δ-Sequences and Evaluation Codes defined by Plane Valuations at Infinity

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ABSTRACT

We introduce the concept of δ-sequence. A δ-sequence Δ generates a well-ordered semigroup S in $\mathbb{Z}^2$ or $\mathbb{R}$. We show how to construct (and compute parameters) for the dual code of any evaluation code associated with a weight function defined by Δ from the polynomial ring in two indeterminates to a semigroup S as above. We prove that this is a simple procedure which can be understood by considering a particular class of valuations of function fields of surfaces, called plane valuations at infinity. We also give algorithms to construct an unlimited number of δ-sequences of the different existing types, and so this paper provides the tools to know and use a new large set of codes.

1. Introduction

BCH and Reed-Solomon codes can be decoded from the sixties by using the Berlekamp-Massey algorithm [**6**, **25**]. A paper by Sakata [**33**] allowed to derive the Berlekamp-Massey-Sakata algorithm which can be used to efficiently decode certain Algebraic Geometry codes on curves [**39**]. Indeed, this last algorithm allows us to get fast implementations of the modified algorithm of [**22**, **37**] (see [**23**, **21**]) and of the majority voting scheme for unknown syndromes of Feng and Rao [**11**] (see [**23**, **24**, **34**, **35**]). For a survey on the decoding of Algebraic Geometry codes one can see [**20**].

The so-called order functions were introduced in [**19**], which in this initial stage had as image set a sub-semigroup of the set of nonnegative integers. An order function defines a filtration of vector spaces contained in its definition domain which, together with an evaluation map, provide two families of error correcting codes (evaluation codes and their duals). We note that the one-point geometric Goppa codes or weighted Reed-Muller codes can be regarded as codes given by order functions.

A type of particularly useful order functions are the weight functions. Goppa distance and Feng-Rao distances (also called order bounds) are lower bounds for the minimum distance of their associated dual codes which can be decoded by using the mentioned Berlekamp-Massey-Sakata algorithm, correcting a number of errors that depends on the above bounds [**19**, **40**]. Furthermore, Matsumoto in [**26**] proved that their associated order domains are affine coordinate rings of algebraic curves with exactly one place at infinity.

Recently in [**16**], the concept of order function (and the related ones of weight function and order domain) have been enlarged by admitting that the image of those functions can be a well-ordered semigroup. Order domains are close to Groebner algebras and they allow to use the theory of Groebner basis [**5**]. This enlargement provides a greater variety of evaluation codes and it has the same advantages (bounds of minimum distance and fast decoding) that we had with the first defined concept.

Weight functions were introduced to give an elementary treatment to some Algebraic Geometry codes. Nevertheless, a deeper study of these functions with the help of Algebraic Geometry may derive in obtaining new good linear codes. This is the line of this paper. The main notion we introduce is an extension to elements in $\mathbb{Z}^2$, $\mathbb{Q}$ and $\mathbb{R}$ of the classical concept of δ-sequence defined by the Abhyankar-Moh conditions. These δ-sequences provide valuations in function fields of surfaces (related to curves with only one place at infinity) and, for that reason, weight functions. This allows us to focus our development to an application in Coding Theory that consists of studying the evaluation codes given by those weight functions.

Valuations and weight functions are very close objects as one can see in [40]. Apart from the simpler case of curves, it is only available a classification of valuations of function fields of nonsingular surfaces (also called plane valuations) [43, 38, 17], that allows us to decide which of them are suitable for providing, in an explicit manner, order domains and evaluation codes. The first examples that use that classification were given in [40] and a more systematic development can be found in [15]. Although both papers have the same background, they provide different types of examples and it seems that examples in [40] cannot be obtained from the development in [15]. In this paper we give theoretic foundations to provide families of weight functions that contain as particular cases the examples concerning plane valuations given in [40]. This leads us to a deeper knowledge of the involved evaluation codes what allows us to get explicitly a large set of codes (and associated parameters) from a simple input which can be easily determined.

More explicitly, in [15] it is assumed that R is a 2-dimensional Noetherian regular local ring with quotient field K and that R has an algebraically closed coefficient field k of arbitrary characteristic. By picking a plane valuation ν (of K centered at R) belonging to any type of the above mentioned classification, except the so called divisorial valuations, it is found an order domain D attached to the weight function $-\nu$, whose image semigroup is the value semigroup $S := \{ f \in R \setminus \{0\} | \nu(f) \geq 0 \}$ of ν. Moreover, in that paper parametric equations and examples of those weight functions, and also bounds for the minimum distance of the corresponding dual evaluation codes are given. In this paper, we shall consider a different point of view. This is to look for weight functions, whose order domain is the polynomial ring in two indeterminates $T := k[x, y]$, derived from valuations of the quotient field $k(x, y)$ of T, where, now, k needs not be algebraically closed. The mentioned paper [40] provides some examples of this type with monomial and non-monomial associated ordering, but no additional explanation is supplied.

We shall show that the key to get this last type of weight functions (with value semigroup generated by the so-called δ-sequences) is to use for their construction certain class of plane valuations, that we shall name plane valuations at infinity (see Definition 4.5 and Proposition 4.10). These valuations are those given by certain families of infinitely many plane curves, all of them having only one place at infinity. This constitutes, in some sense, an extension of the above result by Matsumoto.

Our development allows us to reach our main goal, which consists of explicitly constructing weight functions from T, attached to plane valuations at infinity, for which it is easy to determine its image semigroup and the filtration of vector spaces in T we need to construct the codes, and either to bound or to determine the corresponding Feng-Rao and Goppa distances (see Theorems 4.9 and 5.3 and Proposition 5.4). The unique input to compute those data is the so-called δ-sequence of the weight function which also determines it. It is also worth adding that the obtained image semigroups have a behavior close to the one of telescopic semigroups and that the so-called approximates of the valuation at infinity (see Definition 4.6) make easy the computation of the mentioned filtration.

The implementation of these new codes is very simple since to compute the data to use them is straightforward from the δ-sequences that define the weight functions; this paper provides algorithms (if necessary) to get an unlimited number of δ-sequences of any existing type. In fact, to provide a family of codes $\{E_\alpha\}_{\alpha \in S}$ and its dual family $\{C_\alpha\}_{\alpha \in S}$ with alphabet code k

(k is any finite field), we only need to pick a δ-sequence Δ (Definition 4.7) and a set $\mathfrak{A}$ of n different points in k^2. Δ can be finite (included in $\mathbb{Z}^2$ or $\mathbb{R}$) or infinite (included in $\mathbb{Q}$); in the last case, we need only a finite subset of Δ that depends on k and $\mathfrak{A}$. To obtain δ-sequences is easy following the procedure given in Section 4.3. Moreover S is the semigroup spanned by Δ and must be ordered lexicographically when it is in $\mathbb{Z}^2$ and usually ordered otherwise. E_α is the image $ev(O_\alpha)$ of a vector space O_α of $k[x, y]$, where ev is the map $ev : k[x, y] \to k^n$ that evaluates polynomials at the points in $\mathfrak{A}$. C_α is the dual vector space of E_α. Finally, bases for the spaces O_α can be computed without difficulty since they are formed by polynomials as in (2.3) (see the beginning of Section 4.3), where the exponents are obtained from Δ by an algorithm similar to Euclid's one (see the beginning of Section 4.2).

Next, we briefly describe the content of the paper. Valuations on function fields of surfaces and their classification are fundamental for understanding the development of this paper and Section 3 is devoted to provide a short summary for it. We only use a particular subset of these valuations, plane valuations at infinity. Although the definition is given in Section 4, to manipulate these valuations we need some knowledge about plane curves with only one place at infinity and this information is displayed in Section 2. The development of this section shortens Section 3 due to the closeness between them. Section 4 contains the core of the paper: we give the definition of δ-sequence and algorithms for constructing δ-sequences. We also explain the way to use them to get weight functions associated with valuations at infinity and how to compute bases of the vector spaces filtration needed to obtain the corresponding codes. There are δ-sequences giving rise to valuations at infinity only for three of the five types of the above mentioned classification for plane valuations. These δ-sequences provide weight functions with monomial and non-monomial corresponding orderings. Some examples are also supplied in this section. Finally, Section 5 studies the Feng-Rao and Goppa distances for dual codes given by δ-sequences (see Theorem 5.3 and Proposition 5.4). A useful property to study these distances is that δ-sequences generate semigroups similar to telescopic ones, which we call generalized telescopic semigroups. Moreover, the δ-sequences which are in $\mathbb{Z}^2$ span simplicial semigroups and thus we can compute the least Feng-Rao distance using an algorithm by Ruano [**32**]. We end this paper proving, in Proposition 5.6, that Reed-Solomon codes are a particular case of codes given by δ-sequences included in $\mathbb{Z}^2$ and giving several examples of dual codes (showing their parameters) defined by δ-sequences of all mentioned types.

2. *Plane curves with only one place at infinity*

We devote this section to summarize several known results concerning plane curves with only one place at infinity because this geometric concept supports the definition of those weight functions that will be useful for our purposes. References for the subject are [**4**, **3**, **36**, **29**, **41**, **13**]. Some of the results concerning this type of curves have been used by Campillo and Farrán in [**8**] for computing the Weierstrass semigroup and the least Feng-Rao distance of the corresponding Goppa codes attached to singular plane models for curves with only one place at infinity. We begin with the definition of a key concept for this paper. Denote by $\mathbb{N}_{>(\geq)0}$ the set of positive (nonnegative) integers.

DEFINITION 2.1. A *δ-sequence in* $\mathbb{N}_{>0}$ is a finite sequence of positive integers $\Delta = \{\delta_i\}_{i=0}^g$, $g \geq 0$, satisfying the following three conditions
(1) If $d_i = \gcd(\delta_0, \delta_1, \ldots, \delta_{i-1})$, for $1 \leq i \leq g+1$, and $n_i = d_i/d_{i+1}$, $1 \leq i \leq g$, then $d_{g+1} = 1$ and $n_i > 1$ for $1 \leq i \leq g$.
(2) For $1 \leq i \leq g$, $n_i\delta_i$ belongs to the semigroup generated by $\delta_0, \delta_1, \ldots, \delta_{i-1}$, that we usually denote $\langle \delta_0, \delta_1, \ldots, \delta_{i-1} \rangle$.
(3) $\delta_0 > \delta_1$ and $\delta_i < \delta_{i-1}n_{i-1}$ for $i = 2, 3, \ldots, g$.

Above conditions are usually called Abhyankar-Moh conditions. We denote by S_Δ the sub-semigroup of $\mathbb{N}_{\geq 0}$ spanned by Δ.

Along this paper $\mathbb{P}^2_k$ (or $\mathbb{P}^2$ for short) stands for the projective plane over a field k of arbitrary characteristic. Now, let us state the definition of plane curve with only one place at infinity.

DEFINITION 2.2.   Let L be the line at infinity in the compactification of the affine plane to $\mathbb{P}^2$. Let C be a projective absolutely irreducible curve of $\mathbb{P}^2$ (i.e., irreducible as a curve in $\mathbb{P}^2_{\bar{k}}$, $\bar{k}$ being the algebraic closure of k). We shall say that C *has only one place at infinity* if the intersection $C \cap L$ is a single point p (the one at infinity) and C has only one branch at p which is rational (that is, defined over k).

Set C a curve with only one place at infinity. Denote by K the quotient field of the local ring $\mathcal{O}_{C,p}$; the germ of C at p defines a discrete valuation on K that we set $\nu_{C,p}$, which allows us to state the following

DEFINITION 2.3.   Let C be a curve with only one place at infinity given by p. The *semigroup at infinity* of C is the following sub-semigroup of $\mathbb{N}_{\geq 0}$:

$$S_{C,\infty} := \{-\nu_{C,p}(h) | h \in T\},$$

where T is the k-algebra $\mathcal{O}_C(C \setminus \{p\})$.

For any curve C with only one place at infinity, except when the characteristic of k divides the degree of C, it can be proved that there is a δ-sequence in $\mathbb{N}_{>0}$, Δ, such that $S_{C,\infty} = S_\Delta$ [4]. Conversely, as we shall precise later, for any field k and any δ-sequence in $\mathbb{N}_{>0}$, there exists a plane curve C with only one place at infinity such that $S_\Delta = S_{C,\infty}$ and δ_0 is the degree of C.

C has a singularity at p, except when $g = 0$. Consider the infinite sequence of morphisms

$$\cdots \to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0 := \mathbb{P}^2, \tag{2.1}$$

where $X_1 \to X_0$ is the blowing-up at $p_0 := p$ (the point at infinity) and, for each $i \geq 1$, $X_{i+1} \to X_i$ denotes the blowing-up of X_i at the unique point p_i which lies on the strict transform of C and the exceptional divisor created by the preceding blowing-up; notice that p_i is defined over k, since the branch of C at p is rational. It is well-known that there exists a minimum integer n such that, if $\pi : X_n \to \mathbb{P}^2$ denotes the composition of the first n blowing-ups, the germ of the strict transform of C by π at p_n becomes regular and transversal to the exceptional divisor. This gives the (minimal embedded) resolution of the germ of C at p. The essential information, the (topological) equisingularity class of the germ, can be given in terms of its sequence of *Newton polygons* [7, III.4] or by means of its *dual graph* (see [10] within a more general setting or [13] for a slightly different version). This information basically provides the number and the position of blowing-up centers of π: these can be placed either on a free point (not an intersection of two exceptional divisors) or on a satellite point. In this last case, it is also important to know whether, or not, the blowing-up center belongs to the last but one created exceptional divisor. Thinking of blowing-up centers, we shall say that a center p_i is proximate to other p_j whenever p_i is on any strict transform of the divisor created after blowing-up at p_j.

The complete information of the resolution process can be described by means of the so-called Hamburger-Noether expansion (HNE for short) [7, 9], being this expansion specially useful when the characteristic of the field k is positive. More explicitly, let $\{u', v'\}$ be local

coordinates of the local ring $\mathcal{O}_{C,p}$; in those coordinates the HNE of C at p has the form

$$
\begin{aligned}
v' &= a_{01}u' + a_{02}u'^2 + \cdots + a_{0h_0}u'^{h_0} + u'^{h_0}w_1 \\
u' &= w_1^{h_1}w_2 \\
&\vdots \qquad\qquad \vdots \\
w_{s_1-2} &= w_{s_1-1}^{h_{s_1-1}}w_{s_1} \\
w_{s_1-1} &= a_{s_1k_1}w_{s_1}^{k_1} + \cdots + a_{s_1h_{s_1}}w_{s_1}^{h_{s_1}} + w_{s_1}^{h_{s_1}}w_{s_1+1} \\
&\vdots \qquad\qquad \vdots \\
w_{s_g-1} &= a_{s_gk_g}w_{s_g}^{k_g} + \cdots
\end{aligned}
$$

where the family $\{s_i\}_{i=0}^g$, $s_0 = 0$, of nonnegative integers is the set of indices corresponding to the free rows of the expression, that is those rows that express the blowing-ups at free points (they are those that have some nonzero $a_{jl} \in k$) and the main goal (of the HNE) is that it gives local coordinates of the transform of the germ of C at p in each center of blowing-up. The local coordinates after $\{u', v'\}$ are $\{u', (v'/u') - a_{01}\}$ and so on.

The dual graph Γ associated with the above germ of curve is a tree such that each vertex represents an exceptional divisor of the sequence π and two vertices are joined by an edge whenever the corresponding divisors intersect. Additionally, we label each vertex with the minimal number of blowing-ups needed to create its corresponding exceptional divisor. The dual graph can be done by gluing by their vertices st_i subgraphs Γ_i $(1 \le i \le g)$ corresponding to blocks of data $B_i = \{h_{s_{i-1}} - k_{i-1} + 1, h_{s_{i-1}+1}, h_{s_{i-1}+2}, \ldots, h_{s_i-1}, k_i\}$ (with $k_0 = 0$), which represent the divisors involved in the part of HNE of the germ between two free rows. That is Γ_i contains divisors corresponding to $h_{s_{i-1}} - k_{i-1} + 1$ free points and to sets of h_j $(s_{i-1} + 1 \le j \le s_i - 1)$ and k_i proximate points to satellite ones. Each subgraph Γ_i starts in the vertex st_{i-1} and ends in st_i containing, among others, the vertex ρ_i. So, the dual graph has the shape depicted in Figure 1.
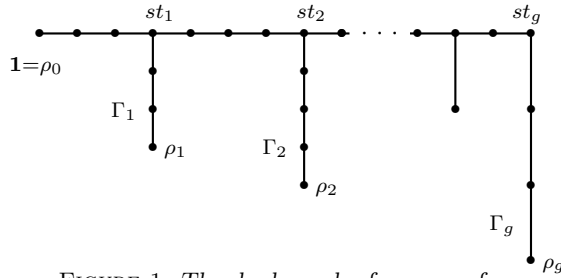

FIGURE 1. *The dual graph of a germ of curve*

Set E_{s_i} $(1 \le i \le g)$ the exceptional divisor obtained after blowing-up the last free point corresponding to the subgraph Γ_i. It corresponds to the vertex ρ_i in the dual graph. An irreducible germ of curve at p, ψ, is said to *have maximal contact of genus i* with the germ of C at p, if the strict transform of ψ in the (corresponding germ of the) surface containing E_{s_i} is not singular and meets transversely E_{s_i} and no other exceptional curves.

For convenience, throughout this paper we fix homogeneous coordinates $(X : Y : Z)$ on $\mathbb{P}^2$. $Z = 0$ will be the line at infinity and $p = (1 : 0 : 0)$. Set (x, y) coordinates in the chart $Z \ne 0$ and $(u = y/x, v = 1/x)$ coordinates around the point at infinity. We shall assume that the curve C is defined by a monic polynomial $f(x, y)$ in the indeterminate y with coefficients in $k[x]$.

The so-called approximate roots of C [3] are an important tool to get a δ-sequence Δ in $\mathbb{N}_{>0}$ such that (for suitable characteristic of k) $S_\Delta = S_{C,\infty}$. One can see in [8] an algorithm for computing them (see also [13] for the complex case). We need not use approximate roots for the

development of this paper but a close weaker concept which is given in the following definition (see [**30**]). Its main advantage (explicit description for certain explicit curves obtained only from the δ-sequence in $\mathbb{N}_{>0}$) is showed in Proposition 2.5.

DEFINITION 2.4.   Assuming the above notation, a sequence of polynomials in $k[x, y]$

$$q_0^*(x, y), q_1^*(x, y), \ldots, q_g^*(x, y)$$

is a *family of approximates* for the curve C given by $f(x, y)$ if the following conditions hold:
   (i) $q_0^*(x, y) = x$, $q_1^*(x, y) = y$, $\delta_0^* := -\nu_{C,p}(q_0^*) = \deg_y(f)$ and $\delta_1^* := -\nu_{C,p}(q_1^*)$.
   (ii) $q_i^*(x, y)$ $(1 < i \leq g)$ has degree δ_0^*/d_i and it is monic in the indeterminate y, where $d_i = \gcd(\delta_0^*, \delta_1^*, \ldots, \delta_{i-1}^*)$, being $\delta_i^* := -\nu_{C,p}(q_i^*)$.
   (iii) The germ of curve at p given by the local expression of $q_i^*(x, y)$ $(1 < i \leq g)$ in the coordinates (u, v) has maximal contact with the germ of C at p, of genus i when $\delta_0^* - \delta_1^*$ does not divide δ_0^* and of genus $i - 1$ otherwise.

By an abuse of notation, when we set $-\nu_{C,p}(q_i^*)$, q_i^* stands for the element in the fraction field of $\mathcal{O}_{C,p}$ that it defines. On the other hand, under the conditions of Abhyankar-Moh Theorem, that is, the characteristic of k does not divide the degree of the curve C, approximate roots are a family of approximates for C.

Now, let $\Delta = \{\delta_i\}_{i=0}^g$ be a δ-sequence in $\mathbb{N}_{>0}$. It is well-known the existence of a unique expression of the form

$$n_i\delta_i = \sum_{j=0}^{i-1} a_{ij}\delta_j, \tag{2.2}$$

where $a_{i0} \geq 0$ and $0 \leq a_{ij} < n_j$, for $1 \leq j \leq i - 1$. Set $q_0 := x$ $q_1 := y$ and, for $1 \leq i \leq g$,

$$q_{i+1} := q_i^{n_i} - t_i \prod_{j=0}^{i-1} q_j^{a_{ij}}, \tag{2.3}$$

where $t_i \in k \setminus \{0\}$ are arbitrary. Although all the results in this paper concerning these polynomials hold for any family of parameters $\{t_i\}_{i=1}^g$, we fix for convenience $t_i = 1$ for all i. Then, by applying the algorithms relative to Newton polygons of a germ of curve given by Campillo in [**7**, III.4] to the germ given by q_{g+1}, it holds the following result (see [**30**, Section 4] for more details), where we notice that there is no restriction for the characteristic of the field k.

PROPOSITION 2.5.   *The equality $q_{g+1} = 0$ defines a plane curve C with only one place at infinity such that $S_{C,\infty} = S_\Delta$ and the set $\{q_i\}_{i=0}^g$ is a family of a approximates for C such that $-\nu_{C,p}(q_i) = \delta_i$ for all $i = 0, 1, \ldots, g$.*

The sequence of Newton polygons and the dual graph of the germ of a curve with only one place at infinity can be recovered from a δ-sequence in $\mathbb{N}_{>0}$, $\Delta = \{\delta_0, \delta_1, \ldots, \delta_s\}$, associated with it. We assume that the Newton polygons are given by segments P_i $(0 \leq i \leq g - 1)$ joining the points $(0, e_i)$ and $(m_i, 0)$, $e_i, m_i \in \mathbb{N}_{>0}$. If $\delta_0 - \delta_1$ does not divide δ_0 then $s = g$ and

$$e_0 = \delta_0 - \delta_1, \quad e_i = d_{i+1}$$

$$m_0 = \delta_0, \quad m_i = n_i\delta_i - \delta_{i+1}$$

for $1 \le i \le s - 1$. Otherwise, $s = g + 1$ and

$$e_0 = d_2 = \delta_0 - \delta_1, \quad e_i = d_{i+2}$$

$$m_0 = \delta_0 + n_1 \delta_1 - \delta_2, \quad m_i = n_{i+1} \delta_{i+1} - \delta_{i+2}$$

for $1 \le i \le s - 2$. These formulae can be deduced from the knowledge of the sequence of Newton polygons associated with the singularity at infinity of the curve defined by the equation $q_{g+1}(x, y) = 0$ and results in [7, IV.3].

With respect to the dual graph or the blocks in the HNE of the germ, one gets

$$\frac{m_{j-1}}{e_{j-1}} + k_{j-1} = h_{s_{j-1}} + \cfrac{1}{h_{s_{j-1}+1} + \cfrac{\ddots}{\ddots + \cfrac{1}{h_{s_j - 1} + \frac{1}{k_j}}}}, \tag{2.4}$$

for $j = 1, 2, \ldots, g$, where $s_0 = k_0 = 0$ (see [7, III.4]).

We end this section by collecting recent information about a particularly simple set of curves with only one place at infinity.

DEFINITION 2.6.   An *Abhyankar-Moh-Suzuki* (*AMS* for short) curve C is a plane curve with only one place at infinity such that it is rational and nonsingular in its affine part.

The advantage of this type of curves is that their dual graphs are easily described [12] and their associated δ-sequences in $\mathbb{N}_{>0}$ are also very easy to compute: they are those sequences $\{\delta_i\}_{i=0}^g$ of distinct positive integers such that δ_i divides δ_{i-1} for all $i = 1, 2, \ldots, g$ and $\delta_g = 1$ (this can be deduced from the proof of [27, Proposition 2]).

3.   Valuations of function fields of surfaces

In this paper, we are concerned with weight functions given by certain type of valuations close to curves with only one place at infinity. For this reason, we devote this section to state the main facts related to valuations we shall need. See [42, 43, 1, 2, 38, 17] for some significant applications of valuation theory in algebraic geometry. Additional details to this section, following the same line of this paper, can be found in Section 3 of [15].

From now on, we shall set p a point in $\mathbb{P}^2$, $R := \mathcal{O}_{\mathbb{P}^2, p}$ and K the quotient field of R.

DEFINITION 3.1.   A *valuation* of the field K is a mapping

$$\nu : K^*(:= K \setminus \{0\}) \to G,$$

where G is a totally ordered group, such that it satisfies $\nu(f + g) \ge \min\{\nu(f), \nu(g)\}$ and $\nu(fg) = \nu(f) + \nu(g)$, f and g being elements in K^*.

A valuation as above is said to be centered at R whenever $R \subseteq R_\nu := \{f \in K^* | \nu(f) \ge 0\} \cup \{0\}$ and $R \cap m_\nu (:= \{f \in K^* | \nu(f) > 0\} \cup \{0\})$ coincides with the maximal ideal m of R. We call this type of valuations *plane valuations*. Assume for a while that the field k is algebraically closed. Plane valuations have a deep geometrical meaning as the following result proves (see [38]):

THEOREM 3.2.   *There is a one to one correspondence between the set of plane valuations (of K centered at R) and the set of simple sequences of quadratic transformations of the scheme* Spec R.

What Theorem 3.2 says is that, attached to a plane valuation ν, there is a unique sequence of point blowing-ups

$$\cdots X_{N+1} \xrightarrow{\pi_{N+1}} X_N \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X = \operatorname{Spec} R, \tag{3.1}$$

where π_{i+1} is the blowing-up of X_i at the unique closed point p_i of the exceptional divisor E_i (obtained after the blowing-up π_i) satisfying that ν is centered at the local ring $\mathcal{O}_{X_i,p_i}$ $(:= R_i)$. Conversely, each sequence as in (3.1) provides a unique plane valuation.

This fact reflects the closeness between plane valuations and germs of plane curves. It is clear that in a similar way to that explained in Section 2, we can provide a dual graph and also a HNE (with respect to a fixed regular system of parameters $\{u, v\}$ of the ring R) for each plane valuation. Notice that, in this case, the sequence (3.1) can be either infinite or finite (what does not happen for germs of curves, although usually it is only showed the important part, that is the blowing-ups we must do until the germ is resolved). Attending to the structure of the dual graph of the sequence (3.1), Spivakovsky in [**38**] classifies the plane valuations in five types (this refines a previous classification by Zariski and it can also be refined [**14**]). We briefly recall the essential of this classification, since it will be useful for us. For further reference in the line of this paper, the reader can see [**15**, 3.3], where this classification is given in terms of the HNE of the valuations. Notice that the HNE has the advantage that it is suitable for positive characteristic and provides parametric equations of the valuations.

– Valuations whose associated sequence (3.1) is finite are called of **TYPE A**.

– A plane valuation whose sequence (3.1) consists, from one blowing-up on, only of blowing-ups at free points is named of **TYPE B**. Sequences (3.1) for these valuations behave as those that resolve germs of plane curves.

– **TYPE C** valuations are those such that their attached sequence (3.1) can contain finitely many blowing-ups as above, that is, with blocks of free and satellite points alternatively but it ends with infinitely many satellite blowing-ups, all of them with center at the strict transform of the same exceptional divisor.

– When the sequence (3.1) is as above, that is, it ends with infinitely many satellite blowing-ups, but they are not ever centered at the strict transform of the same exceptional divisor, we get **TYPE D** valuations.

– Finally, a valuation whose corresponding sequence (3.1) alternates indefinitely blowing-ups at (blocks of) free and satellite points is named to be a **TYPE E** valuation.

An important fact for the development of this paper is that any plane valuation can be regarded as a limit of a sequence $\{\nu_i\}_{i=1}^{\infty}$ of type A valuations. The valuations ν_i correspond to the divisors created by the sequence (3.1) attached to ν. The proof is based on the fact that the ring R_ν is the direct limit of the sequence of rings R_i.

When the valuation is centered at a two-dimensional regular local ring $\mathfrak{R}$ whose residue field is not algebraically closed, the above procedure works similarly. Indeed, the valuation ring $\mathfrak{R}_\nu$ of a type A valuation ν is a local ring that dominates $\mathfrak{R}$ and has the same quotient field as $\mathfrak{R}$. Between $\mathfrak{R}$ and $\mathfrak{R}_\nu$ there exists a uniquely determined sequence of mutually dominated local rings

$$\mathfrak{R} \subset \mathfrak{R}_1 \subset \cdots \subset \mathfrak{R}_{N+1} = \mathfrak{R}_\nu,$$

which provides the blowing-up sequence. As above, the residue field of $\mathfrak{R}_\nu$ is a transcendental extension of the one of $\mathfrak{R}$ and the difference consists on the fact that the residue field of $\mathfrak{R}_i$

$(1 \leq i \leq N)$ needs not coincide with the one of $\mathfrak{R}$. However, in this paper we only consider valuations where the above residue fields always coincide and therefore we can handle our valuations as in the algebraically closed case.

Returning to the sequence $\{\nu_i\}_{i=1}^{\infty}$ of type A valuations converging to a plane valuation ν, when we deal with a type D or E valuation ν, its value group G is a subset of the set of real numbers $\mathbb{R}$ and, if we consider the normalization $\nu_i' := \nu_i/\nu_i(m)$ of the valuations ν_i, m being the maximal ideal of R and $\nu_i(m)$ the minimum of the values $\nu_i(g)$ when g runs over $m \setminus \{0\}$, then $\nu(f) = \lim_{i \to \infty} \nu_i'(f)$, for all $f \in K^*$. For type B or C valuations, G is included in $\mathbb{Z}^2$, $\mathbb{Z}$ denoting the integer numbers, and one can understand the limit by using the Noether formula. Indeed, let ν be a type A valuation, with attached sequence (3.1) which ends at X_{N+1} and $f \in R$ an analytically irreducible element. Set m_i the maximal ideal of the local ring R_i in the above sequence and assume that $p_0, p_1, \ldots, p_r$, $r \leq N$, are the common infinitely near points for the sequence (3.1) and the resolution of the germ of curve given by f. Then $\nu(m_i) > 0$ can be easily computed from the dual graph or the HNE of ν (as in the case of germs of curves) and

$$\nu(f) = \sum_{i=0}^{r} \nu(m_i)e(p_i), \tag{3.2}$$

where $e(p_i)$ is the multiplicity of the germ given by f at the point p_i. When ν is of type B, we get the same equality, but here $N = \infty$ and $\nu(m_i) \in \mathbb{Z}^2$ although its first coordinate vanishes; the limit appears when $r = \infty$ because then we set $\nu(f) = (1, 0)$. Finally, Formula (3.2) also holds whenever ν is a type C valuation; the concept of limit appears in the values $\nu(m_i)$ in the following way: set p_{i_0} that blowing-up center in (3.1) having infinitely many proximate points p_i, $i > i_0$, then $\nu(m_i) = (0, 1)$ when $i > i_0$, $\nu(m_{i_0}) = (1, 0)$ (the "sum" of infinitely many $(0, 1)$) and the remaining values $\nu(m_i)$ can be computed from the above ones as in the case of type A valuations.

4. Weight functions given by plane valuations at infinity

4.1. Weight functions and plane valuations at infinity

To give the definition of weight function, first we recall some concepts related to semigroups. Assume that α, β, γ are arbitrary elements in a commutative semigroup with zero Γ. If $\leq$ is an ordering on Γ, $\leq$ is said to be *admissible* if $0 \leq \alpha$ and, moreover, $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$. On the other hand, Γ is named *cancellative* whenever from the equality $\alpha + \beta = \alpha + \gamma$ one can deduce $\beta = \gamma$. Finally for stating the mentioned definition, stand Γ for a cancellative well-ordered commutative with zero and with admissible ordering semigroup and $\Gamma \cup \{-\infty\}$ for the above semigroup together with a new minimal element denoted by $-\infty$, which satisfies $\alpha + (-\infty) = -\infty$ for all $\alpha \in \Gamma \cup \{-\infty\}$.

DEFINITION 4.1.   A *weight function* from a k-algebra A onto a semigroup $\Gamma \cup \{-\infty\}$ as above is a mapping $w : A \longrightarrow \Gamma \cup \{-\infty\}$ such that, for $p, q \in A$, the following statements must be satisfied:

   (i) $w(p) = -\infty$ if and only if $p = 0$;
   (ii) $w(ap) = w(p)$ for all nonzero element $a \in k$;
   (iii) $w(p + q) \leq \max\{w(p), w(q)\}$;
   (iv) If $w(p) = w(q)$, then there exists a nonzero element $a \in k^*$ such that $w(p - aq) < w(q)$;
   (v) $w(pq) = w(p) + w(q)$.

When the last condition is not imposed, we get the definition of *order function*. It is clear that if w is a weight function, then the triple (A, w, Γ) is an order domain over k (see, for instance, [**16**] for the definition of order domain).

The next result, proved in [**15**, Proposition 2.2], shows how to get weight functions from valuations.

PROPOSITION 4.2. *Let $\mathfrak{K}$ be the quotient field of a Noetherian regular local domain $\mathfrak{R}$ with maximal $\mathfrak{m}$. Let $\nu : \mathfrak{K}^* \to \mathfrak{G}$ be a valuation of $\mathfrak{K}$ which is centered at $\mathfrak{R}$. Assume that the canonical embedding of the field $\mathfrak{k} = \mathfrak{R}/\mathfrak{m}$ into the field $\mathfrak{R}_\nu/\mathfrak{m}_\nu$ is an isomorphism.*

Set $w : \mathfrak{K}^* \to \mathfrak{G}$ the mapping given by $w(f) = -\nu(f)$, $f \in \mathfrak{K}^*$. If $\mathfrak{A} \subseteq \mathfrak{K}^*$ is a $\mathfrak{k}$-algebra such that $w(\mathfrak{A})$ is a cancellative, commutative, free of torsion, well-ordered semigroup with zero, Γ, where the associated ordering is admissible, then $w : \mathfrak{A} \longrightarrow w(\mathfrak{A}) \cup \{-\infty\}$, $w(0) = -\infty$, is a weight function.

Recall that $p \in \mathbb{P}^2$, $R = \mathcal{O}_{\mathbb{P}^2,p}$ and K is the quotient field of R. The isomorphism $R/m \cong R_\nu/m_\nu$ happens for any plane valuation ν except for those of type A.

Now, we are going to introduce another fundamental concept for us: *plane valuation at infinity*. To do it, we start by stating the concept of general element of a type A plane valuation. As we have said, these valuations are the unique whose corresponding sequence of point blowing-ups is finite. In fact, they are defined by the last created exceptional divisor and, by this reason, they are also named divisorial valuations. Concretely, with the notation in Section 3, if π_{N+1} is the last blowing-up in the sequence (3.1) given by a divisorial valuation ν, then ν is the m_N-adic valuation, m_N being the maximal ideal of the ring R_N.

DEFINITION 4.3. Let ν be a divisorial valuation. An element f in the maximal ideal of R is called to be a *general element of ν* if the germ of curve given by f is analytically irreducible, its strict transform in X_{N+1} is smooth and meets E_{N+1} transversely at a non-singular point of the exceptional divisor of the sequence (3.1) attached to ν.

General elements are useful to compute plane valuations. Indeed, if $f \in R$, then

$$\nu(f) = \min \{(f,g) | g \text{ is a general element of } \nu\},$$

where (f, g) stands for the intersection multiplicity of the germs of curve given by f and g.

DEFINITION 4.4. A *plane divisorial valuation at infinity* is a plane divisorial valuation of K centered at R that admits, as a general element, an element in R providing the germ at p of some curve with only one place at infinity (p being its point at infinity).

DEFINITION 4.5. A plane *valuation ν of K centered at R* is said to be *at infinity* whenever it is a limit of plane divisorial valuations at infinity. More explicitly, set $\{\nu_i\}_{i=1}^\infty$ the set of plane divisorial valuations, corresponding to divisors E_i, that appear in the sequence (3.1) given by ν. ν will be at infinity if, for any index i_0, there is some $i > i_0$ such that ν_i is a plane divisorial valuation at infinity.

Afterwards, we shall see that, as it happens for type A valuations, type B ones are not suitable for our purposes. So we exclude them of the next definition.

DEFINITION 4.6.   Let ν be a plane valuation at infinity that is neither of type A nor of type B. A sequence of polynomials $P = \{q_i(x,y)\}_{i \geq 0}$ in $k[x,y]$ is a *family of approximates* for ν whenever each plane curve C with only one place at infinity providing a general element of some of the plane divisorial valuations at infinity converging to ν admits some subset of P as a family of approximates and P is minimal with this property.

4.2. δ-sequences

Now, we introduce the most important concept of this paper: the one of δ-sequence. First of all, notice that if S is an ordered commutative semigroup in $\mathbb{R}$ (naturally ordered) or $\mathbb{Z}^2$ (with the lexicographical ordering, that is $\alpha > \beta$ if and only if the left-most nonzero entry of $\alpha - \beta$ is positive) and $\gamma_0 > \gamma_1 > 0$ are in S such that $j\gamma_1 > \gamma_0$ for some integer number $j > 0$, then we can set $\gamma_0 = m_1 \gamma_1 + \gamma_2$ with $m_1 \in \mathbb{N}_{>0}$, γ_2 in $\mathbb{R}$ or $\mathbb{Z}^2$ and $0 \leq \gamma_2 < \gamma_1$ (note that m_1 and γ_2 are uniquely determined by these conditions). Repeating this procedure, one gets $\gamma_1 = m_2 \gamma_2 + \gamma_3$ whenever $j\gamma_2 > \gamma_1$ for some $j > 0$. If we iterate, there are three possibilities: *Case* 1) The algorithm stops because we get $\gamma_i = 0$ for some index i; in this case we can speak about the *greatest common divisor* γ_{i-1} of γ_0 and γ_1. *Case* 2) The algorithm does not stop. *Case* 3) In certain step i, there is no $j \in \mathbb{N}_{>0}$ such that $j\gamma_i > \gamma_{i-1}$. The concept of δ-sequence is motivated by these three possibilities, that is, we shall consider sequences of positive elements $\Delta = \{\delta_0, \delta_1, \dots, \delta_i, \dots\}$ in $\mathbb{R}$ or in $\mathbb{Z}^2$ and, reproducing the computations in pages 6 and 7 with the help of the above procedure (that is the Euclidian algorithm adapted to our data), we shall arrive to different situations which allow us to enlarge in a natural way the concept of δ-sequence in $\mathbb{N}_{>0}$.

For a start, a *normalized δ-sequence in* $\mathbb{N}_{>0}$ is an ordered finite set of rational numbers $\overline{\Delta} = \{\overline{\delta}_0, \overline{\delta}_1, \dots, \overline{\delta}_g\}$ such that there is a δ-sequence in $\mathbb{N}_{>0}$, $\Delta = \{\delta_0, \delta_1, \dots, \delta_g\}$, satisfying $\overline{\delta}_i = \delta_i/\delta_1$ for $0 \leq i \leq g$. Notice that from $\overline{\Delta}$, by writing $\overline{\delta}_i = r_i/s_i$ as a quotient of relatively prime elements, one can recover Δ since $\delta_i = \overline{\delta}_i \mathrm{lcm}(s_i)_{0 \leq i \leq g}$. From now on, we set $C_\Delta = C_{\overline{\Delta}}$ the curve with only one place at infinity given by Δ that provides Proposition 2.5. The dual graph of the resolution of the singularity of C_Δ at the point at infinity will be named *the dual graph given by* Δ *or* $\overline{\Delta}$.

DEFINITION 4.7.   A δ-*sequence* in $\mathbb{Z}^2$ (respectively, $\mathbb{Q}$) (respectively, $\mathbb{R}$) is a sequence $\Delta = \{\delta_0, \delta_1, \dots, \delta_i, \dots\}$ of elements in $\mathbb{Z}^2$ (respectively, $\mathbb{Q}$) (respectively, $\mathbb{R}$) such that it generates a well-ordered sub-semigroup of $\mathbb{Z}^2$ (respectively, $\mathbb{Q}$) (respectively, $\mathbb{R}$) and
($\mathbb{Z}^2$) $\Delta = \{\delta_0, \delta_1, \dots, \delta_g\} \subset \mathbb{Z}^2$ is finite, $g \geq 2$ (respectively, ≥ 3) and there exists a δ-sequence in $\mathbb{N}_{>0}$, $\Delta^* = \{\delta_0^*, \delta_1^*, \dots, \delta_g^*\}$, such that $\delta_0^* - \delta_1^*$ does not divide (respectively, divides) δ_0^* and

$$\delta_i = \frac{\delta_i^*}{Aa_t + B}(A, B) \quad (0 \leq i \leq g-1) \quad \text{and}$$

$$\delta_g = \frac{\delta_g^* + A'a_t + B'}{Aa_t + B}(A, B) - (A', B'),$$

where $\langle a_1; a_2, \dots, a_t \rangle$, $a_t \geq 2$, is the continued fraction expansion of the quotient m_{g-1}/e_{g-1} (respectively, m_{g-2}/e_{g-2}) given by Δ^* (see (2.4)) and, considering the finite recurrence relation $\underline{y}_i = a_{t-i}\underline{y}_{i-1} + \underline{y}_{i-2}$, $\underline{y}_{-1} = (0,1)$, $\underline{y}_0 = (1,0)$, then $(A,B) := \underline{y}_{t-2}$ and $(A', B') := \underline{y}_{t-3}$. We complete this definition by adding that $\Delta = \{\delta_0, \delta_1\}$ (respectively, $\Delta = \{\delta_0, \delta_1, \delta_2\}$) is a δ-sequence in $\mathbb{Z}^2$ whenever $\delta_0 = \underline{y}_{t-1}$ and $\delta_0 - \delta_1 = \underline{y}_{t-2}$ (respectively, $\delta_0 = j\underline{y}_{t-2}$, $\delta_0 - \delta_1 = \underline{y}_{t-2}$ and $\delta_0 + n_1\delta_1 - \delta_2 = \underline{y}_{t-1}$) for the above recurrence attached to a δ-sequence in $\mathbb{N}_{>0}$, $\Delta^* = \{\delta_0^*, \delta_1^*\}$ (respectively, $\Delta^* = \{\delta_0^*, \delta_1^*, \delta_2^*\}$, such that $j := \delta_0^*/(\delta_0^* - \delta_1^*) \in \mathbb{N}_{\geq 0}$ and $n_1 := \delta_0^*/\gcd(\delta_0^*, \delta_1^*)$).

($\mathbb{Q}$) (Respectively, $\Delta = \{\delta_0, \delta_1, \ldots, \delta_i, \ldots\} \subset \mathbb{Q}$ is infinite and any ordered subset $\Delta_j = \{\delta_0, \delta_1, \ldots, \delta_j\}$ is a normalized δ-sequence in $\mathbb{N}_{>0}$).

($\mathbb{R}$) (Respectively, $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\} \subset \mathbb{R}$ is finite, $g \geq 2$, δ_i is a positive rational number for $0 \leq i \leq g-1$, δ_g is non-rational, and there exists a sequence

$$\left\{ \overline{\Delta}_j = \{\delta_0^j, \delta_1^j, \ldots, \delta_g^j\} \right\}_{j \geq 1}$$

of normalized δ-sequences in $\mathbb{N}_{>0}$ such that $\delta_i^j = \delta_i$ for $0 \leq i \leq g-1$ and any j and $\delta_g = \lim_{j \to \infty} \delta_g^j$. We complete this definition by adding that $\Delta = \{\tau, 1\}$, $\tau > 1$ being a non-rational number, is also a δ-sequence in $\mathbb{R}$).

For simplicity's sake, we shall do our theoretical development only for δ-sequences (in $\mathbb{Z}^2$, $\mathbb{Q}$ or $\mathbb{R}$) verifying that there is no positive integer j such that $\delta_0 = j(\delta_0 - \delta_1)$. However all the results in this paper are true for any δ-sequence because the reasonings in the non-considered case are similar taking into account that m_{g-2} and e_{g-2} must be used instead m_{g-1} and e_{g-1}.

Let us see that δ-sequences Δ in $\mathbb{Z}^2$ are intimately related to type C plane valuations at infinity. Consider the curve $C := C_{\Delta^*}$ and the set of associated polynomials $\{q_i(x,y)\}_{i=0}^{g+1}$ of Proposition 2.5. Then, $\delta_i^* = -\nu_{C,p}(q_i(x,y))$ for all $i = 0, 1, \ldots, g$. We have defined the values δ_i in the same form, but associated to a certain valuation at infinity of type C obtained from C. Indeed, let $\mathcal{D} := \{p_0 = p, p_1, p_2, \ldots\}$ be the sequence of centers of the blowing-ups associated with the curve C given in (2.1) and set i_0 the maximum among the positive integers j such that p_j admits more than a point proximate to it; our definition uses the relationship between $-\nu_{C,p}(q_i(x,y)) = \delta_i^*$ and $-\nu(q_i(x,y))$, which we name δ_i, where ν is the valuation of type C defined by the infinite sequence of quadratic transformations of the scheme Spec $\mathcal{O}_{\mathbb{P}^2,p}$ centered at the closed points in the set $\mathcal{C} := \{r_j\}_{j \in \mathbb{N}}$, where $r_j := p_j$ whenever $j \leq i_0 + 1$ and, for each $j > i_0 + 1$, r_j is the unique point of the blowing-up centered at r_{j-1} which is proximate to $r_{i_0} = p_{i_0}$.

The concrete relation that we give in the definition can be deduced as follows. The integer a_t is the number of points in $\mathcal{D}$ which are proximate to r_{i_0}; then $e_{i_0+1}(C) = 1$ and $e_{i_0}(C) = a_t$, where $e_j(C)$ denotes the multiplicity at r_j of the strict transform of the germ (C,p) at r_j. The remaining multiplicities can be obtained using recurrent relations, the so-called *proximity equalities*, that is, $e_j(C) = \sum_k e_k(C)$ for all $j \geq 0$, where k runs over the set of indexes such that r_k is proximate to r_j. The values $\nu(m_j)$ satisfy the same relations, but with different initial values: $\nu(m_j) = (0,1)$ for all $j > i_0$, $\nu(m_{i_0}) = (1,0)$ and $\nu(m_j) = \sum_k \nu(m_k)$ for $j < i_0$, with k running over the same set as before (recall the last paragraph of Section 3). From these facts it is easy to deduce that, if i_1 denotes the maximum index such that r_{i_1} is a free point but r_{i_1+1} is not so, there exist natural numbers A', B', A, B such that $e_{i_1}(C) = A'a_t + B'$, $e_{i_1-1}(C) = Aa_t + B$, $\nu(m_{i_1}) = (A', B')$ and $\nu(m_{i_1-1}) = (A, B)$. The first two values are the integers y_{t-3} and y_{t-2} (y_{t-2} and y_{t-1} in the case $g = 1$) obtained from the recurrence relation given in Definition 4.7 by taking the initial values $y_{-1} = 1$ and $y_0 = a_t$ (see [**7**, III.4]); so, the last two values can be obtained as in the definition. Taking coordinates (u,v) around the point at infinity p (as in Definition 2.4) one has that $q_0(x,y) = v^{-1}$ and, for $i \geq 1$,

$$q_i(x,y) = v^{-\delta_0^*/d_i^*} \overline{q}_i(u,v), \tag{4.1}$$

$\overline{q}_i(u,v)$ being the local expression around p of the curve given by $q_i(x,y)$ and $d_i^* = \gcd(\delta_0^*, \delta_1^*, \ldots, \delta_{i-1}^*)$. Therefore,

$$\delta_0 = \nu(v) \text{ and } \delta_i = -\nu(q_i(x,y)) = \frac{\delta_0^*}{d_i^*}\delta_0 - \nu(\overline{q}_i(u,v)) \text{ for } i \geq 1. \tag{4.2}$$

From the above information and Formula (3.2) applied to the germs defined by $\overline{q}_i(u,v)$ it can be deduced the existence of natural numbers $\{b_i\}_{i=0}^g$ such that $\nu_{C,p}(\overline{q}_g(u,v)) = b_g(Aa_t + B) +$

$A'a_t + B'$, $\nu(\overline{q}_g(u,v)) = b_g(A,B) + (A',B')$ and, for each $i \le g-1$, $\nu_{C,p}(\overline{q}_i(u,v)) = b_i(Aa_t + B)$ and $\nu(\overline{q}_i(u,v)) = b_i(A,B)$. As a consequence, the relations between δ_i and δ_i^* given in the Definition 4.7 hold by taking (4.2) into account.

We note that if $\Delta^* = \{\delta_0^*, \delta_1^*, \ldots, \delta_g^*\}$ and $\Delta' = \{\delta_0', \delta_1', \ldots, \delta_g'\}$ are δ-sequences in $\mathbb{N}_{>0}$ such that $\delta_i^*/\delta_1^* = \delta_i'/\delta_1'$ $(0 \le i < g)$ and the continued fraction expansions $\langle a_1; a_2, \ldots, a_t \rangle$ of the respective quotients m_{g-1}/e_{g-1} (see (2.4)) have the same length and only differ in the last value a_t, then the δ-sequences in $\mathbb{Z}^2$ defined by Δ^* and Δ' will be equal. The reason is that the above numbers b_i depend only on the proximity relations among the points r_j with $j \le i_0 + 1$ (notice that the index i_0 is clearly the same for Δ^* and Δ') and these coincide for both δ-sequences in $\mathbb{N}_{>0}$.

To guarantee that our valuation is a plane valuation at infinity, we must prove that if $\Delta^* = \{\delta_0^*, \delta_1^*, \ldots, \delta_g^*\}$ is a δ-sequence in $\mathbb{N}_{>0}$, with attached continued fraction $\langle a_1; a_2, \ldots, a_t \rangle$ (see Definition 4.7), then there exists another δ-sequence in $\mathbb{N}_{>0}$, $\Delta' = \{\delta_0', \delta_1', \ldots, \delta_g'\}$ such that $\delta_i^*/\delta_1^* = \delta_i'/\delta_1'$ $(0 \le i < g)$ and the associated continued fraction $\langle a_1'; a_2', \ldots, a_t' \rangle$ satisfies $a_i = a_i'$, $1 \le i < t$ and $a_t < a_t'$ (note that the first condition implies that C_{Δ^*} and $C_{\Delta'}$ can be chosen with the same set of approximates; this fact and the second condition guarantee that the divisorial valuations associated to the resolution of the singularity of C_{Δ^*} at infinity are also associated to the one of $C_{\Delta'}$). Let us see how to get Δ'.

Set $\{\check{\delta}_0, \check{\delta}_1, \ldots, \check{\delta}_{g-1}\}$ the δ-sequence in $\mathbb{N}_{>0}$ corresponding to the normalized sequence $\{\delta_i^*/\delta_1^*\}_{i=0}^{g-1}$ and define $\delta_i' = z\check{\delta}_i$, $0 \le i \le g-1$, for some $z \in \mathbb{N}_{>0}$ to be defined later. Consider the sequence of convergents [**28**, Chapter 7] $\{h_n, k_n\}$ for the continued fraction $\langle a_1; a_2, \ldots, a_t \rangle$. For $a \in \mathbb{N}_{>0}$, one gets

$$\langle a_1; a_2, \ldots, a_{t-1}, a \rangle = \frac{ah_{t-1} + h_{t-2}}{ak_{t-1} + k_{t-2}}.$$

Then

$$\frac{ah_{t-1} + h_{t-2}}{ak_{t-1} + k_{t-2}} = \frac{m_{g-1}'}{e_{g-1}'} = \frac{m_{g-1}'}{z},$$

m_{g-1}' and e_{g-1}' being the values defined in page 6 for our tentative Δ' and

$$\delta_g' = n_{g-1}' z \check{\delta}_{g-1} - m_{g-1}' =$$

$$= n_{g-1}'(ak_{t-1} + k_{t-2})\check{\delta}_{g-1} - (ah_{t-1} + h_{t-2}) =$$

$$= k_{t-1}(n_{g-1}'\check{\delta}_{g-1} - \frac{h_{t-1}}{k_{t-1}})a + k_{t-2}(n_{g-1}'\check{\delta}_{g-1} - \frac{h_{t-2}}{k_{t-2}}).$$

So, we only need to pick z large enough to that $a_t' = a > a_t$ and $\delta_g \in \langle \check{\delta}_0, \check{\delta}_1, \ldots, \check{\delta}_{g-1} \rangle$, which is possible since $\gcd(\check{\delta}_0, \check{\delta}_1, \ldots, \check{\delta}_{g-1}) = 1$ and the semigroup that these elements generate $\langle \check{\delta}_0, \check{\delta}_1, \ldots, \check{\delta}_{g-1} \rangle$ has a conductor.

δ-sequences in $\mathbb{Q}$ (respectively, in $\mathbb{R}$) are related with valuations at infinity of type E (respectively, D). To see it, it suffices to recall the definition and, in the first case, to consider the valuation given by the sequence of infinitely near points associated with the curves given by the polynomials q_i mentioned in page 6 for i large enough. In the second case (assuming $g \ge 2$), one must consider one of the normalized δ-sequences in $\mathbb{N}_{>0}$ $\overline{\Delta}_j$ of the definition and its corresponding δ-sequence in $\mathbb{N}_{>0}$, say $\Delta_j = \{\delta_0', \delta_1', \ldots, \delta_g'\}$; the related valuation of type D is determined by the sequence of infinitely near points associated with the resolution of the singularity at infinity of the curve defined by the approximate $q_g(x,y)$ of C_{Δ_j}, and the infinitely many satellite points whose corresponding block B_g is determined by the continued fraction expansion of the non-rational number $\frac{n_{g-1}\delta_{g-1} - \delta_g}{e_{g-1}/\delta_1'}$ (see pages 6 and 7). Observe that

the numbers n_{g-1} and e_{g-1}/δ_1' do not depend on the chosen δ-sequence Δ_j; in fact, the above mentioned non-rational number is the limit of the sequence of quotients $\frac{m_{g-1}}{e_{g-1}}$ associated with the δ-sequences Δ_j. With respect to the case $g = 2$ notice that, for whichever non-rational number $\tau > 1$, from the convergents of the continued fraction given by $\tau/(\tau - 1)$ we can derive normalized δ-sequences $\Delta^j = \{\delta_0^j, 1\}$ approaching the δ-sequence in $\mathbb{R}$, $\{\tau, 1\}$. The described valuations associated to δ-sequences in $\mathbb{Q}$ are clearly valuations at infinity and this is also for the real case by similar reasonings to those given for δ-sequences in $\mathbb{Z}^2$.

As a consequence of the last paragraphs, we have proved the following

PROPOSITION 4.8.   *Let $\Delta = \{\delta_i\}_{i=0}^r, r \leq \infty$, be a δ-sequence in $\mathbb{Z}^2$ (respectively, in $\mathbb{Q}$) (respectively, in $\mathbb{R}$). Then, there exists a plane valuation at infinity ν_Δ of type C (respectively, E) (respectively, D) and a family $\{q_i(x, y)\}_{i=0}^r$ of approximates for ν_Δ such that $-\nu_\Delta(q_i(x, y)) = \delta_i$ for all index i.*

Notice that we have chosen a concrete valuation ν_Δ, but this election needs not be unique since we could take another suitable families of approximates. Moreover, even when we have fixed certain type of approximates, we have infinitely many possibilities for our q_i, $1 < i < r$, according the parameters t_i we set in (2.3).

REMARK.   In this paper, generically, we shall name δ-sequence to any of the δ-sequences (in $\mathbb{Z}^2$, $\mathbb{Q}$ or $\mathbb{R}$) above defined. Due to their definition, we can apply the Euclidian algorithm as we described at the beginning of this subsection to the values m_j, e_j, $0 \leq j < g$, $(g = \infty$ in the case in $\mathbb{Q}$) defined as in page 6 and computed from the δ-sequences (newly with the Euclidian algorithm). Case 3) happens for δ-sequences in $\mathbb{Z}^2$, since we get $\gamma_{i-1} = (1, 0)$ and $\gamma_i = (0, 1)$, Case 2) holds for δ-sequences in $\mathbb{R}$ and, for δ-sequences in $\mathbb{Q}$, we have an indefinite iteration of Case 1) . Along this paper, we shall use freely the notation m_j, e_j, d_j, $0 \leq j \leq g - 1$, adapted to the corresponding δ-sequence.

Table 1 summarizes briefly the relation among the different types of δ-sequences and their corresponding families of approximates and valuations.

TABLE 1. *Summarizing Table*

Type of δ-sequence Δ	Case in Euclidian algorithm	Family of approximates (given by Δ using the Euclidian algorithm)	Type of valuation (at infinity)
$\mathbb{Z}^2$	3	Finite	C
$\mathbb{R}$	2	Finite	D
$\mathbb{Q}$	1	Infinite	E

REMARK.   Notice that we have no definition of δ-sequence related to type B valuations. This concept could be defined following the same line of this paper; however it would not be useful for us, since we would get δ-sequences $\Delta = \{\delta_0, \delta_1, \ldots, \delta_{g+1}\}$ where $\delta_i \in \{0\} \oplus \mathbb{N}_{>0}$ $(0 \leq i \leq g)$ and $\delta_{g+1} = (-1, a)$, $a \in \mathbb{N}_{>0}$, and then S_Δ would not be well-ordered (for the lexicographical ordering).

Next, we state the main result of the paper.

THEOREM 4.9.   Let $\Delta = \{\delta_i\}_{i=0}^r$, $r \leq \infty$, be a δ-sequence. Set $k[x,y]$ the polynomial ring in two indeterminates over an arbitrary field k. Then,
(a) There exists a weight function $w_\Delta : k[x,y] \longrightarrow S_\Delta \cup \{-\infty\}$.
(b) The map $-w_\Delta : k(x,y) \to G(S_\Delta) \cup \{\infty\}$, $G(S_\Delta)$ being the group generated by S_Δ, is a plane valuation at infinity.
(c) If $\{q_i\}_{i=0}^r$ denotes a family of approximates for the valuation $-w_\Delta$ then, for any $\alpha \in S_\Delta$, the vector spaces

$$O_\alpha := \{p \in k[x,y] \mid w_\Delta(p) \leq \alpha\}$$

are spanned by the set of polynomials $\prod_{i=0}^m q_i^{\gamma_i}$ such that $0 \leq m < r + 1$, $\beta := \sum_{i=0}^m \gamma_i \delta_i$ runs over the unique expression of the values $\beta \in S_\Delta$ satisfying $\beta \leq \alpha$, $\gamma_0 \geq 0$, $0 \leq \gamma_i < n_i$, whenever $1 \leq i < m$ and $\gamma_m \geq 0$ if $m = r$ and otherwise $0 \leq \gamma_m < n_m$.

Proof.   Let $\nu = \nu_\Delta$ be the plane valuation defined in Proposition 4.8. Set $w_\Delta := -\nu$. Since ν is of type C, D or E, to prove (a) and (b), it suffices to show that $S_\Delta \cup \{\infty\}$ is the image of $k[x,y]$ by w_Δ. To do it, recall that the polynomials q_i of Proposition 4.8 described before it satisfy the equalities given in (4.1). Consider the same notation and pick $f(x,y) \in k[x,y]$; then $f(x,y) = v^{-\deg(f)}\overline{f}(u,v)$ and, since the set $\{v\} \cup \{\overline{q}_i(u,v)\}_{i=1}^r$ is a minimal generating sequence for the valuation ν (see [**38**]), factoring $\overline{f}(u,v)$ into product of analytically irreducible elements, one has that there exists a polynomial of the form $m(u,v) = u^{s_0} v^{s_1} \prod_{i=2}^j \overline{q}_i^{s_i}(u,v)$, $s_i \geq 0$, $0 \leq i \leq j$, $j < r + 1$ such that $\nu(\overline{f}(u,v)) = \nu(m(u,v))$ and $s_0 + s_1 + \sum_{i=2}^j s_i \deg(\overline{q}_i) \leq \deg(f)$. Therefore $v^{-\deg(f)}\overline{f}(u,v)$ has the same valuation as $x^{\deg(f)-(s_0+s_1+\sum_{i=2}^j s_i \deg(\overline{q}_i))} y^{s_0} \prod_{i=2}^j q_i^{s_i}(x,y)$ and so $-\nu(f) \in S_\Delta$.

Finally, (c) is clear from the forthcoming Proposition 5.2 and the fact that w_Δ is a weight function and the images by w_Δ of the considered products $\prod_{i=0}^m q_i^{\gamma_i}$ give exactly once each value $\beta \in S_\Delta$ such that $\beta \leq \alpha$. $\square$

A *δ-sequence* Δ will also be said of *type* C, D or E whenever the corresponding valuation $-w_\Delta$ is of that type. We end this subsection by proving that if one considers suitable value semigroups, then any weight function of the polynomial ring $k[x,y]$ comes from a valuation at infinity.

PROPOSITION 4.10.   Let $w : k[x,y] \to S$ be a weight function on a semigroup S such that $S = S_\Delta$ for some δ-sequence Δ. Then, there exists a plane valuation at infinity $\nu : k(x,y) \to G$ such that $-\nu$ and w coincide on the ring $k[x,y]$.

Proof.   Consider coordinates in $\mathbb{P}^2$ and local coordinates (x,y) and (u,v) as we gave before Definition 2.4. It holds that the natural extension $-w = \nu : k(x,y) \to G(S)$, where $G(S)$ is the group generated by S, is a plane valuation of type C, D or E centered at the local ring $k[u,v]_{(u,v)}$. Pick polynomials $q_i(x,y)$ such that $w(q_i) = \delta_i$, $0 \leq i < r + 1$. When ν is of type E, the curves in $\mathbb{P}^2$ given by q_i guarantee that ν is as desired. Finally, in cases C and D, if we consider δ-sequences in $\mathbb{N}_{>0}$, $\Delta_j = \{\delta_0, \delta_1, \ldots, \delta_g\}$, approaching Δ (as we have described in Definition 4.7 for type D valuations and after that definition for type C ones), then the curves in $\mathbb{P}^2$ given by the polynomials $q_g^{n_g} - \prod_{i=0}^{g-1} q_i^{a_{gi}}$, where a_{gi} are the unique coefficients of the expression $n_g \delta_g = \sum_{i=0}^{g-1} a_{gi} \delta_i$, $0 \leq a_{gi} < n_i$ ($1 \leq i \leq g - 1$), prove that ν is a plane valuation at infinity. $\square$

4.3. *Construction of weight functions attached to valuations at infinity*

To end this section, we provide algorithms to get δ-sequences of every described type. Recall that the ordering of the semigroup S_Δ is given by the lexicographical one in $\mathbb{Z}^2$ for type C weight functions and by the natural ordering in $\mathbb{R}$ (respectively, $\mathbb{Q}$) for type D (respectively, type E) weight functions. To get a basis of the vector space O_α, $\alpha \in S_\Delta$, we only need to compute approximates $\{q_i\}_{i=1}^r$ for $-w_\Delta$ as we have described to prove Proposition 4.8 (see (2.3)) and fix a unique polynomial $q_\beta := \prod_{i=1}^m q_i^{\gamma_i}$ with $\beta = \sum_{i=1}^m \gamma_i \delta_i$ for each $\beta \in S_\Delta$ such that $\beta \leq \alpha$. Then, the set $\{q_\beta\}_{\beta \leq \alpha}$ will be a basis as desired.

4.3.1. *δ-sequences in $\mathbb{N}_{>0}$.* The concept of δ-sequence in $\mathbb{N}_{>0}$ is an important tool in this paper because it supports our general definition of δ-sequence. In [**13**], it can be found a complete list of δ-sequences in $\mathbb{N}_{>0}$ for curves over $\mathbb{C}$ with only one place at infinity and genus ≤ 30 (notice that, in that list, one must interchange δ_0 and δ_1 in order to be coherent with our notation). In the same paper, it is announced the existence of an algorithm for obtaining any δ-sequence in $\mathbb{N}_{>0}$. Next, we present an algorithm whose input is a δ-sequence $\Delta = \{\delta_0, \delta_1, \ldots, \delta_g\}$ in $\mathbb{N}_{>0}$ and whose output is another one $\Delta' = \{\delta_0', \delta_1', \ldots, \delta_{g+1}'\}$ such that $\delta_i/\delta_i' = \delta_j/\delta_j'$ for $0 \leq i, j \leq g$. From Formula (2.4) and Proposition 2.5, it is clear that this implies that the dual graph of the germ of curve $C_{\Delta'}$ that Proposition 2.5 associates with Δ' consists of the one of C_Δ plus a new subgraph Γ_{g+1}. The algorithm has the following steps:

1. Choose a natural number $z \geq 2$.
2. If there exists δ_{g+1}' such that $\delta_{g+1}' \in \langle \delta_0, \delta_1, \ldots, \delta_g \rangle$, $\delta_{g+1}' < z^2 \delta_g$ and $\gcd(z, \delta_{g+1}') = 1$, then return $\Delta' = \{z\delta_0, z\delta_1, \ldots, z\delta_g, \delta_{g+1}'\}$. Else,
3. increase the value of z and go to Step 2.

This procedure ends because the semigroup $\langle \delta_0, \delta_1, \ldots, \delta_g \rangle$ has a conductor.

4.3.2. *Type C weight functions.* Definition 4.7 shows how to obtain δ-sequences in $\mathbb{Z}^2$ (and so type C weight functions). Next, we describe a particularly simple case of this type.

Type C functions coming from AMS curves. This type of weight functions is simple because the easiness for obtaining δ-sequences attached to AMS curves. Indeed, fix a finite set $\{n_i\}_{i=1}^g$ of positive integers, $n_i \geq 2$, $n_g > 2$. We know that the set $\Delta^* := \{\delta_i^* := n_{i+1} n_{i+2} \cdots n_g\}_{i=0}^{g-1} \cup \{1\}$ is a δ-sequence in $\mathbb{N}_{>0}$ associated with an AMS curve. Therefore, setting

$$
\begin{aligned}
\delta_0 &= (n_1 n_2 \cdots n_{g-1}, n_1 n_2 \cdots n_{g-1}) \\
\delta_1 &= (n_2 n_3 \cdots n_{g-1}, n_2 n_3 \cdots n_{g-1}) \\
&\vdots \quad \vdots \qquad\qquad \vdots \\
\delta_{g-1} &= \qquad\qquad (1, 1) \\
\delta_g &= \qquad\qquad (0, 1)
\end{aligned}
$$

we get a δ-sequence that provides a type C weight function. The corresponding infinite dual graph Γ coincides with the one of the AMS curve replacing, as usual, the last set of proximate points by infinitely many ones. Notice that the δ_i's, $0 \leq i \leq g-1$, are in the line $x - y = 0$.

4.3.3. *Type D weight functions.* We are going to explicitly construct δ-sequences suitable for weight functions of type D. We start with a δ-sequence in $\mathbb{N}_{>0}$, $\Delta = \{\delta_0, \ldots, \delta_{g-1}\}$, $g \geq 2$, and a positive non-rational number $a \in \mathbb{R}$ such that

$$a < n_{g-1} \delta_{g-1}, \tag{4.3}$$

n_{g-1} being the usual number for the δ-sequence Δ. Set $\langle a_1; a_2, a_3, \ldots, a_j, \ldots \rangle$ the infinite continued fraction expansion given by a and, for any index $j \geq 2$, let m_{g-1}^j, e_{g-1}^j be the

relatively prime positive integers such that

$$\frac{m_{g-1}^j}{e_{g-1}^j} = \langle a_1; a_2, a_3, \ldots, a_j \rangle.$$

Next, we shall define a family of finite sequences $\left\{ \Delta_j = \{\delta_0^j, \ldots \delta_{g-1}^j, \delta_g^j\} \right\}_{j=2}^{\infty}$ which, under suitable conditions, will give the normalized δ-sequences in $\mathbb{N}_{>0}$ providing our δ-sequence in $\mathbb{R}$. To do it, define $\bar{\delta}_i^j := e_{g-1}^j \delta_i$ $(i = 0, 1, \ldots, g-1)$, $\bar{\delta}_g^j := n_{g-1} e_{g-1}^j \delta_{g-1} - m_{g-1}^j$ and $\delta_i^j := \bar{\delta}_i^j / \gcd(\bar{\delta}_i^j \mid i = 0, 1, \ldots, g)$, $0 \leq i \leq g$.

Notice that, for $j \geq 4$, the equality

$$\bar{\delta}_g^j = \bar{\delta}_g^{j-1} a_j + \bar{\delta}_g^{j-2} \tag{4.4}$$

holds. Indeed, $m_{g-1}^j = a_j m_{g-1}^{j-1} + m_{g-1}^{j-2}$ and $e_{g-1}^j = a_j e_{g-1}^{j-1} + e_{g-1}^{j-2}$ [28]. Therefore

$$\bar{\delta}_g^j = n_{g-1} \left(a_j e_{g-1}^{j-1} + e_{g-1}^{j-2} \right) \delta_{g-1} - \left(a_j m_{g-1}^{j-1} + m_{g-1}^{j-2} \right) =$$

$$\left(n_{g-1} e_{g-1}^{j-1} \delta_{g-1} - m_{g-1}^{j-1} \right) a_j + \left(n_{g-1} e_{g-1}^{j-2} \delta_{g-1} - m_{g-1}^{j-2} \right) = \bar{\delta}_g^{j-1} a_j + \bar{\delta}_g^{j-2},$$

as stated. Soon, we shall give conditions in order to Δ_j be a δ-sequence. Firstly assume that there exists a positive integer s_0 such that $n_{g-1}\delta_{g-1} > \langle a_1; a_2, a_3, \ldots, a_j \rangle$ for $j \in \{s_0, s_0 + 1\}$. Then, the chain of equalities

$$\frac{\bar{\delta}_g^j}{e_{g-1}^j} = n_{g-1}\delta_{g-1} - \frac{m_{g-1}^j}{e_{g-1}^j} = n_{g-1}\delta_{g-1} - \langle a_1; a_2, a_3, \ldots, a_j \rangle$$

and (4.4) prove that $\bar{\delta}_g^j > 0$ for $j \geq s_0$. We notice that the definition of $\bar{\delta}_g^j$ and (4.3) prove that s_0 always exists.

Finally, we are ready to prove that Δ_j is a δ-sequence in $\mathbb{N}_{>0}$ for $j \geq s_1$ whenever $s_1 \geq s_0$ is an index such that the values $\bar{\delta}_g^{s_1}$ and $\bar{\delta}_g^{s_1+1}$ belong to the semigroup in $\mathbb{N}_{>0}$ generated by $\delta_0, \delta_1, \ldots, \delta_{g-1}$ (notice that s_1 exists because this semigroup has a conductor). In fact, we only need to show that $n_g^j \delta_g^j$ belongs to the semigroup generated by $\delta_0^j, \delta_1^j, \ldots, \delta_{g-1}^j$ for all $j \geq s_1$, where n_g^j denotes $\gcd(\delta_0^j, \ldots, \delta_{g-1}^j)$. But this is equivalent to the fact that $\bar{\delta}_g^j$ belongs to the semigroup generated by $\delta_0, \delta_1, \ldots, \delta_{g-1}$. Now, the result follows inductively from the hypothesis and Equality (4.4).

As a consequence, fixed a δ-sequence in $\mathbb{N}_{>0}$, we can determine non-rational real numbers a such that all the above sequences Δ_j are δ-sequences in $\mathbb{N}_{>0}$, giving rise after normalizing to a δ-sequence in $\mathbb{R}$. In fact, we must consider a positive integer $a_1 < n_{g-1}\delta_{g-1} - 1$ and pick $a_2, a_3 \in \mathbb{N}_{>0}$ such that both $e_{g-1}^2(n_{g-1}\delta_{g-1} - \langle a_1; a_2 \rangle)$ and $e_{g-1}^3(n_{g-1}\delta_{g-1} - \langle a_1; a_2, a_3 \rangle)$ are in the semigroup of $\mathbb{N}_{>0}$ generated by $\delta_0, \delta_1, \ldots, \delta_{g-1}$. Then, it suffices to take

$$a = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{b}}},$$

where b is any non-rational positive real number. Thus, with the above election the corresponding δ-sequence in $\mathbb{R}$ is $\left\{ \frac{\delta_0}{\delta_1}, 1, \frac{\delta_2}{\delta_1}, \ldots, \frac{\delta_{g-1}}{\delta_1}, \frac{1}{\delta_1}(n_{g-1}\delta_{g-1} - a) \right\}$.

We conclude this section by showing how to construct type E weight functions and giving some examples.

4.3.4. *Type E weight functions.* A procedure for obtaining this type of weight functions is, starting with a given δ-sequence in $\mathbb{N}_{>0}$, to reproduce indefinitely the procedure given in Subsection 4.3.1 but with the following extra condition in Step 2: $z\delta_g < \delta_{g+1}'$. This condition

assures that, normalizing the successively obtained δ-sequences, one gets an increasing δ-sequence in $\mathbb{Q}$ (with the natural ordering) and, then, it generates a well-ordered semigroup. For instance, considering only values of z which are relatively prime with δ_g and taking $\delta'_{g+1} = (z+1)\delta_g$ in Step 2, the above condition will be satisfied.

4.3.5. *Examples.*   We begin this subsection by noting that the examples of weight functions over the polynomial ring in two indeterminates given by O'Sullivan in [**40**] are of the types described in this paper. Let us start by Example 3.3 in [**40**]. There, up to minor changes of notation, it is considered a pair (r, s) of relatively prime positive integers, $r < s$, and also the corresponding integers p and q such that $pr - qs = 1$, $0 < p < s$. Setting $u' = y^s/x^r$ and $v' = x^q/y^p$, the author considers the valuation given by $\nu(u') = (0, 1)$ and $\nu(v') = (1, 0)$ whose valuation ring is $k[u', v', v'/u', v'/u'^2, \ldots]_{(u')}$. The weight function defined by ν has as order domain $k[x, y]$. Expressed in our language, this is simply a type C weight function given by the δ-sequence $\{\delta_0 := (s, p), \delta_1 := (r, q)\}$. If we consider the continued fraction expansion of $s/(s - r) = \langle a_1; a_2, \ldots, a_r \rangle$, it holds that when we apply the Euclidean algorithm to δ_0 and $\delta_0 - \delta_1$ as described at the beginning of Section 4.2, we get $r - 1$ rows reproducing the case of s and $s - r$ and, afterwards, we cannot continue. So, the dual graph will have infinitely many satellite points over the divisor obtained after reproducing the blowing-up procedure given by the pair $(s, s - r)$.

Also, it is clear that Example 3.4 in [**40**] corresponds to a type D weight function w such that $w(y) = 1$ and $w(x) = \tau$, $\tau > 1$ being an non-rational real number.

Finally, Example 5.2 in [**40**] corresponds to the simplest type E weight function. The HNE of the associated valuation in a regular system of parameters $\{u, v\}$ of the local ring $\mathcal{O}_{\mathbb{P}^2, p}$ is

$$
\begin{aligned}
v &= u^2 + u^2 w_1 \\
u &= w_1^2 + w_1^2 w_2 \\
w_1 &= w_2^2 + w_2^2 w_3 \\
&\vdots \qquad \vdots
\end{aligned}
$$

repeating the above structure indefinitely. The first elements of a family of δ-sequences in $\mathbb{N}_{>0}$ providing the desired δ-sequence would be $\{3, 1\}, \{6, 2, 5\}, \{12, 4, 10, 19\}, \ldots$.

Let us see other examples. $\Delta = \{(10, 10), (3, 3), (24, 25)\}$ is a δ-sequence in $\mathbb{Z}^2$ which comes from the δ-sequence in $\mathbb{N}_{>0}$ $\{40, 12, 97\}$. The continued fraction $\langle a_1; a_2, a_3 \rangle$ is $\langle 5; 1, 3 \rangle$, since $m_1 = 23$ and $e_1 = 4$, so it holds that $(A, B) = (1, 1)$ and $(A', B') = (1, 0)$.

Starting with the δ-sequence in $\mathbb{N}_{>0}$ $\{11, 9\}$, the development in Subsection 4.3.3 shows that $\Delta = \{11/9, 1, (19 - \frac{2\sqrt{3}+1}{3\sqrt{3}+1})/9\}$ is a δ-sequence in $\mathbb{R}$. Indeed, the last element in the δ-sequence can be obtained by setting, with the notation in that subsection, $a_1 = 80, a_2 = 1, a_3 = 2$ and $b = \sqrt{3}$. $\Delta = \{12/8, 1, 1/3, 3/4, 13/24, (11 - \frac{\sqrt{3}+1}{2\sqrt{3}+2})/24\}$ is another example of δ-sequence in $\mathbb{R}$. This is attached to the δ-sequence in $\mathbb{N}_{>0}$, $\{36, 24, 8, 18, 13\}$. Here, $a_1 = 15, a_2 = 2, a_3 = 1$ and also, for convenience, $b = \sqrt{3}$.

5.   *Evaluation codes given by δ-sequences.*

5.1. *Generalized telescopic semigroups*

A semigroup $S \subseteq \mathbb{N}_{\geq 0}$ is called to be *telescopic* if it is spanned by a finite sequence of positive integers $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ (named *telescopic sequence*) such that $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_r) = 1$ and $\alpha_i / \gcd(\alpha_1, \alpha_2, \ldots, \alpha_i)$ belongs to the semigroup generated by

the set $\{\alpha_1/\gcd(\alpha_1,\alpha_2,\ldots,\alpha_{i-1}),\ldots,\alpha_{i-1}/\gcd(\alpha_1,\alpha_2,\ldots,\alpha_{i-1})\}$. An important property of these semigroups is that each element $\alpha \in S$ can be uniquely expressed in the form $\alpha = \sum_{i=1}^{r} a_i\alpha_i$, provided that $a_1 \geq 0$ and $0 \leq a_i < \gcd(\alpha_1,\alpha_2,\ldots,\alpha_{i-1})/\gcd(\alpha_1,\alpha_2,\ldots,\alpha_i)$ $(2 \leq i \leq r)$. This fact has importance when one desires to bound the minimum distance of the dual codes of the evaluation ones given by classical weight functions [**19**].

The following definition is a natural enlargement in such a way that the mentioned property is preserved.

DEFINITION 5.1.   A *generalized telescopic semigroup* in $\mathbb{Z}^2$ (respectively, $\mathbb{Q}$) (respectively, $\mathbb{R}$) is a cancellative well-ordered commutative with zero semigroup spanned by a set $A = \{\alpha_i\}_{i=1}^{r}$ such that:

($\mathbb{Z}^2$) $r < \infty$, $A \subset \mathbb{Z}^2$, which is lexicographically ordered, the points in $\{\alpha_i\}_{i=1}^{r-1}$ belong to the same line L which passes through $(0,0)$, $\alpha_r \notin L$ and there exists a telescopic sequence, $\{\beta_i\}_{i=1}^{r-1}$, such that the morphism of ordered semigroups $\varrho : \langle\alpha_1,\ldots\alpha_{r-1}\rangle \longrightarrow \langle\beta_1,\ldots\beta_{r-1}\rangle$ given by $\varrho(\alpha_i) = \beta_i$ is an isomorphism.

($\mathbb{Q}$) (Respectively, $r = \infty$, $A \subset \mathbb{Q}$ (natural order) and for each $i > 1$ there exists a telescopic sequence, $\{\beta_j\}_{j=1}^{i}$, such that the morphism of ordered semigroups $\rho : \langle\alpha_1,\ldots\alpha_{i-1}\rangle \longrightarrow \langle\beta_1,\ldots\beta_{i-1}\rangle$, $\rho(\alpha_j) = \beta_j$, are isomorphisms.)

($\mathbb{R}$) (Respectively, $r < \infty$, $A \subset \mathbb{R}$ (natural order), $\alpha_i \in \mathbb{Q}, 0 \leq i \leq r-1$, $\alpha_r \in \mathbb{R} \setminus \mathbb{Q}$ and there exists a telescopic sequence, $\{\beta_i\}_{i=1}^{r-1}$, such that the morphism of ordered semigroups ρ defined as in ($\mathbb{Z}^2$) is an isomorphism.)

Generically, a generalized telescopic semigroup in $\mathbb{Z}^2$, $\mathbb{Q}$ or $\mathbb{R}$ will be named simply a generalized telescopic semigroup.

PROPOSITION 5.2.   *Let S be a generalized telescopic semigroup spanned by $\{\alpha_i\}_{i=1}^{r}, r \leq \infty$, as above. Then any element $\alpha \in S$ can be written in a unique way of the form*

$$\alpha = \sum_{i=1}^{s} a_i\alpha_i, \tag{5.1}$$

where $s < r + 1$, $a_1, a_s \geq 0$ and $0 \leq a_i < \gcd(\alpha_1,\alpha_2,\ldots,\alpha_{i-1})/\gcd(\alpha_1,\alpha_2,\ldots,\alpha_i)$ $(2 \leq i < s)$ and we have put $\gcd(\alpha_1,\alpha_2,\ldots,\alpha_j) := \gcd(\beta_1,\beta_2,\ldots,\beta_j)$ $(1 \leq j < r)$, β_j being as in Definition 5.1.

Proof.   First let us see the case when r is finite. Due to the nature of the generators of S, when we set $\alpha = \sum_{i=1}^{r} a_i\alpha_i$, it holds that the value a_r must be unique, since either it is the unique nonnegative integer such that $\alpha - a_r\alpha_r$ is on the line L when $S \subset \mathbb{Z}^2$ or it is the unique nonnegative integer such that $\alpha - a_r\alpha_r$ is a rational value whenever $S \subset \mathbb{R}$. Now the semigroup spanned by $\{\alpha_i\}_{i=1}^{r-1}$ behaves as the one generated by the elements, except the last one, of a telescopic semigroup. Thus, if we set $a = \gcd(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})$, it happens that $\langle\alpha_1/a,\alpha_2/a,\ldots,\alpha_{r-1}/a\rangle$ is like a telescopic semigroup and the fact

$$(\alpha - a_r\alpha_r)/a \in \langle\alpha_1/a,\alpha_2/a,\ldots,\alpha_{r-1}/a\rangle$$

provides the desired property.

When r is infinite, the proof runs similarly. Indeed, assume that $\alpha = \sum_{i=1}^{s} b_i\alpha_i$. If $a = \gcd(\alpha_1,\alpha_2,\ldots,\alpha_s)$, then $\langle\alpha_1/a,\alpha_2/a,\ldots,\alpha_s/a\rangle$ behaves as a telescopic semigroup and, therefore $\alpha/a = \sum_{i=1}^{s} a_i(\alpha_i/a)$, where the set $\{a_i\}_{i=1}^{s}$ satisfies the desired properties and the result is proved.   □

REMARK.   A δ-sequence in $\mathbb{N}_{>0}$ generates a telescopic semigroup. We have just enlarged this last concept in such a way that any δ-sequence Δ generates a generalized telescopic semigroup and so S_Δ satisfies the property given in the above proposition.

5.2. *Evaluation codes*

To construct error-correcting codes from a δ-sequence, we must consider the weight function w_Δ described in Theorem 4.9 and an epimorphism of k-algebras $ev : k[x,y] \to k^n$, for some fixed positive integer n, which usually will consist of evaluating n previously picked points p_i ($1 \leq i \leq n$) in k^2. The family of defined evaluation codes will be $\{E_\alpha := ev(O_\alpha)\}_{\alpha \in S_\Delta}$, O_α as in Theorem 4.9. The dual spaces of the vector spaces E_α will be denoted by C_α and they are the elements in the family of dual codes of the evaluation ones. Fixed α it suffices to compute the family $\{ev(\prod_{i=0}^m q_i^{\gamma_i})\}$, where $\prod_{i=0}^m q_i^{\gamma_i}$ runs over the set of polynomials described at the beginning of Subsection 4.3 for obtaining a generator set of E_α. Using linear algebra, it is easy to compute bases for E_α and C_α. Depending on n, there is a positive integer Ω_n such that the vector spaces C_α vanish if and only if $\alpha \geq \Omega_n$.

On the other hand, and as in the case of evaluation codes coming from order functions on $\mathbb{N}_{\geq 0}$, denote by β an element in S_Δ and set

$$\omega_\beta := \operatorname{card}\{(\beta_1, \beta_2) \in S_\Delta^2 \mid \beta_1 + \beta_2 = \beta\}.$$

The values

$$d(\alpha) := \min\{\omega_\beta | \alpha < \beta \in S_\Delta\}$$

and

$$d_{ev}(\alpha) := \min\{\omega_\beta | \alpha < \beta \in S_\Delta \text{ and } C_\beta \neq C_{\beta^+} \},$$

where $\beta^+ := \min\{\gamma \in S_\Delta | \gamma > \beta\}$ are named *Feng-Rao distances* of C_α. They satisfy $d(C_\alpha) \geq d_{ev}(\alpha) \geq d(\alpha)$, $d(C_\alpha)$ being the minimum distance of the code C_α. Above considerations and Proposition 5.2 prove the following

THEOREM 5.3.   *Let $\Delta = \{\delta_i\}_{i=0}^r$, $r \leq \infty$, be a δ-sequence and $\{E_\alpha\}_{\alpha \in S_\Delta}$ and $\{C_\alpha\}_{\alpha \in S_\Delta}$ the evaluation and dual codes given by Δ and an epimorphism ev. Then, the Feng-Rao distances satisfy $d(\alpha) \leq \min[\prod_{i=0}^s(a_i + 1)] - 2 \leq d_{ev}(\alpha)$, where the integer vectors $(a_0, a_1, \ldots, a_s)$ runs over the unique coefficients of the corresponding expression (5.1) for $\{\delta_i\}_{i=0}^r$ instead of $\{\alpha_i\}_{i=1}^r$ of those elements in S_Δ which are larger than or equal to α and smaller than Ω_n.*

Dual evaluation codes given by classical weight functions admit another lower bound of its minimal distance, called Goppa distance. In this case, the attached semigroup to the weight function S^* is numerical, its elements can be enumerated according the natural ordering by a map $\chi : S^* \to \mathbb{N}_{\geq 0}$ and, if $\alpha^* \in S^*$, the Goppa bound of the dual evaluation code associated with α^* is $d_G(\alpha^*) = \chi(\alpha^*) + 1 - \xi_{S^*}$, ξ_{S^*} being the number of gaps of S^*. Let us see that a similar bound can be given in our case. Let Δ be a δ-sequence. When Δ is finite, for our purposes of ordering S_Δ, we can always write $\Delta = \{\delta_0, \delta_1, \ldots, \delta_{g-1}\} \cup \{\delta_g\}$, where the ordered semigroup spanned by $\{\delta_0, \delta_1, \ldots, \delta_{g-1}\}$ is isomorphic to a telescopic one that we shall write $S^* = \langle \delta_0^*, \delta_1^*, \ldots, \delta_{g-1}^* \rangle$. When Δ is not finite, to each value $i \geq 0$, we shall associate a value $\xi_\Delta(i)$, which will be the number of gaps of the semigroup generated by the δ-sequence in $\mathbb{N}_{>0}$, $\{\delta_0^*, \delta_1^*, \ldots, \delta_i^*\}$, associated with the normalized one $\{\delta_0, \delta_1, \ldots, \delta_i\}$.

Now, let $\alpha \in S_\Delta$. If Δ is finite, set $\alpha = \sum_{i=0}^g a_i \delta_i$ the unique expression of α as given in (5.1). Set B the least positive integer such that $B\delta_g > \alpha$ and for $0 \leq j \leq B$, write

$$A_j + j\delta_g := \min\{z + j\delta_g | z + j\delta_g > \alpha \text{ and } z \in \langle \delta_0, \delta_1, \ldots, \delta_{g-1} \rangle\}.$$

If $A_j = \sum_{i=0}^{g-1} a_i^j \delta_i$ according (5.1), we denote $A_j^* = \sum_{i=0}^{g-1} a_i^j \delta_i^*$ and define the *Goppa distance* of C_α as $d_{S_\Delta}(\alpha) = \min\{d_G(A_j^*)(j+1)|0 \leq j \leq B\}$.

For the infinite case, assuming as above $\alpha = \sum_{i=0}^s a_i \delta_i$, set $\alpha^* = \sum_{i=0}^s a_i \delta_i^*$ and the *Goppa distance* of C_α will be

$$d_{S_\Delta}(\alpha) := \chi(\alpha^*) + 1 - \xi_\Delta(s).$$

Taking into account properties of the classical Goppa distance and of the telescopic semigroups, it holds the following

PROPOSITION 5.4.   *Let* $\Delta = \{\delta_i\}_{i=0}^r$, $r \leq \infty$, *be a* δ-*sequence and* $\{E_\alpha\}_{\alpha \in S_\Delta}$ *and* $\{C_\alpha\}_{\alpha \in S_\Delta}$ *the corresponding evaluation and dual codes for some fixed evaluation morphism* ev. *Then,* $d(\alpha) \geq d_{S_\Delta}(\alpha)$ *and the Goppa distance of* C_α *is*

$$d_{S_\Delta}(\alpha) = \begin{cases} \min_{0 \leq j \leq B} \left\{ \left[\chi(A_j^*) + 1 - \frac{1 + \sum_{i=0}^{g-1}(n_i - 1)\delta_i^*}{2} \right](j+1) \right\} & \text{if } s = g \text{ is finite} \\ \chi(\alpha^*) + 1 - \left(1 + \sum_{i=0}^s (n_i - 1)\delta_i^* \right)/2 & \text{otherwise,} \end{cases}$$

where B *is as above,* $n_0 = 1$ *and the remaining* n_i *are the usual ones for* Δ.

Proof.   First part can be proved by taking into account that when one considers weight functions with values in a semigroup S^* in $\mathbb{N}_{\geq 0}$ spanned by values whose greatest common divisor is one and with g gaps, then the Feng-Rao distance $d(\alpha)$ of the dual evaluation code associated with $\alpha \in S^*$ is larger than or equal to its Goppa distance. Last part is a consequence of the computation given in [**19**] of the number of gaps of a telescopic semigroup. $\square$

Finally, we prove that the semigroups S_Δ corresponding to δ-sequences Δ providing weight functions of type C are simplicial. This one will be a useful property since there exists an algorithm given by Ruano in [**32**] for computing the Feng-Rao distance $d(\alpha)$ of codes C_α associated with order functions with simplicial image semigroup included in $(\mathbb{N}_{\geq 0})^r$, $r \geq 1$. Recall what this concept means. Let S be a semigroup included in $(\mathbb{N}_{\geq 0})^r$, $r \geq 1$. Setting U an indeterminate, the k-algebra $k[S] := \bigoplus_{\alpha \in S} kU^\alpha$, where the product of polynomials is induced by $(aU^\alpha)(bU^\beta) = (ab)U^{\alpha+\beta}$, $a, b \in k$ and $\alpha, \beta \in S$ is named the *k-algebra of the semigroup* S. Now, set C_S the cone in $\mathbb{R}^r$ spanned by S. C_S is a strongly convex cone. We shall say that S is *simplicial* whenever the dimension of the k-algebra $k[S]$ coincides with the number of extremal rays of the cone C_S.

PROPOSITION 5.5.   *The semigroup* S_Δ *of a* δ-*sequence* $\Delta = \{\delta_i\}_{i=0}^g \subseteq (\mathbb{N}_{\geq 0})^2$ *of type C is simplicial.*

Proof.   Since Δ is of type C, the number of extremal rays of C_{S_Δ} is two. Now, set $k[V_0, V_1, \ldots, V_g]$ the polynomial ring in $g + 1$ indeterminates and set

$$\psi : k[V_0, V_1, \ldots, V_g] \to k[S_\Delta]$$

the morphism of k-algebras given by $\psi(V_i) = U^{\delta_i}$. ψ is an epimorphism. For $0 < i < g$, consider the unique expression

$$n_i \delta_i = \sum_{j=0}^{i-1} a_{ij} \delta_j,$$

such that $a_{i0} > 0$ and $0 \leq a_{ij} < n_j$ for the remaining indices j, being n_j the usual number associated with Δ. Then, the proof follows by taking into account that the kernel of ψ, I, is

the ideal of $k[V_0, V_1, \ldots, V_g]$ spanned by the set $G := \{V_i^{n_i} - \prod_{j=0}^{i-1} V_j^{a_{ij}}\}_{0<i<g}$ and therefore $k[S] \cong k[V_0, V_1, \ldots, V_g]/I$ whose dimension is also two.

Only remains to prove that G spans I. To do it, it suffices to notice that I is generated by the binomials $A - B \in k[V_0, V_1, \ldots, V_g]$ such that A and B are homogeneous monomials with coefficient 1 and $\psi(A - B) = 0$. These monomials can be represented by the set $\mathcal{B}$ of pairs $(a, b) \in (\mathbb{N}_{\geq 0}^{g+1})^2$ representing the exponents of both monomials. $\mathcal{B}$ gives rise to a congruence, that is an equivalence binary relation such that if $c \in \mathbb{N}_{\geq 0}^{g+1}$ and $(a, b) \in \mathcal{B}$, then $(a + c, b + c) \in \mathcal{B}$. As the set $\mathcal{D}$ of pairs given by the exponents of the elements in G, satisfies that $\mathcal{B}$ corresponds to the smallest congruence containing $\mathcal{D}$, we get that G spans the ideal I, which concludes the proof (see the proof of [**15**, Theorem 5.2] for a more detailed explanation of a close result). $\square$

5.3. *Examples*

First of all, we prove that Reed-Solomon codes can be regarded as particular cases of evaluation codes attached to type C weight functions.

PROPOSITION 5.6. *Consider the finite field $k = \mathbb{F}_q$, a δ-sequence in $\mathbb{Z}^2$, $\Delta = \{\delta_0 = (p_1, p_2), \delta_1 = (q_1, q_2)\}$, and the epimorphism ev given by evaluating $d \leq q - 1$ points in a line in $\mathbb{F}_q^2$. Then the family of evaluation codes $\{E_\alpha\}_{\alpha \in S_\Delta}$ is the family of length d Reed-Solomon codes associated with $\mathbb{F}_q$.*

Proof. Assume that we evaluate points at the line given by $y - ax - b = 0$; $a, b \in \mathbb{F}_q$. Our approximates are x and y and, in order to span the spaces O_α, we must use monomials in x and y. When we evaluate a monomial $x^m y^n$, this corresponds to evaluate $x^m(ax + b)^n$, that is a polynomial of degree $m + n$ in the indeterminate x. $p_1 \geq q_1$ and, since $\delta_1 < 2\delta_1 < \cdots < l\delta_1$ are elements in S_Δ, when we consider $E_{l\delta_1}$, $l \in \mathbb{N}_{>0}$, one evaluates among others a monomial of degree l in x. So, it suffices to show that no monomial of degree larger than l appears in $O_{l\delta_1}$. That is, we must prove that $r\delta_0 + s\delta_1 < t\delta_1$, $r, s \in \mathbb{N}_{\geq 0}$, implies $r + s \leq t$ and this is true because then $rp_1 + sq_1 \leq tq_1$ and thus $(s - t)q_1 \leq -rp_1 \leq -rq_1$, which concludes the proof when $q_1 > 0$. Otherwise the proof follows because then $r = 0$. $\square$

To finish this paper, we show some parameters of several families of dual codes C_α attached to δ-sequences. Our data are computed using the computer algebra system SINGULAR [**18**].

EXAMPLES 5.7. Fix the field $\mathbb{F}_7$ and the δ-sequences Δ_1, Δ_2 and Δ_3 of types C, D and E respectively given, as we have described, by the sequence in $\mathbb{N}_{>0}$, $\{11, 9\}$. $\Delta_1 = \{(5, 1), (4, 1)\}$. Indeed, with the notation in Definition 4.7, $\delta_0^* = 11, \delta_1^* = 9$, $t = 2, a_1 = 5, a_2 = 2$. So $\delta_0 = \underline{y}_1 = (5, 1)$ and $\delta_1 = \delta_0 - \underline{y}_0 = (4, 1)$. Our election for Δ_2 is $\Delta_2 = \{11/9, 1, (19 - \frac{2\sqrt{3}+1}{3\sqrt{3}+1})/9 \approx 2,031105\}$ (see Subsection 4.3.5). Δ_3, and any δ-sequence of type E given in the sequel, is constructed as we described at the end of Subsection 4.3.4, using the least value z we can choose. In this case the first four elements of Δ_3 are $11/9, 1, 3/2, 9/4$.

Consider the map ev given by evaluating at the points in $\mathbb{F}_7^2$:

$$\{(1, 1), (2, 2), \ldots, (6, 6), (1, 2), (1, 3), \ldots, (1, 6), (2, 1)\}.$$

$q_0 = x, q_1 = y$ are approximates for the corresponding to Δ_1 case and q_0, q_1 and $q_2 = y^{11} - x^9$ otherwise. Tables 2 and 3 show parameters (dimension k, minimum distance $d(C_\alpha)$ and Feng-Rao distance $d_{ev}(\alpha)$) of the successive codes C_α. Furthermore, only for Δ_1, we add the exponents of the approximates we use to get the new generator to add to the previous ones for

obtaining a basis of O_α. That is $O_{(4,1)}$ is generated by 1 and q_1, $O_{(5,1)}$ by 1, q_1 and q_0, $O_{(8,2)}$ by 1, q_1, q_0 and q_1^2, and so on.

TABLE 2. *First Case in Examples 5.7*

α	exp	k	$d_{\Delta_1}(C_\alpha)$	$d_{ev,\Delta_1}(\alpha)$
(4,1)	01	10	2	2
(5,1)	10	9	3	3
(8,2)	02	8	4	3
(9,2)	11	7	4	3
(10,2)	20	6	4	4
(12,3)	03	5	5	5
*(13,3)	12	4	5	5
(16,4)	04	3	6	6
**(17,4)	13	2	6	6
(20,5)	05	1	10	10

TABLE 3. *First Case in Examples 5.7*

k	$d_{\Delta_2}(C_{\alpha'})$	$d_{ev,\Delta_2}(\alpha')$	$d_{\Delta_3}(C_{\alpha''})$	$d_{ev,\Delta_3}(\alpha'')$
10	2	2	2	2
9	3	2	3	2
8	4	2	4	2
7	4	3	4	2
6	4	3	4	2
5	4	4	4	4
4	6	4	5	4
3	6	4	5	5
2	6	4	7	4
1	6	5	6	7

Notice that the length of the code is $n = 12$ and that the classical parameters of the codes in the first three rows cannot be improved (see [**31**, Theorem 5.3.10]). Also, if we set $(14, 3)$ 2 1 or $(15, 3)$ 3 0 as α and the exponents instead of those given in * either $(18, 4)$ 2 2; $(19, 4)$ 3 1 or $(20, 4)$ 4 0 instead of the values given in ** we get the same parameters for the corresponding codes attached to Δ_1.

Let us see another case, where we have used the δ-sequence in $\mathbb{N}_{>0}$, $\{36, 24, 8, 18, 13\}$. Here $\Delta_1 = \{(18, 0), (12, 0), (4, 0), (9, 0), (7, -1)\}$ and our election for Δ_2 is $\{12/8, 1, 1/3, 3/4,$ $13/24, (11 - \frac{\sqrt{3}+1}{2\sqrt{3}+2})/24 \approx 0,441666\}$. The approximates for Δ_1 and Δ_3 are $q_0 = x, q_1 = y, q_2 = -x^2 + y^3$, $q_3 = -x^6 + 3x^4y^3 - 3x^2y^6 + y^9 - y$ and $q_4 = x^{12} + x^{10}y^3 + x^8y^6 + x^6y^9 + 2x^6y + x^4y^{12} + x^4y^4 + x^2y^{15} - x^2y^7 - x + y^{18} - 2y^{10} + y^2$ and we must add another large polynomial q_5 for Δ_2. The table with the same parameters as above is displayed in Table 4.

EXAMPLES 5.8.   Next, we show another examples. First, we use only type C weight functions and, afterwards, we use weight functions of all described types. Consider the finite field $\mathbb{F}_{2^5}$ and ξ a primitive element. Tables 5 and 6 show the same parameters as above corresponding to some dual codes associated with the evaluation at the following 31 points in $\mathbb{F}_{2^5}^2$:

$$\{(\xi, \xi), (\xi, \xi^2), \ldots, (\xi, \xi^{14}), (\xi^2, \xi), (\xi^2, \xi^2), \ldots, (\xi^2, \xi^{14}), (\xi^3, \xi^3), (\xi^4, \xi^4), (\xi^5, \xi^5)\},$$

TABLE 4. *Second Case in Examples 5.7*

k	$d_{\Delta_1}(C)$	d_{ev,Δ_1}	$d_{\Delta_2}(C)$	d_{ev,Δ_2}	$d_{\Delta_3}(C)$	d_{ev,Δ_3}
10	2	2	2	2	2	2
9	3	2	2	2	2	2
8	3	2	3	2	2	2
7	4	3	4	2	4	4
6	4	3	5	3	4	4
5	4	3	5	3	4	4
4	4	4	6	3	4	4
3	7	4	7	4	4	4
2	7	4	9	4	8	5
1	11	5	12	5	8	8

and with the δ-sequences of type C $\Delta_1 = \{(21,0),(15,0),(35,0),(39,-1)\}$, $\Delta_2 = \{(2,1),(1,1)\}$ and $\Delta_3 = \{(5,5),(2,2),(7,8)\}$. Only for Δ_1, we add the values α and the exponents for the approximates which are $q_0 = x, q_1 = y, q_2 = y^7 + x^5$ and $q_3 = x^{15} + x^{10}y^7 + x^{15}y^{14} + x^5 + y^{21}$. k remains valid for the codes of each row.

TABLE 5. *First Case in Examples 5.8*

α	exp	k	$d_{\Delta_1}(C_\alpha)$	$d_{ev,\Delta_1}(\alpha)$
(15,0)	0100	29	2	2
(21,0)	1000	28	3	2
(30,0)	0200	27	3	2
(35,0)	0010	26	4	2
(36,0)	1100	25	4	2
(39,-1)	0001	24	4	3
(42,0)	2000	23	4	3
(45,0)	0300	22	5	3
(50,0)	0110	21	5	3

TABLE 6. *First Case in Examples 5.8*

k	$d_{\Delta_2}(C_{\alpha'})$	$d_{ev,\Delta_2}(\alpha')$	$d_{\Delta_3}(C_{\alpha''})$	$d_{ev,\Delta_3}(\alpha'')$
29	2	2	2	2
28	3	3	2	2
27	3	3	3	2
26	3	3	3	2
25	4	4	3	2
24	5	5	5	3
23	5	5	6	3
22	5	5	6	3
21	6	6	7	3

Now, we consider the same set of evaluation points, but Δ_1, Δ_2 and Δ_3 are δ-sequences of types C, D and E related with the δ-sequence in $\mathbb{N}_{>0}$, $\{36,24,8,18,13\}$. Concretely, $\Delta_1 = \{(18,0),(12,0),(4,0),(9,0),(7,-1)\}$, Δ_3 is as we described at the end of Subsection 4.3.4 and we take as election for Δ_2, $\Delta_2 = \{12/8, 1, 1/3, 3/4, 13/24, (6 - \frac{2\sqrt{3}+1}{11\sqrt{3}+5})/24 \approx 0,242266\}$. A partial table with parameters as above is displayed in Table 7.

TABLE 7. *Second Case in Examples 5.8*

k	$d_{\Delta_1}(C)$	d_{ev,Δ_1}	$d_{\Delta_2}(C)$	d_{ev,Δ_2}	$d_{\Delta_3}(C)$	d_{ev,Δ_3}
29	2	2	2	2	2	2
28	3	2	3	2	3	2
27	3	2	3	2	3	2
26	4	3	3	2	4	2
25	5	3	3	2	5	2
24	5	3	4	2	5	2
23	5	3	6	2	5	2
22	7	3	6	4	7	2
21	7	3	7	4	7	2

EXAMPLES 5.9.   Finally, we consider the same field of Examples 5.8 and the δ-sequences $\Delta_1 = \{(3,1),(2,1)\}$, $\Delta_2 = \{7/5, 1, (7 - \frac{\sqrt{3}+1}{4\sqrt{3}+3})/5 \approx 1,34496\}$ (for convenience our examples of type D valuations always use $b = \sqrt{3}$ according the notation in Subsection 4.3.3) and Δ_3 of type E related with the δ-sequence in $\mathbb{N}_{>0}$, $\{7,5\}$. The family of points to evaluate is

$$\{(\xi,\xi),(\xi,\xi^2),\ldots,(\xi,\xi^{14}),(\xi^6,\xi),(\xi^6,\xi^2),\ldots,(\xi^6,\xi^{10}),(\xi^2,\xi^{11}),(\xi^2,\xi^{12}),(\xi^2,\xi^{13}),$$

$$(\xi^2,\xi^{14}),(\xi^{20},\xi^{20}),(\xi^{21},\xi^{21}),(\xi^{28},\xi^{28})\}.$$

The (partial) corresponding table is given in Table 8.

TABLE 8. *Examples 5.9*

k	$d_{\Delta_1}(C)$	d_{ev,Δ_1}	$d_{\Delta_2}(C)$	d_{ev,Δ_2}	$d_{\Delta_3}(C)$	d_{ev,Δ_3}
29	2	2	2	2	2	2
28	3	3	3	2	3	2
27	4	3	3	3	3	2
26	4	3	4	3	4	2
25	4	4	5	3	4	2
24	5	4	5	3	4	2
23	5	4	6	3	5	2
22	5	4	6	3	5	2
21	6	4	6	4	5	2

References

1. S. S. ABHYANKAR, 'Local uniformization on algebraic surfaces over ground field of characteristic $p \neq 0$', *Ann. Math.* 63 (1956) 491–526.

2. S. S. ABHYANKAR, 'On the valuations centered in a local domain', *Amer. J. Math.* 78 (1956) 321–348.

3. S. S. ABHYANKAR, *Lectures on expansion techniques in Algebraic Geometry*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics 57 (Tata Institute of Fundamental Research, Bombay, 1977).

4. S. S. ABHYANKAR and T. T. MOH, 'Newton-Puiseux expansion and generalized Tschirnhausen transformation', *J. Reine Angew. Math.* 260 (1973) 47–83 and 261 (1973) 29–54.

5. P. BECKMAN and J. STÜCKRAD, 'The concept of Gröbner algebra', *J. Symb. Comput.* 10 (1990) 465–479.

6. E. R. BERLEKAMP, *Algebraic Coding Theory* (McGraw-Hill, New York, 1968).

7. A. CAMPILLO, *Algebroid curves in positive characteristic*, Lecture Notes in Math. 613 (Springer-Verlag, 1980).

8. A. CAMPILLO and J.I. FARRÁN, 'Computing Weierstrass semigroups and the Feng-Rao distance from singular plane models', *Finite Fields Appl.* 6 (2000) 71–92.

9. A. CAMPILLO and J.I. FARRÁN, 'Symbolic Hamburger-Noether expressions of plane curves and applications to AG codes', *Math. Comput.* 71 (2002) 1759–1780.

10. F. DELGADO, C. GALINDO and A. NUÑEZ, 'Saturation for valuations on two-dimensional regular local rings', *Math. Z.* 234 (2000) 519–550.

11. G. L. FENG and T. R. N. RAO, 'Decoding of algebraic geometric codes up to the designed minimum distance', *IEEE Trans. Inform. Theory* 39 (1993) 37–45.

12. J. FERNÁNDEZ DE BOBADILLA, 'Moduli spaces of polynomials in two variables', *Mem. Amer. Math. Soc.* 173 (2005).

13. M. FUJIMOTO and M. SUZUKI, 'Construction of affine plane curves with one place at infinity', *Osaka J. Math.* 39 (2002) 1005–1027.

14. C. GALINDO, 'Plane valuations and their completions', *Comm. Algebra* 23 (6) (1995) 2107–2123.

15. C. GALINDO and M. SANCHIS, 'Evaluation codes and plane valuations', *Des. Codes Crypt.* 41 (2) (2006) 199–219.

16. O. GEIL and R. PELLIKAAN, 'On the structure of order domains', *Finite Fields Appl.* 8 (2002) 369–396.

17. S. GRECO and K. KIYEK, 'General elements in complete ideals and valuations centered at a two-dimensional regular local ring' in *Algebra, Arithmetic and Geometry, with Applications* (Springer, 2003) 381–455.

18. G.M. GREUEL, G. PFISTER and H. SCHÖENEMANN, SINGULAR 3.0. *A Computer Algebra System for Polynomial Computations* (Centre for Computer Algebra, University of Kaiserslautern, 2005) http://www.singular.uni-kl.de.

19. T. HØHOLDT, J.H. VAN LINT and R. PELLIKAAN, 'Algebraic geometry codes' in *Handbook of coding theory 1* (Elsevier, Amsterdam, 1998) 871–961.

20. T. HØHOLDT and R. PELLIKAAN, 'On the decoding of algebraic geometric codes', *IEEE Trans. Inform. Theory* 41 (1995) 1589–1614.

21. C.D JENSEN, 'Fast decoding of codes from algebraic geometry', *IEEE Trans. Inform. Theory* 40 (1994) 223–230.

22. J. JUSTESEN, K.J. LARSEN, H.E. JENSEN, A. HAVEMOSE and T. HØHOLDT, 'Construction and decoding of a class of algebraic geometric codes', *IEEE Trans. Inform. Theory* 35 (1989) 811–821.

23. J. JUSTESEN, K.J. LARSEN, H.E. JENSEN and T. HØHOLDT, 'Fast decoding of codes from algebraic plane curves', *IEEE Trans. Inform. Theory* 38 (1992) 111–119.

24. Y. MADELUNG, *Implementation of a decoding algorithm for AG-codes from the Hermitian curve* (Rep. IT-93-137, The Technical Univ. Denmark, Lyngby, 1993).

25. J. L. MASSEY, 'Shift-register synthesis and BCH decoding', *IEEE Trans. Inform. Theory* 15 (1969) 122–127.

26. R. MATSUMOTO, 'Miura's generalization of one point AG codes is equivalent to Høholdt, van Lint and Pellikaan's generalization', *IEICE Trans. Fundam.* E82-A (10) (1999) 2007–2010.

27. F. MONSERRAT, 'Curves having one place at infinity and linear systems on rational surfaces', *J. Pure Appl. Algebra* 211 (2007) 685–701.

28. I. NIVEN and H. ZUCKERMAN, *An introduction to the theory of numbers* (John Wiley & Sons, 1972).

29. H. PINKHAM, 'Séminaire sur les singularités des surfaces' (Demazure-Pinkham-Teissier), Course donné au Centre de Math. de l'Ecole Polytechnique (1977–1978).

30. A. J. REGUERA, 'Semigroups and clusters at infinity', in *Algebraic geometry and singularities*, La Rábida, 1991, *Progr. Math.* 134 (1996) 339–374.

31. S. ROMAN, *Coding and information theory* (Springer, 1992).

32. D. RUANO, 'Computing the Feng-Rao distances for codes from order domains', *J. Algebra.* 309 (2007) 672–682.

33. S. SAKATA, 'Extension of the Berlekamp-Massey algorithm to N dimensions', *Inform. and Comput.* 84 (1990) 207–239.

34. S. SAKATA, J. JUSTESEN, Y. MADELUNG, H. E. JENSEN and T. HØHOLDT, 'Fast decoding of algebraic geometric codes up to the designed minimum distance', *IEEE Trans. Inform. Theory* 41 (1995) 1672–1677.

35. S. SAKATA, H.E. JENSEN and T. HØHOLDT, 'Generalized Berlekamp-Massey decoding of algebraic geometric codes up to half the Feng-Rao bound', *IEEE Trans. Inform. Theory* 41 (1995) 1762–1768.

36. A. SATHAYE, 'On planar curves', *Amer. J. Math.* 99 (1977) 1105–1135.

37. A. N. SKOROBOGATOV and S. G. VLĂDUT, 'On the decoding of algebraic geometric codes', *IEEE Trans. Inform. Theory* 36 (1990) 1051–1060.

38. M. SPIVAKOVSKY, 'Valuations in function fields of surfaces', *Amer. J. Math.* 112 (1990) 107–156.

39. M. E. O'Sullivan, 'Decoding of codes defined by a single point on a curve', *IEEE Trans. Inform. Theory* 41 (1995) 1709–1719.
40. M. E. O'Sullivan, 'New codes for the Belekamp-Massey-Sakata algorithm', *Finite Fields Appl.* 7 (2001) 293–317.
41. M. Suzuki, 'Affine plane curves with one place at infinity', *Ann. Inst. Fourier* 49 (1999) 375–404.
42. O. Zariski, 'Local uniformization on algebraic varieties', *Ann. Math.* 41 (1940) 852–896.
43. O. Zariski and P. Samuel, *Commutative Algebra.* Vol. II (Springer-Verlag, 1960).

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