

PHASE PORTRAITS OF ABEL QUADRATIC DIFFERENTIAL SYSTEMS OF SECOND KIND

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ABSTRACT. We provide normal forms and the global phase portraits on the Poincaré disk of some Abel quadratic differential equations of the second kind. Moreover, we also provide the bifurcation diagrams for these global phase portraits.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Neils Henrik Abel, one of the most active mathematicians in his time, dedicated himself to integral equations where he defined the Abel integral and worked on methods to solve special integral equations, which were later known as *Abel integral equations* [11]. His research on integral equations and their solutions led him work on differential equations, where he verified the importance of the Wronskian determinant for a differential equation of order two [11]. His study of the theory of elliptic functions was what got him involved into the analysis of special differential equations, which are a generalized Riccati differential equation. Due to his seminal work on these differential equations, they are named Abel differential equations. The Abel differential equation of the first and second kind are both non-homogeneous differential equations of first order and are related between them by a substitution. Despite the fact that Abel equations of first kind are very well studied, this is not the case of Abel differential equations of second kind. In this paper we will focus on them.

An Abel differential equation of second kind has the form

$$y \frac{dy}{dx} = A(x)y^2 + B(x)y + C(x), \quad (1)$$

with $A(x), B(x), C(x) \in \mathbb{R}(x, y)$. This differential equation can be written equivalently as the polynomial differential system

$$\dot{x} = d(x)y, \quad \dot{y} = a(x)y^2 + b(x)y + c(x),$$

where $A(x) = a(x)/d(x)$, $B(x) = b(x)/d(x)$ and $C(x) = c(x)/d(x)$, with polynomials $a(x), b(x), c(x)$ and $d(x)$. In this paper we are interested in studying the *Abel quadratic polynomial differential systems*, i.e. the differential systems of the form

$$\dot{x} = d(x)y := (d_0 + d_1x)y, \quad \dot{y} = a_0y^2 + (b_0 + b_1x)y + c_0 + c_1x + c_2x^2 \quad (2)$$

coming from the Abel differential equation of second kind (1). Here all the parameters are real. We assume that \dot{x} and \dot{y} do not have a common factor, that is $c_0^2 + c_1^2 + c_2^2 \neq 0$. Moreover $a_0 \neq 0$, otherwise this is not the Abel equation of second kind, and $b_1^2 + c_1^2 + c_2^2 + d_1^2 \neq 0$ to have a quadratic system, otherwise the system would not depend on x and hence would not be of our interest.

In this work we provide the global phase portraits of all the Abel differential systems of second kind of degree two given by system (2) with $d_1 = 0$. We assume then that $d_0 \neq 0$.

We shall use the well-known Poincaré compactification of polynomial vector fields. For more details, if needed, see chapter V in [6]. We say that two polynomial vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one

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onto the other which sends orbits to orbits preserving or reversing the direction of the flow. Before stating our theorem, we provide normal forms for system (2).

Proposition 1. *System (2) with $d_1 = 0$ and $d_0 \neq 0$ leads, after an affine change of variables and a scaling of the time, to the normal form $\dot{x} = y$, $\dot{y} = Q(x, y)$, where $Q(x, y)$ is one of the following five quadratic polynomials:*

- (K1) $Q(x, y) = Sx(x - r) + (B_0 + B_1x)y + y^2$;
- (K2) $Q(x, y) = Sx^2 + (B_0 + B_1x)y + y^2$;
- (K3) $Q(x, y) = S((x - a)^2 + b^2) + (B_0 + B_1x)y + y^2$;
- (K4) $Q(x, y) = Sx + (B_0 + B_1x)y + y^2$;
- (K5) $Q(x, y) = S + B_1xy + y^2$;

where $S^2 = 1$, $B_0, B_1, r, a \in \mathbb{R}$ and $b > 0$. Additionally, $B_1 \geq 0$ in all cases; $rS > 0$ in case (K1); and $B_1 \neq 0$ in case (K5).

Observe that once in normal form system (2) is a Kukles quadratic differential system.

We note that there are no intersections among the normal forms because the number and the multiplicity of the finite singular points is different from one normal form to the other. For (K3) and (K5) some phase portrait may coincide because the number of real finite singular points is the same.

The main result in this paper is the following one. The global phase portraits on the Poincaré disk of all the families of Proposition 1 are provided.

Theorem 2. *System (2) with $d_1 = 0$ and $d_0 \neq 0$ has 56 non-topologically equivalent phase portraits, distributed into the families of Proposition 1 as follows:*

- (a) *The family (K1) has 20 non-topologically equivalent phase portraits, which are the phase portraits (1)–(20) of Figure 1.*
- (b) *The family (K2) has 20 non-topologically equivalent phase portraits, which are the phase portraits (21)–(40) of Figure 1.*
- (c) *The family (K3) has 5 non-topologically equivalent phase portraits, which are the phase portraits (41), (42), (43), (55) and (56) of Figure 1.*
- (d) *The family (K4) has 11 non-topologically equivalent phase portraits, which are the phase portraits (44)–(54) of Figure 1.*
- (e) *The family (K5) has 2 non-topologically equivalent phase portraits, which are the phase portraits (55) and (56) of Figure 1.*

Let $R_0^+ = \{x \in \mathbb{R} : x \geq 0\}$. Theorem 2 follows directly after the next theorem.

Theorem 3. *For the normal forms (K1)–(K5) of Theorem 2 the parameter spaces modulo the group action and the corresponding bifurcation diagrams are as follows:*

- (a) *For the family (K1) the parameter space is $(S, B_0, B_1, r) \in \{-1, 1\} \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}$ with $rS > 0$. Let $C_1 := (B_0 + B_1)(B_0 + B_1r) + (r - 1)^2S$ and $C_2 := B_0(B_0 + B_1 + B_1r) + (r + 1)^2S$.*

When $S = 1$ the global phase portrait is:

- (1) *for $B_1 > 2$, $C_1 > 0$ and $B_0 < -(r + 1)$;*
- (2) *for $C_1 = 0$ and $B_0 < -(r + 1)$;*
- (3) *for $C_1 < 0$;*
- (4) *for $C_1 = 0$ and $-(r + 1) < B_0 < 0$;*
- (5) *for $B_1 > 2$, $C_1 > 0$ and $B_0 < 0$;*
- (6) *for $B_1 > 2$ and $B_0 \geq 0$;*
- (7) *for $B_1 = 2$ and $B_0 < -(r + 1)$;*
- (8) *for $B_1 = 2$ and $B_0 = -(r + 1)$;*
- (9) *for $B_1 = 2$ and $-(r + 1) < B_0 < 0$;*
- (10) *for $B_1 = 2$ and $B_0 \geq 0$;*
- (11) *for $0 \leq B_1 < 2$ and $B_0^2 + B_1^2 > 0$;*
- (12) *for $B_0 = B_1 = 0$.*

The bifurcation diagram is shown in Figure 2(a). When $S = -1$ the global phase portrait is:

- (13) for either $B_1 \neq 0, C_1 > 0$ and $B_0 < 0$, or $B_1 \neq 0, C_2 < 0$ and $B_0 \leq 0$, or $B_1 = 0, C_2 < 0, r > -1$ and $B_0 \neq 0$;
- (14) for either $B_1 \neq 0, C_1 = 0$ and $B_0 < 0$, or $B_1 \neq 0, C_2 = 0$ and $r > -1$, or $B_1 = 0, C_2 = 0$ and $r > -1$;
- (15) for either $B_1 \neq 0, C_2 > 0$ and $B_0 < 0$, or $B_1 \neq 0, C_2 < 0$ and $B_0 > 0$, or $B_1 \neq 0, C_2 > 0$ and $B_0 > 0$, or $B_1 = 0$ and $C_2 > 0$;
- (16) for either $B_1 \neq 0, C_2 = 0$ and $r < -1$, or $B_1 = 0, C_2 = 0$ and $r < -1$;
- (17) for $r = -1$ and $B_0 = 0$;
- (18) for either $B_1 \neq 0, C_2 < 0$ and $r < -1$, or $B_1 = 0, C_2 < 0, r < -1$ and $B_0 \neq 0$;
- (19) for $B_1 = 0, r > -1$ and $B_0 = 0$;
- (20) for $B_1 = 0, r < -1$ and $B_0 = 0$.

The bifurcation diagrams are shown in Figures 2(b) and (c), depending on whether $B_1 \neq 0$ or $B_1 = 0$, respectively.

- (b) For the family (K2) the parameter space is $(S, B_0, B_1) \in \{-1, 1\} \times \mathbb{R} \times \mathbb{R}_0^+$. Let $C_1 := B_0(B_0 + B_1) + S$.

When $S = 1$ the global phase portrait is:

- (21) for $B_1 > 2, C_1 > 0$ and $B_0 < -1$;
- (22) for $B_1 > 2, C_1 = 0$ and $B_0 < -1$;
- (23) for $C_1 < 0$;
- (24) for $B_1 > 2, C_1 = 0$ and $B_0 > -1$;
- (25) for $B_1 > 2, C_1 > 0$ and $-1 < B_0 < 0$;
- (26) for $B_1 > 2$ and $B_0 = 0$;
- (27) for $B_1 > 2$ and $B_0 > 0$;
- (28) for $B_1 = 2$ and $B_0 < -1$;
- (29) for $B_1 = 2$ and $B_0 = -1$;
- (30) for $B_1 = 2$ and $-1 < B_0 < -1/2$;
- (31) for $B_1 = 2$ and $B_0 = -1/2$;
- (32) for $B_1 = 2$ and $-1/2 < B_0 < 0$;
- (33) for $B_1 = 2$ and $B_0 = 0$;
- (34) for $B_1 = 2$ and $B_0 > 0$;
- (35) for $B_1 < 2$ and $B_0^2 + B_1^2 \neq 0$;
- (36) for $B_1 < 2$ and $B_0 = 0$.

The bifurcation diagram is shown in Figure 2(d). When $S = -1$ the global phase portrait is:

- (37) for $C_1 > 0$;
- (38) for $C_1 = 0$;
- (39) for $C_1 < 0$ and $B_0 \neq 0$.
- (40) for $B_0 = 0$.

The bifurcation diagram is shown in Figure 2(e).

- (c) For the family (K3) the parameter space is $(S, B_0, B_1, a, b) \in \{-1, 1\} \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R}^+$.

When $S = 1$ the global phase portrait is:

- (41) for $B_1 = 2$ and $B_0 \neq -(2a + 1)$;
- (42) for $B_1 = 2$ and $B_0 = -(2a + 1)$;
- (43) for $B_1 < 2$;
- (55) when $B_1 > 2$.

The bifurcation diagram is shown in Figure 2(f). When $S = -1$ the global global phase portrait is (56).

- (d) For the family (K4) the parameter space is $(S, B_0, B_1) \in \{-1, 1\} \times \mathbb{R} \times \mathbb{R}_0^+$. Let $C_0 := B_1(B_0 + B_1) - S, C_1 := B_0 + B_1/4 - B_1^3/64 + \dots$ (see the proof) and $C_2 := B_0B_1 - S$.

When $S = 1$ the global phase portrait is:

(44) for $C_0 < 0$;

(45) for $C_0 = 0$;

(46) for $C_0 > 0$.

The bifurcation diagram is shown in Figure 2(g). When $S = -1$ the global phase portrait is:

(47) for $C_2 < 0$;

(48) for $C_2 = 0$;

(49) for $C_1 < 0$ and $C_2 > 0$;

(50) for $C_1 = 0$;

(51) for $C_1 > 0$ and $B_0 < 0$;

(52) for $B_0 \geq 0$ and $B_1 > 0$;

(53) for $B_1 = 0$ and $B_0 \neq 0$;

(54) for $B_0 = B_1 = 0$;

The bifurcation diagram is shown in Figure 2(h).

(e) For the family (K5) the parameter space is $(S, B_1) \in \{-1, 1\} \times \mathbb{R}_0^+$. The phase portrait is (55) when $S = 1$ and is (56) when $S = -1$.

We remark that the curve $C_1 = 0$ controlling the bifurcation of the global phase portraits described in Theorem 3 for family (K4) is not algebraic. The rest of the curves controlling the bifurcation of the global phase portraits in Theorem 3 are algebraic.

In section 2 we state some basic results that we shall need later on. Proposition 1 is proved in section 3. The proof of Theorem 2 follows directly from the proof of Theorem 3, which is given in section 4.

2. PRELIMINARY RESULTS

First we introduce the basic definitions, notations and results that we need for the analysis of the local phase portraits of the finite and infinite singular points of the quadratic differential systems and afterwards we define the Poincaré compactification. The results of subsections 2.1 and 2.3 can be found in [4]. The results of subsection 2.2 can be found in [5]. The results of subsection 2.4 can be found in [7].

2.1. Singular points. Consider an analytic planar differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ and its associated vector field $X = (P, Q)$. A point $p \in \mathbb{R}^2$ is a *singular point* of X if $P(p) = Q(p) = 0$. We define, for a singular point $p \in \mathbb{R}^2$, $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p) \in \mathbb{R}$ and $T = P_x(p) + Q_y(p) \in \mathbb{R}$. They correspond, respectively, to the determinant and the trace of the Jacobian matrix $DX(p)$.

The singular point p is *non-degenerated* if $\Delta \neq 0$. It is *degenerated* otherwise. Then, p is an isolated singular point. Moreover p is a *saddle* if $\Delta < 0$, a *node* if $T^2 \geq 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < \Delta$; for more details see [2].

The singular point p is called *hyperbolic* if the two eigenvalues of the Jacobian matrix $DX(p)$ have nonzero real part. So the hyperbolic singular points are the non-degenerate ones except the weak foci and the centers.

A degenerate singular point p such that $T \neq 0$ is called *semi-hyperbolic*, and p is isolated in the set of all singular points. Next we summarize the results on semi-hyperbolic singular points, see Theorem 65 of [2].

Proposition 4. *Let $(0, 0)$ be an isolated singular point of the vector field $(F(x, y), y + G(x, y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in x and y . Let $y = g(x)$ be the solution of the equation $y + G(x, y) = 0$ in a neighborhood of $(0, 0)$. Assume that $f(x) = F(x, g(x)) = \mu x^m + \dots$, where $m \geq 2$ and $\mu \neq 0$. If m is odd then the origin is either an unstable node or a saddle depending on whether $\mu > 0$ or $\mu < 0$, respectively. If m is even then $(0, 0)$ is a saddle-node,*

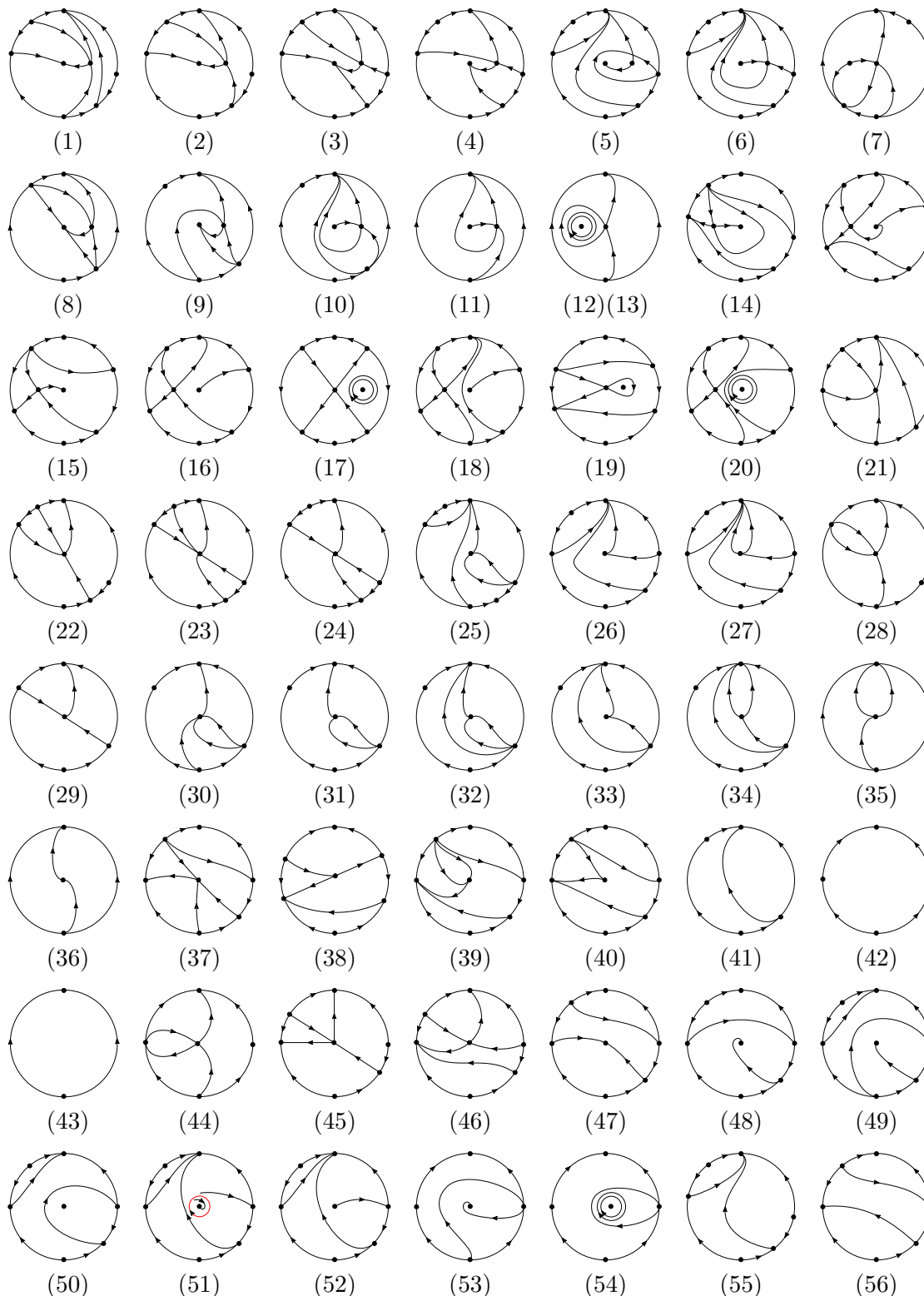


FIGURE 1. Phase portraits of the differential systems appearing in Theorems 2 and 3.

i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector.

The singular points which are non-degenerate or semi-hyperbolic are called *elementary*.

When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is *nilpotent*. Next we summarize some results

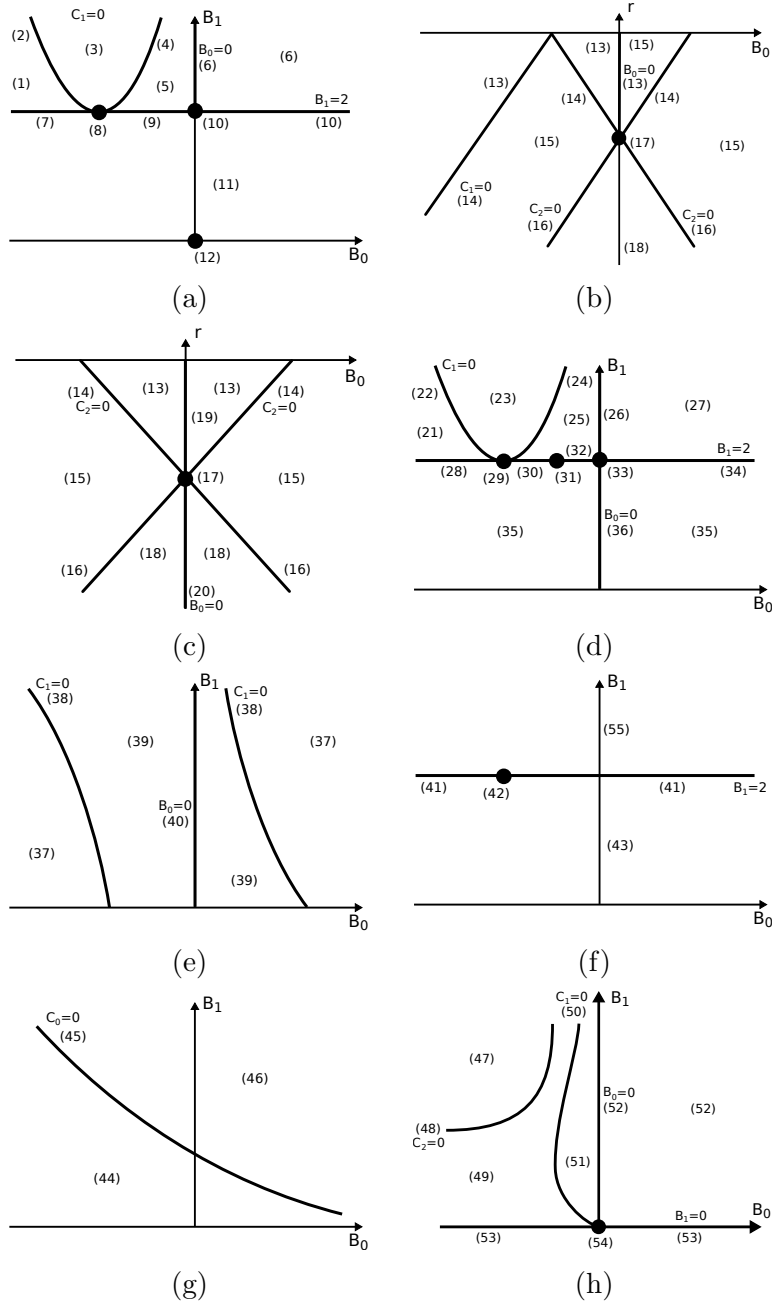


FIGURE 2. Bifurcation diagrams of the normal forms: (a) (K1) in the case $S = 1$ and for all $r > 0$; (b) (K1) in the case $S = -1$ and for all $B_1 \neq 0$; (c) (K1) in the case $S = -1$ and $B_1 = 0$; (d) (K2) in the case $S = 1$; (e) (K2) in the case $S = -1$; (f) (K3) in the case $S = 1$; (g) (K4) in the case $S = 1$; (h) (K4) in the case $S = -1$.

on nilpotent singular points (see Theorems 66 and 67 and the simplified scheme of section 22.3 of [2]).

Proposition 5. *Let $(0, 0)$ be an isolated singular point of the vector field $(y + F(x, y), G(x, y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in x and y . Let $y = f(x)$ be the solution of the equation $y + F(x, y) = 0$ in a neighborhood of $(0, 0)$. Assume that $G(x, f(x)) = Kx^\kappa + \dots$ and $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = Lx^\lambda + \dots$, with $K \neq 0$, $\kappa \geq 2$ and $\lambda \geq 1$. The following statements hold.*

- (i) *If κ is even and*
 - (a) *$\kappa > 2\lambda + 1$ then the origin is a saddle-node.*

- (b) $\kappa < 2\lambda + 1$ or $\Phi \equiv 0$ then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors.
- (ii) If κ is odd and $K > 0$ then the origin is a saddle.
- (iii) If κ is odd, $K < 0$ and
 - (a) λ even, $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ even and $\kappa > 2\lambda + 1$ then the origin is a stable (unstable) node if $L < 0$ ($L > 0$).
 - (b) λ odd, $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ odd and $\kappa > 2\lambda + 1$ then the origin is an elliptic saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector.
 - (c) $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) < 0$, or $\kappa < 2\lambda + 1$ then the origin is a focus or a center, and if $\Phi(x) \equiv 0$ then the origin is a center.

Finally if the Jacobian matrix at the singular point p is identically zero and p is isolated inside the set of all singular points then we say that p is *linearly zero*. The study of its local phase portrait needs a special treatment: the directional blow-ups, see for more details [1, 3]. But if a quadratic vector field has a finite linearly zero singular point then it is equivalent to a homogeneous quadratic differential system doing if necessary a translation of the linearly zero singular point to the origin, and the global phase portraits of the quadratic homogeneous vector fields are well known, see [13].

The definitions of hyperbolic, parabolic and elliptic sectors near a singular point can be found in [2]. Roughly speaking, in a *hyperbolic sector* there are curves through points of the sector which leave the sector with both increasing and decreasing time. A sector such that all curves in a sufficiently small neighborhood of the singular point tend to it as $t \rightarrow +\infty$ ($t \rightarrow -\infty$) and leave the sector as $t \rightarrow -\infty$ ($t \rightarrow +\infty$) is known as a *parabolic sector*. Finally, a sector containing loops, and moreover only nested loops, is known as an *elliptic sector*.

The number of elliptic sectors and the number of hyperbolic sectors in a neighborhood of a singular point are denoted by \mathbf{e} and \mathbf{h} , respectively. The rest of the sectors are parabolic. The *index* of a singular point p is defined as $i(p) = (\mathbf{e} - \mathbf{h})/2 + 1 \in \mathbb{Z}$. For a proof of this formula see [2].

Next we state a result of [9] that allows to distinguish between a center and a focus for a singular point of a quadratic differential system having pure imaginary eigenvalues.

Theorem 6. *Consider the quadratic differential system*

$$\dot{x} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \quad \dot{y} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2. \quad (3)$$

Let

$$\begin{aligned} \omega_1 &= A\alpha - B\beta, \\ \omega_2 &= \gamma(5(A\beta + B\alpha) - \alpha^2 - \beta^2), \\ \omega_3 &= \gamma\delta(A\beta + B\alpha), \end{aligned}$$

where

$$\begin{aligned} A &= a_{20} + a_{02}, \quad B = b_{20} + b_{02}, \quad \alpha = a_{11} + 2b_{02}, \quad \beta = b_{11} + 2a_{20}, \\ \gamma &= b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3, \\ \delta &= a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B. \end{aligned}$$

Then the following statements hold:

- (1) If $\omega_1 = \omega_2 = \omega_3 = 0$ then the origin is a center.
- (2) If there exists $k \in \{1, 2, 3\}$ such that $\omega_j = 0$ for all $1 \leq j < k$ and $\omega_k \neq 0$, then the origin is a focus of order k . The stability of this focus is given by the sign of $\omega_k \neq 0$: when $\omega_k < 0$ the focus is stable and when $\omega_k > 0$ the focus is unstable.

2.2. Separatrices and canonical regions. Consider the planar differential system $\dot{x} = P(x, y), \dot{y} = Q(x, y)$, where $P, Q \in \mathcal{C}^r, r \geq 1$. For this differential system the following three properties are well-known, see for more details [12]:

- (i) For all $p \in U$ there exists an open interval $I_p \subseteq \mathbb{R}$ where the unique maximal solution $\varphi_p : I_p \rightarrow U$ of the system such that $\varphi_p(0) = p$ is defined.
- (ii) If $q = \varphi_p(t)$ and $t \in I_p$ then $I_q = I_p - t = \{r - t : r \in I_p\}$ and $\varphi_q(s) = \varphi_p(t + s)$ for all $s \in I_q$.
- (iii) The set $D = \{(t, p) : p \in U, t \in I_p\}$ is open in \mathbb{R}^3 and the map $\varphi : D \rightarrow U$ defined by $\varphi(t, p) = \varphi_p(t)$ is \mathcal{C}^r .

The map $\varphi : D \rightarrow U$ is a *local flow* of class \mathcal{C}^r on U associated to the system. It verifies:

- (1) $\varphi(0, p) = p$ for all $p \in U$;
- (2) $\varphi(t, \varphi(s, p)) = \varphi(t + s, p)$ for all $p \in U$ and for all s and t such that $s, t + s \in I_p$;
- (3) $\varphi_p(-t) = \varphi_p^{-1}(t)$ for all $p \in U$ such that $t, -t \in I_p$.

We consider \mathcal{C}^r -local flows on \mathbb{R}^2 , with $r \geq 0$. Of course when $r = 0$ the flow is only continuous. Two such flows φ and φ' are \mathcal{C}^k -equivalent, with $k \geq 0$, if there exists a \mathcal{C}^k -diffeomorphism that brings orbits of φ onto orbits of φ' preserving sense (but not necessarily the parametrization).

Let φ be a \mathcal{C}^r -local flow with $r \geq 0$ on \mathbb{R}^2 . We say that φ is \mathcal{C}^k -parallel if it is \mathcal{C}^k -equivalent to one of the following flows:

- (i) \mathbb{R}^2 with the flow defined by $x' = 1, y' = 0$.
- (ii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined (in polar coordinates) by $r' = 0, \theta' = 1$.
- (iii) $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $r' = r, \theta = 0$.

We call these flows *strip*, *annular* and *spiral*, respectively.

Let $p \in \mathbb{R}^2$. We denote by $\gamma(p)$ the *orbit* of the flow φ through p , i.e. $\gamma(p) = \{\varphi_p(t) : t \in I_p\}$. The *positive semiorbit* of p is $\gamma^+(p) = \{\varphi_p(t) : t \in I_p, t \geq 0\}$. In a similar way we define the *negative semiorbit* $\gamma^-(p)$ of p .

We define the α -limit and the ω -limit of p as $(\gamma^\pm(p))$ and let

$$\alpha(p) = \text{cl}(\gamma^-(p)) - \gamma^-(p), \omega(p) = \text{cl}(\gamma^+(p)) - \gamma^+(p),$$

respectively. Here, as usual, cl denotes the closure.

Let $\gamma(p)$ be an orbit of the flow φ . A *parallel neighborhood* of the orbit $\gamma(p)$ is an open neighborhood N of $\gamma(p)$ such that φ is \mathcal{C}^k -equivalent in N to a parallel flow for some $k \geq 0$.

We say that $\gamma(p)$ is a *separatrix* of φ if it is not contained in a parallel neighborhood N satisfying the following two assumptions:

- (i) for any $q \in N$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$;
- (ii) $\text{cl}(N) \setminus N$ consists of $\alpha(p), \omega(p)$ and exactly two orbits $\gamma(a), \gamma(b)$ of φ , with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

We denote by Σ the union of all separatrices of φ . Σ is a closed invariant subset of \mathbb{R}^2 . A component of the complement of Σ in \mathbb{R}^2 , with the restricted flow, is a *canonical region* of φ .

The following lemma can be found in [10].

Lemma 7. *Every canonical region of a local flow φ on \mathbb{R}^2 is \mathcal{C}^0 -parallel.*

2.3. The Poincaré compactification. Let X be a real planar polynomial vector field of degree n . The *Poincaré compactified vector field* $p(X)$ corresponding to X is an analytic vector field induced on \mathbb{S}^2 as follows (see for instance [8]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent plane to \mathbb{S}^2 at a point y . Identify \mathbb{R}^2 with $T_{(0,0,1)}\mathbb{S}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \rightarrow \mathbb{S}^2$. This map defines two copies of X on \mathbb{S}^2 , one in the northern hemisphere and the other in the southern hemisphere. Denote by \bar{X} the vector field $Df \circ X$ defined on \mathbb{S}^2 except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly \mathbb{S}^1 is identified to the *infinity* of \mathbb{R}^2 . Usually, when we talk about the circle of the infinity of X we simply talk about the infinity.

In order to extend \bar{X} to a vector field on \mathbb{S}^2 (including \mathbb{S}^1) it is necessary for X to satisfy suitable conditions. If X is a real polynomial vector field of degree n then $p(X)$ is the only analytic extension of $y_3^{n-1}\bar{X}$ to \mathbb{S}^2 . On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of X , and knowing the behavior of $p(X)$ around \mathbb{S}^1 we know the behavior of X in a neighborhood of the infinity. The Poincaré compactification has the property that \mathbb{S}^1 is invariant under the flow of $p(X)$. The projection of the closed northern hemisphere of \mathbb{S}^2 on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by \mathbb{D}^2 .

Two polynomial vector fields X and Y on \mathbb{R}^2 are *topologically equivalent* if there exists a homeomorphism on \mathbb{S}^2 preserving the infinity \mathbb{S}^1 and carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$.

As \mathbb{S}^2 is a differentiable manifold, for computing the expression of $p(X)$ we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$, $i = 1, 2, 3$. The diffeomorphisms $F_i : U_i \rightarrow \mathbb{R}^2$ and $G_i : V_i \rightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If $z = (u, v)$ is the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration) then we obtain the following expressions for $p(X)$:

$$v^n \Delta(z) \left(Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right), -vP \left(\frac{1}{v}, \frac{u}{v} \right) \right) \quad (4)$$

on U_1 ;

$$v^n \Delta(z) \left(P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right), -vQ \left(\frac{u}{v}, \frac{1}{v} \right) \right) \quad (5)$$

on U_2 ; and

$$\Delta(z) (P(u, v), Q(u, v))$$

on U_3 , where $\Delta(z) = (u^2 + v^2 + 1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates, for $i = 1, 2$, $v = 0$ always denotes the points of \mathbb{S}^1 . We can omit the factor $\Delta(z)$ by scaling the vector field $p(X)$. Thus, the expression of $p(X)$ becomes a polynomial vector field in each local chart.

2.4. A result on heteroclinic connections in planar systems. The following theorem (see [7]) will be used to obtain a curve in the bifurcation diagram of family (K4) when $S = -1$.

Theorem 8. *Consider the system*

$$\dot{x} = y, \quad \dot{y} = -x + my + xy - ny^2$$

with $(m, n) \in \mathbb{R}^2$. Then there exists a function $m = m^*(n)$ such that $m^*(-n) = -m^*(n)$, $nm^*(n) > 0$ for $n \neq 0$, for which:

- (i) *The system has exactly one hyperbolic limit cycle (born from a Hopf bifurcation) if either $n > 0$ and $0 < m < m^*(n)$ or $n < 0$ and $m^*(n) < m < 0$; otherwise the system has no limit cycles.*
- (ii) *The value $m = m^*(n)$ corresponds, for each n , to the value where the limit cycle disappears giving rise to a heteroclinic connection. Indeed,*

$$m^*(n) = \frac{1}{4n} + \frac{1}{64n^3} + \frac{5}{512n^5} + \dots$$

3. PROOF OF PROPOSITION 1

We consider first the case $c_2 \neq 0$, for which system (2) with $d_1 = 0$ has two finite singular points, which are $(-c_1 \pm \sqrt{\Delta})/(2c_2), 0)$, where $\Delta = c_1^2 - 4c_0c_2$. We split this case into three subcases depending on the sign of Δ . If $\Delta > 0$, that is we have two real different singular points, then the change of variables and time

$$(x, y) \mapsto \left(\frac{r(c_1 + 2c_2x + \sqrt{\Delta})}{2\sqrt{\Delta}}, \frac{c_2r\sqrt{d_0rS}}{\Delta^{3/4}}y \right), \quad \frac{dt}{ds} = \frac{\sqrt{d_0}\sqrt[4]{\Delta}}{\sqrt{rS}},$$

where $r \neq 0$ is such that $c_2 d_0 r - a_0 \Delta = 0$ and $S \in \{-1, 1\}$ is the sign of $a_0 c_2 \neq 0$, leads to the first normal form (K1) after renaming

$$b_0 = \frac{\sqrt{d_0}}{\sqrt{\Delta r S}} B_0 + \frac{\sqrt{d_0 r S} (c_1 + \sqrt{\Delta})}{2 \sqrt[4]{\Delta}} B_1, \quad b_1 = \frac{c_2 \sqrt{d_0 r S}}{\sqrt[4]{\Delta}} B_1.$$

If $\Delta = 0$, that is we have a double real singular point, then the change of variables and time

$$(x, y) \mapsto \left(\frac{a_0 (c_1 + 2c_2 x)}{2c_2 d_0}, \frac{a_0 \sqrt{a_0 S}}{\sqrt{c_2 d_0}} y \right), \quad \frac{dt}{ds} = \frac{d_0 \sqrt{c_2}}{\sqrt{a_0 S}},$$

where $S \in \{-1, 1\}$ is again the sign of $a_0 c_2 \neq 0$, leads to the second normal form (K2) after renaming

$$b_0 = \frac{d_0 \sqrt{c_2}}{\sqrt{a_0 S}} B_0 + \frac{c_1 \sqrt{a_0}}{2 \sqrt{c_2 S}} B_1, \quad b_1 = \sqrt{a_0 c_2 S} B_1.$$

If $\Delta < 0$, that is we have two complex conjugate singular points, then the change of variables and time

$$(x, y) \mapsto \left(a + \frac{a_0 (c_1 + 2c_2 x)}{2c_2 d_0}, \frac{a_0 \sqrt{a_0 S}}{\sqrt{c_2 d_0}} y \right), \quad \frac{dt}{ds} = \frac{d_0 \sqrt{c_2}}{\sqrt{a_0 S}},$$

where $a \in \mathbb{R}$ and $b > 0$ is such that $a_0^2 (c_1^2 - 4c_0 c_2) + 4b^2 c_2^2 d_0^2 = 0$ and $S \in \{-1, 1\}$ is the sign of $a_0 c_2 \neq 0$, leads to the third normal form (K3) after renaming

$$b_0 = \frac{d_0 \sqrt{c_2}}{\sqrt{a_0 S}} B_0 + \frac{2ac_2 d_0 + a_0 c_1}{2 \sqrt{a_0 c_2 S}} B_1, \quad b_1 = \sqrt{a_0 c_2 S} B_1.$$

Now we consider the case $c_2 = 0$. If $c_1 \neq 0$ then we have one single finite singular point $(-c_0/c_1, 0)$. If $c_1 = 0$ then we have no finite singular points. We distinguish between these two cases. If $c_1 \neq 0$ then the change of variables and time

$$(x, y) \mapsto \left(\frac{a_0 (c_0 + c_1 x)}{c_1 d_0}, \frac{a_0}{\sqrt{c_1 d_0 S}} y \right), \quad \frac{dt}{ds} = \sqrt{c_1 d_0 S},$$

where $S \in \{-1, 1\}$ is the sign of $c_1 d_0 \neq 0$, leads to the fourth normal form (K4) after renaming

$$b_0 = \sqrt{c_1 d_0 S} B_0 + \frac{a_0 c_0}{\sqrt{c_1 d_0 S}} B_1, \quad b_1 = \frac{a_0 \sqrt{c_1}}{\sqrt{d_0 S}} B_1.$$

We finally consider the case $c_1 = 0$. We must have $c_0 \neq 0$, otherwise $y|(\dot{x}, \dot{y})$, and $b_1 \neq 0$, otherwise the whole system does not depend on x and is not of our interest. The change of variables and time

$$(x, y) \mapsto \left(\frac{a_0 (b_0 + b_1 x)}{b_1 d_0}, \frac{\sqrt{a_0 S}}{\sqrt{c_0}} y \right), \quad \frac{dt}{ds} = \frac{\sqrt{a_0 c_0}}{\sqrt{S}},$$

where $S \in \{-1, 1\}$ is the sign of $a_0 c_0 \neq 0$, leads to the fifth normal form (K5) after renaming

$$b_1 = \frac{a_0^{3/2} \sqrt{c_0}}{d_0 \sqrt{S}} B_1.$$

Finally we deal with the different restrictions on the parameters shown in the proposition. If $B_1 = 0$ in (K5), then x is not appearing in the right-hand side of the system, so we remove this case.

The change of variables $(x, y, t) \mapsto (x, -y, -t)$ together with $B_0 \mapsto -B_0$ and $B_1 \mapsto -B_1$ allows to assume $B_1 \geq 0$ in all cases. Additionally, in the normal form (K1) after the change of variables $(x, y, t) \mapsto (x + r, y, t)$, $B_0 \mapsto B_0 - B_1 r$ and $r \mapsto -r$ we can assume that $rS > 0$. \square

4. PROOF OF THEOREM 3

After Proposition 1, we prove the theorem separately for the five different normal forms, starting from (K5).

Proof of Theorem 3(e). We deal here with the differential system

$$\dot{x} = y, \quad \dot{y} = S + B_1xy + y^2, \tag{6}$$

where $S^2 = 1$ and $B_1 > 0$.

System (6) has no finite singular points. According to the expressions (4) and (5), system (6) at infinity can be studied from the systems

$$\dot{u} = u(B_1 + u) + v(Sv - u^2), \quad \dot{v} = -uv^2$$

and

$$\dot{u} = -u(1 + B_1u) + v(1 - Suv), \quad \dot{v} = -v(1 + B_1u + Sv^2),$$

respectively. The characteristic equation at infinity is $xy(B_1x + y) = 0$, so three different singular points appear at infinity. The first system has a singular point at $(0, 0)$, with eigenvalues $-1, -1$, thus it is a node. The other two singular points can be studied from the second system: they are $(0, 0)$ with eigenvalues $B_1, 0$ and $(-B_1, 0)$ with eigenvalues $-B_1, 0$. Applying Proposition 4 to $(0, 0)$, we have $y = g(x) = -Sx^2 + \dots$ and therefore $F(x, g(x)) = f(x) = B_1Sx^4 + \dots$. Now applying Proposition 4 to $(-B_1, 0)$, we have $y = g(x) = Sx^2 + \dots$ and therefore $F(x, g(x)) = f(x) = -B_1x^2 + \dots$. Since $B_1S \neq 0$, both singular points are saddle-nodes. We get the phase portraits (55) when $S = 1$ and (56) when $S = -1$. Then statement (e) of Theorem 3 follows. \square

Proof of Theorem 3(d). We deal here with the differential system

$$\dot{x} = y, \quad \dot{y} = Sx + B_0y + B_1xy + y^2, \tag{7}$$

where $S^2 = 1$ and $B_1 \geq 0$.

System (7) has a finite singular point at the origin, with eigenvalues $(B_0 \pm \sqrt{B_0^2 + 4S})/2$. Observe that the sign of B_0 determines the stability of nodes and foci, when it is not zero. To study the case $B_0 = 0, S = -1$, the only one for which we have a center-focus, we use Theorem 6: we write system (7) into the form (3) to obtain

$$\dot{x} = -y, \quad \dot{y} = x + B_1xy + y^2.$$

We have $\omega_1 = -B_1$. Since $\gamma = 0$, we have $\omega_2 = \omega_3 = 0$, and thus we have a center if and only if $B_1 = 0$. Moreover, according to Theorem 6, and taking into account that we changed the sign of the time to obtain this system, system (7) has an unstable focus when $B_1 > 0$.

Table 1 summarizes the different possibilities of behavior of this singular point depending on the values of the parameters.

According to the expressions (4) and (5), system (7) at infinity can be studied from the systems

$$\dot{u} = u(B_1 + u) + v(S + B_0u - u^2), \quad \dot{v} = -uv^2 \tag{8}$$

and

$$\dot{u} = -u(1 + B_1u) + v(1 - B_0u - Su^2), \quad \dot{v} = -v(1 + B_1u + B_0v + Suv).$$

As in (K5), the characteristic equation at infinity is $xy(B_1x + y) = 0$, so at most three different singular points appear. The point in the direction $x = 0$ has eigenvalues $-1, -1$, so it is a node. In the case $B_1 \neq 0$, we have two more singular points. The point in the direction $y = 0$ has eigenvalues $B_1, 0$. Applying Proposition 4 to $(0, 0)$, we have $y = g(x) = (B_0B_1 - S)x^2 + \dots$, and hence $F(x, g(x)) = f(x) = B_1^2Sx^3 + \dots$. This means an unstable node if $S = 1$ and a saddle if $S = -1$. Note that we must consider apart the case $B_0B_1 = S$: in this case $y = g(x) = 0$ and $f(x) = B_1^2Sx^3$, so the same result is obtained. However the system has an invariant straight line in this case, so we shall have to take this curve into account in the bifurcation diagram.

Singular point	Conditions	Type of point
(0, 0)	$S = 1$	Saddle
	$B_0^2 \geq 4, S = -1$	Node
	$0 < B_0^2 < 4, S = -1$	Focus
	$B_0 = 0, S = -1, B_1 \neq 0$	Focus
	$B_0 = 0, S = -1, B_1 = 0$	Center
$(0, 0) \in U_1$		Stable Node
$(0, 0) \in U_2$	$B_1 \neq 0, S = 1$	Unstable Node
	$B_1 \neq 0, S = -1$	Saddle
	$B_1 = 0$	Degenerate
$(-B_1, 0) \in U_2$	$B_1 \neq 0$	Saddle-Node

TABLE 1. Behavior of the singular points for family (K4). The sign of $B_0 \neq 0$ determines the stability of the origin: stable if $B_0 < 0$ and unstable if $B_0 > 0$. When $B_0 = 0$, if $B_1 > 0$ we have an unstable focus and if $B_1 < 0$ we have a stable focus.

Still in the case $B_1 \neq 0$ we apply Proposition 4 to the third infinite singular point $(-B_1, 0)$. We have $y = g(x) = -((B_0 + B_1)B_1 - S)Sx^2 + \dots$, and hence $F(x, g(x)) = f(x) = -B_1^2x^2 + \dots$. Thus we have a saddle-node. We must consider apart the case $C_0 := (B_0 + B_1)B_1 - S = 0$: in this case we have $y = g(x) = 0$ and $f(x) = -B_1^2x^2$, so the same result is obtained. However also in this case the system has an invariant straight line, so we shall have to take this curve into account in the bifurcation diagram.

When $B_1 = 0$ the two last singular points are the same (degenerate) singular point. To obtain its local behavior we need to apply the blow-up technique (see [1] for more details). Starting from system (8) with $B_1 = 0$ (which has our point at the origin) we apply the change $v = u^2t$, where t is a new variable, to obtain

$$\dot{u} = -u(-1 - St - B_0tu + tu^2), \quad \dot{t} = t(-2 - 2St - 2B_0tu + tu^2).$$

This system has a saddle at the origin and a semi-hyperbolic singular point at $(0, -S)$. Applying Proposition 4 to this point, we have $y = g(x) = -(2B_0^2 + S)x^2 + \dots$, and hence $F(x, g(x)) = f(x) = Sx^3 + \dots$, so we have an unstable node if $S = 1$ and a saddle if $S = -1$. Note that we must consider apart the case $2B_0^2 = -S = 1$: in this case $y = g(x) = \pm 3/\sqrt{2}x^3 + \dots$ and $f(x) = -x^3 + \dots$, so the same result is obtained. Although we do not obtain any invariant straight line in this case, we shall take this two points into account in the bifurcation diagram.

Table 1 summarizes the different behaviors of the singular points at infinity. Next we draw the bifurcation diagrams.

When $S = 1$ we have the following behaviors: if $C_0 < 0$ then we obtain the phase portrait (44). If $C_0 = 0$ then the system has an invariant algebraic curve, given by $B_1x + y = 0$. This straight line forms the stable separatrices of the saddle at the origin (and the unstable separatrix of the saddle-node at infinity); we have the phase portrait (45). When $C_0 > 0$ we are in (46). We note that the curve $B_0B_1 = 1$ does not play any role in this bifurcation diagram.

When $S = -1$ and $B_1 \neq 0$, we apply Theorem 8 with $m = -B_0$ and $n = 1/B_1$. According to [7], we have the phase portrait (52) when $B_0 \geq 0$. Next we have a phase portrait with a hyperbolic limit cycle that is born from a Hopf bifurcation at the origin when $B_0 < 0$ and $C_1 := B_0 + B_1/4 - B_1^3/64 - 5B_1^5/512 + \dots > 0$. This is phase portrait (51). This limit cycle disappears when $C_1 = 0$ giving rise to a heteroclinic connection, see the phase

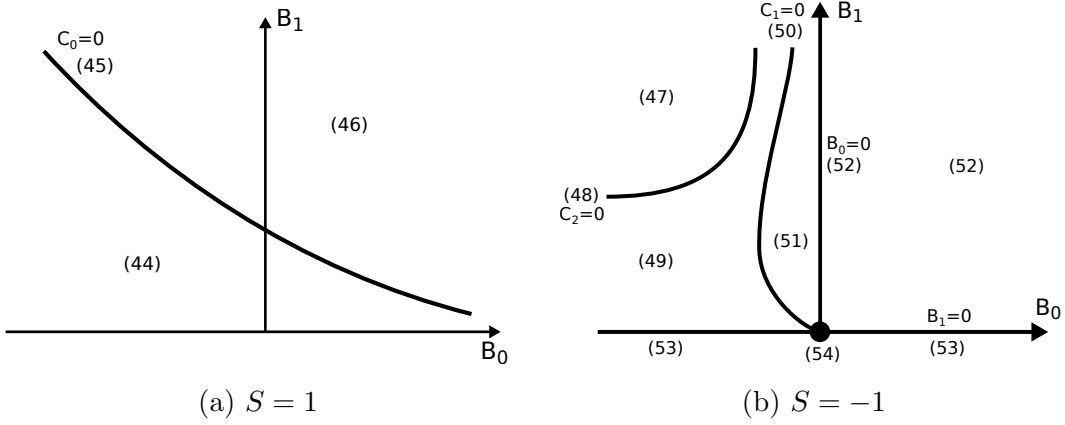


FIGURE 3. Bifurcation diagrams of the normal form (K4) in the half-plane (B_0, B_1) , $B_1 \geq 0$.

portrait (50). This heteroclinic connection is a separatrix of an infinite saddle when $C_1 < 0$ and $C_2 := B_0 B_1 + 1 > 0$, see the phase portrait (49). The two separatrices not through the origin meet when $C_2 = 0$, see the phase portrait (48), and finally they separate again when $C_2 < 0$, see the phase portrait (47). We note that the curve $C_0 = 0$ does not play any role in this bifurcation diagram.

Finally we consider the case $S = -1$ and $B_1 = 0$. When $B_0 = 0$ we have a center at the origin, so we have the phase portrait (54). If $B_0 \neq 0$ then we have either a node or a focus, and so the phase portrait is not changing topologically. We have the phase portrait (53). The case $2B_0^2 = -S = 1$ is included here.

The bifurcation diagrams in Figure 3 show all the different phase portraits depending on the values of B_0 and $B_1 \geq 0$. For $S = -1$, the bifurcation curves in the second quadrant, $C_1 = 0$ and $C_2 = 0$, were computed in [7], where the Hopf bifurcation appearing in this system is studied. Note that we also found C_2 . \square

Proof of Theorem 3(c). We deal now with the differential system

$$\dot{x} = y, \quad \dot{y} = S((x - a)^2 + b^2) + B_0 y + B_1 x y + y^2, \quad (9)$$

where $S^2 = 1$, $B_1 \geq 0$ and $b > 0$.

This system has no finite singular points. According to the expression (5), system (9) at infinity can be studied from the system

$$\begin{aligned} \dot{u} &= -u(1 + B_1 u + S u^2) + v(1 - B_0 u + 2a S u^2 - (a^2 + b^2) S u v), \\ \dot{v} &= -v(1 + B_1 u + S u^2) + (B_0 - 2a S u) v + (a^2 + b^2) S v^2. \end{aligned}$$

We have three infinite singular points: a node at the origin, with eigenvalues $-1, -1$, and two more points at $u = (-B_1 \pm \sqrt{B_1^2 - 4S})/2$. The existence of the last two points depends on the value of the discriminant $B_1^2 - 4S$. Concerning their stability, at least one eigenvalue is always zero. When $B_1^2 - 4S > 0$ we can apply Proposition 4 to obtain two saddle-nodes. When $B_1 = 2$, $S = 1$ and $B_0 \neq -(2a + 1)$ we apply Proposition 5 to get a saddle-node. In the case $B_1 = 2$, $S = 1$ and $B_0 = -(2a + 1)$ we need to apply the blow-up technique because both eigenvalues are zero. In this case we have a singular point at $(-1, 0)$. We move the singular point to the origin and do the change $v = ut$, where t is a new variable. We obtain

$$\dot{u} = -u(-1 + (2a - 1)t - a^2 t^2 - b^2 t^2 + (1 - 2at + a^2 t^2 + b^2 t^2)u), \quad \dot{t} = -t(1 - 2at + a^2 t^2 + b^2 t^2).$$

This system has, on $u = 0$, a unique real singular point, located at the origin. It is a saddle. The other two infinite singular points are complex.

We note that we do not need to study the expression (4) because all the infinite singular points can be studied from the above differential system.

Singular point	Conditions	Type of point
$(0, 0)$		Stable Node
$(\frac{-B_1 + \sqrt{\Delta}}{2S}, 0)$	$\Delta = B_1^2 - 4S > 0$	Saddle-Node
$(\frac{-B_1 - \sqrt{\Delta}}{2S}, 0)$	$\Delta = B_1^2 - 4S > 0$	Saddle-Node
$(-\frac{B_1}{2}, 0)$	$B_1 = 2, S = 1, B_0 \neq -(2a + 1)$	Saddle-Node
	$B_1 = 2, S = 1, B_0 = -(2a + 1)$	Degenerate

TABLE 2. Behavior of the infinite singular points for families (K2) and (K3). In case (K2) we just need to take $a = 0$.

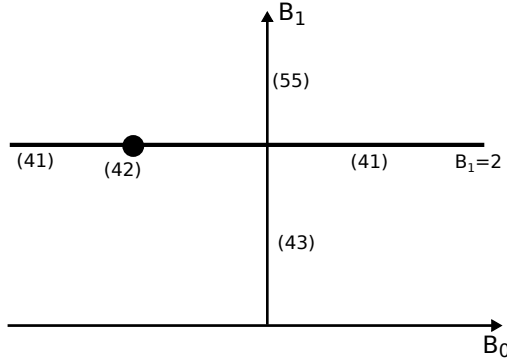


FIGURE 4. Bifurcation diagram of the normal form (K3) in the half-plane (B_0, B_1) , $B_1 \geq 0$, in the case $S = 1$.

Table 2 shows the behavior of all singular points depending on the parameters. Only the phase portrait (55) appears in this case for $S = -1$. When $S = 1$ Figure 4 shows all the possible phase portraits. We have the phase portraits (43) when $B_1 < 2$, (41) when $B_1 = 2$ and $B_0 \neq -(2a + 1)$, (42) when $B_1 = 2$ and $B_0 = -(2a + 1)$, and (55) when $B_1 > 2$. We note that the bifurcation diagram does not depend on b . \square

Proof of Theorem 3(b). We deal now with the differential system

$$\dot{x} = y, \quad \dot{y} = Sx^2 + B_0y + B_1xy + y^2, \quad (10)$$

where $S^2 = 1$ and $B_1 \geq 0$.

This system has a finite singular point at the origin with eigenvalues $0, B_0$. If $B_0 \neq 0$ then according to Proposition 4 it is a saddle-node. If $B_0 = 0$ then according to Proposition 5 it is a cusp. Using the expression (5), system (10) at infinity can be studied from the system

$$\dot{u} = -u(1 + B_1u + Su^2) + v(1 - B_0u), \quad \dot{v} = -v(1 + B_1u + Su^2 + B_0v).$$

The behavior of the infinite singular points is the same as in the case (K3), so Table 2 with $a = 0$ shows their behaviors depending on the parameters. As in (K3), we do not need to study the expression (4) because all the infinite singular points can be studied from the previous differential system.

When applying Proposition 4 to the infinite singular points in the case $B_1^2 - 4S > 0$ we must take into account the case $C_1 := B_0^2 + B_0B_1 + S = 0$. This curve in the bifurcation diagram separates two behaviors, see Figure 5. Indeed straightforward computations show that, in the case $C_1 = 0$, the system has an invariant straight line given by $B_0x - y = 0$.

When $S = 1$, Figure 5(a) shows the bifurcation diagram in the half-plane (B_0, B_1) . We note that phase portrait (29) corresponds to the case $B_0 = -1$ and $B_1 = 2$ and the phase

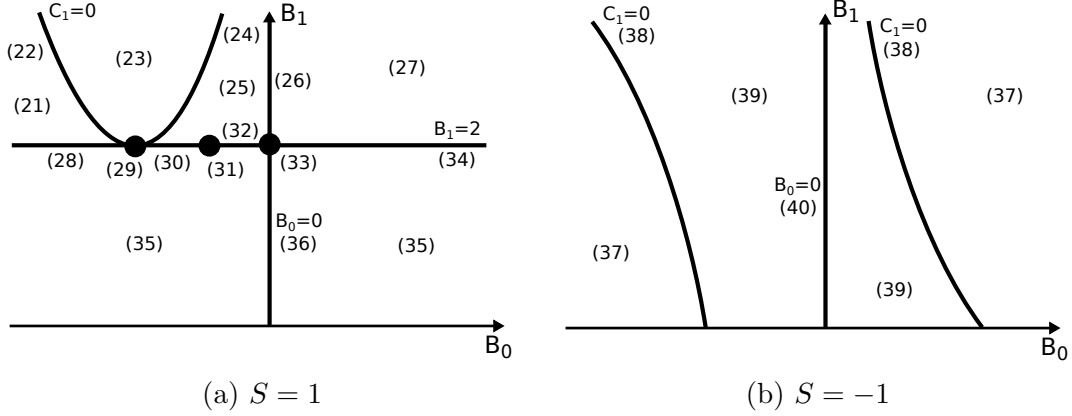


FIGURE 5. Bifurcation diagrams of the normal form (K2) in the half-plane (B_0, B_1) , $B_1 \geq 0$.

portrait (31) to the case $B_0 = -1/2$, $B_1 = 2$. We take into account the curves $C_1 = 0$, $B_0 = 0$ and $B_1 = 2$.

When $S = -1$ we have the bifurcation diagram in Figure 5(b). Here we take into account the curves $B_0^2 + B_0B_1 + S = 0$ and $B_0 = 0$. \square

Proof of Theorem 3(a). We finally deal with the differential system

$$\dot{x} = y, \quad \dot{y} = Sx(x - r) + B_0y + B_1xy + y^2, \quad (11)$$

where $S^2 = 1$, $B_1 \geq 0$ and $rS > 0$.

We have two finite singular points: $(0, 0)$ and $(r, 0)$. The eigenvalues at the origin are $(B_0 \pm \sqrt{B_0^2 - 4rS})/2$. So, when $B_0 \neq 0$, its stability depends on the sign of B_0 and the origin is either a node or a focus depending on whether $B_0^2 - 4rS \geq 0$ or $B_0^2 - 4rS < 0$, respectively. When $B_0 = 0$ we must apply Theorem 6. We first write the system as

$$\dot{x} = -y, \quad \dot{y} = x - \frac{x^2}{r} - \frac{B_1}{\sqrt{rS}}xy - y^2.$$

We have $\omega_1 = -B_1(r + 1)/(r\sqrt{rS})$. Since $\gamma = 0$, we have $\omega_2 = \omega_3 = 0$, hence we have a center if and only if $\omega_1 = 0$. Moreover, the focus is stable when $\omega_1 > 0$ and unstable when $\omega_1 < 0$.

The second singular point has eigenvalues $(B_0 + B_1r \pm \sqrt{(B_0 + B_1r)^2 + 4rS})/2$. Since $rS > 0$, this point is a saddle.

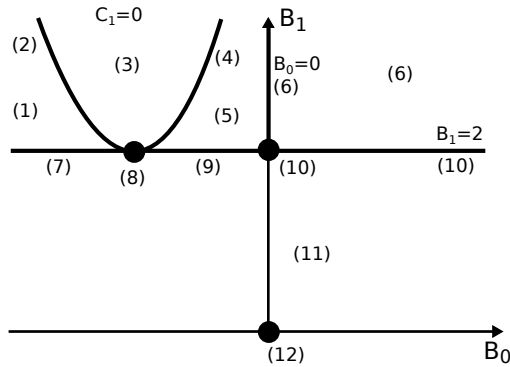


FIGURE 6. Bifurcation diagrams of the normal form (K1) in the half-plane (B_0, B_1) , $B_1 \geq 0$, in the case $S = 1$ and for all $r > 0$.

According to the expression (5), system (11) at infinity becomes

$$\dot{u} = -u(1 + B_1u + Su^2) + v(1 - B_0u + rSu^2), \quad \dot{v} = -v(1 + B_1u + Su^2 + B_0v - rSuv).$$

This system has at most three real singular points on $v = 0$. We note that we do not need to study the expression (4) because all the infinite singular points can be studied from the previous differential system. The origin if this system is a stable node: it has eigenvalues $-1, -1$. The others are located at $u = (-B_1 \pm \sqrt{B_1^2 - 4S})/(2S)$. We have three cases depending on the sign of the discriminant $B_1^2 - 4S$. If it is positive, then applying Proposition 4 we know that we have two saddle-nodes. As in previous cases, we need to take into account some manifolds in the bifurcation diagram: they are $C_1 := (B_0 + B_1)(B_0 + B_1 r) + (r - 1)^2 S = 0$ and $C_2 := B_0(B_0 + B_1 + B_1 r) + (r + 1)^2 S = 0$.

If the discriminant $B_1^2 - 4S$ is zero, i.e. $B_1 = 2$ and $S = 1$, then we must distinguish two subcases: if $B_0 \neq -(r + 1)$ then Proposition 5 shows that we have a saddle-node and if $B_0 = -(r + 1)$ we need to use the blow-up technique. We obtain three singular points, all of them saddles.

Finally, when the discriminant is negative the two singular points are complex. Table 3 shows the singular points and their behavior depending on the parameters.

Singular point	Conditions	Type of point
$(0, 0)$	$B_0^2 > 4rS$	Node
	$B_0^2 = 4rS$	Node
	$0 < B_0^2 < 4rS$	Focus
	$B_0 = 0$ & $(B_1 = 0$ or $r = S = -1)$	Center
	$B_0 = 0$ & Not a Center	Focus
$(r, 0)$		Saddle
$(0, 0) \in U_1$		Stable Node
$(\frac{-B_1 + \sqrt{B_1^2 - 4S}}{2S}, 0) \in U_1$	$B_1^2 - 4S > 0$	Saddle-Node
$(\frac{-B_1 - \sqrt{B_1^2 - 4S}}{2S}, 0) \in U_1$	$B_1^2 - 4S > 0$	Saddle-Node
$(-\frac{2}{B_1}, 0) \in U_1$	$B_1 = \pm 2, S = 1, B_0 \neq \mp(r + 1)$	Saddle-Node
	$B_1 = \pm 2, S = 1, B_0 = \mp(r + 1)$	Degenerate

TABLE 3. Behavior of the singular points for family (K1). The stability of the origin when $B_0 \neq 0$ depends on the sign of B_0 : stable when $B_0 < 0$ and unstable when $B_0 > 0$. When $B_0 = 0$, the stability is given by the sign of $\omega_1 = -B_1(r + 1)/r \neq 0$ (stable for positive, unstable for negative).

20 new phase portraits appear for system (11). The bifurcation diagrams are shown in Figures 6 and 7. The diagram in 6 corresponds to the case $S = 1$. It is drawn on the half-plane (B_0, B_1) , because the value of $r > 0$ does not affect the diagram. The diagram in Figure 7(a) corresponds to $S = -1$ with $B_1 > 0$ and it is drawn on the half-plane (B_0, r) . The diagram in Figure 7(b) corresponds to $S = -1$ with $B_1 = 0$ and it is also drawn on the half-plane (B_0, r) .

We note that $C_2 = 0$ does not play any role in the bifurcation diagram when $S = 1$. The point with the phase portrait (8) in Figure 6 corresponds to $B_1 = 2, B_0 = -(r + 1)$. \square

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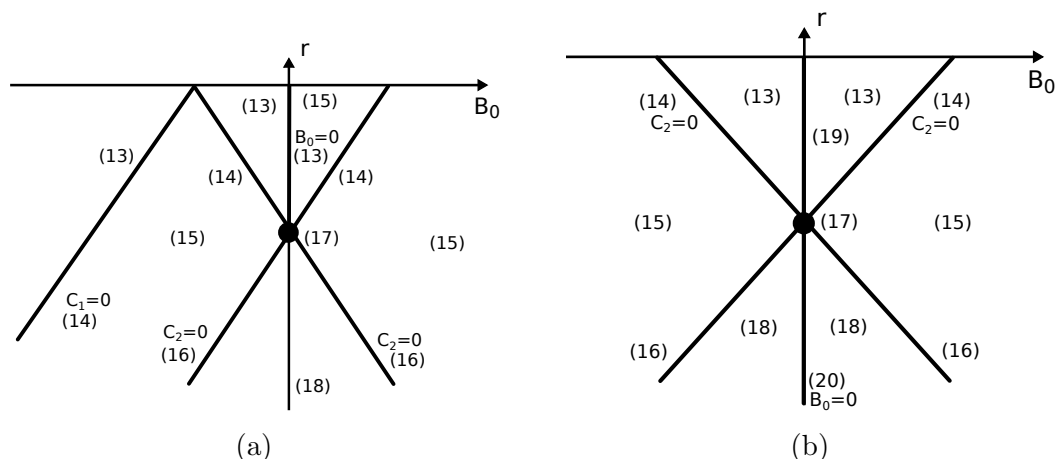


FIGURE 7. Bifurcation diagrams of the normal form (K1) in the case $S = -1$: (a) in the half-plane (B_0, r) , $r < 0$ and for all $B_1 \neq 0$; and (b) in the half-plane (B_0, r) , $r < 0$ and for $B_1 = 0$.

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