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Completeness of Hutton [0, 1]-quasi-uniformities induced by a crisp uniformity

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Abstract

In [25], Katsaras introduced a method for constructing a Hutton $[0, 1]$-quasi-uniformity from a crisp uniformity. In this paper we present other different methods for making this based mainly in the concept of a fuzzy uniform structure. Furthermore, we prove that some of these methods preserve the completeness property of the quasi-uniformity. Moreover, we also show that Katsaras' construction allows to develop a theory of completion of a Hutton [0,1]-quasi-uniform space obtained from a uniform space.

Keywords: Uniformity, fuzzy uniform structure, Hutton $[0, 1]$-quasi-uniformity, completeness

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1 Introduction

The problem of finding appropriate notions for topological concepts in the fuzzy context as well as the study of their relationship with crisp topological concepts has been a fruitful influential area of research. In particular, the quest for finding suitable notions of fuzzy metric, fuzzy uniformity and fuzzy proximity has deserved a lot of attention during the last decades [1, 3, 7, 12, 24, 25, 27, 26, 28, 20, 19, 33, 38, 41], etc. Nowadays, the probably most widespread notion of fuzzy metric is that due to George and Veeramani [7] based on that of Kramosil and Michaleck [27] which is directly motivated by the probabilistic metrics [29, 18]. George and Veeramani showed [7] how to obtain a fuzzy metric starting from a metric d (see also Remark 2.7). One of these fuzzy metrics is usually called the standard fuzzy metric of d, a terminology which has been proved to be very suitable since both metrics share some properties: they generate the same topology and the same uniformity [7, 11]; d is complete if and only if its standard fuzzy metric so is [9]; every standard fuzzy metric space has a (up to isometry) unique fuzzy metric completion which is the standard fuzzy metric space of the completion of d [11]; etc. The converse problem of defining a metric from a fuzzy metric has also been considered by some authors (see [4, 30]).

On the other hand, we can find different notions of uniformities in the fuzzy context as Lowen uniformities [28], probabilistic uniformities [18], Hutton uniformities [23], fuzzy uniform structures [16], fuzzifying uniformities [36], etc. (see [40]). Several attempts have been made in order to provide a theory for unifying various of these different concepts of uniformity [3, 41]. Nevertheless, there is not too many results about how to reconcile the theory of fuzzy metric spaces with that of fuzzy uniform spaces. In crisp theory, there is a standard procedure which allows to construct a uniformity by means of a metric having a good behaviour as from a categorical point of view as with respect to some uniform properties like precompactness and completeness. However, this procedure is not clear at all in the fuzzy theory.

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Some authors have considered the question of constructing a uniformity in the fuzzy context from a fuzzy metric. In this way, in [18, 19] Höhle gave a method to construct a probabilistic uniformity and a Lowen uniformity from a probabilistic pseudometric. This technique was used in [34] to characterize probabilistic and Lowen uniformities by means of certain families of fuzzy pseudometrics. Later on, in [13] it is given a method to endow a fuzzy metric space with a Hutton $[0,1]$-quasi-uniformity (see also [37]) meanwhile in [39] it is constructed a fuzzifying uniformity in the same context. Recently [15] different procedures to endow a fuzzy metric space with a probabilistic uniformity or a Hutton [0,1]-quasi-uniformity have been studied. The categorical behaviour of these constructions was analyzed as well as their induced fuzzy topologies. From that study we can deduce that some constructions have not appropriate properties since, for example, they don't preserve fuzzy uniformly continuous functions.

Here, we explore the related problem of defining a Hutton [0,1]-quasi-uniformity starting from a crisp uniformity. To the best of our knowledge, the only method which has appeared in the literature is the one due to Katsaras [25] who established covariant functors between the categories of (quasi-)uniform spaces and Hutton [0,1]-(quasi-)uniform spaces (see also [16, Remark 4.5]). Based on the constructions of Hutton [0,1]-(quasi-)uniformities from a fuzzy quasi-metric provided in [15] and the isomorphism between the categories of uniform spaces and fuzzy uniform spaces as introduced in [16] (see Theorem 3.15), we give other methods to associate a Hutton [0,1]-quasi-uniformity with a classic uniformity (see Proposition 4.7). These methods generalize those provided in [13, 37] in the context of fuzzy metric spaces.

In order to check the usefulness of these new constructions we will study one of the main properties considered in the context of uniform spaces: completeness. In this way, we analyze how completeness is transferred from a uniform space to the different Hutton [0,1]-quasi-uniform spaces that we consider (see Theorem 5.14). For uniform spaces, we use the classical notion of completeness. For its part, in the context of Hutton [0,1]-quasi-uniform spaces we use the completeness notion introduced in [14] by means of [0,1]-filters (see Section 5). Nevertheless, since two of our constructions of Hutton [0,1]-quasi-uniform spaces are based on the isomorphism between uniform spaces and fuzzy uniform spaces [16], we first introduce completeness in this context providing a characterization of this concept in terms of nets. From our results we can infer which methods of inducing a Hutton [0,1]-quasi-uniformity from a crisp uniformity are more appropriate in the sense that they preserve completeness. In particular, we will demonstrate that Katsaras' method preserves completeness and it also allows to construct a suitable theory of completion.

The structure of the paper is as follows. We present in Section 2 basic facts about fuzzy metric spaces. Section 3 is devoted to recall the definition of a uniform space and a fuzzy uniform space as well as the isomorphism between the categories formed by these objects. After recalling in Section 4 some theory related with Hutton [0,1]-quasi-uniform spaces [23] we present several methods, extending those of [15], for constructing a Hutton [0,1]-quasi-uniformity from a uniformity, via fuzzy uniform structures. In Section 5 we study, by using the concept of completeness introduced in [14] for Hutton [0,1]-quasi-uniform spaces, whether completeness of a uniformity is transferred to its associated Hutton [0,1]-quasi-uniformities. We obtain a positive result for two of our constructions but the question for the other two is open. In the last section we develop a theory of completion for Hutton $[0,1]$-quasi-uniformities. A first attempt to obtain this theory was made in [14] but unfortunately the authors used a result of [13] which is no longer true (see [15]). Here we present a positive result by using the Hutton $[0,1]$-quasi-uniformity induced by a quasi-uniformity as introduced by Katsaras [25].

2 Fuzzy metric spaces

In the following, we will use the notations: $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. Given a nonempty set X, 1_A will denote the characteristic function of a subset A of X.

Definition 2.1. [35] *A binary operation* $*\colon [0,1] \times [0,1] \to [0,1]$ *is called a* triangular norm *or a* t-norm *if* $([0,1], *)$ *is an Abelian monoid with unit* 1*, such that* $\alpha * \beta \leq \gamma * \delta$ *whenever* $\alpha \leq \gamma$ *and* $\beta \leq \delta$*, with* $\alpha, \beta, \gamma, \delta \in [0,1]$*.*

If $*$ *is also continuous, then we will say that it is a* continuous t-norm*.*

Example 2.2. *Three distinguished examples of continuous t-norms are* $\wedge$*,* $\cdot$ *and* $*_L$ *(the Lukasiewicz t-norm) which are defined as*

$$\alpha \wedge \beta = \min\{\alpha, \beta\}, \quad \alpha \cdot \beta = \alpha\beta \quad \textit{and} \quad \alpha *_L \beta = \max\{\alpha + \beta - 1, 0\}.$$

It is well-known [2, Definition 2.5.1] that any t-norm has an associated implication given by $x \overset{*}{\to} y = \bigvee\{z \in I : x * z \leq y\}$, for all $x, y \in I$, named the $*$-*residuated implication* of $*$ (we will omit the superscript $*$ if no confusion arises). Notice that if $*$ is left-continuous, then the following residuated axiom is satisfied:

$$x * y \leq z \Longleftrightarrow y \leq x \overset{*}{\to} z$$

for all $x, y, z \in I$ (see [2, Proposition 2.5.2]). This means that given $x \in I$, the pair of isotone maps $T_*(x, \cdot) : I \to I$ and $I_*(x, \cdot) : I \to I$ forms a Galois connection where $T_*(x, y) = x * y$ and $I_*(x, y) = x \overset{*}{\to} y$ for all $y \in I$.

Example 2.3. *The residuated implications of the t-norms of the previous example are*

$$\alpha \overset{\wedge}{\to} \beta = \begin{cases} 1, & \text{if } \alpha \le \beta; \\ \beta, & \text{if } \beta < \alpha; \end{cases} \quad \alpha \overset{\cdot}{\to} \beta = \begin{cases} 1, & \text{if } \alpha \le \beta; \\ \frac{\beta}{\alpha}, & \text{if } \beta < \alpha; \end{cases} \quad \text{and} \quad \alpha \overset{*_L}{\to} \beta = \min\{1 - \alpha + \beta, 1\},$$

for all $\alpha, \beta \in [0, 1]$.

Definition 2.4. [16] *A* fuzzy pseudometric *(in the sense of Kramosil and Michalek) on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that*

$(FM1)$ $M(x, y, 0) = 0$;
$(FM2)$ $M(x, x, t) = 1$;
$(FM3)$ $M(x, y, t) = M(y, x, t)$;
$(FM4)$ $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
$(FM5)$ $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous;*
for every $x, y, z \in X$ and $t, s > 0$.

If the fuzzy pseudometric $(M, *)$ also satisfies:

(FM2') $M(x, y, t) = 1$ *for all $t > 0$ if and only if $x = y$,*

then $(M, *)$ is said to be a fuzzy metric *on X* [27].

A fuzzy (pseudo)metric *space is a triple $(X, M, *)$ such that X is a nonempty set and $(M, *)$ is a fuzzy (pseudo)metric on X.*

Remark 2.5. *Every fuzzy pseudometric $(M, *)$ on a nonempty set X generates a topology $\tau(M)$ on X which has as a base the family $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in I_0, t > 0\}$ where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$. Furthermore (cf.* [10]*) every fuzzy (pseudo)metric space $(X, M, *)$ is (pseudo)metrizable. A compatible (pseudo)metric d_M can be obtained by applying the Kelley metrization lemma to the uniformity $\mathcal{U}_M$ compatible with $\tau(M)$ which has a countable base given by*

$$U_n^M = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$$

(we will omit the superscript M if no confusion arises).

Definition 2.6. [8] *A function $f : (X, M, *) \to (Y, N, \star)$ between two fuzzy pseudometric spaces is said to be* uniformly continuous *if for every $\varepsilon \in I_0$ and $t > 0$ there exist $\delta \in I_0$ and $s > 0$ such that if $M(x, y, s) > 1 - \delta$ then $N(f(x), f(y), t) > 1 - \varepsilon$, where $x, y \in X$.*

This is equivalent to assert that $f : (X, \mathcal{U}_M) \to (Y, \mathcal{U}_N)$ is uniformly.

Remark 2.7. ([16], [7]) *Let (X, d) be a pseudometric space and let M_d be the fuzzy set on $X \times X \times [0, \infty)$ given by*

$$M_d(x, y, t) = \begin{cases} \frac{t}{t + d(x, y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d, *)$ is a fuzzy pseudometric on X which is called the standard fuzzy pseudometric induced by d.

Furthermore, we notice that $\mathcal{U}_d = \mathcal{U}_{M_d}$ (cf. [11, Lemma 5]*) where $\mathcal{U}_d$ is the uniformity generated by d. Hence $\tau(d) = \tau(\mathcal{U}_d) = \tau(\mathcal{U}_{M_d}) = \tau(M_d)$.*

The converse problem of constructing a metric from a fuzzy metric has also been studied in the literature (see for example [4, 30]*).*

3 Uniform spaces and fuzzy uniform spaces

Uniform spaces admit several equivalent definitions among which we can emphasize the following two: by entourages of the diagonal or by pseudometrics. We recall these concepts.

Definition 3.1. *A* uniformity *on a nonempty set X is a filter $\mathcal{U}$ on $X \times X$ such that:*

(U1) *if* $U \in \mathcal{U}$ *then* $U^{-1} \in \mathcal{U}$, *where* $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$;

(U2) $\Delta \subseteq U$ *for all* $U \in \mathcal{U}$, *where* $\Delta = \{(x, x) : x \in X\}$;

(U3) *given* $U \in \mathcal{U}$ *there exists* $V \in \mathcal{U}$ *such that* $V^2 \subseteq U$ *where* $V^2 = \{(x, y) \in X \times X :$ *there exists* $z \in X$ *such that* $(x, z), (z, y) \in V\}$.

The pair $(X, \mathcal{U})$ *is said to be a* uniform space.

Example 3.2. [6] *Let us consider a pseudo space* (X, d). *Then the filter* $\mathcal{U}_d$ *generated by all the sets of the form* $U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$, *where* $\varepsilon > 0$, *is a uniformity on* X *called the pseudometric uniformity generated by* d.

We will denote by Unif the topological category whose objects are the uniform spaces and whose morphisms are the uniformly continuous functions (a function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is said to be uniformly continuous if $(f \times f)^{-1}(V) \in \mathcal{U}$, for all $V \in \mathcal{V}$).

Uniformities can be defined alternatively by means of a family of pseudometrics as follows:

Definition 3.3. *Let* X *be a nonempty set. A* uniform structure *on* X *is a nonempty family* $\mathcal{D}$ *of pseudometrics on* X *such that:*

(US1) *if* $d, q \in \mathcal{D}$, *then* $\max\{d, q\} \in \mathcal{D}$;

(US2) *if* e *is a pseudometric on* X *and for each* $\varepsilon > 0$ *there exist* $d \in \mathcal{D}$ *and* $\delta > 0$ *such that* $d(x, y) < \delta$ *implies* $e(x, y) < \varepsilon$ *for all* $x, y \in X$, *then* $e \in \mathcal{D}$.

Remark 3.4. *If* X *is a nonempty set, then we can define easily two uniform structures on* X :

- *the* discrete uniform structure *formed by all the pseudometrics which can be defined on* X;
- *the* trivial uniform structure *formed only by the pseudometric* $t(x, y) = 0$ *for all* $x, y \in X$.

Remark 3.5. *If* $\mathcal{D}$ *is a uniform structure on* X, *condition (US2) is equivalent to* $e \in \mathcal{D}$ *whenever* $\mathcal{U}_e \subseteq \bigvee_{d \in \mathcal{D}} \mathcal{U}_d$.

Remark 3.6. *Uniform structures are frequently called* gauges *(see* [5]*). An asymmetric version of gauges was considered by Reilly* [31].

Definition 3.7. *A* base for a uniform structure *on a nonempty set* X *is a nonempty family* $\mathcal{B}$ *of pseudometrics on* X *satisfying (US1).*

A base for a uniform structure $\mathcal{B}$ *generates a uniform structure on* X *given by all the pseudometrics* e *on* X *such that for each* $\varepsilon > 0$ *there exist* $d \in \mathcal{B}$ *and* $\delta > 0$ *such that* $d(x, y) < \delta$ *implies* $e(x, y) < \varepsilon$ *for all* $x, y \in X$.

Example 3.8. [6] *Let* (X, τ) *be a Tychonoff space. Let us denote by* $C(X)$ *the family of all real-valued continuous functions on* X. *Given* $f_1, \ldots, f_k \in C(X)$, *define* $d_{f_1,\ldots,f_k} : X \times X \to [0, \infty)$ *as*

$$d_{f_1,\ldots,f_k}(x, y) = \max\{|f_1(x) - f_1(y)|, \ldots, |f_k(x) - f_k(y)|\}$$

for every $x, y \in X$. *It is easy to see that* $d_{f_1,\ldots,f_k}$ *is a pseudometric on* X *and the family* $\{d_{f_1,\ldots,f_k} : f_1, \ldots, f_k \in C(X), k \in \mathbb{N}\}$ *is a base for a uniform structure on* X.

Let us consider that a function $f : (X, \mathcal{D}) \to (Y, \mathcal{Q})$ between two spaces endowed with a uniform structure is uniformly continuous whenever $f : (X, \bigvee_{d \in \mathcal{D}} \mathcal{U}_d) \to (Y, \bigvee_{q \in \mathcal{Q}} \mathcal{U}_q)$ is uniformly continuous (cf. [5]). Then we can define the category SUnif whose objects are the spaces endowed with a uniform structure and whose morphisms are the uniformly continuous functions. It is well-known that Unif and SUnif are isomorphic categories as the next theorem shows.

Theorem 3.9. *Let* $\mathcal{U}$ *and* $\mathcal{D}$ *be a uniformity and a uniform structure on a nonempty set* X *respectively. Define:*

- $\mathcal{D}_\mathcal{U}$ *as the family of all pseudometrics* d *on* X *such that* $\mathcal{U}_d \subseteq \mathcal{U}$;
- $\mathcal{U}_\mathcal{D}$ *as the uniformity* $\bigvee_{d \in \mathcal{D}} \mathcal{U}_d$.

Then the mappings:

- $\Delta : \mathsf{Unif} \to \mathsf{SUnif}$ *given by* $\Delta((X,\mathcal{U})) = (X, \mathcal{D}_{\mathcal{U}})$;
- $\Lambda : \mathsf{SUnif} \to \mathsf{Unif}$ *given by* $\Lambda((X,\mathcal{D})) = (X, \mathcal{U}_{\mathcal{D}})$;

which leave morphisms unchanged are covariant functors such that $\Delta \circ \Lambda = 1_{\mathsf{SUnif}}$ *and* $\Lambda \circ \Delta = 1_{\mathsf{Unif}}$.

In [16] the authors studied a fuzzy notion of the concept of uniform structure giving a new category isomorphic to Unif. We recall the necessary notions to establish this isomorphism.

Definition 3.10 ([16, Definition 3.1])**.** *Let $\mathcal{M}$ be a family of fuzzy pseudometrics on X with respect to a fixed continuous t-norm $*$. The pair $(\mathcal{M},*)$ is said to be a* fuzzy uniform structure *on X if it satisfies:*

$(FSU1)$ $(M \wedge N, *) \in \mathcal{M}$ whenever $(M,*),(N,*) \in \mathcal{M}$;

$(FSU2)$ if $(M,*)$ is a fuzzy pseudometric on X such that for all $\varepsilon \in I_0$ and $t > 0$ there exist $(N,*) \in \mathcal{M}$, $\delta \in I_0$, and $s > 0$ such that for all $x,y \in X$, $N(x,y,s) > 1-\delta$ implies $M(x,y,t) > 1-\varepsilon$, then $(M,*) \in \mathcal{M}$.

The triple $(X,\mathcal{M},*)$ is called a fuzzy uniform space.

Example 3.11. *If X is a nonempty set and $*$ is a continuous t-norm, then we can define easily two fuzzy uniform structures on X :*

- *the* $*-$discrete uniform structure *formed by all the fuzzy pseudometrics $(M,*)$ which can be defined on X;*
- *the* $*-$trivial uniform structure *formed only by the fuzzy pseudometric*

$$M(x,y,t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x,y \in X$ and all $t \geq 0$.

Remark 3.12. *Every fuzzy uniform structure $(\mathcal{M},*)$ on a nonempty set X induces a uniformity $\mathcal{U}_{\mathcal{M}}$ on X given by $\mathcal{U}_{\mathcal{M}} = \bigvee_{(M,*)\in\mathcal{M}} \mathcal{U}_M$. This uniformity has as a base the family $\{U_{M,\varepsilon,t} : (M,*) \in \mathcal{M}, \varepsilon \in I_0, t > 0\}$ where $U_{M,\varepsilon,t} = \{(x,y) \in X \times X : M(x,y,t) > 1-\varepsilon\}$ (cf.*[16, Proposition 3.4]*).*

A pair $(\mathcal{B},*)$ where $*$ is a continuous t-norm and $\mathcal{B}$ is a nonempty family of fuzzy pseudometrics on X with respect to $*$ satisfying (FUS1) is a *base for a fuzzy uniform structure* $(\mathcal{M}_{\mathcal{B}},*)$ where $\mathcal{M}_{\mathcal{B}}$ is the family of all fuzzy pseudometrics $(M,*)$ on X such that for each $\varepsilon \in I_0$ and each $t > 0$ there exist $(N,*) \in \mathcal{M}$, $\delta \in I_0$ and $s > 0$ such that $N(x,y,s) > 1-\delta$ implies $M(x,y,t) > 1-\varepsilon$. Observe that in this case $\mathcal{U}_{\mathcal{B}} := \bigvee_{(M,*)\in\mathcal{B}} \mathcal{U}_M = \mathcal{U}_{\mathcal{M}_{\mathcal{B}}}$.

Definition 3.13. [16] *Let $(X,\mathcal{M},*)$ and $(Y,\mathcal{N},\star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be* uniformly continuous *if for each $N \in \mathcal{N}$, $\varepsilon \in I_0$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in I_0$ and $s > 0$ such that $N(f(x),f(y),t) > 1-\varepsilon$ whenever $M(x,y,s) > 1-\delta$.*

Remark 3.14. *Notice that a function $f : (X,\mathcal{M},*) \to (Y,\mathcal{N},\star)$ between two fuzzy uniform spaces is uniformly continuous if and only if $f : (X,\mathcal{U}_{\mathcal{M}}) \to (Y,\mathcal{U}_{\mathcal{N}})$ so is. Furthermore, if $(\mathcal{B},*)$ is a base for a fuzzy uniform structure on X, then $(\mathcal{M}_{\mathcal{B}},*)$ is the largest fuzzy uniform structure on X such that* $\mathrm{id} : (X,\mathcal{U}_{\mathcal{B}}) \to (X,\mathcal{U}_{\mathcal{M}_{\mathcal{B}}})$ *is uniformly continuous.*

Then we can consider the category FUnif whose objects are the fuzzy uniform spaces and whose morphisms are the uniformly continuous functions. Besides, if $*$ is a continuous t-norm, we denote by $\mathsf{FUnif}(*)$ the full subcategory of FUnif whose objects are the fuzzy uniform spaces of the form $(X,\mathcal{M},*)$. This is a topological category [16, Corollary 3.15]. Moreover, in [16] it is proved that the category $\mathsf{FUnif}(*)$ is isomorphic to Unif as follows:

Theorem 3.15 ([16, Theorem 3.14])**.** *Let $(X,\mathcal{U})$ be a uniform space and $(X,\mathcal{M},*)$ be a fuzzy uniform space. Let us consider:*

- *$(\delta_*(\mathcal{U}),*)$ the fuzzy uniform structure on X which has as a base the family $\{(M_d,*) : d \in \mathcal{D}_{\mathcal{U}}\}$ where $\mathcal{D}_{\mathcal{U}}$ is the uniform structure of $\mathcal{U}$, i. e. $\delta_*(\mathcal{U}) = \{(M,*) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \mathcal{U}\}$;*
- *$\lambda(\mathcal{M})$ is the family of all pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}_{\mathcal{M}}$.*

Then:

(i) $\Delta^F_* : \mathsf{Unif} \to \mathsf{FUnif}(*)$ *is a covariant functor sending each $(X,\mathcal{U})$ to $(X,\delta_*(\mathcal{U}),*)$;*

(ii) $\Lambda^F : \mathsf{FUnif}(*) \to \mathsf{Unif}$ *is a covariant functor sending each $(X,\mathcal{M},*)$ to $(X,\mathcal{U}_{\mathcal{M}}) = (X,\mathcal{U}_{\lambda(\mathcal{M})})$;*

(iii) $\Delta_*^F \circ \Lambda^F = 1_{\mathsf{FUnif}(*)}$ *and* $\Lambda^F \circ \Delta_*^F = 1_{\mathsf{Unif}}$.

Remark 3.16. *We also notice that if* $(X, \mathcal{U})$ *is a uniform space, then*

$$\tau(\mathcal{U}) = \bigvee_{d \in \mathcal{D}_\mathcal{U}} \tau(d) = \bigvee_{d \in \mathcal{D}_\mathcal{U}} \tau(M_d) = \bigvee_{(M,*) \in \delta_*(\mathcal{D}_\mathcal{U})} \tau(M).$$

Completeness of fuzzy uniform spaces in terms of filters was studied in [16, 17]. We recall this concept.

Definition 3.17. [16] *A filter* $\mathcal{F}$ *in a fuzzy uniform space* $(X, \mathcal{M}, *)$ *is said to be a* Cauchy filter *if for each* $(M, *) \in \mathcal{M}$*, each* $\varepsilon \in I_0$ *and each* $t > 0$ *there is* $x \in X$ *such that* $B_M(x, \varepsilon, t) \in \mathcal{F}$.

A fuzzy uniform space $(X, \mathcal{M}, *)$ *is said to be* complete *if every Cauchy filter in* $(X, \mathcal{M}, *)$ *converges in* $\tau(\mathcal{U}_\mathcal{M})$.

It is well-known that in a uniform space, completeness can be equivalently described in terms of filters or nets. Since fuzzy uniform spaces are isomorphic to crisp uniform spaces, this equivalence can be also obtained in this context. For the sake of completeness, we present here this equivalence by considering the following natural notion of Cauchy net in a fuzzy uniform space.

Definition 3.18. *A net* $(x_\lambda)_{\lambda \in \Lambda}$ *in a fuzzy uniform space* $(X, \mathcal{M}, *)$ *is said to be* Cauchy *if for each* $(M, *) \in \mathcal{M}$*, each* $\varepsilon \in I_0$ *and each* $t > 0$ *there exists* $\lambda_\varepsilon \in \Lambda$ *such that* $M(x_\lambda, x_\alpha, t) > 1 - \varepsilon$ *for all* $\lambda, \alpha \geq \lambda_\varepsilon$.

Proposition 3.19. *A fuzzy uniform space* $(X, \mathcal{M}, *)$ *is complete if and only if every Cauchy net in* $(X, \mathcal{M}, *)$ *is convergent with respect to* $\tau(\mathcal{U}_\mathcal{M})$.

Proof. It follows a similar structure of the proof of the equivalence between the completeness of a metric space in terms of nets or filters. Nevertheless, we include it here.

Suppose that $(X, \mathcal{M}, *)$ is complete and let $(x_\lambda)_{\lambda \in \Lambda}$ be a Cauchy net in $(X, \mathcal{M}, *)$. Let us consider the filter $\mathcal{F}$ having as base the family $\{\{x_\alpha : \alpha \geq \lambda\} : \lambda \in \Lambda\}$. Given $(M, *) \in \mathcal{M}$, $\varepsilon \in I_0$ and $t > 0$ we can find by assumption $\lambda_\varepsilon \in \Lambda$ that $M(x_\lambda, x_\alpha, t) > 1 - \varepsilon$ for all $\lambda, \alpha \geq \lambda_\varepsilon$. Hence $\{x_\lambda : \lambda \geq \lambda_\varepsilon\} \subseteq B_M(x_{\lambda_\varepsilon}, \varepsilon, t) \in \mathcal{F}$. Consequently, $\mathcal{F}$ is a Cauchy filter in $(X, \mathcal{M}, *)$ so it converges to a point x which is also a limit point of $(x_\lambda)_{\lambda \in \Lambda}$ [6, Theorem 1.6.12].

Conversely, let $\mathcal{F}$ be a Cauchy filter in $(X, \mathcal{M}, *)$ and let us consider the set $\{(x, F) : x \in F \in \mathcal{F}\}$ directed by $(x_1, F_1) \leq (x_2, F_2)$ if $F_2 \subseteq F_1$. Then $(x_{(x,F)})_{(x,F) \in \Lambda}$ is the usual associated net of $\mathcal{F}$ [6, Theorem 1.6.13]. Let us consider $(M, *) \in \mathcal{M}$, $\varepsilon \in I_0$ and $t > 0$. Find $\delta \in I_0$ such that $1 - \varepsilon < (1 - \delta) * (1 - \delta)$. By hypothesis there exists $x \in X$ such that $B_M(x, \delta, t/2) \in \mathcal{F}$. Consequently, if $(x_1, F_1), (x_2, F_2) \geq (x, B_M(x, \delta, t/2))$, then $M(x, x_1, t/2) > 1 - \delta$ and $M(x, x_2, t/2) > 1 - \delta$ so

$$M(x_1, x_2, t) \geq M(x_1, x, t/2) * M(x, x_2, t/2) > (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Therefore, $(x_{(x,F)})_{(x,F) \in \Lambda}$ is Cauchy so $\tau(\mathcal{U}_\mathcal{M})$-convergent to a point $x \in X$. Then $\mathcal{F}$ is also convergent to x [6, Theorem 1.6.13]. □

Remark 3.20. [16] *Notice that a filter* $\mathcal{F}$ *in a fuzzy uniform space* $(X, \mathcal{M}, *)$ *is Cauchy if and only if it is Cauchy in* $(X, \mathcal{U}_\mathcal{M})$ *which in turn equals to be Cauchy in* $(X, \mathcal{U}_M)$ *for all* $(M, *) \in \mathcal{M}$*. Furthermore,* $\mathcal{F}$ *is a Cauchy filter in a uniform space* $(X, \mathcal{U})$ *if and only if it is a Cauchy filter in* $(X, \delta_*(\mathcal{U}), *)$.

Nevertheless, completeness of $(X, \mathcal{U}_\mathcal{M})$ is not equivalent to completeness of $(X, M, *)$ for all $(M, *) \in \mathcal{M}$, where $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.

For example, consider the set of rational numbers $\mathbb{Q}$ with the usual metric d and consider $(\mathcal{M}, \cdot)$ the fuzzy uniform structure on $\mathbb{Q}$ associated to the fine uniformity $\mathcal{FN}$ of $\tau(d)$ as in Theorem 3.15. Then $\mathcal{U}_\mathcal{M}$ is complete (i. e. $(\mathbb{Q}, \tau(d))$ is Dieudonné complete) since $(\mathbb{Q}, d)$ is paracompact (see [6]). Nevertheless, d is not complete (in fact $(\mathbb{Q}, \tau(d))$ is not completely metrizable) so M_d is not complete either. Furthermore, $d \in \mathcal{D}_{\mathcal{FN}}$ so $(M_d, \cdot) \in \mathcal{M}$.

On the other hand, let us consider on the set of natural numbers $\mathbb{N}$ the family of fuzzy pseudometrics $\mathcal{B} = \{(M_{d_k}, \cdot) : k \in \mathbb{N}\}$ where d_k is the pseudometric on $\mathbb{N}$ given by

$$d_k(n, m) = \begin{cases} 0 & \text{if } n = m \text{ or } n \wedge m \geq k \\ 1 & \text{otherwise} \end{cases}.$$

It is clear that $(\mathcal{B}, \cdot)$ is a base for a fuzzy uniform structure since $(M_{d_n} \wedge M_{d_m}, \cdot) = (M_{d_n \vee d_m}, \cdot) = (M_{d_{n \wedge m}}, \cdot) \in \mathcal{B}$. Obviously $(\mathbb{N}, d_k)$ is complete for every $k \in \mathbb{N}$ so $(\mathbb{N}, M_{d_k}, \cdot)$ is also complete (see [11, Proposition 1]). Nevertheless, $(\mathbb{N}, \mathcal{U}_\mathcal{B})$ is not complete. In fact, let us consider the filter $\mathcal{F}$ on $\mathbb{N}$ having as a base the family $\{\{k \in \mathbb{N} : k \geq n\} : n \in \mathbb{N}\}$. It is clear that $\mathcal{F}$ is Cauchy in $(X, M_{d_k}, \cdot)$ for all $(M_{d_k}, \cdot) \in \mathcal{B}$ so it is Cauchy in $(X, \mathcal{U}_\mathcal{B}) = (X, \mathcal{U}_{\mathcal{M}_\mathcal{B}})$. However it is not $\tau(\mathcal{U}_\mathcal{B})$ convergent. In fact, $U_{d_k, \frac{1}{2}} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : d_k(n, m) < \frac{1}{2}\} \in \mathcal{U}_\mathcal{B}$ for all $k \in \mathbb{N}$ but given $n_0 \in \mathbb{N}$ then $U_{d_k, \frac{1}{2}}(n_0) = \{n_0\} \notin \mathcal{F}$ for all $k > n_0$ so $\mathcal{F}$ is not convergent to n_0 in $\tau(\mathcal{U}_\mathcal{B})$.

4 Hutton $[0,1]$-quasi-uniformities on a uniform space

In [15], the authors studied four methods for constructing a Hutton $[0,1]$-quasi-uniformity starting from a fuzzy metric (see also [13, 37]). Here, we extend these methods to the context of a uniform space by means of the equivalence between uniform spaces and fuzzy uniform spaces given in Theorem 3.15. We start recalling basic notions of Hutton [0,1]-quasi-uniformities.

Let $\mathcal{H}(X)$ denote the family of all $W\colon [0,1]^X \to [0,1]^X$ such that:

(1) $W(a) \geq a$ for all $a \in [0,1]^X$;

(2) $W(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} W(a_i)$ for all $\{a_i : i \in I\} \subseteq [0,1]^X$ and $W(1_\varnothing) = 1_\varnothing$.

Since $a = \bigvee_{x\in X} a(x) * 1_{\{x\}}$ for each $a \in [0,1]^X$ it follows that every $W \in \mathcal{H}(X)$ is completely determined by the family

$$\{W(\alpha * 1_{\{x\}}) : \alpha \in I_0,\ x \in X\}.$$

Definition 4.1 (cf. [23])**.** *A* Hutton $[0,1]$-quasi-uniformity *on a nonempty set X is a nonempty subset $\mathfrak{U}$ of $\mathcal{H}(X)$ such that*

$(HU1)$ *if $U \in \mathfrak{U}$, $U \leq V$ and $V \in \mathcal{H}(X)$, then $V \in \mathfrak{U}$;*
$(HU2)$ *if $U, V \in \mathfrak{U}$ there exists $W \in \mathfrak{U}$ such that $W \leq U$ and $W \leq V$;*
$(HU3)$ *if $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \circ V \leq U$.*
If $\mathfrak{U}$ also satisfies
$(HU4)$ *if $U \in \mathfrak{U}$ then $U^{-1} \in \mathfrak{U}$ where $U^{-1}(a) = \bigwedge\{b \in [0,1]^X : U(1_X - b) \leq 1_X - a\}$, $\quad a \in [0,1]^X$,*
then it is called a Hutton $[0,1]$-uniformity. *The pair $(X,\mathfrak{U})$ is called a* Hutton $[0,1]$-(quasi-)uniform space.
A function $f\colon (X,\mathfrak{U}) \to (Y,\mathsf{V})$ between two Hutton $[0,1]$-(quasi-)uniform spaces is said to be (quasi-)uniformly continuous *if for every $V \in \mathsf{V}$ there exists $U \in \mathfrak{U}$ such that*

$$U(a) \leq V\big(\bigvee_{x\in X} a(x) * 1_{\{f(x)\}}\big) \circ f, \qquad a \in [0,1]^X.$$

We denote by $\mathsf{H(Q)Unif}$ the category of Hutton $[0,1]$-(quasi-)uniform spaces and (quasi-)uniformly continuous functions.

Definition 4.2. *A* base *for a Hutton $[0,1]$-(quasi-)uniformity $\mathfrak{U}$ on X is a nonempty subset $\mathfrak{B}$ of $\mathcal{H}(X)$ such that for each $U \in \mathfrak{U}$ there exists $B \in \mathfrak{B}$ such that $B \leq U$.*

If $\mathfrak{B}$ is a nonempty subset of $\mathcal{H}(X)$ verifying:
$(BH1)$ *if $B_1, B_2, \in \mathfrak{B}$ there exists $B_3 \in \mathfrak{B}$ such that $B_3 \leq B_1$ and $B_3 \leq B_2$;*
$(BH2)$ *if $B_1 \in \mathfrak{B}$ there exists $B_2 \in \mathfrak{B}$ such that $B_2 \circ B_2 \leq B_1$,*
then we say that it is a basis for a Hutton $[0,1]$-(quasi-)uniformity on X given by

$$\mathfrak{U}_{\mathfrak{B}} = \{U \in \mathcal{H}(X) : \text{ there exists } B \in \mathfrak{B} \text{ such that } B \leq U\}.$$

A Hutton $[0,1]$-(quasi-)uniformity is said to be stratified *if it has a base $\mathfrak{B}$ satisfying*

$$\alpha * U(1_{\{x\}}) \leq U(\alpha * 1_{\{x\}}) \quad \forall U \in \mathfrak{B}, \forall \alpha \in [0,1].$$

The following is a classical example of a Hutton $[0,1]$-quasi-uniformity

Example 4.3 ([23, 32, 33])**.** *Let us consider the family $\mathcal{F}$ of fuzzy sets λ on $\mathbb{R}$ such that λ is antitone, $\bigvee_{t\in\mathbb{R}} \lambda(t) = 1$ and $\bigwedge_{t\in\mathbb{R}} \lambda(t) = 0$. Consider the equivalence relation $\sim$ on $\mathcal{F}$ given by $\lambda \sim \mu$ whenever $\bigvee_{s>t} \lambda(s) = \bigvee_{s>t} \mu(s)$ for all $t \in \mathbb{R}$. The $[0,1]$-fuzzy real line $\mathbb{R}([0,1])$ is the quotient set $\mathcal{F}/\sim$.*

Given $\varepsilon > 0$, define $B_\varepsilon : [0,1]^{\mathbb{R}([0,1])} \to [0,1]^{\mathbb{R}([0,1])}$ by

$$B_\varepsilon(a)([\lambda]) = \bigvee_{s>t_a-\varepsilon} \lambda(s) \quad \text{where } t_a = \bigvee\{t \in \mathbb{R} : a([\mu]) \leq \bigwedge_{s<t} \mu(s) \ \ \forall[\mu] \in \mathbb{R}([0,1])\}.$$

Then $\{B_\varepsilon : \varepsilon > 0\}$ is a base for a Hutton [0,1]-quasi-uniformity $\mathfrak{B}$ on $\mathbb{R}([0,1])$ called the right-handed Hutton [0,1]-quasi-uniformity. This construction was used by Rodabaugh [32] to obtain an appropriate Hutton quasi-uniformity on the fuzzy real line which allows to obtain uniformly isomorphic fuzzy real lines for isomorphic completely distributive lattices with an order reversing involution.

In 1984, Katsaras established the following relations between the categories of (quasi-)uniform spaces and Hutton [0,1]-(quasi-)uniform spaces providing a method to construct a Hutton [0,1]-quasi-uniformity from a crisp quasi-uniformity.

Proposition 4.4. [25] *Let* $(X,\mathcal{U})$ *be a (quasi-)uniform space,* $(Y,\mathfrak{U})$ *be a Hutton* $[0,1]$*-(quasi-)uniform space and* $*$ *a continuous t-norm. Let* $\mathfrak{U}_{\mathcal{U}}^K := \Phi(\mathcal{U})$ *be the Hutton* $[0,1]$*-(quasi-)uniformity generated by* $\{\phi(U) : U \in \mathcal{U}\}$*, where* $\phi(U)(a)(x) = \bigvee_{y\in X} a(y) * 1_{U(y)}(x) = \bigvee_{x\in U(y)} a(y)$ *for each* $U \in \mathcal{U}$*,* $a \in [0,1]^X$ *and* $x \in X$*. Let* $\Psi(\mathfrak{U})$ *be the (quasi-)uniformity generated by* $\{\psi(U) : U \in \mathfrak{U}\}$*, where* $\psi(U) = \{(x,y) \in X \times X : \forall a \in [0,1]^X,\ a(x) \leq U(a)(y)\}$ *for each* $U \in \mathfrak{U}$*. Then*:

(1) $\Phi\colon \mathsf{(Q)Unif} \to \mathsf{H(Q)Unif}$ *is a functor sending each* $(X,\mathcal{U})$ *to* $(X,\Phi(\mathcal{U}))$ *and leaving morphisms unchanged*;

(2) $\Psi\colon \mathsf{H(Q)Unif} \to \mathsf{(Q)Unif}$ *is a functor sending each* $(X,\mathfrak{U})$ *to* $(X,\Psi(\mathfrak{U}))$ *and leaving morphisms unchanged*;

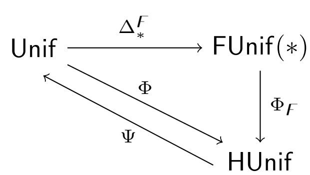
(3) $\Psi(\Phi(\mathcal{U})) = \mathcal{U}$;

(4) $\mathfrak{U} \subseteq \Phi(\Psi(\mathfrak{U}))$;

(5) Ψ *is a right adjoint of* Φ.

Remark 4.5. *We can consider a functor* $\Phi_F : \mathsf{FUnif}(*) \to \mathsf{HUnif}$ *leaving morphisms unchanged and transforming a fuzzy uniform space* $(X,\mathcal{M},*)$ *into the Hutton [0,1]-uniform space* $(X, \bigvee_{(M,*)\in\mathcal{M}} \mathfrak{U}_{\mathcal{U}_M}^K)$.

Then it is easy to check that the following diagram commutes:



As a consequence, if $(X,\mathcal{U})$ *is a uniform space and* $(\delta_*(\mathcal{U}),*)$ *is the fuzzy uniform structure associated with* $\mathcal{U}$*, then*

$$\mathfrak{U}_{\mathcal{U}}^K = \bigvee_{(M,*)\in\delta_*(\mathcal{U})} \mathfrak{U}_{\mathcal{U}_M}^K.$$

In [13] the authors provided a method to construct a Hutton $[0,1]$-quasi-uniformity from a fuzzy metric space in the sense of George and Veeramani. This method is also valid in the context of fuzzy metrics in the sense of Kramosil and Michalek [15]. We recall it.

Proposition 4.6 ([13, Proposition 14])**.** *Let* $(X,M,*)$ *be a fuzzy metric space. Then the family* $\{W_{\varepsilon,t}^M : \varepsilon \in I_0,\ t > 0\}$ *is a base for a Hutton* $[0,1]$*-quasi-uniformity* $\mathfrak{U}_M$ *on* X *where* $W_{\varepsilon,t}^M\colon I^X \to I^X$ *is given by*

$$W_{\varepsilon,t}^M(a)(y) = \bigvee_{x\in X} a(x) * ((1-\varepsilon) \to M(x,y,t)), \qquad a \in I^X,\ y \in X.$$

This construction gives a factorization via Hutton $[0,1]$-quasi-uniformities of the association of a uniformity to each fuzzy metric space.

In the next proposition, we propose the construction of three Hutton [0,1]-quasi-uniformities induced by a crisp uniformity. The first construction is a version of the Hutton [0,1]-quasi-uniformity $\mathfrak{U}_M$ of the previous proposition but in this case it is induced by a uniformity. The second one is motivated by another different Hutton [0,1]-quasi-uniformity induced by a fuzzy metric as developed in [37] (see also [15]). The last one makes use of the Lowen functor ω [28], which allows to transform a uniformity into a Lowen uniformity, and it also uses the Höhle construction of a Hutton [0,1]-quasi-uniformity from a probabilistic uniformity [18].

Proposition 4.7. *Let* $(X,\mathcal{U})$ *be a uniform space and* $*$ *be a continuous t-norm. Let* $(\delta_*(\mathcal{U}),*)$ *be the fuzzy uniform structure associated with* $\mathcal{U}$*. Then*

(1) the family $\{W_{\varepsilon,t}^M : \varepsilon \in I_0,\ t > 0, (M,*) \in \delta_*(\mathcal{U})\}$ *is a base for a stratified Hutton [0,1]-quasi-uniformity* $\mathfrak{U}_{\mathcal{U}}$*;*

(2) the family $\{W_t^M : t > 0, (M,*) \in \delta_*(\mathcal{U})\}$ *is a base for a stratified Hutton [0,1]-quasi-uniformity* $\mathfrak{U}_{\mathcal{U}}^H$ *where* $W_t^M\colon [0,1]^X \to [0,1]^X$ *is given by* $W_t^M(a)(y) = \bigvee_{x\in X} a(x) * M(x,y,t)$*, for* $a \in [0,1]^X,\ y \in X$*;*

(3) *the family* $\{W_{\mathsf{U}} : \mathsf{U} \in [0,1]^{X\times X}$ *and* $\mathsf{U}^{-1}(\varepsilon,1] \in \mathcal{U}$ *for all* $\varepsilon \in [0,1[\}$ *is a base for a stratified Hutton [0,1]-quasi-uniformity* $\mathfrak{U}^{\omega}_{\mathcal{U}}$ *where* $W_{\mathsf{U}} : [0,1]^X \to [0,1]^X$ *is given by* $W_{\mathsf{U}}(a)(y) = \bigvee_{x\in X} a(x) * \mathsf{U}(x,y)$, *for* $a \in [0,1]^X$, $y \in X$.

Proof.

(1) This can be shown following the techniques of the proof of [13, Proposition 14]. In fact, the same proof allows to obtain that $W^M_{\varepsilon,t} \in \mathcal{H}(X)$ and $W^M_{\delta,\frac{t}{2}} \circ W^M_{\delta,\frac{t}{2}} \le W^M_{\varepsilon,t}$ for every $(M,*) \in \mathcal{M}_{\mathcal{U}}$, $t > 0$, $\varepsilon \in (0,1]$ and $\delta \in (0,1]$ verifying that $(1-\delta) * (1-\delta) \ge (1-\varepsilon)$.

We only prove (BH2). Let $(M,*),(N,*) \in \mathcal{M}_{\mathcal{U}}$, $t,s > 0$ and $\varepsilon,\delta \in I_0$. Since $\mathcal{M}_{\mathcal{U}}$ is a fuzzy uniform structure then $(M \wedge N,*) \in \mathcal{M}_{\mathcal{U}}$. We claim that $W^{M\wedge N}_{\varepsilon\wedge\delta,t\wedge s} \le W^M_{\varepsilon,t} \wedge W^N_{\delta,s}$.

In fact, $(M\wedge N)(x,y,t\wedge s) = M(x,y,t\wedge s) \wedge N(x,y,t\wedge s) \le M(x,y,t) \wedge N(x,y,s)$ so

$$\begin{aligned}(1-\varepsilon\wedge\delta) \to (M\wedge N)(x,y,t\wedge s) &\le [(1-\varepsilon) \to (M\wedge N)(x,y,t\wedge s)] \wedge [(1-\delta) \to (M\wedge N)(x,y,t\wedge s)]\\ &\le [(1-\varepsilon) \to M(x,y,t)] \wedge [(1-\delta) \to N(x,y,s)]\end{aligned}$$

Hence, given $a \in [0,1]^X$ and $y \in X$

$$\begin{aligned}W^{M\wedge N}_{\varepsilon\wedge\delta,t\wedge s}(a)(y) &= \bigvee_{x\in X} a(x) * ((1-\varepsilon\wedge\delta) \to (M\wedge N)(x,y,t\wedge s))\\ &\le \left[\bigvee_{x\in X} a(x) * ((1-\varepsilon) \to M(x,y,t))\right] \wedge \left[\bigvee_{x\in X} a(x) * ((1-\delta) \to N(x,y,s))\right]\\ &= W^M_{\varepsilon,t}(a)(y) \wedge W^N_{\delta,s}(a)(y).\end{aligned}$$

It is obvious that $\mathfrak{U}_{\mathcal{U}}$ is stratified since $\alpha * W^M_{\varepsilon,t}(1_{\{x\}}) = W^M_{\varepsilon,t}(\alpha * 1_{\{x\}})$ for every $\alpha \in [0,1]$, $(M,*) \in \mathcal{M}_{\mathcal{U}}$, $t > 0$, $\varepsilon \in I_0$.

(2) This is straightforward.

(3) This is immediate since $\mathfrak{U}^{\omega}_{\mathcal{U}}$ is the Hutton [0,1]-quasi-uniformity associated with the Lowen uniformity $\omega(\mathcal{U})$ [28] by means of Höhle construction [18] (see also [41]).

□

Example 4.8. *Let us consider the real line* $\mathbb{R}$ *and let* $\mathcal{U}_e$ *be the uniformity induced by the Euclidean metric* e *on* $\mathbb{R}$. *Considering the product t-norm* $\cdot$, *we can construct the fuzzy uniform structure* $(\delta.(\mathcal{U}_e),\cdot)$ *having as base the family* $\{(M_d,\cdot) : d \in \mathcal{D}_{\mathcal{U}_e}\}$ *where* $\mathcal{D}_{\mathcal{U}_e}$ *is the uniform structure of* $\mathcal{U}_e$. *For example, given* $t \in \mathbb{R}$, *the fuzzy metric* $(M_{d_t},\cdot)$ *belongs to* $\delta.(\mathcal{U}_e)$ *where* $d_t(x,y) = t|x-y|$ *for all* $x,y \in \mathbb{R}$.

(1) *The Hutton [0,1]-quasi-uniformity* $\mathfrak{U}_{\mathcal{U}_e}$ *has as a base* $\{W^d_{\varepsilon,t} : \varepsilon \in I_0,\ t > 0, d \in \mathcal{D}_{\mathcal{U}_e}\}$ *where, for all* $\alpha \in I_0$, $x,y \in \mathbb{R}$,

$$\begin{aligned}W^d_{\varepsilon,t}(\alpha\cdot 1_{\{x\}})(y) &= \bigvee_{z\in\mathbb{R}} \alpha\cdot 1_{\{x\}}(z)\cdot((1-\varepsilon) \to M_d(z,y,t))\\ &= \alpha((1-\varepsilon) \to M_d(x,y,t)) = \begin{cases} \alpha & \text{if } 1-\varepsilon \le \frac{t}{t+d(x,y)}\\ \frac{\alpha t}{(1-\varepsilon)(t+d(x,y))} & \text{if } 1-\varepsilon > \frac{t}{t+d(x,y)}\end{cases}\end{aligned}$$

(2) *The Hutton [0,1]-quasi-uniformity* $\mathfrak{U}^H_{\mathcal{U}_e}$ *has as a base* $\{W^d_t : t > 0, d \in \mathcal{U}_e\}$ *where, for all* $\alpha \in I_0$, $x,y \in \mathbb{R}$,

$$W^d_t(\alpha\cdot 1_{\{x\}})(y) = \bigvee_{z\in X} \alpha\cdot 1_{\{x\}}(z)\cdot M_d(z,y,t) = \frac{\alpha t}{t+d(x,y)}.$$

(3) *The Hutton [0,1]-quasi-uniformity* $\mathfrak{U}^{\omega}_{\mathcal{U}_e}$ *has as a base the family* $\{W_{\mathsf{U}} : \mathsf{U} \in [0,1]^{\mathbb{R}\times\mathbb{R}}$ *and* $\mathsf{U}^{-1}(\varepsilon,1] \in \mathcal{U}_e$ *for all* $\varepsilon \in [0,1[\}$. *We can construct some elements of this family. For example, if* $d \in \mathcal{D}_{\mathcal{U}_e}$ *is bounded by 1, then it is obvious that* $\mathsf{U}_d^{-1}(\varepsilon,1] \in \mathcal{U}_e$ *for all* $\varepsilon \in [0,1[$, *where* $\mathsf{U}_d(x,y) = 1-d(x,y)$. *In this case, for all* $\alpha \in I_0$, $x,y \in \mathbb{R}$, *we have,*

$$W_{\mathsf{U}_d}(\alpha\cdot 1_{\{x\}})(y) = \bigvee_{z\in X} \alpha\cdot 1_{\{x\}}(z)\cdot \mathsf{U}_d(z,y) = \alpha(1-d(x,y)).$$

On the other hand, if $U \in \mathcal{U}_e$ *then trivially* $1_U^{-1}(\varepsilon,1] \in \mathcal{U}_e$ *for all* $\varepsilon \in [0,1[$. *Now, for all* $\alpha \in I_0$, $x,y \in \mathbb{R}$, *we obtain that,*

$$W_{1_U}(\alpha\cdot 1_{\{x\}})(y) = \bigvee_{z\in X} \alpha\cdot 1_{\{x\}}(z)\cdot 1_U(z,y) = \alpha\cdot 1_U(x,y),$$

Remark 4.9. *It is not difficult to see (cf. [15]) that*

$$\mathfrak{U}_{\mathcal{U}} \subseteq \mathfrak{U}_{\mathcal{U}}^{H} \wedge \mathfrak{U}_{\mathcal{U}}^{K} \subseteq \mathfrak{U}_{\mathcal{U}}^{H} \vee \mathfrak{U}_{\mathcal{U}}^{K} \subseteq \mathfrak{U}_{\mathcal{U}}^{\omega}.$$

5 Completeness of Hutton [0,1]-quasi-uniformities

In [14] it was introduced a concept of completeness for Hutton $[0,1]$-quasi-uniformities by means of a kind of $[0,1]$-filters. The authors proved the equivalence between completeness of any fuzzy metric space $(X, M, *)$ and completeness of the induced Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}_M$ as defined in [13] (see Proposition 4.6).

In the previous section we have considered several different Hutton $[0,1]$-quasi-uniformities induced by a uniformity. This section is devoted to study whether the result of [14] is also true in this context, i. e. whether completeness of a uniformity is transferred to the different Hutton [0,1]-quasi-uniformities induced by it.

First, we begin with some preliminaries in order to recall the completeness notion of a Hutton $[0,1]$-quasi-uniformity considered in [14]. Although this theory can be developed for an arbitrary strictly two-sided commutative quantale we restrict ourselves to $[0,1]$.

Definition 5.1 ([22, Section 5.1])**.** *A* $[0,1]$-topology *on a nonempty set X is a subset $\mathfrak{T}$ of* $[0,1]^X$ *such that:*

(1) $1_{\varnothing}, 1_X \in \mathfrak{T}$*;*

(2) $a \wedge b \in \mathfrak{T}$ *whenever* $a, b \in \mathfrak{T}$;

(3) $\bigvee_{i\in\Lambda} a_i \in \mathfrak{T}$ *whenever* $\{a_i : i \in \Lambda\} \subseteq \mathfrak{T}$.

In this case, the pair $(X, \mathfrak{T})$ is called a $[0,1]$-topological space.

Furthermore, if $\alpha * 1_X \in \mathfrak{T}$ for all $\alpha \in [0,1]$, then $\mathfrak{T}$ is called a weakly stratified $[0,1]$-topology.

Moreover, if $\alpha * a \in \mathfrak{T}$ whenever $\alpha \in [0,1]$ and $a \in \mathfrak{T}$, then $\mathfrak{T}$ is said to be stratified.

A function $f\colon (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ between two $[0,1]$-topological spaces is continuous *if $f^{-1}(a) \in \mathfrak{T}_X$ for each $a \in \mathfrak{T}_Y$.*

The category of (weakly) stratified $[0,1]$-topological spaces is a coreflective subcategory of the category of $[0,1]$-topological spaces.

Definition 5.2 ([22, Section 6.1])**.** *Let $\mathcal{N} : X \to [0,1]^{([0,1]^X)}$ be a map and denote for each $x \in X$ the image of x under $\mathcal{N}$ by μ_x. $\mathcal{N}$ is said to be a* $[0,1]$-neighborhood system *on X if for all $x \in X$ and for all $a, a_1, a_2 \in [0,1]^X$, μ_x satisfies:*

$(NS1)$ $\mu_x(1_X) = 1$;

$(NS2)$ *if* $a_1 \leq a_2 \in [0,1]^X$, *then* $\mu_x(a_1) \leq \mu_x(a_2)$;

$(NS3)$ $\mu_x(a_1) \wedge \mu_x(a_2) \leq \mu_x(a_1 \wedge a_2)$;

$(NS4)$ $\mu_x(a) \leq a(x)$;

$(NS5)$ $\mu_x(a) \leq \bigvee\{\mu_x(b) : b(y) \leq \mu_y(a) \forall y \in X\}$.

A $[0,1]$-neighborhood space *is a nonempty set X endowed with a $[0,1]$-neighborhood system.*

It is proved in [22] that $[0,1]$-topologies and $[0,1]$-neighborhood systems are equivalent concepts. In fact, if $\mathfrak{T}$ is a $[0,1]$-topology on X, then it possesses a $[0,1]$-neighborhood system $\mathcal{N}_{\mathfrak{T}} : X \to [0,1]^{([0,1]^X)}$ such that for all $x \in X, a \in [0,1]^X$ it is given by

$$\mathcal{N}_{\mathfrak{T}}(x)(a) = (\bigvee\{b \in \mathfrak{T} : b \leq a\})(x).$$

We denote by $\mu_x^{\mathfrak{T}}$ the evaluation of $\mathcal{N}_{\mathfrak{T}}$ at x (the superscript $\mathfrak{T}$ will be omitted if no confusion arises).

Conversely, if $\mathcal{N}$ is a neighborhood system on X, we can define the following $[0,1]$-topology on X:

$$\mathfrak{T}_{\mathcal{N}} = \{a \in [0,1]^X : a(x) \leq \mu_x(a) \text{ for all } x \in X\}.$$

On the other hand, if $\mathfrak{U}$ is a Hutton $[0,1]$-quasi-uniformity, then it generates a weakly stratified $[0,1]$-topology given by

$$\mathfrak{T}(\mathfrak{U}) = \big\{a \in [0,1]^X : a \leq \bigvee\{b \in [0,1]^X : \text{ there exists } U \in \mathfrak{U} \text{ such that } U(b) \leq a\}\big\}.$$

Observe that if $\mathfrak{U}, \mathfrak{W}$ are two Hutton $[0,1]$-quasi-uniformities on X such that $\mathfrak{U} \subseteq \mathfrak{W}$, then $\mathfrak{T}(\mathfrak{U}) \subseteq \mathfrak{T}(\mathfrak{W})$.

Definition 5.3 ([22, Definition 6.1.4]). *Let X be a nonempty set. A map $\nu : [0,1]^X \to [0,1]$ is called a* $[0,1]$-filter *if for all $a_1, a_2 \in [0,1]^X$, ν verifies the following properties:*
$(LF1)$ $\nu(1_X) = 1$;
$(LF2)$ *if* $a_1 \leq a_2$, *then* $\nu(a_1) \leq \nu(a_2)$;
$(LF3)$ $\nu(a_1) \wedge \nu(a_2) \leq \nu(a_1 \wedge a_2)$;
$(LF4)$ $\nu(1_\varnothing) = 0$;
An $[0,1]$-filter ν is stratified *if for all $\alpha \in [0,1], a \in [0,1]^X$,*
$(LF5)$ $\alpha * \nu(a) \leq \nu(\alpha * a)$.
A $[0,1]$-filter ν is tight *if for all $\alpha \in [0,1]$,*
$(LF6)$ $\nu(\alpha * 1_X) = \alpha$.

Example 5.4. [12] *Let X be a nonempty set.*

- *The map $\nu : [0,1]^X \to [0,1]$ given by $\nu(a) = \inf_{x \in X} a(x)$ is a stratified tight $[0,1]$-filter on X;*
- *fixing $x_0 \in X$, the map $\nu_{x_0} : [0,1]^X \to [0,1]$ given by $\nu_{x_0}(a) = a(x_0)$ is a stratified tight $[0,1]$-filter on X.*

Remark 5.5 ([21, Subsection 4.4]). *If $\mathcal{F}$ is a filter on a nonempty set X, then it induces a stratified and tight $[0,1]$-filter $\nu_\mathcal{F}$ on X for every $a \in [0,1]^X$, it is given by $\nu_\mathcal{F}(a) = \bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} a(x)$.*

Furthermore, it is obvious that $\varnothing \neq a^{-1}(\varepsilon, 1] \in \mathcal{F}$ for all $\varepsilon \in I_1$ if and only if $\nu_\mathcal{F}(a) = \bigvee_{x \in X} a(x)$. Hence, $a^{-1}(\varepsilon, 1] \in \mathcal{F}$ for all $\varepsilon \in I_1$ if and only if $\nu_\mathcal{F}(a) = 1$.

Furthermore, if $(x_n)_{n \in \mathbb{N}}$ is a sequence on X, then we can define a stratified and tight $[0,1]$-filter on X as follows:

$$\nu_{x_n}(a) = \liminf_{n \in \mathbb{N}} a(x_n).$$

Remark 5.6. *Observe that if ν is a tight $[0,1]$-filter on X and $\nu(a) = 1$, then $a^{-1}(\varepsilon, 1] \neq \varnothing$ for all $\varepsilon \in I_1$. In fact, if $a^{-1}(\varepsilon, 1] = \varnothing$ for some $\varepsilon \in I_1$, then $a \leq \varepsilon * 1_X$ so $\nu(a) \leq \nu(\varepsilon * 1_X) = \varepsilon$ which is a contradiction.*

Definition 5.7 ([22, Definition 6.4.1]). *Let $(X, \mathfrak{T})$ be a $[0,1]$-topological space and $\{\mu_x\}_{x \in X}$ be the corresponding $[0,1]$-neighborhood system. Let ν be a $[0,1]$-filter on X. A point $x \in X$ is said to be a* limit point *of ν if $\mu_x(a) \leq \nu(a)$ for all $a \in [0,1]^X$. In this case we say that ν is a convergent $[0,1]$-filter in $(X, \mathfrak{T})$.*

Notice that if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two $[0,1]$-topologies on X such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$, then $\mu_x^{\mathfrak{T}_1} \leq \mu_x^{\mathfrak{T}_2}$ for every $x \in X$. Hence if a $[0,1]$-filter ν converges to $x \in X$ in $(X, \mathfrak{T}_2)$, then it also converges to x in $(X, \mathfrak{T}_1)$.

Convergence of $[0,1]$-filters in Hutton $[0,1]$-quasi-uniform spaces admits the following characterization:

Proposition 5.8. [14] *Let $(X, \mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space, ν a $[0,1]$-filter and $x \in X$. Then ν is convergent to x in $(X, \mathfrak{T}(\mathfrak{U}))$ if and only if $\alpha \leq \nu(U(\alpha * 1_{\{x\}}))$ for all $U \in \mathfrak{U}$ and all $\alpha \in [0,1]$.*

Furthermore, if $\mathfrak{U}$ is stratified then ν is convergent to x in $(X, \mathfrak{T}(\mathfrak{U}))$ if and only if $\nu(U(1_{\{x\}})) = 1$ for all $U \in \mathfrak{U}$.

Definition 5.9. [14] *Let $(X, \mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space. A $[0,1]$-filter ν is a* Cauchy $[0,1]$-filter *in $(X, \mathfrak{U})$ if*

$$\text{for all } U \in \mathfrak{U}, \alpha \in (0,1], \text{ there exists } x_U \in X \text{ such that } \alpha \leq \nu(U(\alpha * 1_{\{x_U\}})).$$

Moreover, if $\mathfrak{U}$ is stratified, then ν is a Cauchy $[0,1]$-filter if and only if for all $U \in \mathfrak{U}$ there exists $x_U \in X$ such that $\nu(U(1_{\{x_U\}})) = 1$.

Definition 5.10. [14] *A Hutton $[0,1]$-quasi-uniform space $(X, \mathfrak{U})$ is said to be* complete *if any stratified and tight Cauchy $[0,1]$-filter on X is convergent.*

We turn now to the problem addressed in this section: the relationship between the completeness of a uniformity and the completeness of the four Hutton $[0,1]$-quasi-uniformities that we have considered in the previous section.

Proposition 5.11 (cf. [14, Proposition 23]). *Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{F}$ be a filter on X. The following are equivalent:*

(1) $\mathcal{F}$ is convergent in $(X, \tau(\mathcal{U}))$;

(2) $\nu_\mathcal{F}$ is convergent in $(X, \mathfrak{T}(\mathfrak{U}_\mathcal{U}^\omega))$;

(3) $\nu_\mathcal{F}$ is convergent in $(X, \mathfrak{T}(\mathfrak{U}_\mathcal{U}^K))$;

(4) $\nu_{\mathcal{F}}$ *is convergent in* $(X, \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^H))$;

(5) $\nu_{\mathcal{F}}$ *is convergent in* $(X, \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}))$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathcal{F}$ is a filter convergent to an element $x_0 \in X$ with respect to $\tau(\mathcal{U})$. Let $\mathsf{U} \in [0,1]^{X\times X}$ such that $\mathsf{U}^{-1}(\varepsilon, 1] \in \mathcal{U}$ for all $\varepsilon \in [0,1)$. Since $\mathcal{F}$ is convergent to x_0 then $U^{-1}(\varepsilon,1](x_0) \in \mathcal{F}$ so given $\varepsilon \in [0,1)$ we have that

$$\nu_{\mathcal{F}}(W_{\mathsf{U}}(1_{\{x_0\}})) = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} W_{\mathsf{U}}(1_{\{x_0\}})(x) \geq \bigwedge_{x\in U^{-1}(\varepsilon,1](x_0)} W_{\mathsf{U}}(1_{\{x_0\}})(x) = \bigwedge_{x\in U^{-1}(\varepsilon,1](x_0)} \mathsf{U}(x_0,x) \geq \varepsilon.$$

Since ε is arbitrary then $\nu_{\mathcal{F}}(W_{\mathsf{U}}(1_{\{x_0\}})) = 1$ so $\nu_{\mathcal{F}}$ is convergent to x_0 in $(X, \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^\omega))$ by Proposition 5.8.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5) are obvious by Remark 4.9.

(5) $\Rightarrow$ (1) Let $\delta_*(\mathcal{U})$ be the fuzzy uniform structure associated with $\mathcal{U}$ (Theorem 3.15). If $(M,*) \in \delta_*(\mathcal{U})$, then $\mathfrak{U}_M \subseteq \mathfrak{U}_{\mathcal{U}}$ so $\mathfrak{T}(\mathfrak{U}_M) \leq \mathfrak{T}(\mathfrak{U}_{\mathcal{U}})$. Hence if $\nu_{\mathcal{F}}$ is convergent in $(X, \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}))$ to x_0, then it is also convergent to x_0 in $(X, \mathfrak{T}(\mathfrak{U}_M))$ for all $(M,*) \in \delta_*(\mathcal{U})$. By [14, Proposition 23] $\mathcal{F}$ is convergent to x_0 with respect to $\tau(M)$ for all $(M,*) \in \delta_*(\mathcal{U})$. Hence $\mathcal{F}$ is convergent to x_0 in $(X, \tau(\mathcal{U}))$ (see Remark 3.16). □

Proposition 5.12 (cf. [14, Proposition 23])**.** *Let* $\mathcal{F}$ *be a filter on a uniform space* $(X,\mathcal{U})$ *and let* $*$ *be a continuous t-norm. The following are equivalent:*

(1) $\mathcal{F}$ *is a Cauchy filter in* $(X,\mathcal{U})$;

(2) $\nu_{\mathcal{F}}$ *is a Cauchy* $[0,1]$*-filter in* $(X, \mathfrak{U}_{\mathcal{U}}^K)$;

(3) $\nu_{\mathcal{F}}$ *is a Cauchy* $[0,1]$*-filter in* $(X, \mathfrak{U}_{\mathcal{U}})$.

Proof. (1) $\Rightarrow$ (2) Suppose $\mathcal{F}$ is a Cauchy filter in $(X,\mathcal{U})$. Given $U \in \mathcal{U}$ we can find $x_U \in X$ such that $U(x_U) \in \mathcal{F}$. Then:

$$\nu_{\mathcal{F}}(W_{1_U}(1_{\{x_U\}})) = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} W_{1_U}(1_{\{x_U\}})(x) \geq \bigwedge_{x\in U(x_U)} W_{1_U}(1_{\{x_U\}})(x) = \bigwedge_{x\in U(x_U)} \bigvee_{y\in X} 1_{\{x_U\}}(y) * 1_U(y,x) = \bigwedge_{x\in U(x_U)} 1_U(x_U,x) = 1$$

so $\nu_{\mathcal{F}}$ is a Cauchy $[0,1]$-filter in $(X, \mathfrak{U}_{\mathcal{U}}^K)$.

(2) $\Rightarrow$ (3) This is obvious by Remark 4.9.

(3) $\Rightarrow$ (1) Let $(\delta_*(\mathcal{U}), *)$ be the fuzzy uniform structure associated with $\mathcal{U}$. Then $\mathfrak{U}_M \subseteq \mathfrak{U}_{\mathcal{U}}$ for every $(M,*) \in \delta_*(\mathcal{U})$. Hence, $\nu_{\mathcal{F}}$ is a Cauchy $[0,1]$-filter in $(X, \mathfrak{U}_M)$ for every $(M,*) \in \delta_*(\mathcal{U})$. By [14, Proposition 23] $\mathcal{F}$ is a Cauchy filter in $(X, M, *)$ so in $(X, \mathcal{U}_M)$ for every $(M,*) \in \delta_*(\mathcal{U})$. Then by Remark 3.20 $\mathcal{F}$ is a Cauchy filter in $(X,\mathcal{U})$. □

Notice that if $\nu_{\mathcal{F}}$ is a Cauchy $[0,1]$-filter in $(X, \mathfrak{U}_{\mathcal{U}}^\omega)$ or in $(X, \mathfrak{U}_{\mathcal{U}}^H)$, then, by the above result, $\mathcal{F}$ is a Cauchy filter in $(X,\mathcal{U})$. Nevertheless the converse is not true as the next example shows.

Example 5.13. *Let us consider on* $X = (0,1)$ *the Euclidean metric* e *and its associated uniformity* $\mathcal{U}_e$. *Let* $(\delta_*(\mathcal{U}_e), *_L)$ *be the fuzzy uniform structure induced by* $\mathcal{U}_e$. *Then it is easy to see that the pair* $(M, *_L)$ *is a fuzzy metric on* X *where, for every* $x, y \in X$, $M(x,y,t) = 1 - e(x,y)$ *if* $t > 0$ *meanwhile* $M(x,y,0) = 0$. *Furthermore,* $\mathcal{U}_M = \mathcal{U}_e$ *so* $(M,*) \in \delta_*(\mathcal{U}_e)$. *On the other hand the sequence* $(\frac{1}{2n})_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $(X,\mathcal{U}_e)$. *Let* $\mathcal{F}$ *be its associated filter having as base the collection* $F_n = \{\frac{1}{2k} : k \geq n\}$ *for all* $n \in \mathbb{N}$. *We show that* $\nu_{\mathcal{F}}$ *is not a Cauchy* $[0,1]$*-filter in* $(X, \mathfrak{U}_{\mathcal{U}_e}^H)$ *(so in* $(X, \mathfrak{U}_{\mathcal{U}_e}^\omega)$*). Fix* $t > 0$. *Then given* $p \in X$ *we have*

$$\nu_{\mathcal{F}}(W_t^M(1_{\{p\}})) = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} W_t^M(1_{\{p\}})(x) = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} M(p,x,t) = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} (1 - e(p,x))$$
$$\leq \bigvee_{n\in\mathbb{N}} \bigwedge_{x\in F_n} (1 - e(p,x)) \leq \bigvee_{n\in\mathbb{N}, p\neq\frac{1}{2n}} (1 - e(p, \tfrac{1}{2n})) < 1.$$

Then $\nu_{\mathcal{F}}$ *is not a Cauchy* $[0,1]$*-filter in* $(X, \mathfrak{U}_{\mathcal{U}_e}^H)$.

Theorem 5.14 (cf. [14, Theorem 24])**.** *Let* $(X,\mathcal{U})$ *be a uniform space and let* $*$ *be a continuous t-norm. The following statements are equivalent:*

(1) $(X,\mathcal{U})$ *is complete;*

(2) $(X, \mathfrak{U}_{\mathcal{U}})$ *is complete;*

(3) $(X, \mathfrak{U}_{\mathcal{U}}^K)$ is complete.

Proof. (1) $\Rightarrow$ (2) The proof of this implication follows the same ideas of the proof of [14, Theorem 24]. Nevertheless, since the first part requires a different construction we include it here.

Let ν be a stratified and tight Cauchy $[0,1]$-filter in $(X, \mathfrak{U}_{\mathcal{U}})$. Let us consider the set $\delta_*(\mathcal{U}) \times \mathbb{N}$ directed by the partial order $\sqsubseteq$ given by $(M,n) \sqsubseteq (N,m)$ whenever $N(x,y,t) \leq M(x,y,t)$ for all $x,y \in X$, $t > 0$ and $n \leq m$. Then for each $(M,n) \in \delta_*(\mathcal{U}) \times \mathbb{N}$ we can find $x_{(M,n)} \in X$ such that $\nu(W^M_{\frac{1}{n},\frac{1}{n}}(1_{\{x_{(M,n)}\}}) = 1$. Let us check that the net $(x_{(M,n)})_{(M,n)\in\mathcal{B}\times\mathbb{N}}$ is a Cauchy net on $(X, \delta_*(\mathcal{U}), *)$. Fix $(F,*) \in \delta_*(\mathcal{U})$, $\varepsilon \in I_1$ and $t > 0$. Let $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{t}{2}$ and $\left(1-\frac{1}{n_0}\right) * \left(1-\frac{1}{n_0}\right) * \left(1-\frac{1}{n_0}\right) * \left(1-\frac{1}{n_0}\right) > \varepsilon$. Given $(M,n),(N,m) \in \delta_*(\mathcal{U}) \times \mathbb{N}$ then $\nu(W^M_{\frac{1}{n},\frac{1}{n}}(1_{\{x_{(M,n)}\}}) \wedge W^N_{\frac{1}{m},\frac{1}{m}}(1_{\{x_{(N,m)}\}})) = 1$. Hence, by Remark 5.6 we can find $x_{(M,n),(N,m)} \in X$ such that

$$W^M_{\frac{1}{n},\frac{1}{n}}(1_{\{x_{(M,n)}\}})(x_{(M,n),(N,m)}) \wedge W^N_{\frac{1}{m},\frac{1}{m}}(1_{\{x_{(N,m)}\}})(x_{(M,n),(N,m)}) > 1 - \frac{1}{n_0},$$

i. e.

$$\left(1-\frac{1}{n}\right) \to M(x_{(M,n)}, x_{(M,n),(N,m)}, \tfrac{1}{n})) > 1 - \frac{1}{n_0},$$

$$\left(1-\frac{1}{n}\right) \to N(x_{(N,m)}, x_{(M,n),(N,m)}, \tfrac{1}{m}) > 1 - \frac{1}{n_0}.$$

If $(M,n),(N,m) \sqsupseteq (F,n_0)$, then

$$\begin{aligned} F(x_{(M,n)}, x_{(N,m)}, t) &\geq F(x_{(M,n)}, x_{(M,n),(N,m)}, \tfrac{t}{2}) * F(x_{(N,m)}, x_{(M,n),(N,m)}, \tfrac{t}{2}) \\ &\geq M(x_{(M,n)}, x_{(M,n),(N,m)}, \tfrac{1}{n}) * N(x_{(N,m)}, x_{(M,n),(N,m)}, \tfrac{1}{m}) \\ &\geq \left(1-\frac{1}{n}\right) * \left(1-\frac{1}{n_0}\right) * \left(1-\frac{1}{n}\right) * \left(1-\frac{1}{n_0}\right) > \varepsilon. \end{aligned}$$

Therefore, $(x_{(M,n)})_{(M,n)\in\mathcal{B}\times\mathbb{N}}$ is a Cauchy net in $(X, \delta_*(\mathcal{U}), *)$ so it has a limit point x with respect to $\tau(\mathcal{U})$. To prove x is also a limit point of ν in $\mathfrak{T}(\mathfrak{U}_{\mathcal{U}})$ we can follow the proof of [14, Theorem 24] because it only needs a slight modification.

(2)$\Rightarrow$ (1) This follows from Propositions 5.11 and 5.12.

(1)$\Rightarrow$ (3) If ν is a stratified and tight Cauchy $[0,1]$-filter in $(X, \mathfrak{U}_{\mathcal{U}}^K)$, then for each $U \in \mathcal{U}$ we can find $x_U \in X$ such that $\nu(W_{1_U}(1_{\{x_U\}})) = 1$. Then by (LF3) we have that $\nu(W_{1_U}(1_{\{x_U\}}) \wedge W_{1_V}(1_{\{x_V\}})) = 1$ for all $U,V \in \mathcal{U}$. Since ν is tight then $(W_{1_U}(1_{\{x_U\}}) \wedge W_{1_V}(1_{\{x_V\}}))^{-1}(t,1] \neq \varnothing$ for all $t \in I_1$ so we can find $y_{UV} \in X$ such that $W_{1_U}(1_{\{x_U\}})(y_{UV}) = W_{1_V}(1_{\{x_V\}})(y_{UV}) = 1$, i. e. $y_{UV} \in U(x_U) \cap V(x_V)$.

Let $U,V \in \mathcal{U}$ such that $V^2 \subseteq U$. Given $W_1, W_2 \in \mathcal{U}$ with $W_1 \cup W_2 \subseteq V$ then $(x_{W_1}, y_{W_1W_2}) \in W_1 \subseteq V$ and $(x_{W_2}, y_{W_1W_2}) \in W_2 \subseteq V$ so $(x_U, x_V) \in V^2 \subseteq U$. Hence $(x_U)_{U\in\mathcal{U}}$ is a Cauchy sequence in $(X,\mathcal{U})$ and convergents to some point $x \in X$. Let us check that x is a limit point of ν in $\mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^K)$. Given $U_0 \in \mathcal{U}$ we consider as above $V_0 \in \mathcal{U}$ such that $V_0^2 \subseteq U_0$. Furthermore, we can find $W_0 \in \mathcal{U}$ with $W_0 \subseteq V_0$ such that $x_U \in V_0(x)$ for all $U \subseteq W_0$. Then $W_{1_{W_0}}(1_{\{x_{W_0}\}}) \leq W_{1_U}(1_{\{x\}})$. In fact, we have that

$$W_{1_U}(1_{\{x\}})(y) = \begin{cases} 1 & \text{if } (x,y) \in U \\ 0 & \text{otherwise} \end{cases} \text{ and } W_{1_{W_0}}(1_{\{x_{W_0}\}})(y) = \begin{cases} 1 & \text{if } (x_{W_0},y) \in W_0 \\ 0 & \text{otherwise} \end{cases}.$$

Since $(x, x_{W_0}) \in V_0$, $W_0 \subseteq V_0$ and $V_0^2 \subseteq U_0$ we deduce that $(x,y) \in U$ whenever $(x_{W_0}, y) \in W_0$ which proves the assertion.

Since $\nu(W_{1_{W_0}}(1_{\{x_{W_0}\}})) = 1$ then $\nu(W_{1_U}(1_{\{x\}})) = 1$ so ν is convergent to x in $(X, \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^K))$.

(3)$\Rightarrow$ (1) This follows from Propositions 5.11 and 5.12. □

Remark 5.15. *Notice that in general $\mathfrak{T}(\mathfrak{U}_{\mathcal{U}}) \neq \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^K)$ [15, Example 6.14]. Hence, implication (3) $\Rightarrow$ (2) of the above theorem is not obvious.*

Remark 5.16. *Let $(X,\mathcal{U})$ be a uniform space. Taking into account [15, Proposition 6.13] and the construction of $\mathfrak{U}_{\mathcal{U}}^{\omega}$ and $\mathfrak{U}_{\mathcal{U}}^K$ given in Proposition 4.7, we easily deduce that $\mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^{\omega}) = \mathfrak{T}(\mathfrak{U}_{\mathcal{U}}^K)$.*

Hence, and using the previous theorem, we obtain that completeness of $(X, \mathfrak{U}_{\mathcal{U}}^{\omega})$ implies completeness of $(X,\mathcal{U})$. However, we don't know whether the converse is true.

6 Completion of a Hutton quasi-uniformity induced by a uniformity

In [14, Theorem 27] it is constructed a completion for the Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}_M$ (see Proposition 4.6) of a fuzzy metric space $(X,M,*)$. Nevertheless, this construction makes use of a result of [13] which is not true (see [15]) so the problem of constructing a completion for a Hutton $[0,1]$-quasi-uniformity induced by a fuzzy metric space remains open. Here, we provide such a construction but in the more general context of a uniform space $(X,\mathcal{U})$ and considering the Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}_{\mathcal{U}}^K$ instead of $\mathfrak{U}_{\mathcal{U}}$, which is a uniform version of $\mathfrak{U}_M$. To achieve this, we will make use of the concept of density for a Hutton [0,1]-quasi-uniformity introduced in [14] adapted here to an arbitrary Hutton $[0,1]$-quasi-uniform space.

Definition 6.1 (cf. [14])**.** *A subset Z of a Hutton $[0,1]$-quasi-uniform space $(X,\mathfrak{U})$ is called* dense *if for each $x\in X,\alpha\in[0,1]$ and $U\in\mathfrak{U}$ there is $z\in Z$ such that $\alpha*1_{\{z\}}\leq U(\alpha*1_{\{x\}})$.*

Remark 6.2. *Notice that in the above definition we can replace $\mathfrak{U}$ by one of its bases.*

Moreover, if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are two Hutton $[0,1]$-quasi-uniformities on X and $\mathfrak{U}_1\subseteq\mathfrak{U}_2$, then if Z is dense in $(X,\mathfrak{U}_2)$ so is in $(X,\mathfrak{U}_1)$.

Proposition 6.3. *Let $(X,\mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space. Then $Z\subseteq X$ is dense in $(X,\mathfrak{U})$ if given $a\in[0,1]^X$, $U\in\mathfrak{U}$ and $x\in X$ we can find $z\in Z$ such that $a(x)\leq U(a)(z)$.*

Proof. Let $a\in[0,1]^X$. Fix $x_0\in X$ and $U\in\mathfrak{U}$. Let $\alpha=a(x_0)$. By assumption there exists $z\in Z$ such that $\alpha*1_{\{z\}}\leq U(\alpha*1_{\{x_0\}})$, i. e. $\alpha\leq U(\alpha*1_{\{x_0\}})(z)$. Then

$$U(a)(z)=U(\bigvee_{x\in X}a(x)*1_{\{x\}})(z)=\bigvee_{x\in X}U(a(x)*1_{\{x\}})(z)\geq U(a(x_0)*1_{\{x_0\}})(z)=U(\alpha*1_{\{x_0\}})(z)\geq\alpha=a(x_0).$$

Conversely, let $x\in X$, $\alpha\in[0,1]$ and $U\in\mathfrak{U}$. By assumption there exists $z\in Z$ with $(\alpha*1_{\{x\}})(x)\leq U(\alpha*1_{\{x\}})(z)$, i. e. $\alpha\leq U(\alpha*1_{\{x\}})(z)$ so $\alpha*1_{\{z\}}\leq U(\alpha*1_{\{x\}})$. □

Proposition 6.4. *Let $(X,\mathcal{U})$ be a uniform space and $Z\subseteq X$. The following statements are equivalent:*

(1) Z is dense in $\tau(\mathcal{U})$;

(2) Z is dense in $(X,\mathfrak{U}_{\mathcal{U}}^K)$;

(3) Z is dense in $(X,\mathfrak{U}_{\mathcal{U}})$.

Proof. (1)⇒(2) Let $\alpha\in I_0$, $U\in\mathcal{U}$ and $x\in X$. By assumption there exists $z\in Z$ such that $(x,z)\in U$. Hence $W_{1_U}(\alpha*1_{\{x\}})(z)=\alpha*1_U(x,z)=\alpha$ so $\alpha*1_{\{z\}}\leq W_{1_U}(\alpha*1_{\{x\}})$.

(2)⇒(3) Obvious since $\mathfrak{U}_{\mathcal{U}}\subseteq\mathfrak{U}_{\mathcal{U}}^K$.

(3)⇒(1) Let $U\in\mathcal{U}$. Then we can find $(M,*)\in\delta_*(\mathcal{U})$, $\varepsilon\in I_1$ and $t>0$ such that $\{(x,y)\in X\times X:M(x,y,t)>\varepsilon\}\subseteq U$. Let $\alpha\in I_1$ such that $\varepsilon<\alpha*\alpha$. Given $x\in X$ we can find by assumption $z\in Z$ such that $\alpha*1_{\{z\}}\leq W_{1-\alpha,t}^M(\alpha*1_{\{x\}})$. So $\alpha\leq W_{1-\alpha,t}^M(\alpha*1_{\{x\}})(z)=\alpha*(\alpha\to M(x,z,t))$. Hence $\alpha\leq\alpha\to M(x,z,t)$ so that $\varepsilon<\alpha*\alpha\leq M(x,z,t)$, i. e. $z\in U(x)$. Consequently, Z is dense in $\tau(\mathcal{U})$. □

We observe that the above lemma is not true when we use the Hutton $[0,1]$-quasi-uniformities $\mathfrak{U}_{\mathcal{U}}^H$ and $\mathfrak{U}_{\mathcal{U}}^\omega$. See the next example.

Example 6.5. *Let us consider the real line $\mathbb{R}$ endowed with the uniformity $\mathcal{U}_e$ associated with the Euclidean metric e. It is well-known that $\mathbb{Q}$ is dense in $\tau(\mathcal{U}_e)$. Nevertheless $\mathbb{Q}$ is not dense in $(X,\mathfrak{U}_{\mathcal{U}_e}^H)$ (so it is not dense in $(X,\mathfrak{U}_{\mathcal{U}_e}^\omega)$). Indeed, since $(M_e,\cdot)\in\delta.(\mathcal{U}_e)$, given $x_0\in\mathbb{R}\backslash\mathbb{Q}$, $\alpha\in I_0$ and $t>0$ then $\alpha>W_t^{M_e}(\alpha\cdot 1_{\{x_0\}})(z)=\alpha\cdot M_e(x_0,z,t)$ for all $z\in\mathbb{Q}$ since $M_e(x_0,z,t)<1$. Hence $\alpha*1_{\{z\}}\not\leq W_t^{M_e}(\alpha*1_{\{x\}})$ for all $z\in\mathbb{Q}$.*

Theorem 6.6 (cf. [14, Theorem 27])**.** *Let $(X,\mathcal{U})$ be a uniform space. Then there is a unique (up to uniform isomorphism) complete uniform space $(Y,\mathcal{V})$ having a subset Z dense in the complete Hutton [0,1]-quasi-uniform space $(Y,\mathfrak{U}_{\mathcal{V}}^K)$ and such that the Hutton [0,1]-quasi-uniform spaces $(X,\mathfrak{U}_{\mathcal{U}}^K)$ and $(Z,\mathfrak{U}_{\mathcal{V}}^K)$ are uniformly isomorphic.*

Proof. Let $(\widetilde{X},\widetilde{\mathcal{U}})$ be the uniform completion of $(X,\mathcal{U})$ and let Z be a dense subset of $\widetilde{X}$. If $i:(X,\mathcal{U})\to(Z,\widetilde{\mathcal{U}})$ is a uniform isomorphism, then by [15, Proposition 3.4] and the construction of $\mathfrak{U}_{\mathcal{U}}^K$, $i:(X,\mathfrak{U}_{\mathcal{U}}^K)\to(Z,\mathfrak{U}_{\widetilde{\mathcal{U}}}^K)$ is a uniform isomorphism. Furthermore, by Theorem 5.14, $(\widetilde{X},\mathfrak{U}_{\widetilde{\mathcal{U}}}^K)$ is complete. Then $(\widetilde{X},\widetilde{\mathcal{U}})$ is the desired space $(Y,\mathcal{V})$ since by Proposition 6.4 Z is dense in $(\widetilde{X},\mathfrak{U}_{\widetilde{\mathcal{U}}}^K)$. □

It is an open question whether the above theorem remains true if we consider the $[0,1]$-Hutton quasi-uniformities $\mathfrak{U}_{\mathcal{U}}$, $\mathfrak{U}_{\mathcal{U}}^H$ and $\mathfrak{U}_{\mathcal{U}}^\omega$.

7 Conclusions

In fuzzy theory, there exist several methods to construct a fuzzy concept from its corresponding crisp concept. For example, we can find very different constructions of the fuzzy concept of uniformity being one of the most prominent that of Hutton uniformity. Although Katsaras was the first to establish a relationship between Hutton uniformities and classical uniformities [25], subsequent researches [12, 15, 18, 28, 37] have motivated the study of new ways of inducing a Hutton quasi-uniformity from a classic one. In this paper we have considered some new methods for making this construction.

Furthermore, the problem of studying whether completeness is preserved in this process seems to not have been considered previously in the literature. Then we have studied this problem for the four different methods of inducing a Hutton quasi-uniformity by means of a crisp uniformity presented in this paper. In this way we have been able to establish that two of these Hutton [0,1]-quasi-uniformities preserve the completeness property meanwhile the other two seems to have a bad behaviour.

Moreover, we have provided a construction of the completion of one of the Hutton [0,1]-quasi-uniformities induced by a crisp uniformity. This improves a result of [14] which had a gap.

Our results can be considered as a contribution to the development of standard methods for inducing fuzzy topological structures from crisp ones by selecting only those which preserve prominent properties. We will investigate further the relationship between the completeness of probabilistic uniformities generated by methods similar to the ones presented in this paper and the Hutton quasi-uniformities induced by them by means of Höhle's construction [18].

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