Corrigendum to: "A coprime action version of a solubility criterion of Deskins"

Antonio Beltrán
Departamento de Matemáticas,
Universidad Jaume I, 12071 Castellón, Spain
e-mail: abeltran@mat.uji.es

Changguo Shao
School of Mathematical Science,
University of Jinan, 250022, Shandong, China,
e-mail: shaoguozi@163.com

Abstract

In this Corrigendum we correct a missed case in the statement of Theorem 2.4 and a subsequent mistake in the proof of the main result in "A coprime action version of a solubility criterion of Deskins", Monatsh. Math- **188**. 461-466 (2019).

Keywords: Soluble groups, Maximal subgroups, Coprime action, Group action on groups.

Mathematics Subject Classification (2010): 20D20, 20D15.

The main result of [1] is a coprime action version of a theorem of B. Huppert: If a finite group G has a maximal subgroup that is nilpotent with Sylow 2-subgroup of nilpotency class at most 2, then G is soluble (Satz IV.7.4 of [4]). This theorem is the completion of previous results by Huppert [5], J.G. Thompson [8], W.E. Deskins [2] and Z. Janko [6]. Professor M.D. Pérez Ramos noticed and informed us that there are some mistakes and inaccuracies in the last part of the proof of the main theorem of [1]. Thus the goal of this note is to correct them.

First, the proof of the main theorem uses two classification theorems due to Kondrat'ev [7] and to Gilman and Gorenstein [3], respectively. However, there is one simple group missed in the statement of Theorem 2.4 in [1], which joins both classifications. The correct statement is the following.

Theorem 2.4 *Let G be a finite non-abelian simple group and P a Sylow 2-subgroup of G. If $\mathbf{N}_G(P) = P$ and P has class at most 2, then $G \cong \mathrm{PSL}(2, q)$, where $q \equiv 7, 9 \pmod{16}$ or $G \cong A_7$.*

Proof. This is a consequence of combining the main result of [7] and Theorems 7.1 and 7.4 of [3]. □

As a consequence of this correction, several modifications in the proof of Step 4 of the main theorem of [1] are necessary. Furthermore, in lines 28-29, page 465 it is claimed that the subgroup K is normalised by an element of order 3 lying in S. This is not true. For the reader's convenience we rewrite the whole proof of Step 4.

Proof of Step 4 Let N be a minimal A-invariant normal subgroup of G. We can assume that N is not soluble; otherwise by Step 1, N is not contained in M, and by maximality we obtain $NM = G$. As a consequence, G would be soluble and the proof is finished. Therefore, we can write $N = S_1 \times \ldots \times S_n$ where S_i are isomorphic non-abelian simple groups (possibly $n = 1$). Put $S = S_1$. Notice that A permutes the $S_i's$, but not necessarily this action is transitive. Let $B = \mathbf{N}_A(S)$ and let T be a transversal of B in A. On the other hand, since M is maximal in G, we have $\mathbf{N}_G(M \cap N) = M$, so in particular $\mathbf{N}_N(M \cap N) = M \cap N$. Further, as $M \cap N$ is a Sylow 2-subgroup of N, we have $M \cap N = M \cap S \times \ldots \times M \cap S_n$, so we conclude that $M \cap S$ is self-normalising in S. Also, it has nilpotency class exactly 2 by Lemma 2.1 and Step 3. Then by applying Theorem 2.4, we obtain $S \cong \mathrm{PSL}(2, q)$ with $q \equiv 7, 9 \pmod{16}$ or $S \cong A_7$. We distinguish separately these cases.

Assume first that $q \equiv 9 \pmod{16}$, with $q > 9$. Then we can certainly choose an odd prime $r \mid (q-1)/2$ and R to be a B-invariant Sylow r-subgroup of S. By Lemma 2.5(3), we know that $|\mathbf{N}_S(R)| = q - 1$, so $\mathbf{N}_S(R)$ has odd index in S and contains properly a Sylow 2-subgroup of S. Analogously, if $q \equiv 7 \pmod{16}$, with $q > 7$, there exists an odd prime $r \mid (q+1)/2$ and we take R to be a B-invariant Sylow r-subgroup of S. Again by Lemma 2.5(2), we know that $|\mathbf{N}_S(R)| = (q+1)$, so $\mathbf{N}_S(R)$ has odd index in S and hence, it contains properly a Sylow 2-subgroup of S. In both cases, we put $R_1 = \prod_{t \in T} R^t$, which is an A-invariant Sylow r-subgroup of $\prod_{t \in T} S_1^t$. We can argue similarly to construct an A-invariant Sylow r-subgroup for each orbit of the action of A on the $S_i's$. Hence, we can construct $R_0 = R_1 \times \ldots \times R_t$, where t denotes the number of orbits of A on the $S_i's$, and this is certainly an A-invariant Sylow 2-subgroup of N. We conclude that $|N : \mathbf{N}_N(R_0)| = |S : \mathbf{N}_S(R)|^n$ is odd too. Now, by the Frattini argument, $G = N\mathbf{N}_G(R_0)$ and thus, $|G : \mathbf{N}_G(R_0)| = |N : \mathbf{N}_N(R_0)|$. We conclude that $\mathbf{N}_G(R_0)$ properly contains an A-invariant Sylow 2-subgroup of G, contradicting the maximality of M.

Finally, suppose that $S \cong \mathrm{PSL}(2, 9), \mathrm{PSL}(2, 7)$ or A_7. In all cases, the Sylow 2-subgroups of S are dihedral groups of order 8. Now, $M \cap N$ is an A-invariant Sylow 2-subgroup of N, which is the direct product of n copies of a dihedral group, say D, of S. As M has nilpotence class two, then $[M, M \cap N] \leq M' \leq$

$\mathbf{Z}(M)$, and since $M \cap N \trianglelefteq M$ it follows that $[M, M \cap N] \leq \mathbf{Z}(M) \cap (M \cap N) \leq \mathbf{Z}(M \cap N)$. This implies that every subgroup of $M \cap N$ containing $\mathbf{Z}(M \cap N)$ must be normal in M. We will use this property later. Now let K be one of the two subgroups of D isomorphic to the 4-Klein group, which obviously satisfies $\mathbf{Z}(D) \leq K$ and set $K^A = \langle K^a \mid a \in A \rangle$. By the coprime action hypothesis we have that $|A|$ is odd, and then the fact that D has exactly two subgroups isomorphic to the 4-Klein group implies that for every $a \in A$, either $K^a = K$, or K^a lies in some other distinct copy of S. Furthermore, K^A is a direct product of certain copies of K, each of which lies in a different copy of S. Now, if the action of A on the $S_i's$ is transitive, we will just consider the subgroup K^A, but if the action is not transitive, then we proceed as follows. For each of the orbits of the action of A on the S_i, we choose j with S_j in the orbit, and choose a 4-Klein subgroup $K_j \leq D_j$, where D_j is the corresponding isomorphic copy of D appearing in $M \cap N$. Then we define the subgroup K_j^A similarly as K^A. Set K_0 to be the direct product of these subgroups, one for each orbit of the action of A on the S_i. We can write $K_0 = K_1 \times \ldots \times K_n$, where each K_i is a 4-Klein group lying in S_i. By construction, K_0 is trivially A-invariant and, moreover, $K_0 \trianglelefteq M$, because $\mathbf{Z}(M \cap N) \leq K_0 \leq M \cap N$. By the above proved property, we get $M \leq \mathbf{N}_G(K_0)$, which is also A-invariant. Now $\mathbf{N}_G(K_0) = M\mathbf{N}_N(K_0)$ and $\mathbf{N}_N(K_0) = \prod_{i=1}^n \mathbf{N}_{S_i}(K_i)$. In fact, one can easily check that when $S \cong \mathrm{PSL}(2,9)$ or $\mathrm{PSL}(2,7)$ then $\mathbf{N}_S(K) \cong S_4$, and when $S \cong A_7$, then $\mathbf{N}_S(K) \cong (A_4 \times C_3) \ltimes C_2$. In all cases we get a contradiction with the maximality of M.

Remark. It is possible to give a simpler argument for the case $S \cong A_7$ by using that A_7 possesses a unique conjugacy class of $\{2,3\}$-Hall subgroups. In this case, by Glauberman's Lemma, there exists an A-invariant $\{2,3\}$-Hall subgroup of N, say H. Then the Frattini argument gives $G = N\mathbf{N}_G(H)$, so $|G : \mathbf{N}_G(H)|$ is a $\{2,3\}'$-number. This implies that the A-invariant subgroup $\mathbf{N}_G(H)$ properly contains an A-invariant Sylow 2-subgroup of G, contradicting the maximality of such Sylow 2-subgroup (Step 2).

Acknowledgements

We would like to thank M.D. Pérez-Ramos for many helpful conversations on the subject. The first author was partially supported by Ministerio de Ciencia, Innovación y Universidades, Proyecto PGC2018-096872-B-100 and also by Proyecto UJI-B2019-03. The second author was supported by the Nature Science Fund of Shandong Province (No. ZR2019MA044) and the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2018QZJ04).

References

[1] Beltrán, A., Shao, C.G.: A coprime action version of a solubility criterion of Deskins. Monat. Math. **188**, 461-466 (2019)

[2] Deskins, W.E.: A condition for the solvability of a finite group. Illinois J. Math. **5**, 306-313 (1961)

[3] Gorenstein, D.: The Classification of finite simple groups. Simple groups and local analysis, Bull. Amer. Math. Soc. (New Series), **1**(1), 43-199 (1979)

[4] Huppert, B.: Endliche Gruppen. I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer, Berlin, (1967)

[5] Huppert, B.: Normalteiler und maximale Untergruppen endlicher Gruppen. (German) Math. Z. **60**, 409-434 (1954)

[6] Janko, Z.: Verallgemeinerung eines Satzes von B. Huppert und J.G. Thompson. (German) Arch. Math. (Basel) **12**, 280-281 (1961)

[7] Kondrat'ev, A.S.: Normalizers of the Sylow 2-subgroups in finite simple groups. Math. Zametki **78**(3), 368-376 (2005); translation in Math. Notes **78**(3-4), 338-346 (2005) (**Russian**)

[8] Thompson, J.: Finite groups with fixed-point-free automorphisms of prime order. Proc. Nat. Acad. Sci. U.S.A. **45**, 578-581 (1959)