NEWTON-OKOUNKOV BODIES OF EXCEPTIONAL CURVE PLANE VALUATIONS NON-POSITIVE AT INFINITY

CARLOS GALINDO, FRANCISCO MONSERRAT, JULIO JOSÉ MOYANO-FERNÁNDEZ, AND MATTHIAS NICKEL

1. Introduction

Newton-Okounkov bodies were introduced by Okounkov [19, 20, 21] and independently developed in greater generality by Lazarsfeld and Mustaţă [18], on the one hand, and Kaveh and Khovanskii [13], on the other.

The key idea is to associate a convex body to a big divisor on a smooth irreducible normal projective variety X, with respect to a specific flag of subvarieties of X, via the corresponding valuation on the function field of X. This turns out to be a good way to relate the convex geometry of that object with positivity aspects on the side of the algebraic geometry. More specifically, Newton-Okounkov bodies seem to be suitable to explain, from their convex structure, the asymptotic behavior of the linear systems given by the divisor and the valuation, as well as the structure of the Mori cone of X and positivity properties of divisors on X [2, 14, 15, 16, 17].

The computation of Newton-Okounkov bodies is a very hard task and, sometimes, their behaviour is unexpected, see Küronya and Lozovanu [17]. The case when the underlying variety is a surface is also very hard but there exist some known results which can help. We know that they are polygons with rational slopes and can be computed from Zariski decompositions of divisors.
Very recently, Ciliberto et al. studied in [4] the Newton-Okounkov bodies with respect to exceptional curve plane valuations ν, defined by divisorial valuations ν' with only one Puiseux exponent, centered at a point p in $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$, where $\mathbb{C}$ stands for the complex numbers. These valuations have both rank and rational rank equal to 2 and their transcendence degree equals zero. It is proved in [4] that the Newton-Okounkov bodies of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$, with respect to exceptional curve plane valuations, are triangles or quadrilaterals, where the vertices are given by the defining Puiseux exponent β', an asymptotic multiplicity $\hat{\mu}$ corresponding with ν' and a value in the segment $[0, \hat{\mu}/\beta']$. The asymptotic multiplicity $\hat{\mu}$ can also be used to formulate a generalization of Nagata's conjecture [6], see also [11]. The exact value of $\hat{\mu}$ is only known in some cases, including when $\beta' < 7 + 1/9$.

In this note we only announce a result which determines the Newton-Okounkov bodies of the previously mentioned line bundle with respect to *exceptional curve*

2010 *Mathematics Subject Classification.* Primary: 14C20.

Key words and phrases. Newton-Okounkov bodies; Flags; Exceptional curve plane valuations; Plane valuations non-positive at infinity .

The first three authors were partially supported by the Spanish Government Ministerio de Economía y Competitividad (MINECO), grants MTM2012-36917-C03-03 and MTM2015-65764-C3-2-P, as well as by Universitat Jaume I, grant P1-1B2015-02.

plane valuations non-positive at infinity——the proof and additional details will be published in a forthcoming paper.

Exceptional curve plane valuations non-positive at infinity are a large class of exceptional curve plane valuations, that can have any number of Puiseux exponents and are defined by flags $X \supset E \supset \{q\}$, where E is the last exceptional divisor obtained after a simple finite sequence of point blowing-ups starting at $\mathbb{P}^2$, and defines a plane divisorial valuation ν_E which is non-positive at infinity, cf. Galindo and Monserrat [10]. The valuations ν_E are centered at infinity (see Favre and Jonsson [9]) and present a behavior close to that of plane curves with only one place at infinity (see Abhyankar and Moh [1], and Campillo, Piltant and Reguera [3]).

We finish this introduction being more specific and saying that all the mentioned Newton-Okounkov bodies are triangles and we will give their vertices explicitly. Moreover, the anticanonical Iitaka dimension of infinitely many of the considered surfaces X is $-\infty$ and, in addition, their Picard numbers are arbitrarily large.

Recall that the number of vertices of the Newton-Okounkov body defined by a flag and a big divisor on a surface X is bounded by $2\rho + 2$, ρ being the Picard number of X [17], but the above mentioned results in [4] suggest that the bound could be applied even if we consider the flag on a projective model dominating X (and the Newton-Okounkov body associated to the pull-back of a big divisor on X). Our result can be regarded as new evidence supporting this conjecture.

The results presented in this short note were obtained during a visit of the fourth author to the University Jaume I. Previous studies and a large number of computations were done during the workshop *Positivity and valuations* held on February 2016 at the CRM in Barcelona. The authors wish to thank J. Roé and A. Küronya for stimulating their interest in Newton-Okounkov bodies as well as for their helpful comments and for pointing out a more customary name for our valuations.

2. The general setting

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. We will write $K(X)$ for the function field of X. Let us fix a flag of subvarieties

$$Y_\bullet := \{X = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_n = \{q\}\}$$

such that each $Y_i \subset X$ is irreducible, of codimension i and smooth at q. The point $q \in X$ is called the *center* of the flag.

One may associate to the flag $Y_\bullet$ a discrete valuation of rank n as follows. First, let $g_i = 0$ be the equation of Y_i in Y_{i-1} in a Zariski open set containg q, which is possible since Y_i has codimension i. Then, for $f \in K(X)$ we define

$$\nu_1(f) := \mathrm{ord}_{Y_1}(f), \quad f_1 = \left.\frac{f}{g_1^{\nu_1(f)}}\right|_{Y_1}$$

and, for $2 \leq i \leq n$,

$$\nu_i(f) := \mathrm{ord}_{Y_i}(f_{i-1}), \quad \text{where } f_i = f/g_{i-1}^{\nu_{i-1}(f_{i-1})}\Big|_{Y_i}.$$

Then the map $\nu_{Y_\bullet} : K(X) \setminus \{0\} \to \mathbb{Z}^n_{\mathrm{lex}}$ defined by the sequence of maps ν_i, $1 \leq i \leq n$, as $\nu_{Y_\bullet} := (\nu_1, \ldots, \nu_n)$ is a rank n discrete valuation and any maximal rank valuation comes from a flag [4, Th. 2.9]. Given a flag $Y_\bullet$ and a Cartier divisor D on X, the following subset of $\mathbb{R}^n_+$:

$$\Delta_{Y_\bullet}(D) := \overline{\bigcup_{m\geq 1} \left\{ \frac{\nu_{Y_\bullet}(f)}{m} \mid f \in H^0(X, mD) \setminus \{0\} \right\}},$$

where $\overline{\{\cdot\}}$ stands for the closed convex hull, is called to be the *Newton-Okounkov body* of D with respect to $Y_\bullet$.

Newton-Okounkov bodies are convex bodies such that

$$\mathrm{vol}_X(D) = n! \, \mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{Y_\bullet}(D)\right),$$

where $\mathrm{vol}_{\mathbb{R}^n}$ means Euclidean volume and

$$\mathrm{vol}_X(D) := \lim_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Moreover, given $D \neq D'$ two big divisors on X, they are numerically equivalent if and only if the associated Newton-Okounkov bodies coincide for all admissible flags on X [12]. Furthermore, in the case of surfaces, D and D' are numerically equivalent (up to negative components in the Zariski decomposition that do not go through q) if and only if the associated Newton-Okounkov bodies coincide for all flags centered at q, cf. Roé [22].

3. Exceptional curve plane valuations and Newton-Okounkov bodies

In this section, we introduce the family of flags for which we are interested in computing Newton-Okounkov bodies. Let $\mathbb{P}^2$ be the complex projective plane, and p any point in $\mathbb{P}^2$. Let R be the local ring of $\mathbb{P}^2$ at p, and write F for the field of fractions of R. Valuations ν of F centered at R are in one-to-one correspondence with simple sequences of point blowing-ups whose first center is p [23, p. 121].

$$\eta : \quad \cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \mathbb{P}_2.$$

The *cluster of centers* of η will be denoted by $\mathcal{C} = \{p = p_1, p_2, \ldots\}$ and we say that a point p_i is proximate to p_j, $i > j$, written $p_i \to p_j$, whenever p_i belongs to the strict transform of the exceptional divisor E_j obtained by blowing-up p_j. These valuations were classified by Spivakovsky in [23]. We are interested in the class of exceptional curve valuations (in the terminology of Favre and Jonsson [8]) which corresponds to Case 3 in [23] and to type C of Delgado, Galindo and Núñez [5]. These valuations are characterized by the fact that there exists a point $p_r \in \mathcal{C}$ such that $p_i \to p_r$ for all $i > r$.

Notice that if we consider the surface X_r obtained after blowing-up p_r and the flag

$$E_\bullet := \{X = X_r \supset E_r \supset \{q := p_{r+1}\}\},$$

then the valuation ν is just $\nu_{E_\bullet}$. According to the above mentioned, for an element $f \in R = \mathcal{O}_{\mathbb{P}^2_p}$, we have $\nu(f) = (\nu_1(f), \nu_2(f))$ with $\nu_1(f) = \nu_{E_r}(f)$ and

$\nu_2(f) = \mathrm{ord}_q\left(\pi^*(f)/z_r^{\nu_1(f)}\right)$, where $\pi : X_r \to X_0$ is the composition of the first r point blowing-ups in η, $z_r = 0$ a local equation for E_r and $\pi^*(f)/z_r^{\nu_1(f)}$ may be seen as a function on E_r. Notice that

$$\nu_2(f) = \left(\pi^*(f)/z_r^{\nu_1(f)}, E_r\right)_q,$$

where $(\cdot\ ,\cdot)_q$ denotes the intersection multiplicity at q.

The divisor E_r is defined by a map $\pi : X_r \to \mathbb{P}^2$. The intersections of the strict transforms of the exceptional divisors in X_r are represented by the so-called dual graph of π (or of ν_{E_r}). The *geodesic of the dual graph* is defined to be the set of edges (and vertices) in the path joining the vertices corresponding to E_1 and E_r. Additionally, for $i = 1, \ldots, r$, *φ_i will denote an analytically irreducible germ of curve at p whose strict transform is transversal to E_i at a nonsingular point of the exceptional locus.*

In spite of their importance, very few explicit examples of Newton-Okounkov bodies can be found in the literature. We are interested in an explicit computation of the Newton-Okounkov bodies of flags $E_\bullet$ defined by exceptional curve plane valuations ν with respect to the divisor class H given by the pull-back of the line-bundle $\mathcal{O}_{\mathbb{P}^2}(1)$, which we will denote by $\Delta_\nu(H)$. These Newton-Okounkov bodies were studied in [4] for valuations with only one Puiseux exponent [5]. We devote the next section to announce a result which provides an explicit computation of bodies $\Delta_\nu(H)$ for a large class of valuations ν as above which can have an arbitrary number of Puiseux exponents. Its proof and further details will appear elsewhere.

4. The result

For a start and without loss of generality, we set $(X : Y : Z)$ projective coordinates in $\mathbb{P}^2$, L the line $Z = 0$, which we call the line at infinity, and assume that p is the point with projective coordinates $(1 : 0 : 0)$. Consider also coordinates $x = X/Z$ and $y = Y/Z$ in the affine chart defined by $Z \neq 0$ and local coordinates $u = Y/X$ and $v = Z/X$ around p. With the previous notation, set ν_r a divisorial valuation of the fraction field $K = K(\mathbb{P}^2)$, given by a finite sequence of point blowing-ups $\pi : X_r \to X_0 = \mathbb{P}^2$, whose first blowing-up is at p and is defined by the exceptional divisor E_r. We say that ν_r is *non-positive at infinity* whenever $r \geq 2$, L passes through $p_1 = p$ and p_2 and $\nu_r(f) \leq 0$ for all $f \in \mathbb{C}[x, y] \setminus \{0\}$. Notice that our valuations are valuations centered at infinity [9].

Definition 4.1. An exceptional curve plane valuation ν of K centered at R is said to be *non-positive at infinity* whenever it is given by a flag

$$E_\bullet := \{X = X_r \supset E_r \supset \{q := p_{r+1}\}\}$$

such that $\nu = \nu_{E_r}$ is non-positive at infinity.

Recall from [7] that the volume of a valuation ν_r as above is defined as

$$\mathrm{vol}(\nu_r) := \lim_{m\to\infty} \frac{\dim_{\mathbb{C}}(R/P_m)}{m^2/2},$$

where $P_m = \{f \in R | \nu_r(f) \geq m\} \cup \{0\}$. Divisorial valuations non-positive at infinity have been studied in [10] and admit an easy characterization:

Theorem 4.2. *Let ν_r be a divisorial valuation of K centered at p. The valuation ν_r is non-positive at infinity if and only if*

$$\nu_r(v)^2 \geq [\mathrm{vol}\,(\nu_r)]^{-1}.$$

From the previous condition, it is clear that one can find valuations non-positive at infinity with as many Puiseux exponents as one desires. We should also notice the existence of families of surfaces X_r, defined by valuations ν_r as above, whose anticanonical Iitaka dimension is $-\infty$ [10].

To conclude, we state our result on the Newton-Okounkov bodies $\Delta_\nu(H)$ corresponding to exceptional curve plane valuations non-positive at infinity. Before that, we notice that the proof is based on the fact that Zariski decompositions of certain divisors describe Newton-Okounkov bodies in the case of surfaces [18] and we are able to provide an explicit description of the Zariski decomposition of those divisors, which are $H - tE_r$, where H is the total transform on X_r of a line in $\mathbb{P}^2$ that does not pass through p, and $t \in [0, \nu_r(v)]$.

We will set $\nu_i = \nu_{E_i}$ for $1 \leq i \leq r$.

Theorem 4.3. *Let ν be an exceptional curve plane valuation non-positive at infinity and consider its corresponding flag*

$$E_\bullet := \{X = X_r \supset E_r \supset \{q := p_{r+1}\}\}.$$

Then the Newton-Okounkov body $\Delta_\nu(H)$ is a triangle; more precisely:

(a) *If $\nu_r(v)^2 > [\mathrm{vol}\,(\nu_r)]^{-1}$, then $\Delta_\nu(H)$ is:*

  (1) *A triangle with vertices*

$$(0,0), (\nu_r(v), 0), \left(\frac{1}{\mathrm{vol}(\nu_r)\nu_r(v)}, \frac{1}{\nu_r(v)}\right),$$

  whenever q is a free point in E_r.

  (2) *A triangle with vertices*

$$(0,0), (\nu_r(v), \nu_\ell(v)), \left(\frac{1}{\mathrm{vol}(\nu_r)\nu_r(v)}, \frac{\nu_r(\varphi_\ell)}{\nu_r(v)}\right),$$

  whenever q is a satellite point in $E_\ell \cap E_r$, $\ell < r$ and the vertex given by E_ℓ in the dual graph of ν_r belongs to the geodesic.

  (3) *A triangle with vertices*

$$(0,0), (\nu_r(v), \nu_\ell(v)), \left(\frac{1}{\mathrm{vol}(\nu_r)\nu_r(v)}, \frac{\nu_r(\varphi_\ell)+1}{\nu_r(v)}\right),$$

  otherwise.

(b) *If $\nu_r(v)^2 = [\mathrm{vol}\,(\nu_r)]^{-1}$, then $\Delta_\nu(H)$ is:*

  (1) *A triangle with vertices*

$$(0,0), (\nu_r(v), 0), \left(\nu_r(v), \frac{1}{\nu_r(v)},\right),$$

  whenever q is a free point in E_r.

(2) *A triangle with vertices*

$$(0,0)\,,\left(\nu_r(v),\frac{1-\mathrm{vol}\,(\nu_r)}{\mathrm{vol}\,(\nu_r)\,\nu_r(v)}\right),\left(\nu_r(v),\frac{1}{\mathrm{vol}\,(\nu_r)\,\nu_r(v)}\right),$$

otherwise.

References

[1] S.S. Abhyankar, T.T. Moh, Newton-Puiseux expansion and generalized Tschirnhausen transformation I and II, *J. reine angew. Math.* **260** (1973) 47–83 and **261** (1973) 29–54.
[2] S. Boucksom, A. Küronya, C. Maclean, T. Szemberg, Vanishing sequences and Okounkov bodies, *Math. Ann.* **361** (2015) 811–834.
[3] A. Campillo, O. Piltant, A. Reguera, Curves and divisors on surfaces associated to plane curves with one place at infinity, *Proc. London Math. Soc.* **84** (2002) 559–580.
[4] C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, and C. Shramov, Newton-Okounkov bodies sprouting on the valuative tree, preprint, arXiv: 1602.0207.
[5] F. Delgado, C. Galindo, A. Núñez, Saturation for valuations on two-dimensional regular local rings, *Math. Z.* **234** (2000) 519–550.
[6] M. Dumnicki, B. Harbourne, A. Küronya, J. Roé, T. Szemberg, Very general monomial valuations of $\mathbb{P}^2$ and a Nagata type conjecture, preprint, arxiv:1312.5549.
[7] L. Ein, R. Lazarsfeld and K. Smith, Uniform approximation of Abhyankar valuations in smooth function fields, *Amer. J. Math.* **125** (2003) 409–440.
[8] C. Favre, M. Jonsson, *The valuative tree,* Lect. Notes Math. **1853**. Springer-Verlag, 2004.
[9] C. Favre, M. Jonsson, Eigenvaluations, *Ann. Sci. Éc. Norm. Sup.* **40** (2007) 309–349.
[10] C. Galindo, F. Monserrat, The cone of curves and the Cox ring of a rational surface given by divisorial valuations, *Adv. Math.* **290** (2016) 1040–1061.
[11] C. Galindo, F. Monserrat, J.J. Moyano-Fernández, Minimal plane valuations, preprint, arXiv:1609.05236.
[12] S-Y. Jow, Okounkov bodies and restricted volumes along very general curves, *Adv. Math.* **223** (2010) 1356–1371.
[13] K. Kaveh, A. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, *Ann. Math.* **176** (2012) 925–978.
[14] A. Küronya, V. Lozovanu, Local positivity of linear series on surfaces, arXiv:1411.6205.
[15] A. Küronya, V. Lozovanu, Positivity of line bundles and Newton-Okounkov bodies, preprint, arXiv:1506.06525.
[16] A. Küronya, V. Lozovanu, Infinitesimal Newton-Okounkov bodies and jet separation, preprint, arXiv:1507.04339.
[17] A. Küronya, V. Lozovanu, C. Maclean, Convex bodies appearing as Okounkov bodies of divisors, *Adv. Math.* **229** (2012) 2622-2639.
[18] R. Lazarsfeld, M. Mustaţă, Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Sup.* **42** (2009), 783–835
[19] A. Okounkov, Brunn-Minkowski inequality for multiplicities, *Invent. Math.* **125** (1996) 405–411.
[20] A. Okounkov, Note on the Hilbert polynomial of a spherical variety, *Funct. Anal. Appl.* **31** (1997) 138–140.
[21] A. Okounkov, Why would multiplicities be log-concave?, in *The orbit method in geometry and physics*, Progr. Math. **213** (2003) 329–347.
[22] J. Roé, Local positivity in terms of Newton Okounkov bodies, preprint, arXiv:1505.02051.
[23] M. Spivakovsky, Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990) 107–156.
[24] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, *Ann. Math.* **76** (1962), 560–615.

Universitat Jaume I, Campus de Riu Sec, Departamento de Matemáticas & Institut Universitari de Matemàtiques i Aplicacions de Castelló, 12071 Castellón de la Plana, Spain
E-mail address: galindo@uji.es
E-mail address: moyano@uji.es

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia (Spain).
E-mail address: framonde@mat.upv.es

Goethe-Universität Frankfurt, FB Informatik und Mathematik, 60054 Frankfurt am Main, Germany
E-mail address: nickel@math.uni-frankfurt.de