COMPARISON OF PORTFOLIO OPTIMIZATION THROUGH THE MARKOWITZ'S APPROACH AND VALUE AT RISK AS THE RISK MEASURE

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ABTRACT

This study compares several approaches to portfolio optimization, the Markowitz's (1952) approach and the approach based on the normal distribution of Value at Risk, with the different levels of confidence as a measure of risk, 90%, 95%, and 99%. To comply with these approaches, we obtain the real data of the prices of the assets of seven different companies that belong to the list of the Ibex 35, in order to obtain the optimal portfolios of both approaches. To calculate the different portfolios we have used the Excel program and the Solver, a tool found in Excel. The results are quite equal so we try to compare both approaches following a normal distribution through a normality test and the realization of different plots that affirm that the returns of the assets follow a normal distribution.

Keywords: Portfolio optimization, Value at Risk, Markowitz, normal distribution, volatility

GEL CODE: G-11

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1. INTRODUCTION

The goal of this project is the comparison of portfolio optimization through two different approaches. On the one hand, the Markowitz's (1952) approach, which uses variance of returns as a measure of risk. On the other hand, an alternative approach that considers Value at Risk (VaR) as the risk measure. More precisely, employing stock prices of seven companies from the IBEX35, 13 optimal portfolios in the Markowitz's (1952) are obtained. Given the expected returns of these 13 portfolios, the portfolios that minimize risk measured by VaR are obtained for each expected return, so that both approaches to portfolio optimization can be compared.

Papers such as Campbell, Huisman, and Koedijk (2001), Benati and Rizzi (2007) and Yoshida (2009) have studied portfolio optimization under VaR as risk measure. Nevertheless, their objective is to obtain the efficient frontier of portfolios (namely, all the set of portfolios that for a given expected return minimize the risk), not to compare these optimal portfolios with the ones that would have been obtained if the variance of portfolio's returns were considered as a risk measure.

The project is organised as follows. In section 1, I introduce the project's introduction.

In section 2, I introduce Markowitz's biography, his portfolio selection theory and explain the procedure which Markowitz follows to obtain the results of his model. It also includes the problems of his model and the properties that a good risk meter should meet.

In section 3, I introduce the Value at Risk model, starting with its history and how this measure of risk began to be introduced in the world of finance. This chapter also includes how the Value at Risk is applied in the Markowitz's (1952) model with its different types of confidence level. Finally, it also focuses on the problems posed by the Value at Risk model and its possible solutions to these problems, which are the application of 3 different methods of Value at Risk.

In section 4, I introduce the data used to make the Markowitz's (1952) approach and Value at Risk as the risk measure. In this Chapter are the companies that I used to calculate both models and all the detailed data that have been used to realize all the calculations, in addition it is also within this chapter the period of time that has been used for the comparison of the results obtained.

In section 5, I introduce the methodology used to obtain the results, I explain in detail the procedures used to calculate the Markowitz's (1952) approach and Value at Risk as the risk measure. In addition, the chapter divides into two sub sections, one of which is the calculation process of the Markowitz's (1952) model and the other is the calculation process of the Value at Risk.

In section 6, I introduce the comparison of the results obtained by the Markowitz's (1952) model and the results obtained by the Value at Risk as the risk measure.

In section, I introduce a normality contrast with both models; due to the similarity of the results I also calculate the contrast of normality of the models to have more options to compare both models. The model used to calculate the normality contrast is that of Jarque-Bera

Finally in the 8^{th} and last section I introduce the conclusion obtained from the comparison of both models.

2. MARKOWITZ MODEL

2.1. MARKOWITZ BIOGRAPHY

Harry Max Markowitz, (Chicago 1927-) is an American economist known as being the one who devised the optimization of portfolios that later explained in his article "Portfolio Selection" and which was published in the Journal of Finance in 1952. Markowitz's theory is based on the importance of portfolios, always seeking to optimize the return with the minimum possible risk diversifying portfolios. People changed their way of investing thanks to the work that Markowitz did with the collaboration of 3 other great economists, Merton H. Miller and William F. Sharpe. In 1990 Markowitz's work was rewarded with the Nobel Prize for Economics.

When he was young he studied at the University of Chicago, where he studied economics, focusing mainly on the "Economics of Uncertainly", the Von Neumann and Morgenstern and Marschak arguments about the expected utility, the Friedman-Savage utility function and L.J Savage's defense of personal probability. When he had to choose a theme for his dissertation he proposed the possibility of applying mathematical methods to the stock market and through that he began to devise his theory of portfolio optimization.

In 1952 he joined the RAND Corporation in which he met George Dantzig, thanks to George he learned the techniques of optimization on the fast computation of mean-variance frontiers in which he subsequently used in his work. His article "Selection of portfolio" appeared in 1952 and focused on the application of mathematical or computer techniques to practical problems, particularly problems of business decisions under uncertainly.

In 1989 he was awarded the Von Neumann Prize in Operations Research Theory by the Operations Research Society of America and The Institute of Management Sciences. Nowadays Markowitz is a professor of finance at the Rady School of Management at the University of California, San Diego.

2.2. PORTFOLIO SELECTION THEORY

Markowitz tries to explain in the portfolios selection is that investors seek to maximize or optimize their future returns based on a given level of market risk, that is, build several market portfolios by minimizing risk and obtaining optimal future return.

According to Markowitz (1952), the portfolio with the maximum expected return is not the one with the minimum variance, that is, there is a rate that allows the investor to obtain the expected return assuming the variance, or reduce variance by giving up the expected return.

The portfolio selection theory is based on 3 stages:

1) Determine the efficient portfolios that the investor wants to invest.
2) Determine the investor's attitude towards risk.
3) Determine the optimal portfolio.

In addition relies on different starting assumptions:

- The return of a financial asset must be a random variable with a known distribution function.
- The risk of a portfolio is measured through the variance or typical deviation.
- The investor will always want greater return and lower risk.

The first stage of the Markowitz's (1952) theory is to determine the most efficient portfolios in which the investor is willing to invest. For a portfolio to be efficient, it must have the same return with a lower risk or have the same level of risk in order to obtain greater return.

First of all what Markowitz is looking for is the efficient frontier, the set of the most efficient portfolios in the market, and for this what he needs is to look for a mathematical problem that allows him to maximize the return.

$$R = \sum R_i X_i$$

Being:

R = portfolio yield.

Ri = are considered to be a random variables.

Xi = are fixed by the investor.

To carry up this mathematical problem Markowitz is based on different parametric and budgetary constraints. After several assumptions, at the end Markowitz concluded that the total sum of weight of each portfolio value multiplied by the covariance is equal to the estimated variance of the portfolio. In addition, the value of V will have a different portfolio composition.

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X$$

Being:

V = the variance

Oij = the covariance

As we said previously, Markowitz was based on two constraints, the parametric constraint we just explained; now we are going to explain the budget constraint. The budget constraint used by Markowitz is quite simple since it tells us that the sum of all the weighs of each value of the portfolio cannot be greater than 1, that is, the sum of the values must be equal to 1.

$$\sum_{i=1}^{3} X_i = 1$$

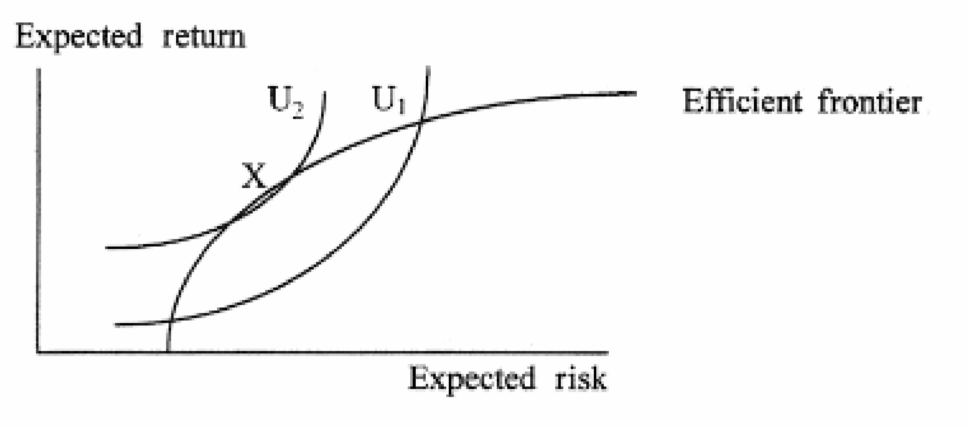
In addition to the constraints, Markowitz establishes a condition of non-negativity, which means that the weights of the portfolio cannot be negative, that is, they have to be equal to or greater than 0.

$$X_i \geqslant 0 \quad \text{for} \quad i = 1,\ 2,\ 3\ .$$

The second stage is to determine the investor's attitude towards risk. Thus, each investor has a different risk aversion and will demand a certain return for each level of risk that is willing to assume. The attitude of the investor against the risk will depend on his map of indifference, that is, a set of concave curves that represents the preferences of the investor. The curves are concave because risk aversion is increasing, if the yield increases the risk too.

The third and final stage is to determine which the optimal portfolio is; this is at the tangent point between one of the indifference curves of the investor and the efficient frontier. All those portfolios that are above the point, are not feasible and the portfolios that are below the point, will obtain a lower satisfaction than the portfolio that is at the tangent point, which will be the optimal point.

Figure 1: Finding the best portfolio



Source: Rattiner, J.H, 2003.p.154.

2.3. PROBLEMS TO APPLY THE MARKOWITZ MODEL

One of the main problems that the Markowitz's (1952) model has had is the mathematical complexity of the model. On the one hand, being a parametric quadratic program, the resolution algorithm was complex. On the other hand, the number of estimates and expected returns, variances and covariance that had to be made was very high.

A solution to the problems we have previously named was given by William Sharpe with a model that tried to simplify the Markowitz method of diversification of portfolios. Sharpe proposes in its model to simplify the process of Markowitz's (1952) model by reducing data substantially. He came to the conclusion that the values do not only have an individual reason, but are related to each other through some indexes.

According to Bhardwa (n.d), Sharpe in his model proposes to relate the evolution of the return of each financial asset with a certain index. Furthermore he showed that the return of each security is correlated by securities markets in the U.S.A.

According to Bhardwa (n.d) many researchers have taken into consideration the Sharpe Index Models. They have preferred the stock price index to the economic indexes in finding out the full covariance frontier of Markowitz for stake of simplicity.

2.4. IS THE VARIANCE A COHERENT RISK MEASURE?

Financial risk can be measured in different ways, but for this risk measure is a good measure it must be coherent. A measure of risk is coherent if it has a series of properties that a measure of risk must have to be considered a good risk measurer, in other words, it is a function that satisfies the properties of monotonicity, sub-additivity, homogeneity and translational invariance.

In this case we will focus on checking if the variance or standard deviation, which is the measures of the risk of the Markowitz mean-variance optimization problem, has the properties to be a good risk measurer. We know that neither the variance nor the standard deviation can be negative.

Positive homogeneity means that if you double your portfolio then you double your risk, $p(\lambda X) = \lambda p(X)$ for $\lambda \geq 0$

Sub-additivity means that the risk of two portfolios together cannot be worse than adding the risk of the portfolios separately, $p(X1 + X2) \leq p(X1) + p(X2)$ Therefore diversifying is beneficial.

Normalized means that the portfolio's risk, which has not assets is equal to 0. $p(0) = 0$

Convexity means that the risk measure prefers to diversify because it is more beneficial, therefore, it would be preferable to have a portfolio with more assets than a portfolio with less assets. $p(\lambda X1+(1 - \lambda)X2) \leq \lambda p(X1)+(1 - \lambda) p(X2)$, for any real $\lambda \in [0,1]$

As we can see the variance is not coherent because it is neither translation invariance nor monotonicity.

Monotonicity means that if we have two portfolios (X1, X2) $X2 \leq X1$ then $p(X1) \leq p(X2)$. That is, if portfolio 1 has more future returns than portfolio 2, the risk of portfolio 1 will be less than the risk of portfolio 2. In other words, portfolio X2 will be more volatile than portfolio X1.

Translation Invariance means that if we add a constant (c) to a random variable (X), in terms of financial risk management, the amount of capital that has been added reduces the risk by the same amount, $p(X + c) = p(X) - p$. If we add a constant to a random variable with the properties of the standard deviation (p) it does not change, i.e. $p(X + c) = p(X)$.

The Value at Risk satisfies the properties of translation invariance, monotonicity and positive homogeneity. Therefore the property that does not satisfy and makes it not a coherent risk measure is the property of sub-additivity.

The objective that we are looking for with this is how two different risk measures we can apply them in the portfolio's optimization and compares them later.

3. VALUE AT RISK

3.1. HISTORY OF VALUE AT RISK

The origins of the name "value at risk" were a bit complicated. The economists used endless names and did not know which fitted perfectly. In the 1990s, names such as “capital at risk” (CaR), “dollars at risk” (DaR), “gains at risk” (EaR), “income at risk” (IaR) and “value at risk” (VaR). The first 4 names did not fit exactly in the procedures that measured the VaR, this meant that the name “value at risk” was the one that had more relationship, perhaps the vagueness of the label “value” is what made it more attractive.

According to Guldimann (2000) the “value at risk” originated before 1985: “*We learned that “fully hedged” in a bank with fully matched funding can have two meanings. We could either invest the Bank’s net equity in long bonds and generate stable interest earnings, or we could invest it in Fed funds and keep the market value constant. We decided to focus on value and assume a target duration investors assigned to the bank’s equity. Thus value-at-risk was born*”

We can define the Value at Risk as a category of probabilistic measures of market risk. If we consider a portfolio with fixed shares which we have full knowledge of its market value and we have a random variable such as could be the case of a week in the future, we can attribute a probability distribution to this portfolio. The necessary Value at Risk metrics are the variance of the yield, the standard deviation and 0.95, 0.90, 0.99-quantile of loss. Therefore, a measure of the Value at Risk, following the necessary metrics for its procedure, is to assign values in these metrics to the portfolios that we have calculated. The first measures of the Value at Risk was developed one in the theory of the portfolio, whose main authors are Bernstein (1992) and Markowitz (1999), and the other was capital adequacy computations.

Markowitz in 1952 and 3 months later Roy in 1952 too, published really similar Value at Risk measures, in addition, their publications were totally independent. Each worked on the development of a means to select portfolios that in one way or another would optimize the reward for a given level of risk. In each of the measures, they incorporated covariance between the risk factors in order to reflect the effects of coverage and diversification. Although mathematically the measurements were similar, the metrics used by each were different Markowitz used a simple variation metric while Roy used a metric of shortfall risk.

Both avoided the problems of their probabilistic assumptions, Roy’s measure required a mean vector and a covariance matrix for the risk factors should be estimated from information about the past. The measure of Markowitz only required the covariance matrix and proposed that it be constructed through procedures that would be called “Bayesian”.[1]

[1] “Bayesian” represents a level of certainly relating to a potential outcome or idea. This is in contrast to a frequents probability that represents the frequency with which a particular outcome will occur over any number of trials.

3.2. VALUE AT RISK APPLIED IN MARKOWITZ PORTFOLIO OPTIMIZATION

Some academics developed new models as a solution to the problems of portfolio optimization. These models are direct extensions of the Markowitz's (1952) model; they consist of replacing the variance by some lower tail return distribution function statistics. The Value at Risk is the "*a*" quantile of the return distribution function, "*a* € (0.1)", in which the value of "*a*" is usually 0.01, 0.05 or 0.1, depending on the percentage of risk that it is facing. Value at Risk is not an attractive model, from the mathematical point of view, since they can have a remarkable number of maximums and minimums and do not possess lineal or convex mathematical properties. Furthermore, has a rather negative point, because of this it can prevent the portfolio from diversifying and what stands out in the Markowitz model is the diversification of portfolios.

The method of calculation will be the same as if we calculate the return using the Markowitz's (1952) model with the only difference that in this case instead of taking the variance as a measure of risk we take the VaR, it is necessary that two fundamental parameters for the calculation, the probability "aVaR" and the return "rVaR" are set in the decision making process. The expected return must be less than the return and at the same time this has to be less than the probability. X<= rVaR <= aVaR.

The constraints that are used to calculate the expected return are the following:

- Maximize the expected return of the portfolio.
- Each convex combination of assets must be greater than 0.
- The sum of these convex combinations of assets is greater than 1 unit.
- The return of the Markowitz portfolio is equal to the random variable.
- The probability to go under rVaR is less or equal to the threshold aVaR

Assuming that Z is a random variable and "a € (0.1)", aVaR (Z) can be estimated from the parametric method and non-parametric method, this first is applied supposing that Z is distributed as a known family of functions, e.g., normal, t-student, etc. In this case, once the estimation of the parameters of the distribution function is obtained, the VaR would be calculated. For example, assuming that the returns on stocks have a normal distribution, the Markowitz (1952) mean-variance model and the mean-VaR as the risk measure would not have a significant effect. This is because both are determined by the same parameters.

3.3. PROBLEMS WITH THE VALUE-AT-RISK MODEL AND ITS SOLUTIONS

There are also problems with the VaR method; its main problem is that the normal distribution curve assumes that each event is completely random. When the stock price falls, people start selling, which provoke the stock price go to extremes faster than the curve actually points.

A similar case would be when the share price increases. To avoid this problem, the parametric approach must be left behind and an empirical approach used to calculate portfolio risk. For this, there are 3 simple methods to calculate the VaR; these methods are the variance and covariance method, the historical simulation method and the Monte Carlo simulation method.

The variance-covariance method for calculating the VaR of a portfolio uses information on the correlation of the stocks and on the volatility of the assets. The volatility of an asset consists of measuring the past fluctuations of prices and, therefore, they are expected to occur in the future. The correlation consists of the relationship between the prices of an asset with the price of another asset.

This calculation method of VaR became famous thanks to the investment bank J.P. Morgan, which in 1994 had the objective of standardizing the measurement of financial risk throughout the industry, the system it used was called RiskMetrics. J.P. Morgan published information on the volatility and correlation of the shares traded in the main markets of the industry. However, in 1998 RiskMetrics separated from J.P. Morgan due to the high demands for improvements and advice they received on a daily basis.

The steps to calculate this model are the following:

- The first step is to specify the level of confidence, and construct the volatility matrix by multiplying the diagonal matrix of the standard deviations of the returns of the assets of a portfolio by the level of the confidence interval of a normal distribution.
- The second step is to calculate the VaR with a matrix that represents the correlation of the return of each of the assets of a portfolio and the use of a column vector that represents the weighting of each asset in the portfolio.

It can be said that this method is one of the most popular and used because its calculation is not difficult to calculate and the way to implement it is quite simple. However, this model is not entirely perfect, it can have several limitations, that is, the level of confidence that is applied to the model follows a normal distribution, and therefore it cannot be entirely realistic. In practice, when obtaining volatility and correlation matrices all institutions connect their administrative systems to the RiskMetrics source.

Because of the large amount of data that is stored, RiskMetrics cannot provide volatility information on a daily basis, therefore, it must be provided for certain periods of time. In addition, if any institution is interested to calculate the VaR in a different period from the one that RiskMetrics provides, it must estimate the data through the mapping function of the available data. In this way a vast amount of information is lost.

The historical simulation method is simpler than the method we have explained previously, it is consist in the loss of data that a bank would have suffered in a given portfolio in a previous period. The Value at Risk of a portfolio is the loss that in X% cases will not be exceeded, that is, the lowest % of returns (1-X) where X is the confidence level. However one of the disadvantages that it has is that it needs a high amount of historical simulation.

One of the reasons why this method is used is because the information provided is totally realistic, in other words, the information is totally accurate. Another advantage that this method provides us is to choose the desired time horizon; this is because the institution maintains its own historical data. Unlike the method explained above, the variance and covariance method, mapping is not required.

One problem we can have with this method, apart from the one we have already explained, is that if the composition of the portfolio varies over time this method will not work, but there is a solution to this problem that is to use a historical simulation approach using the performance data of historical assets.

This simulation uses the composition of the portfolio to calculate the VaR in the different periods of time covered by the historical data and using the historical observations of the asset's return values. Therefore, the current VaR would be the highest VaR of the lowest (1-X) % of VaRs calculated by the historical simulation method.

The Monte Carlo simulation method works in the same way as the historical simulation method with the difference that it uses a process with a difficult prediction since it is based on the intervention of random variables. It is a technique that is basically used to understand the impact that risk can have.

One of the advantages of this model as well says Kaura (2006) in his article: "The method has the advantage of allowing users to tailor ideas about future patterns that differ from historical patterns. Rather than just calculating the VaR of a portfolio, we wish to use the VaR formulation as the objective function and aim to minimise it with respect to a portfolio of stocks". (Vinay Kaura, p.7)

However, if there are advantages there are also disadvantages and in this case the Value at Risk has some of them. A disadvantage that the Value at Risk has with this model is that it is non-sub-addictive and non-convex, that is, you can build two portfolios of X and Y and the sum of the VaR of X,Y is greater than the sum of the VaR of X and the VaR of Y.

This process is contradictory because if we diversify the portfolios in theory the risk should be lower and in this case is the contrary, the risk is greater. In addition, because the VaR has non-convexity, the VaR has multiple local minima that make the problem of optimization have several difficulties when it comes to solving the problem, since it needs to find the global minimum.

4. DATA USED FOR THE CALCULATIONS OF BOTH APPROACHES

To implement both approaches to portfolio optimization, the Markowitz's (1952) model and the approach considering VaR as risk measure, the first thing we done has been to obtain the historical quotes of 7 companies that belong to the IBEX 35. These are the companies that we have used to study the risk and return moderation globally: Abertis, Bankinter, Caixabank, Iag, Indra, Merlin Properties, and Arcelormittal.

- ABERTIS: Leader in managing toll roads
- BANKINTER: Online banking and financial services
- CAIXABANK: Spanish financial services
- IAG: Is one of the world's largest groups
- INDRA: Is a Spanish information technology and defense systems company.
- MERLIN PROPERTIES: It is dedicated to the acquisition and management of commercial assets in the Iberian Peninsula.
- ARCELORMITTAL: Is the world's leading integrated steel and mining company.

We have chosen these companies in a period of time of 3 years, exactly from 03/02/2015 to 05/02/2018. This period of time has been chosen because these data were obtained and treated for a previous study in which Markowitz's (1952) approach to portfolio optimization was applied. For instance, it was ensured that each asset's expected return was positive in the period considered.

All the calculations have been made using the spread sheet Excel. In particular, with the quotes obtained we have created a table that consist of the date, the closing quote and the return, the latter is calculated through the formula LN(closing quote in t/ closing quote in t-1). After calculating the daily return of each company, we can obtain the expectation with the function of "AVERAGE" and the variance of the returns of each company with the function of "VARP" to later elaborate the variance and covariance matrix.

Table 1: Results obtained from the data

	EXPECTATION	VARIANCE
ABERTIS	0,000141437	0,00015735
BANKINTER	0,000477062	0,00022085
CAIXABANK	0,000089963	0,000446717
IAG	0,000003519	0,000607552
INDRA	0,000321327	0,000436459
MERLIN PROPERTIES	0,000080760	0,000254238
ARCELORMITTAL	0,000042622	0,004661894

Source: Own development.

The covariance of an asset with itself is the variance; therefore the diagonal of the matrix will be the variance of each of the companies. Then we will combine the covariance with each of the other companies and thus we will complete the matrix.

Table 2: Covariance matrix

	ABERTIS	BANKINTER	CAIXABANK	IAG	INDRA	MERLIN PROPERTIES	ARCELORMITTAL
ABERTIS	0,000157349922	0,000093907689	0,000111120963	0,000139633844	0,000093055332	0,000081896205	0,000074972369
BANKINTER		0,000220849834	0,000240770292	0,000180062826	0,000118974865	0,000105008977	0,000181360973
CAIXABANK			0,000446717484	0,000247501936	0,000161413238	0,000140867543	0,000330384256
IAG				0,000607552226	0,000192496984	0,000165681667	0,000234081929
INDRA					0,000436459091	0,000096867967	0,000253847504
MERLIN PROPERTIES						0,000254238012	0,000155684499
ARCELORMITTAL							0,004661894035

Source: Own development

5. METHODOLOGY

As stated in the Introduction, the objective of the present study is the comparison of portfolio optimization through, on the one hand, the Markowitz's (1952) approach which uses variance of returns as a measure of risk, and, on the other hand, an alternative approach that considers Value at Risk (VaR) as the risk measure.

To achieve this objective, the study applies a three-stage methodology. In the first stage, 13 optimal portfolios in the Markowitz's (1952) sense are obtained. For each expected return of these 13 portfolios, the portfolios that minimize the risk measured by VaR at three levels of confidence (90%, 95% an 99%) are obtained. The third stage compares the optimal portfolios from both perspectives. The next two subsections lead with the first two stages of the methodology, while the third one is discussed in the next section of the study.

Before starting the discussion of the first two stages of the methodology, it is important to specify that all the optimal portfolios (both according to the Markowitz's (1952) approach and considering VaR as risk measure) have been obtained using the spread sheet of Excel and its solver tool, which allows to solve complex optimization problems. The details of the procedure to obtain these optimal portfolios are discussed in the following two subsections. A details that is important to mention here is that a minimum weight of 0.01 for each asset in a portfolio was established as a restriction in order to avoid possible errors in the calculation.

5.1. <u>CALCULATE OPTIMAL PORTFOLIO ACCORDING TO MARKOWITZ MODEL</u>

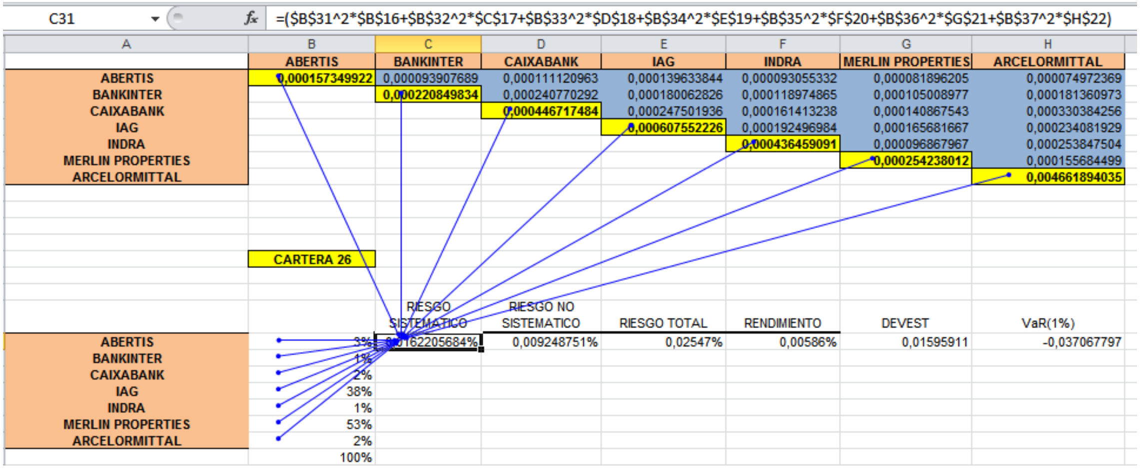
This subsection discusses the first stage of the methodology. In particular, it discusses the construction of the optimal portfolios in the Markowitz's (1952) sense, namely, considering the variance of the portfolio's returns as risk measure,

The variance of the returns of a portfolio of risky assets as a measure of its risk is defined by two components, one representing the non-systematic risk of the portfolio and the other one representing systematic risk, as can be observed in the following expression.

$$V[R_p] = \sigma_p^2 = \underbrace{\sum_{i=1}^{N} X_i^2 \sigma_i^2}_{\text{Systematic risk}} + \underbrace{\sum_{\substack{i,j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{i,j}}_{\text{Non-Systematic risk}}$$

The systematic risk is calculated by the sum of the square of the relative weight of each company in each of the portfolios multiplied by the variance of the portfolio. The following figure shows how a portfolio's systematic risk can be calculated using the spread sheet of Excel.

Figure 2: Calculation process of Systematic risk



C31 =($B$31^2*$B$16+$B$32^2*$C$17+$B$33^2*$D$18+$B$34^2*$E$19+$B$35^2*$F$20+$B$36^2*$G$21+$B$37^2*$H$22)

	ABERTIS	BANKINTER	CAIXABANK	IAG	INDRA	MERLIN PROPERTIES	ARCELORMITTAL
ABERTIS	0,000157349922	0,000093907689	0,000111120963	0,000139633844	0,000093055332	0,000081896205	0,000074972369
BANKINTER		0,000220849834	0,000240770292	0,000180062826	0,000118974865	0,000105008977	0,000181360973
CAIXABANK			0,000446717484	0,000247501936	0,000161413238	0,000140867543	0,000330384256
IAG				0,000607552226	0,000192496984	0,000165681667	0,000234081929
INDRA					0,000436459091	0,000096867967	0,000253847504
MERLIN PROPERTIES						0,000254238012	0,000155684499
ARCELORMITTAL							0,004661894035

	CARTERA 26	RIESGO SISTEMATICO	RIESGO NO SISTEMATICO	RIESGO TOTAL	RENDIMIENTO	DEVEST	VaR(1%)
ABERTIS	3%	0,0162205684%	0,009248751%	0,02547%	0,00586%	0,01595911	-0,037067797
BANKINTER	1%						
CAIXABANK	2%						
IAG	38%						
INDRA	1%						
MERLIN PROPERTIES	53%						
ARCELORMITTAL	2%						
	100%						

Source: Own development

In order to calculate the non-systematic risk in Excel we have needed to elaborate a table with all the necessary calculations to obtain the non-systematic risk of each portfolio. To calculate the non-systematic risk more easily, we have decided to divide the equation in parts. In the first case Xi is the weight of ALBERTIS in the portfolio and Xj is the BANKINTER's weight. One is multiplied by two times the other and the result is multiplied by the covariance of the returns of both assets. This process is done with each asset in the portfolio and the sum of all the results is the portfolio's non-systematic risk. The following table shows this process carried out in Excel.

Table 3: Non-systematic risk calculation process

	Xi*Xj	2*(Xi*Xj)	COVARIANZA	2*Xi*Xj*COVARIANZA
ABERTIS-BANKINTER	0,000255920794	0,000511841588	0,000093907689	0,00000004807
ABERTISCAIXABANK	0,000612755916	0,001225511831	0,000111120963	0,00000013618
ABERTIS-IAG	0,009791782746	0,019583565492	0,000139633844	0,00000273453
ABERTIS-INDRA	0,000255920794	0,000511841588	0,000093055332	0,00000004763
ABERTIS-MERLIN PROPERTIES	0,013548295517	0,027096591035	0,000081896205	0,00000221911
ABERTIS-ARCELORMITTAL	0,000472449082	0,000944898163	0,000074972369	0,00000007084
BANKINTER-CAIXABANK	0,000239431860	0,000478863719	0,000240770292	0,00000011530
BANKINTER- IAG	0,003826098928	0,007652197856	0,000180062826	0,00000137788
BANKINTER-INDRA	0,000100000000	0,000200000000	0,000118974865	0,00000002379
BANKINTER-MERLIN PROPERTIES	0,005293940879	0,010587881757	0,000105008977	0,00000111182
BANKINTER-ARCELORMITTAL	0,000184607540	0,000369215080	0,000181360973	0,00000006696
CAIXABANK-IAG	0,009160899815	0,018321799630	0,000247501936	0,00000453468
CAIXABANK-INDRA	0,000239431860	0,000478863719	0,000161413238	0,00000007729
CAIXABANK-MERLIN PROPERTIES	0,012675381094	0,025350762189	0,000140867543	0,00000357110
CAIXABANK-ARCELORMITTAL	0,000442009266	0,000884018532	0,000330384256	0,00000029207
IAG-INDRA	0,003826098928	0,007652197856	0,000192496984	0,00000147303
IAG-MERLIN PROPERTIES	0,202551415206	0,405102830412	0,000165681667	0,00006711811
IAG-ARCELORMITTAL	0,007063267108	0,014126534215	0,000234081929	0,00000330677
INDRA-MERLIN PROPERTIES	0,005293940879	0,010587881757	0,000096867967	0,00000102563
INDRA-ARCELORMITTAL	0,000184607540	0,000369215080	0,000253847504	0,00000009372
MERLIN PROPERTIES-ARCELORMITTAL	0,009773014024	0,019546028047	0,000155684499	0,00000304301
				0,00009248751

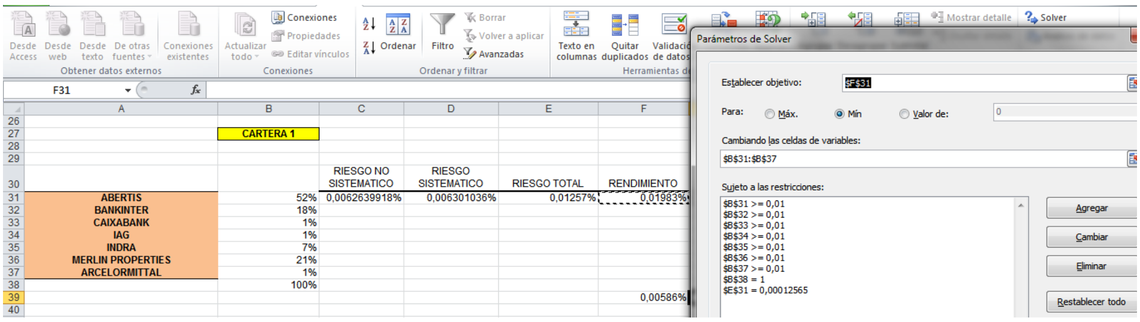
Source: Own development

The global risk of the portfolio will be the sum of non-systematic risk and systematic risk.

Having calculated the total risk of the portfolio, now we need to calculate its expected return. This can be easily calculated as the total sum of the percentage that is invested in each asset of the portfolio (the weight of each asset in the portfolio) multiplied by the expected return on each asset.

Once we have calculated the risk and return, through the Excel's tool Solver, we can calculate the percentages that are invested in each asset in an optimal portfolio in the Markowitz's (1952) sense, namely considering the variance of the portfolio's returns as its risk measure. The perspective applied to calculate these optimal portfolios has been to fix a particular level of risk (namely, a particular value for the variance of the portfolio's returns) and to maximize the expected returns of the portfolio with respect to the weights of the assets in the portfolio. The following figure shows the formulation of the optimization problem with Solver and Excel.

Figure 3: Calculations with solver (Markowitz)



Source: Own development

In the figure above we can see the restrictions we use to get the percentages of the assets in an optimal portfolio in Markowitz's (1952) sense with a variance of its returns equal to 0.012565%. We start marking the target cell, which in this case is the expected return of the portfolio, then we choose the cells which we want to change, the percentages. To finish we put the following restrictions: that each percentage must be grater or equal to 0.01 since it can never be negative and to avoid calculations problems (as stated above) we have to put an amount greater than 0, in addition the sum of the percentages must be equal to 1 (100%) and in the end we put the risk we want to assume.

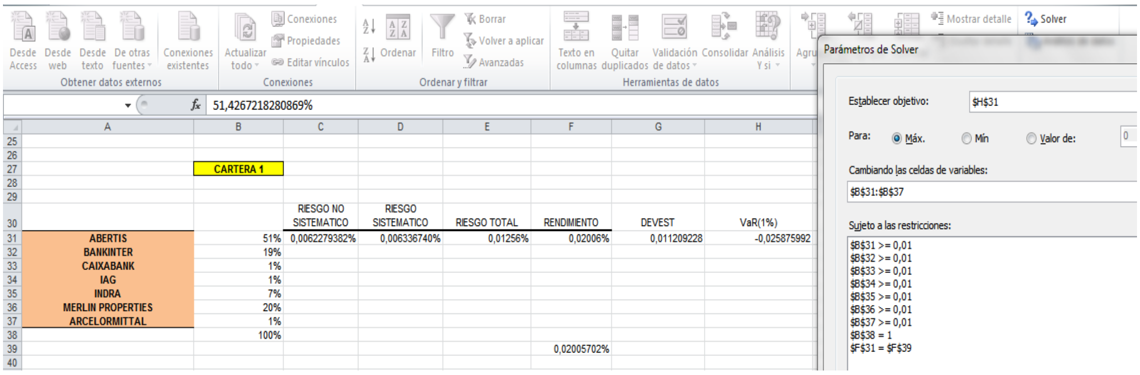
This is, in general, the procedure that should be used to calculate optimal portfolios in Markowitz's (1952) sense in Excel. Thirteen optimal portfolios in the Markowitz's (1952) sense are obtained solving the corresponding optimization problems, each of them for a different level of risk (namely, for a different variance of the portfolio's return).

5.2. CALCULATE OPTIMAL PORTFOLIO UNDER VALUE AT RISK AS RISK MEASURE

Once we have already calculated the Markowitz's (1952) sense, we now focus on calculating optimal portfolios under Value at Risk as risk measure. For this we have to calculate the square root of the variance of the portfolio's returns to obtain the standard deviation. In Excel it can be done with the function RCUAD. After calculating the standard deviation we can obtain the value at risk using the Excel's function INV.NORM. This function consists of returning the inverse of the normal cumulative distribution at a specified probability for the mean and specific standard deviation of the distribution; we use this function since we focus on calculating Value at risk assuming that portfolio's returns follow a normal distribution.

To calculate the Value at Risk at a particular confidence level, we take into account the expected return and the standard deviation of each of the thirteen optimal portfolios in the Markowitz's (1952) sense calculated in the previous stage. Once a cell on the Excel's sheet is defined to calculate portfolio's VaR in this way, the optimization problem to calculate portfolios that minimize risk measures by VaR is formulated in Solver. The following figure shows an example of optimization problem formulated in Solver in order to calculate a portfolio that minimizes risk under VaR as risk measure.

Figure 4: Calculations with solver (VaR)



Source: Own development

In this case the target cell is the one in which value at risk is defined. In particular, in order to obtain portfolios that are comparable with the ones obtained in the previous stage (which are optimal in the Markowitz's (1952) sense) and given that the variance of a portfolio's returns as risk measure is not directly comparable with Value at Risk, the objective of the optimization problems in this stage is to minimize risk measured by Value at Risk for each expected return of the portfolios obtained in the previous stage. The other restrictions of the optimization problem are the same as when obtiaining the optimal portfolios in Markowitz's (1952) sense.

Therefore 13 portfolios are calculated un the same way only varying the return corresponding to each portfolio of the Markowitz's (1952) sense so that in the future we can compare both approaches. This is done considering Value at Risk at three different levels of confidence (90%, 95% and 99%). To calculate Value at Risk at different levels of confidence the only thing that we must do is go to the function of Value at Risk that we have defined previously in the corresponding cell and change the parameter that establishes this level of confidence.

Once we have the 13 optimal portfolios in the Markowitz's (1952) sense and another 13 portfolios with the same expected returns as the previous ones but minimizing their risk measures by Value at Risk at the three levels of confidence, we can start comparing the results and discuss how the different risk measures affect portfolio optimization.

6. COMPARISON OF THE RESULTS OBTAINED WITH THE MARKOWITZ MODEL AND THE VALUE AT RISK MODEL WITH ITS DIFFERENT LEVELS OF CONFIDENCE

After having calculated the 13 portfolios to know which one is more optimal through the Markowitz's (1952) model and the Value at Risk as the risk measure, with its different levels of confidence, we have concluded that the results presented by both models have a high degree of similarity. As a representation of this result, we have developed the following tables of different portfolios for four different expected returns in which we compare the results obtained through the Markowitz's (1952) model and the results obtained by applying, as a risk measure, the VaR with its different levels of confidence.

Table 4: Portfolios with expected return 0.02334%

	EXPECTED RETURN			
	0,02334%			
	MARKOWITZ	VaR (90%)	VaR (95%)	VaR (99%)
ABERTIS	46,309%	46,306%	46,306%	46,306%
BANKINTER	27,280%	27,280%	27,280%	27,280%
CAIXABANK	1,000%	1,000%	1,000%	1,000%
IAG	1,000%	1,000%	1,000%	1,000%
INDRA	7,280%	7,282%	7,282%	7,282%
MERLIN PROPERTIES	16,131%	16,133%	16,133%	16,133%
ARCELORMITTAL	1,000%	1,000%	1,000%	1,000%

Source: Own development

Table 5: Portfolios with expected return 0.03086%

	EXPECTED RETURN			
	0,03086%			
	MARKOWITZ	VaR (90%)	VaR (95%)	VaR (99%)
ABERTIS	34,585%	34,584%	34,584%	34,584%
BANKINTER	47,090%	47,089%	47,089%	47,089%
CAIXABANK	1,000%	1,000%	1,000%	1,000%
IAG	1,000%	1,000%	1,000%	1,000%
INDRA	8,861%	8,863%	8,863%	8,863%
MERLIN PROPERTIES	6,464%	6,464%	6,464%	6,464%
ARCELORMITTAL	1,000%	1,000%	1,000%	1,000%

Source: Own development

Table 6: Portfolios with expected return 0.03790%

	EXPECTED RETURN			
	0,03790%			
	MARKOWITZ	VaR (90%)	VaR (95%)	VaR (99%)
ABERTIS	19,525%	19,529%	19,529%	19,529%
BANKINTER	66,421%	66,427%	66,427%	66,427%
CAIXABANK	1,000%	1,000%	1,000%	1,000%
IAG	1,000%	1,000%	1,000%	1,000%
INDRA	10,054%	10,044%	10,044%	10,044%
MERLIN PROPERTIES	1,000%	1,000%	1,000%	1,000%
ARCELORMITTAL	1,000%	1,000%	1,000%	1,000%

Source: Own development

Table 7: Portfolios with expected return 0.04083%

	EXPECTED RETURN			
	0,04083%			
	MARKOWITZ	VaR (90%)	VaR (95%)	VaR (99%)
ABERTIS	10,635%	10,651%	10,651%	10,651%
BANKINTER	74,984%	75,003%	75,003%	75,003%
CAIXABANK	1,000%	1,000%	1,000%	1,000%
IAG	1,000%	1,000%	1,000%	1,000%
INDRA	10,381%	10,346%	10,346%	10,346%
MERLIN PROPERTIES	1,000%	1,000%	1,000%	1,000%
ARCELORMITTAL	1,000%	1,000%	1,000%	1,000%

Source: Own development

As can observed in the above tables, the optimal weights of the different assets when considering the different risk measures (the variance of the portfolio's returns and the Value at Risk at the three different levels of confidence) hardly vary, not to say that they are almost the same. And the same result is obtained for all the levels expected return considered in the present study.

Following Benati and Rizzi (2007) a possible explanation for this result would be that the series of stock daily returns are normally distributed. According to Benati and Rizzi (2007), if that is the case portfolio optimization in the Markowitz's (1952) sense (under the variance of the portfolio's returns as risk measures) and under Value at Risk as risk measure leads to basically the same results. Therefore, the next natural step would be to test the normality of the daily returns of the stocks employed in this study.

7. TESTING OF NORMALITY OF STOCK RETURNS

A normality test consists of calculating the probability that the sample has been extracted from a normal population. There are two hypotheses:

- The null hypothesis. It tells us that the sample data are not significantly different from the normal population.
- The alternative hypothesis. It tells us that the sample data are significantly different from the normal population.

What is sought in this type of test is find differences between groups. One of the objectives of this test is that the data in the sample are not different from the normal, that is, we accept the null hypothesis. To accept the null hypothesis what we must happen is that the calculated result must be greater than 0.05. If the percentage is less than 0.05 the data will not be normal.

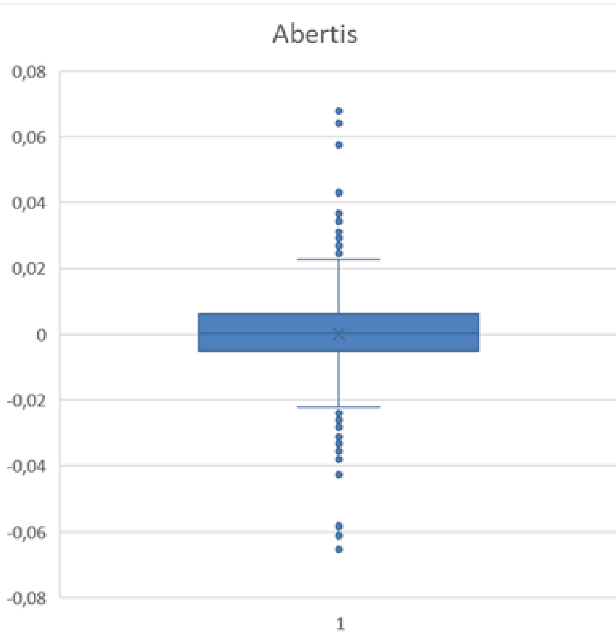
There are several ways to calculate a contrast of normality, we have used Jarque-Bera contrast because it is easy to calculate and quite accurate. To calculate the Jarque-Bera contrast it is necessary to know the size of the sample, as well as the skewness and kurtosis of it.

The number of observations that we have used have been exactly 765, which corresponded to the daily returns of each one of the chosen assets. Then, we have developed a table, in an Excel sheet, to calculate the skewness and the kurtosis that we need for each asset. In each of the assets we have taken the percentage that corresponded in each portfolio and we have added it to the table, later we have calculated the skewness with the function “SKEW” and we have selected all the data of the asset.

To calculate the kurtosis we have used the “KURT” function, also selecting the asset data.

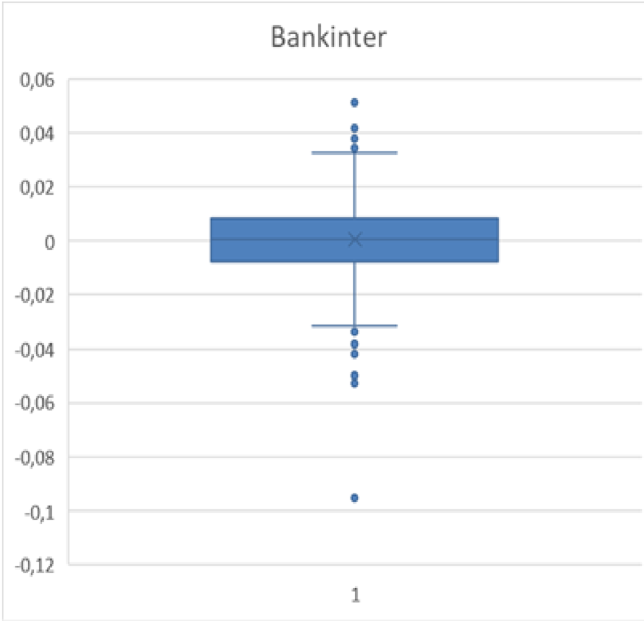
Once we have the skewness, the kurtosis and the number of observations, through the equation that Jarque-Bera uses, we calculate the contrast of normality. After seeing that the results are not adequate, we are going to develop a box plot of the daily returns of each asset to analyse why the results are not what we expect.

Figure 5: Abertis's box-plot



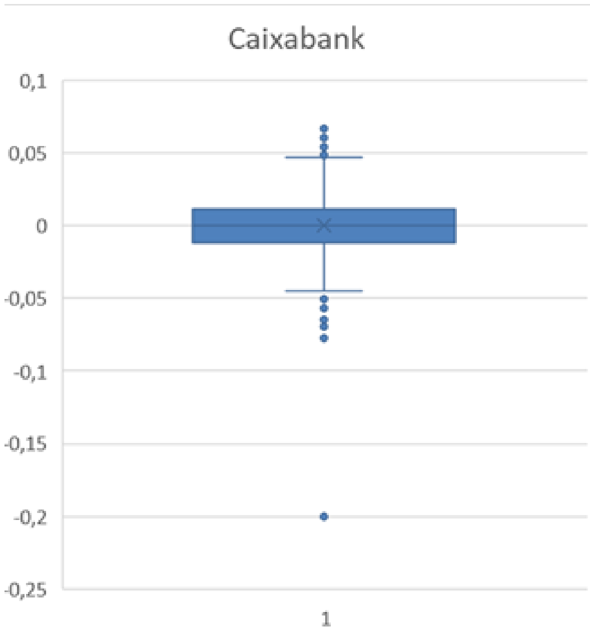
Source: Own development

Figure 6: Bankinter's box-plot



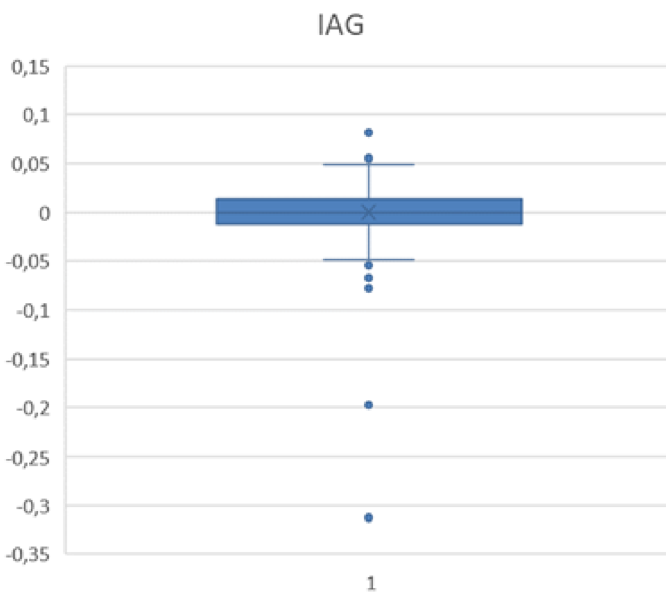
Source: Own development

Figure 7: Caixabank's box-plot



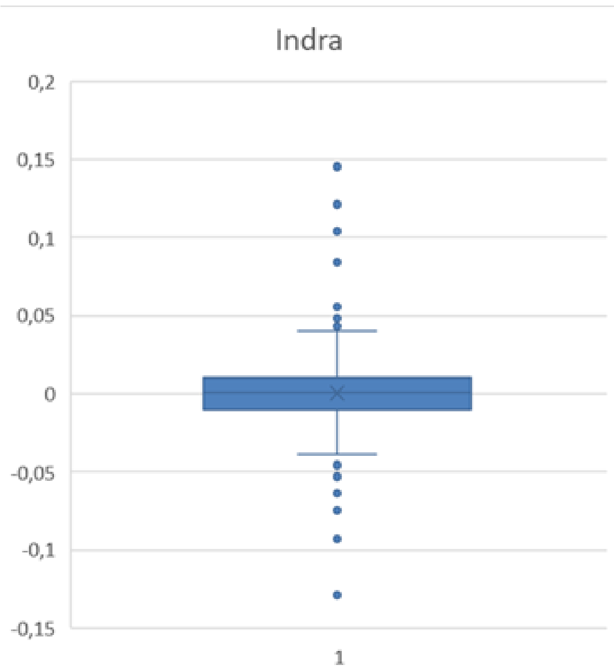
Source: Own development

Figure 8: Iag's box-plot



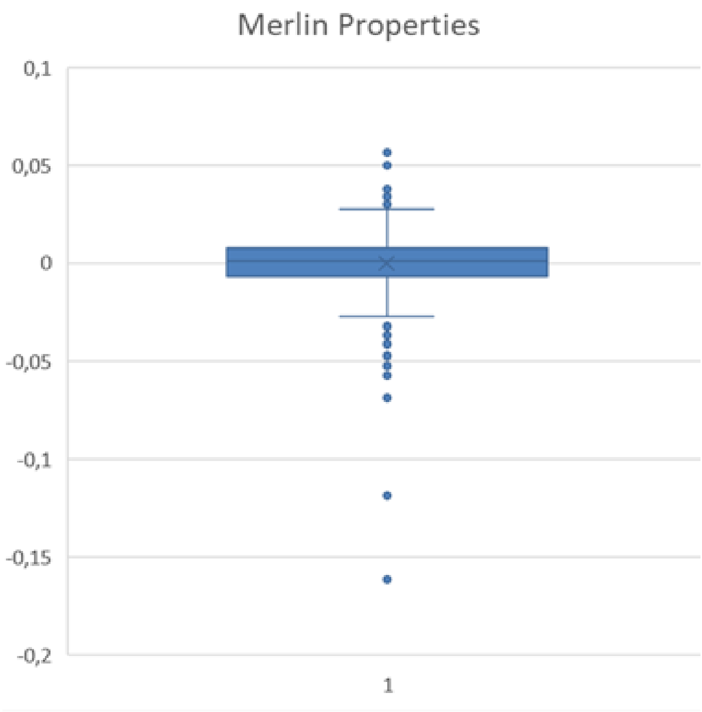
Source: Own development

Figure 9: Indra's box-plot



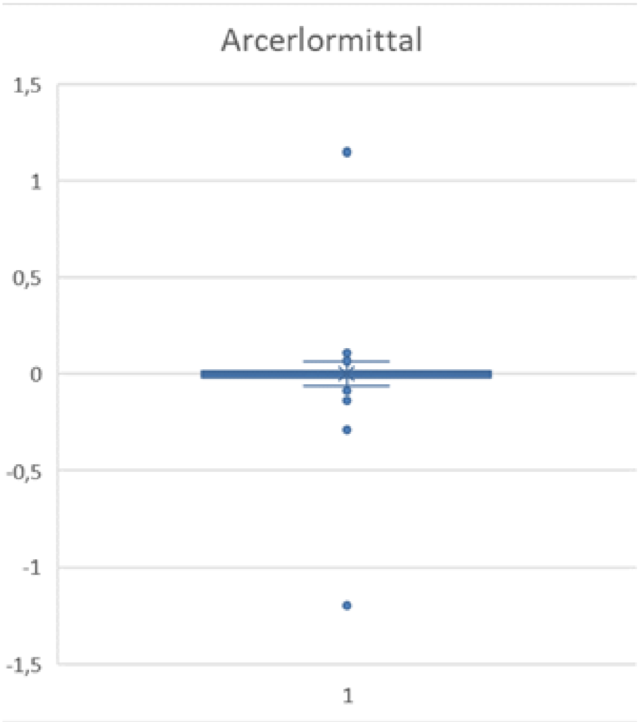
Source: Own development

Figure 10: Merlin properties's box-plot



Source: Own development

Figure 11: Arcelormittal's box-plot



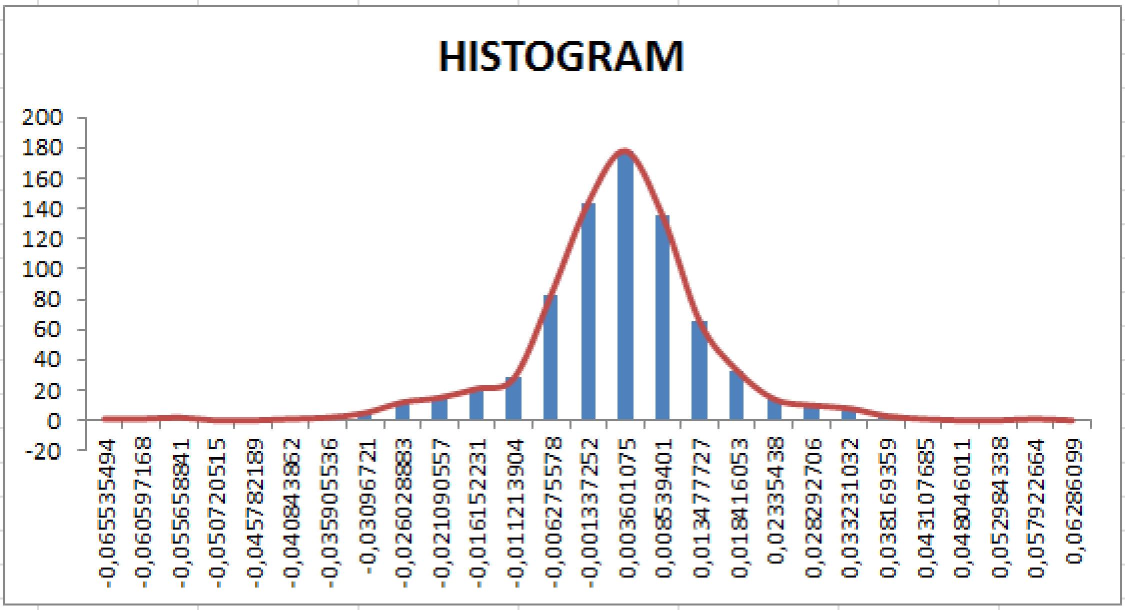
Source: Own development

As can see with the Box-plot of each of the daily returns of each asset, they follow a normal distribution, since inside the box we see the median, which is well proportioned both above and below, that is, it has the same proportion above as below the median. In addition we can also detect, through the external part of the box, that there is a considerable number of atypical observations, therefore we affirm that the normality test could not be performed adequately due to this determined number of atypical values.

To make sure that the assets satisfy a normal distribution, we proceed to elaborate a histogram of each of them, as well as a QQ-plot. This graphic consists in comparing the quantiles of the observed distribution with the quantiles of a normal distribution, if the data are more or less aligned with the line it will be considered that follows a normal distribution. However if the data are separated from the line it will be considered that I does not follow a normal distribution.

7.1. ABERTIS

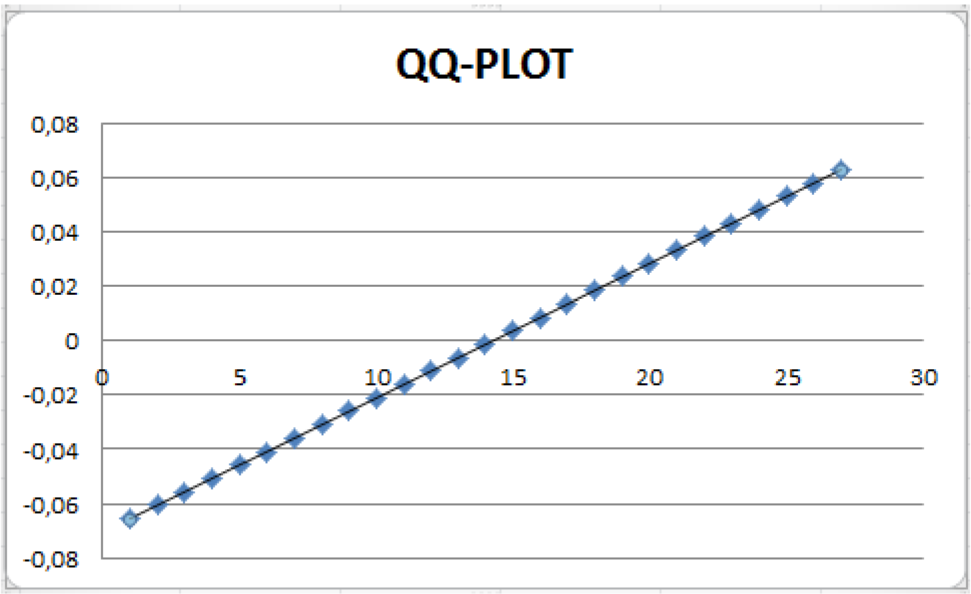
Figure 12: Abertis's histogram



Source: Own development

We can see that due to the skewness (-0,200), it tends to have the left tail more accentuated, although the difference is very small, because the skewness is very close to 0. In addition, the data is not concentrated so much in a single zone, but they are more spread out because their kurtosis is 5,382 and is not significantly high. We can say that Abertis follows a normal distribution.

Figure 13: Abertis's QQ-plot

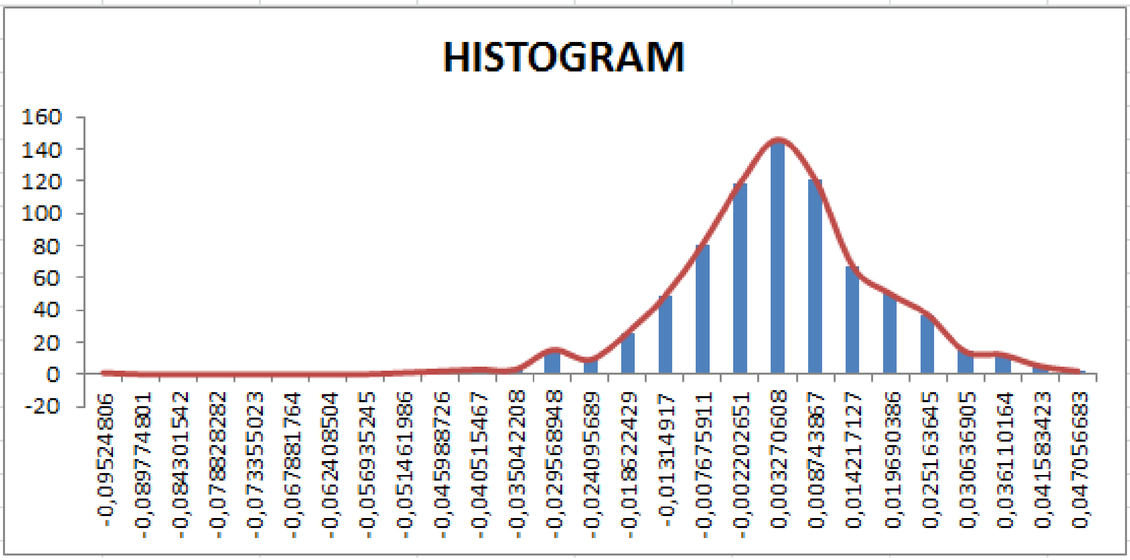


Source: Own development

As we can see, the QQ-plot shows us that Abertis follows a normal distribution, since all the values are associated to the tangent line.

7.2. <u>BANKINTER</u>

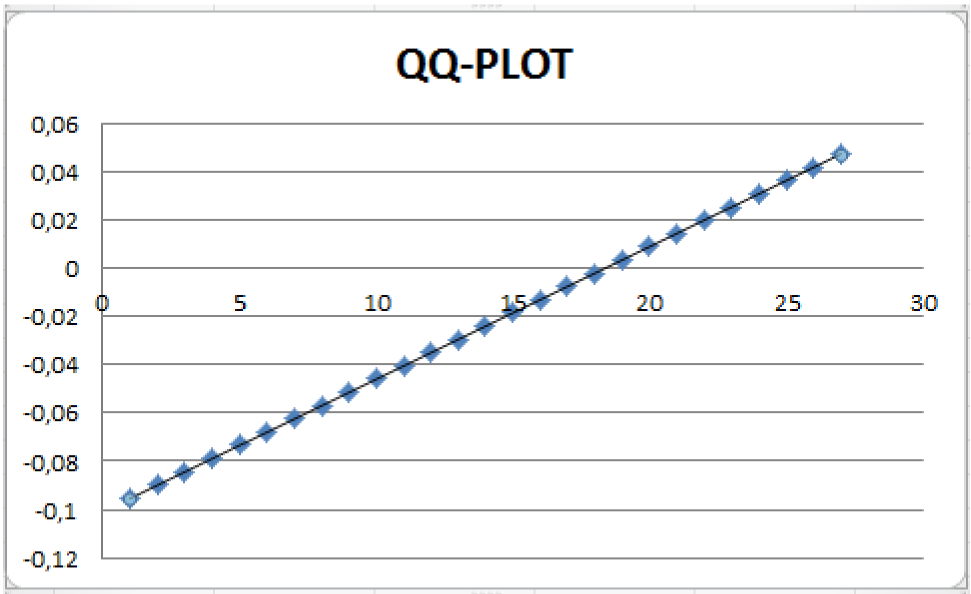
Figure 14: Bankinter's QQ-plot



Source: Own development

Observing the graphic, we can also say that Bankinter follows a normal distribution. Its skewness is -0,370, it is not a very different number than 0, so it also tends to have the tail a bit to the left. The kurtosis is 2,826, therefore being a small amount, the data is quite distributed. Like Abertis, Bankinter also follows a normal distribution.

Figure 15: Bankinter's QQ-plot

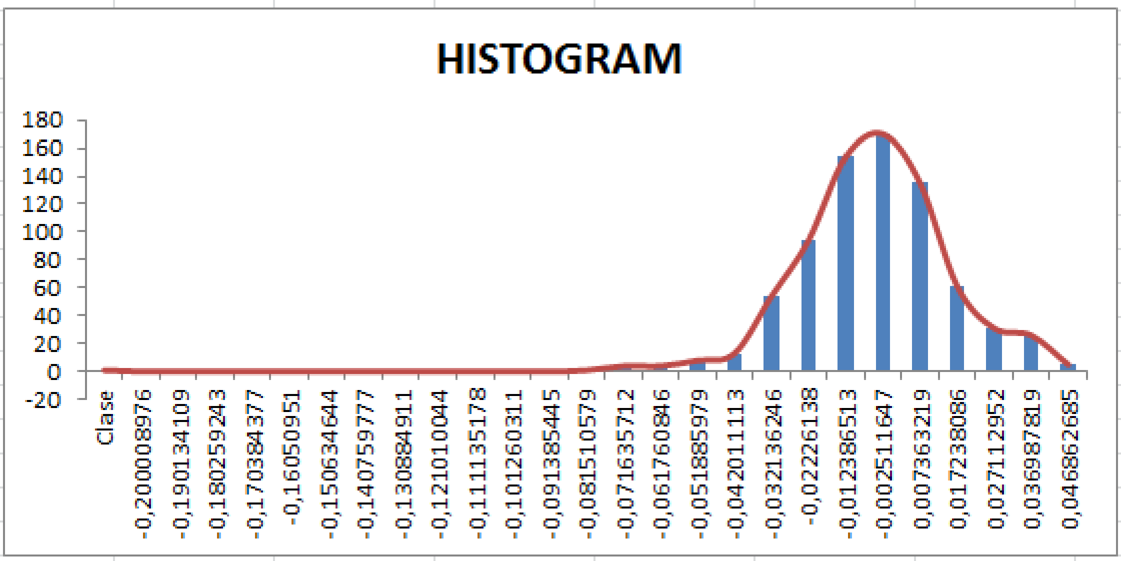


Source: Own development

Bankinter also follows a normal distribution because as we can see in the QQ-plot, the values are very close to the tangent line.

7.3. CAIXABANK

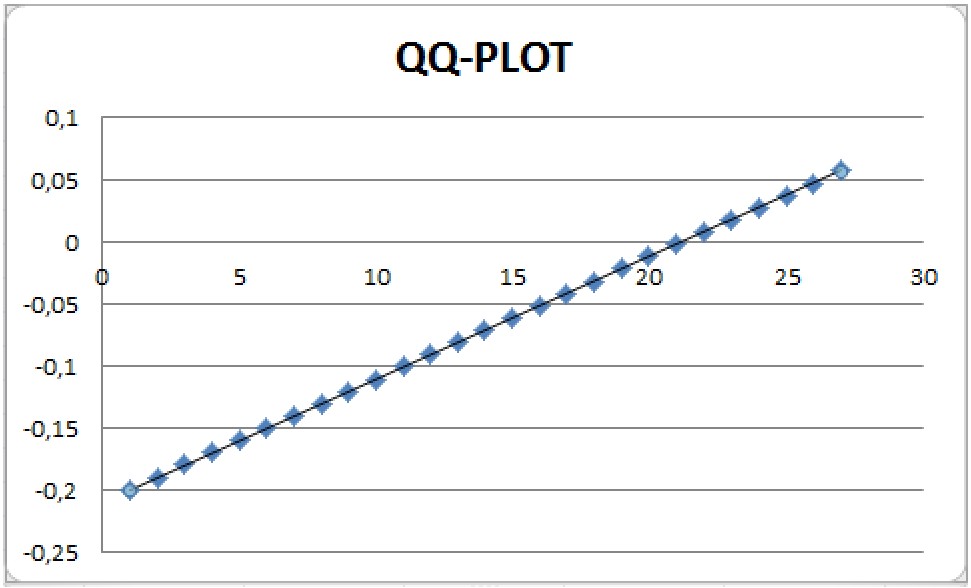
Figure 16: Caixabank's histogram



Source: Own development

In this case, Caixabank due to its skewness (-1,129), has the left tail more marked than the previous histograms, in addition the data are more concentrated because of its kurtosis (10,668). Caixabank also follows a normal distribution.

Figure 17: Caixabank's QQ-plot

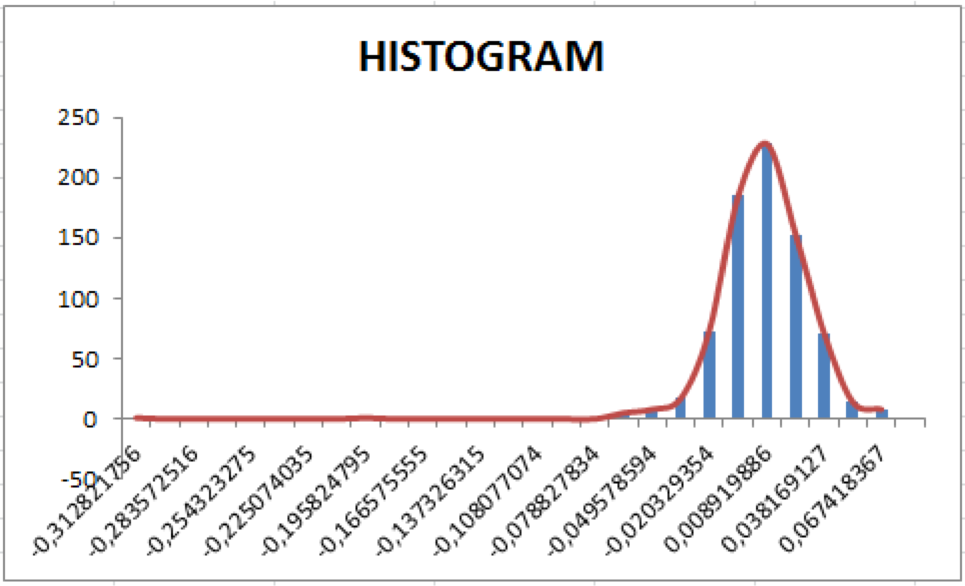


Source: Own development

Observing the QQ-plot, Caixabank follows a normal distribution, because its values are very close to the tangent line.

7.4. IAG

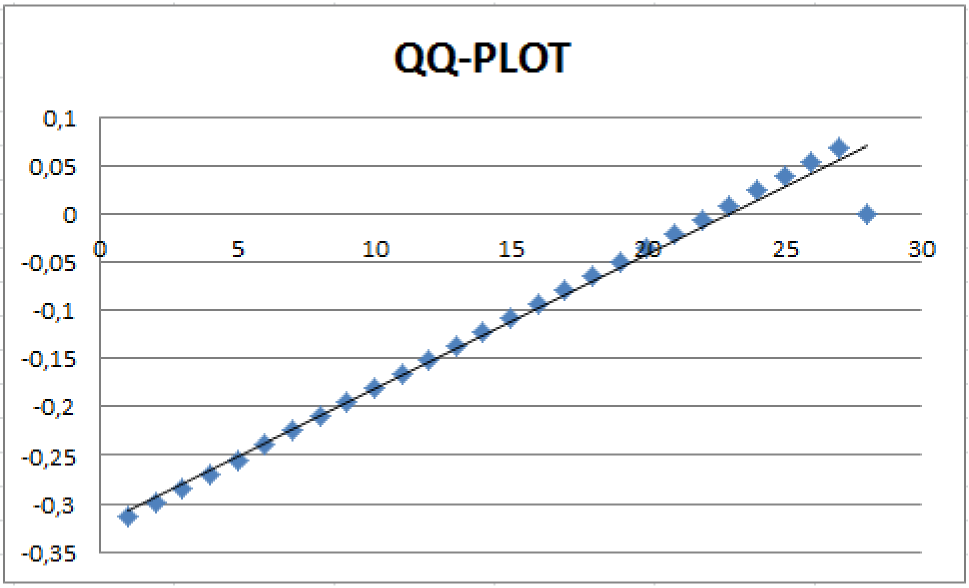
Figure 18: Iag's histogram



Source: Own development

Iag compared to the assets we have seen previously, has a higher skewness (-3,359) which makes the tail to the left is more marked and also has a fairly high kurtosis (38,551), which causes the data to concentrate a lot in one area.

Figure 19: Iag's QQ-plot

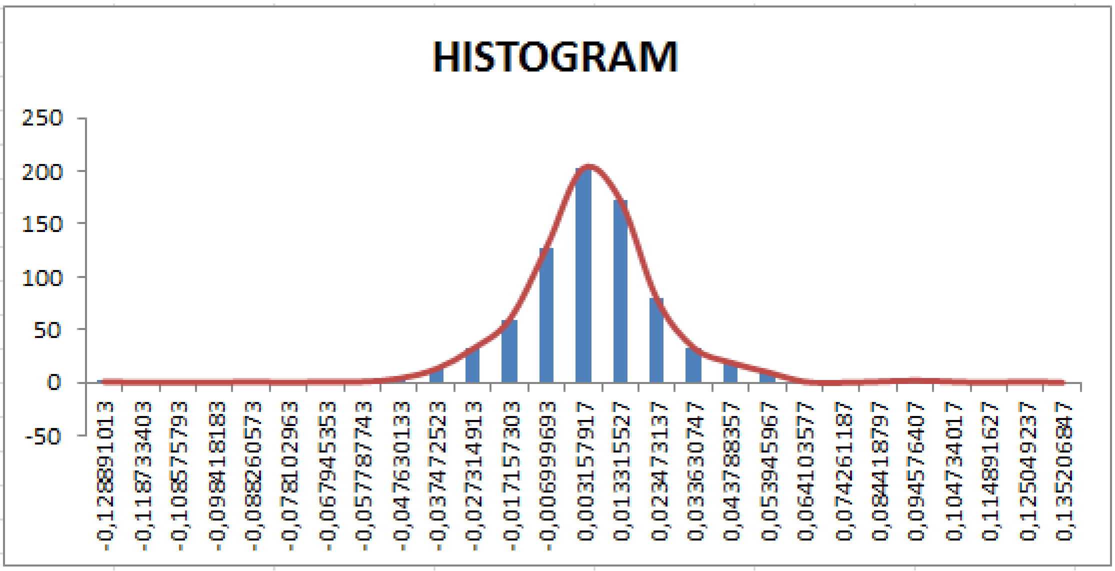


Source: Own development

Iag's QQ-plot shows that it follows a normal distribution, although unlike the previous QQ-plots, the values are not as centred as the others, but still follows a normal distribution.

7.5. <u>INDRA</u>

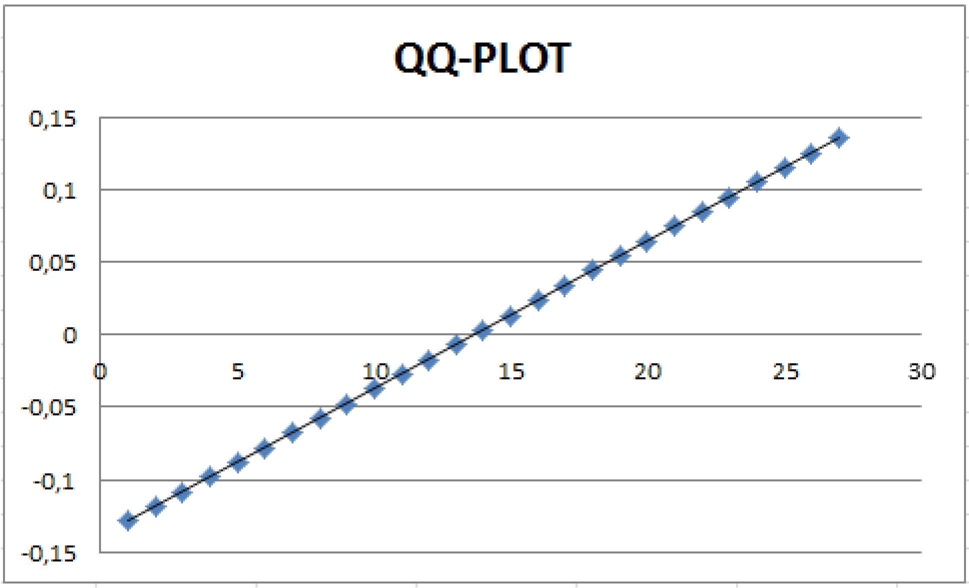
Figure 20: Indra's histogram



Source: Own development

Indra could say that it is a clear example that follows a normal distribution almost perfectly. By having skewness (0,591) so close to 0 and it is also positive; it has a little more right tail, although it is not very marked. Its kurtosis (7,961) is normal, neither very high nor very low, reason why the data are well distributed.

Figure 21: Indra's QQ-plot

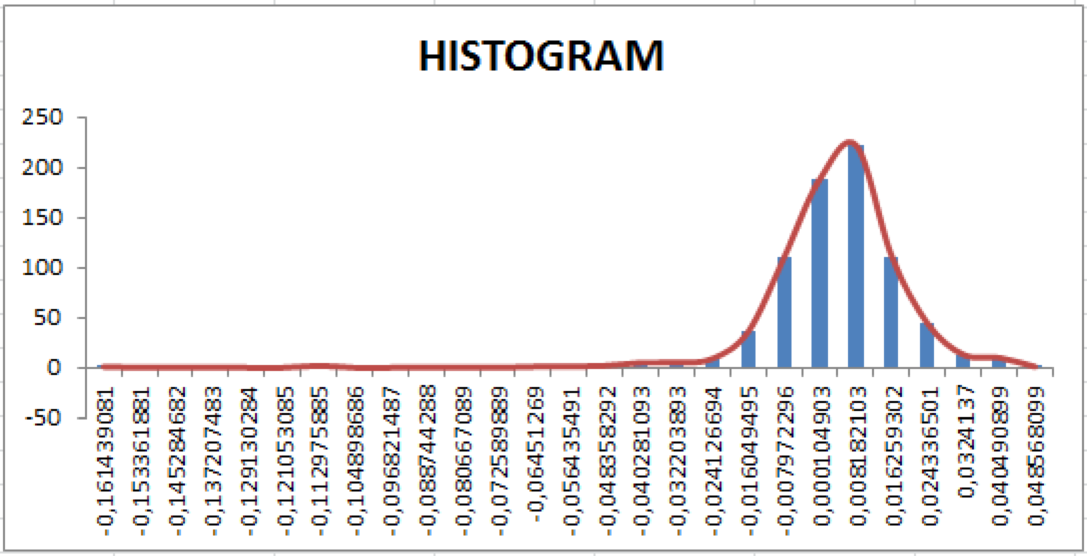


Source: Own development

According to Indra's QQ-plot we can say that it follows a normal distribution, because the values are very close to the tangent line.

7.6. <u>MERLIN PROPERTIES</u>

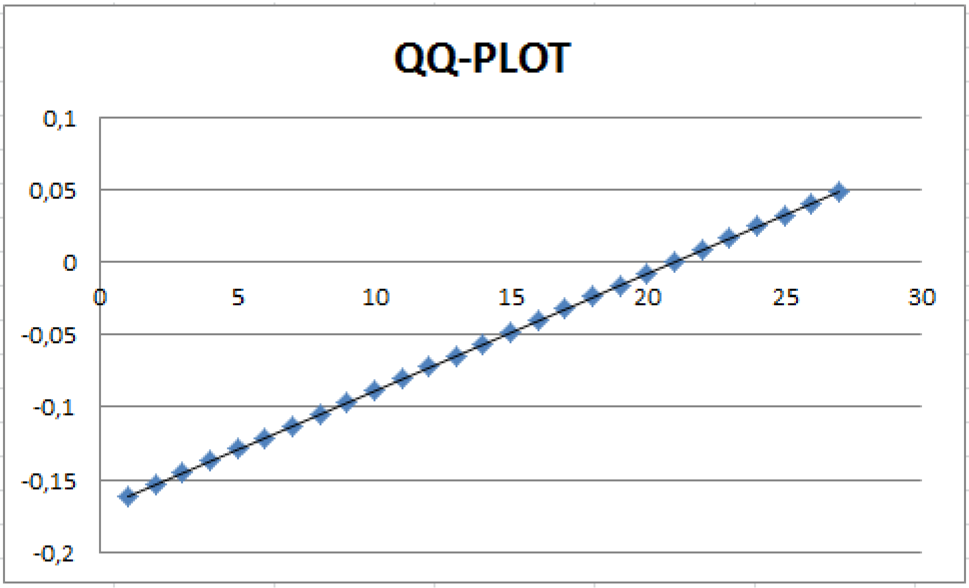
Figure 22: Merlin properties's histogram



Source: Own development

Merlin properties also tends to have more left tail, due to its skewness (-2,539) which is quite separated to 0. Its kurtosis is quite high, so the data is concerted in an area. We can also say that Merlin Properties follows a normal distribution.

Figure 23: Merlin properties's QQ-plot

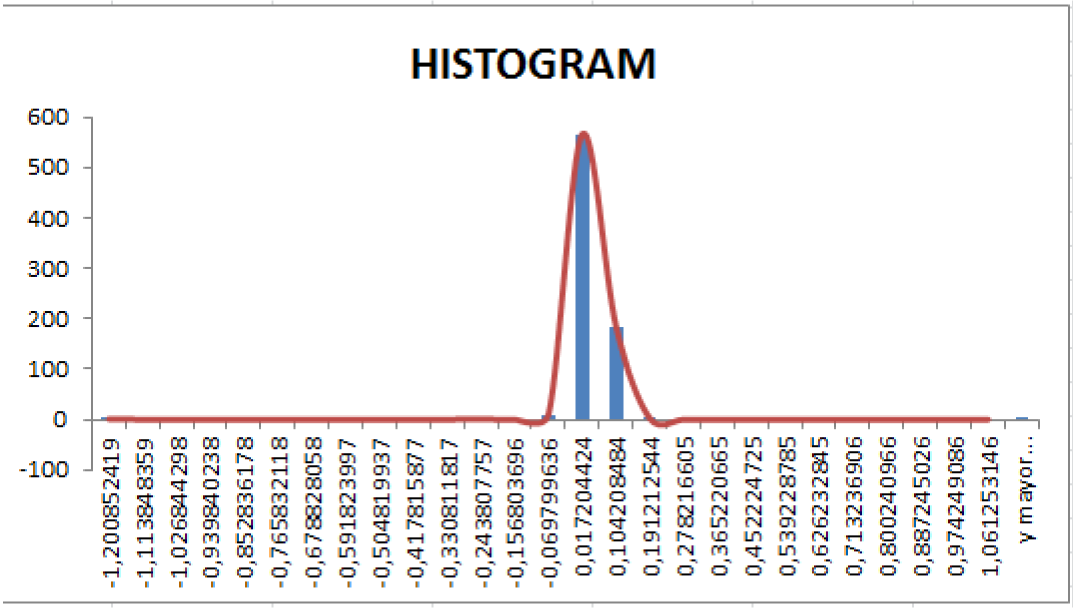


Source: Own development

Merlin Properties also follows a normal distribution, since the values are very close to the tangent line.

7.7. ARCELORMITTAL

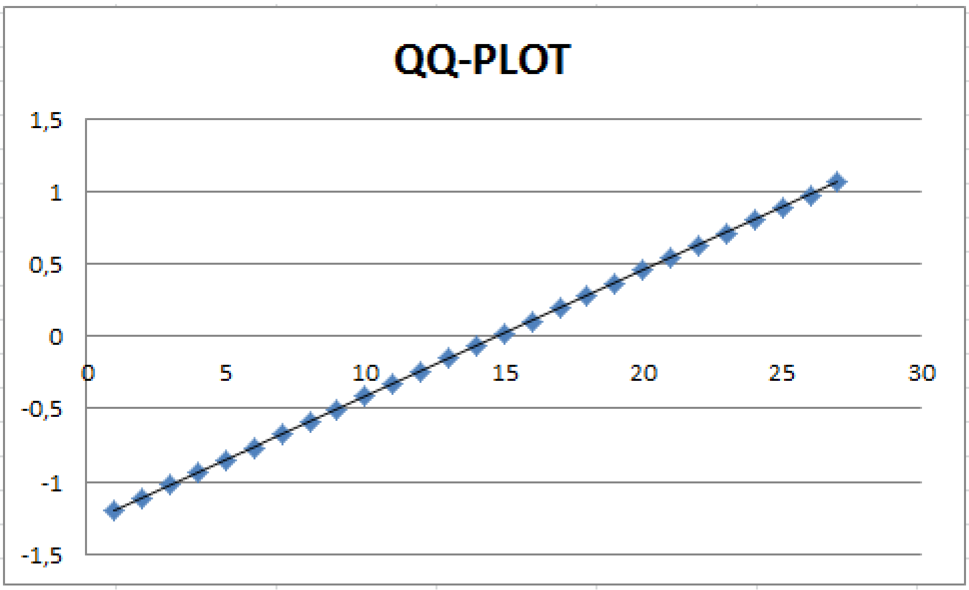
Figure 24: Arcelormittal's histogram



Source: Own development

Looking at the histogram we can see that its skewness (-0,985) is close to 0 and it is a negative number, therefore it has a left tail, although we hardly see any difference. Instead, its kurtosis (228,745) is very high, which means that almost all the data is concentred in an exaggerated way. As in the other cases Arcelormittal also follows a normal distribution.

Figure 25: Arcelormittal's QQ-plot



Source: Own development

According to Arcelormittal's QQ-plot we can say that it follows a normal distribution, because the return's values are very close to the tangent line.

CONCLUSION

To conclude with the final degree dissertation, first, we have sought information about Markowitz (1952), as well as his biography, as well as the theory he proposed in 1952, about optimal portfolio's, which brought out a book about his theory. The theory of the modern portfolio of Markowitz (1952) marked a before and after in the global economy.

Second, we also looked for information on the Value at Risk, since the information was necessary to be able to apply the Value at Risk in the Markowitz's (1952) model and be able to draw our own conclusions from the proposed discussion. Thirdly, through Excel, we made all the relevant calculations to subsequently be able to compare the Markowitz's (1952) model with the Value at Risk approach as a measure of risk and with the different levels of confidence that Value at Risk provides us.

Fourth, we set out to compare the results of the Markowitz's (1952) model with the results acquired by the Value at Risk approach as a measure of risk, following a normal distribution. Unfortunately, we were not able to compare the results previously mentioned, since according to Benati and Rizzi (2007), following a normal distribution, the results of Markowitz's (1952) model and the results with the Value at Risk approach are almost equal, so we had to look for other alternatives to be able to compare them. A possible alternative was to perform a normality test of each of the daily returns of the assets, although at the time of making the calculations, thanks to the development of a box plot of each of the daily returns of the assets we were able to observe that there are a lot of atypical values, therefore they give us too atypical numbers in the normality test.

Finally, to demonstrate that each of the daily returns of the assets follows a normal distribution, we develop a histogram of each one of them and also we develop a QQ-plot of the daily returns of each asset to reaffirm that theory.

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