomorphism $\psi : Y \to X$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : \mathrm{Lip(X)} \to \mathbb{C}$ such that either $Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f)$ for all $f \in \mathrm{Lip(X)}$ and $y \in Y$, or $Tf(y) = T0(y) + \lambda\overline{f(\psi(y))} + L(f)$ for all $f \in \mathrm{Lip(X)}$ and $y \in Y$. Now, by a standard method, we show that $\psi$ is a bi-Lipschitz function. From the given representation for $T$, it follows that for each $f \in \mathrm{Lip(X)}$, the function $f \circ \psi$ belongs to $\mathrm{Lip(Y)}$. Then we can define the complex-linear map $\tilde{T} : (\mathrm{Lip(X)}, \|\cdot\|_{\mathcal{L}}) \longrightarrow (\mathrm{Lip(Y)}, \|\cdot\|_{\mathcal{L}})$ by

$$\tilde{T}f = f \circ \psi \quad (f \in \mathrm{Lip(X)}),$$

where the complete norm $\|\cdot\|_{\mathcal{L}}$ is the addition of the supremum norm and the Lipschitz constant of the function. By the Closed Graph theorem, $\tilde{T}$ is a continuous complex-linear map. Hence, there exists $k > 0$ such that

$$\|\tilde{T}f\|_{\mathcal{L}} \leq k\|f\|_{\mathcal{L}} \quad (f \in \mathrm{Lip(X)}).$$

Let $y, y' \in Y$. Consider the function $f_y : X \longrightarrow \mathbb{R}$ defined by $f_y(z) = d(\psi(y), z)$ for all $z \in X$. It is easy to see $\mathcal{L}(f_y) \leq 1$ and consequently $\|f_y\|_{\mathcal{L}} \leq k'$, where $k' = 1+\mathrm{diam}(X)$. Thus, from the above relations it follows that $\|\tilde{T}f_y\|_{\mathcal{L}} \leq kk'$, and so

$$d(\psi(y), \psi(y')) = |f_y(\psi(y')) - f_y(\psi(y))| = |\tilde{T}(f_y)(y') - \tilde{T}(f_y)(y)| \leq kk'd(y, y').$$

Therefore, the Lipschitz constant $\mathcal{L}(\psi) \leq kk'$, which says that $\psi$ is a Lipschitz function on $Y$. A similar argument shows that $\psi^{-1}$ is a Lipschitz function on $X$. □

Let us recall that for a compact metric space $(X, d)$ and $\alpha \in (0, 1)$, the little Lipschitz algebra of order $\alpha$, $\mathrm{lip}_\alpha(X, d)$ (or simply, $\mathrm{lip}_\alpha(X)$), is the algebra consisting of all scalar-valued Lipschitz functions $f$ such that $\lim\limits_{d(x,x')\to 0} \frac{|f(x)-f(x')|}{d^\alpha(x,x')} = 0$. We remark that a similar result, with the same proof, is valid for the little Lipschitz algebras of the same order. Of course, this is not a restriction since, for $0 < \alpha < \beta < 1$, $\mathrm{lip}_\alpha(X, d_1) = \mathrm{lip}_\beta(X, d_2)$, where $(X, d_2) = (X, d_1^{\frac{\alpha}{\beta}})$.

The next corollary shows that our result also holds for a large class of function spaces and, particularly, includes the results from [16].

**Corollary 3.6.** *Let $A$ and $B$ be dense function spaces in $(C(X), \|.\|_\infty)$ and $(C(Y), \|.\|_\infty)$, respectively. If $T : A \to B$ is a (not necessarily linear) surjective diameter preserving map, then there are a homeomorphism $\psi : Y \to X$, a scalar $\lambda \in \mathbb{T}$ and a functional $L : A \to \mathbb{C}$ such that either $Tf(y) = T0(y) + \lambda f(\psi(y)) + L(f)$ for all $f \in A$ and $y \in Y$, or $Tf(y) = T0(y) + \lambda\overline{f(\psi(y))} + L(f)$ for all $f \in A$ and $y \in Y$.*

*Proof.* Since $A$ is uniformly dense in $C(X)$ and $B$ is uniformly dense in $C(Y)$, then $A_d$ is dense in $C(X)_d$ and $B_d$ is dense in $C(Y)_d$. Hence, the induced surjective real-linear isometry $\widetilde{T_d} : A_d \to B_d$ can be extended to a surjective real-linear isometry $\widetilde{T_d} : C(X)_d \to C(Y)_d$, and then, by the same proof as in Theorem 3.1, we can conclude the result. □

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