POINTWISE CONVERGENCE TOPOLOGY AND FUNCTION SPACES IN FUZZY ANALYSIS

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Abstract. We study the space of all continuous fuzzy-valued functions from a space X into the space of fuzzy numbers $(\mathbb{E}^1, d_\infty)$ endowed with the pointwise convergence topology. Our results generalize the classical ones for continuous real-valued functions. The field of applications of this approach seems to be large, since the classical case allows many known devices to be fitted to general topology, functional analysis, coding theory, Boolean rings, etc.

1. **Introduction and preliminaries**

Fuzzy Analysis has developed a growing interest in the last decades. It embraces a wide variety not only of theoretical aspects, but also of significant applications in fuzzy optimization, fuzzy decision making, etc. Among the literature devoted to this topic we can cite, for instance, [7, 8, 12, 20, 22, 25, 26, 27]. Fuzzy analysis is based on the notion of fuzzy number. The underlying idea is the following. Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the reals. For $u \in F(\mathbb{R})$ and $\lambda \in [0,1]$, the λ-level set of u is defined by

$$[u]^\lambda := \{x \in \mathbb{R} : u(x) \geq \lambda\}, \quad \lambda \in]0,1], \quad [u]^0 := \mathrm{cl}_{\mathbb{R}} \{x \in \mathbb{R} : u(x) > 0\}.$$

Let $\mathbb{E}^1$ be the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

(1) u is normal, i.e., there exists $x \in \mathbb{R}$ with $u(x) = 1$;
(2) u is convex, i.e., for all $x, y \in \mathbb{R}$, $u(z) \geq \min\{u(x), u(y)\}$ for all $x \leq z \leq y$;
(3) u is upper-semicontinuous;
(4) $[u]^0$ is a compact set in $\mathbb{R}$.

Notice that if $u \in \mathbb{E}^1$, then the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0,1]$. We also denote $[u]^\lambda$ by $[u^-(\lambda), u^+(\lambda)]$. Notice that each real number $r \in \mathbb{R}$ can be regarded as an element of $\mathbb{E}^1$ since r can be identified with the element of $\mathbb{E}^1$ $\tilde{r}$ defined as

$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

$\mathbb{E}^1$ is the so-called set of the fuzzy numbers, which were introduced by Dubois and Prade ([10]) to provide formalized tools to deal with non-precise quantities.

Notice that we could consider $\mathbb{E}^1$ as a set endowed with a family of representable interval orders indexed in $]0,1]$. Indeed, for any $\lambda \in]0,1]$, the element $u \in \mathbb{E}^1$ is

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represented by the interval $[u^-(\lambda), u^+(\lambda)]$. Then the binary relation $\prec$ defined on $\mathbb{E}^1$ by declaring $u \prec v$ if and only if $u^+(\lambda) < v^-(\lambda)$ is a representable interval order for any $\lambda \in]0,1]$. (See e.g. Ch. 6 in [5], or [4] for details).

Goetschel and Voxman proposed an equivalent representation of such numbers in a topological vector space setting, which eased the development of the theory and applications of fuzzy numbers (see [15]).

Theorem 1.1. *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

(i) *$u^-(\lambda)$ is a bounded left-continuous non-decreasing function on $]0,1]$;*
(ii) *$u^+(\lambda)$ is a bounded left-continuous non-increasing function on $]0,1]$;*
(iii) *$u^-(\lambda)$ and $u^+(\lambda)$ are right-continuous at $\lambda = 0$;*
(iv) *$u^-(1) \leq u^+(1)$.*

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ from $[0,1]$ into $\mathbb{R}$ satisfy the above conditions (i)-(iv)*, then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.*

The previous result allows us to consider different topologies on $\mathbb{E}^1$ defined by means of different types of convergence on families of functions. From now on, we endow $\mathbb{E}^1$ with the topology of the uniform convergence, that is, a net $(u_\alpha)_{\alpha\in I} \subset \mathbb{E}^1$ converges to $u \in \mathbb{E}^1$ if the net $(u^-_\alpha)_{\alpha\in I}$ converges uniformly to u^- and the net $(u^+_\alpha)_{\alpha\in I}$ converges uniformly to u^+. Equivalently, the topology of uniform convergence is induced by the supremum metric d_∞ defined by using the Hausdorff distance on the hyperspace of all nonempty compact intervals ([9, 15]), that is, if $u, v \in \mathbb{E}^1$, then

$$d_\infty(u,v) = \sup_{\lambda\in[0,1]} \max\left\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\right\}.$$

It is a well-known fact that $(\mathbb{E}^1, d_\infty)$ is a nonseparable, complete metric space. It is worth noting that the set of real numbers equipped with its usual topology induced by the Euclidean metric d_e is a closed subspace of $(\mathbb{E}^1, d_\infty)$. Moreover, since the cardinal of the set of all monotone real-valued functions on $[0,1]$ is the continuum, a consequence of Goetschel–Voxman's theorem is that the cardinality of $\mathbb{E}^1$ is the continuum. As usual, $B_r(x)$ denotes the ball of center x and radius r of $(\mathbb{E}^1, d_\infty)$.

In this paper we deal with C_p–theory in the setting of fuzzy analysis. In the classical case, the pointwise topology is a powerful tool in itself and in its applications to general topology, functional analysis, coding theory, Boolean rings, etc. (see for instance, [2, 3, 6, 18, 23, 24]). Our aim is to make the starting point of a similar theory for fuzziness. Throughout all spaces are assumed to be Tychonoff, that is, completely regular and Hausdorff. Given two spaces X and Y, $C_p(X,Y)$ stands for the space of all continuous functions from X to Y endowed with the pointwise convergence topology which is generated by the sets of the form

$$[x_1, \ldots, x_n; U_1, \ldots, U_n] = \{f \in C_p(X,Y) : f(x_k) \in U_k,\ k = 1, 2, \ldots, n\}$$

where $x_k \in X$ and U_k is an open set of Y $(k = 1, 2, \ldots, n)$. In other words, the topology of $C_p(X,Y)$ is the one induced by the product topology on Y^X. When

$Y = (\mathbb{E}^1, d_\infty)$ (respectively, $Y = (\mathbb{R}, d_e)$) we write simply $C_p(X, \mathbb{E}^1)$ (respectively, $C_p(X)$). Notice that a neighborhood base of a function f for the topology of $C_p(X, \mathbb{E}^1)$ is the family of all the sets of the form

$$\langle f; x_1, \dots, x_n; \epsilon \rangle = \left\{ g \in C_p(X, \mathbb{E}^1) : d_\infty(f(x_k), g(x_k)) < \epsilon \right\} \quad k = 1, 2, \dots, n$$

for all $\varepsilon > 0$ and $x_1, x_2, \dots, x_n \in X$ for all $n \in \mathbb{N}$. Closedness of $(\mathbb{R}, d_e)$ in $(\mathbb{E}^1, d_\infty)$ implies that $C_p(X)$ is a closed subspace of $C_p(X, \mathbb{E}^1)$.

The paper is organized as follows. In Section 2 we introduce some basic properties of the space $(\mathbb{E}^1, d_\infty)$ including the fact that the addition and multiplication are continuous. Section 3 is devoted to the properties of the space $C_p(X, \mathbb{E}^1)$. In Section 4 we deal with several properties related to compactness in $C_p(X, \mathbb{E}^1)$. In particular, a version of the celebrated Grothendieck's theorem on compactness of countably compact subsets of $C_p(X)$ is achieved.

Although our notation and terminology is standard, some comments are in order. A cardinal function is a function Γ assigning to every topological space X a cardinal number $\Gamma(X)$ such that $\Gamma(X) = \Gamma(Y)$ for any pair X, Y of homeomorphic spaces. For a subset A of a space X, we denote by $\overline{A}$ the closure of A in X. The cardinality of a set X is denoted by $|X|$. As usual, the continuum is denoted by $\mathfrak{c}$. $\mathbb{N}$ stands for the natural numbers and $\aleph_0$ for the cardinality of $\mathbb{N}$. The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that every family of pairwise disjoint nonempty open sets of X has cardinality $\leq \mathfrak{m}$ is called the Souslin number (or cellularity) of the space X and it is denoted by $c(X)$. If $c(X) = \aleph_0$, we say that the space X has the Souslin property.

Given a space X, the smallest cardinality of a base for the topology of X (respectively, of a dense subset of X) is called the weight (respectively, the density) of X and it is denoted by $w(X)$ (respectively, by $d(X)$). The character of a point x in X is defined as the smallest cardinal number of a neighborhood base for X at the point x; this cardinal number is denoted by $\chi(x, X)$. The character $\chi(X)$ of a topological space X is defined as the supremum of all numbers $\chi(x, X)$ for $x \in X$.

For a given space X, a family $\mathcal{N}$ of subsets of X is called a network of X if for any open set U of X there is $\mathcal{M} \subset \mathcal{N}$ such that $\bigcup \mathcal{M} = U$. The cardinal $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network of } X\}$ is called the network weight of X.

Recall that a function $f : X \to Y$ is called a condensation if it is a continuous bijection. Let $iw(X) = \min \{|\kappa| : \text{there is a condensation of } X \text{ onto a space of weight } \leq \kappa\}$. The cardinal $iw(X)$ is called the i-weight of X. The tightness $t(X)$ of a space X is the smallest cardinal such that for each set $A \subset X$ and any point x in the closure of A there is a set $B \subset A$ for which $|B| \leq t(X)$ and x belongs to the closure of B. We refer the reader to [11] for further information on these topics.

2. **The Space** $(\mathbb{E}^1, d_\infty)$

Given two fuzzy numbers u and v, we define its addition $u + v$ and its multiplication uv by means of the typical interval operations. To be precise, for each $\lambda \in [0, 1]$, the λ-level of $u + v$ and uv are defined, respectively, by the rules (see [10])

$$(u + v)(\lambda) = [u^-(\lambda) + v^-(\lambda), u^+(\lambda) + v^+(\lambda)]$$

and

$$(uv)(\lambda) = \\ [\min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}, \\ \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}].$$

The following result is probably known but we were not aware of any suitable reference.

Proposition 2.1. *If $u, v, w \in \mathbb{E}^1$ and $k \in \mathbb{R}$, then*

(i) $d_\infty(u, v) = d_\infty(u + w, v + w)$;
(ii) $d_\infty(ku, kv) = |k|d_\infty(u, v)$;
(iii) $d_\infty(wu, wv) \leq \max\{|w^-(0)|, |w^+(0)|\}d_\infty(u, v)$;
(iv) $uv = 0$ *if and only if* $u = 0$ *or* $v = 0$;
(v) *the equation* $u + x = 0$ *has solution if and only if* $u \in \mathbb{R}$.

Proof. Notice that claims (i) and (ii) hold true by the definition of the metric d_∞.

(iii) Let r denote $\max\{|w^-(0)|, |(w)^+(0)|\}$. For any $t \in [0, 1]$, we have $-r \leq w^-(t) \leq w^+(t) \leq w^+(0) \leq r$ so that $|w^-(t)| \leq r$ and $|w^+(t)| \leq r$. Thus, for any $t \in [0, 1]$, the following inequalities hold

$$|w^-(t)u^-(t) - w^-(t)v^-(t)| \leq r|u^-(t) - v^-(t)|,$$
$$|w^+(t)u^-(t) - w^+(t)v^-(t)| \leq r|u^-(t) - v^-(t)|,$$
$$|w^-(t)u^+(t) - w^-(t)v^+(t)| \leq r|u^+(t) - v^+(t)|,$$
$$|w^+(t)u^+(t) - w^+(t)v^+(t)| \leq r|u^+(t) - v^+(t)|.$$

Therefore

$$|(wu)^-(t) - (wv)^-(t)| \leq r\max\{|u^-(t) - v^-(t)|, |u^+(t) - v^+(t)|\},$$
$$|(wu)^+(t) - (wv)^+(t)| \leq r\max\{|u^-(t) - v^-(t)|, |u^+(t) - v^+(t)|\},$$
$$\max\{|(wu)^-(t) - (wv)^-(t)|, |(wu)^+(t) - (wv)^+(t)|\} \leq \\ \leq r\max\{|u^-(t) - v^-(t)|, |u^+(t) - v^+(t)|\},$$

for any $t \in [0, 1]$. We have just shown that

$$d_\infty(wu, wv) \leq \max\{|w^-(0)|, |w^+(0)|\}d_\infty(u, v).$$

(iv) Assume, without loss of generality, that both u and v are different from zero. Take distinct $\lambda_1, \lambda_2 \in [0, 1]$ such that

(1) either $u^-(\lambda_1) \neq 0$ or $u^+(\lambda_1) \neq 0$, and
(2) either $v^-(\lambda_2) \neq 0$ or $v^+(\lambda_2) \neq 0$.

It follows from (1) (respectively, from (2)) that $v^-(\lambda_1) = 0$ and $v^+(\lambda_1) = 0$ (respectively, $u^-(\lambda_2) = 0$ and $u^+(\lambda_2) = 0$). This leads us to a contradiction because u^- is not decreasing and v^+ is not increasing. Thus, the proof is complete.

(v) Let $u = [u^-(\lambda), u^+(\lambda)]$ and $x = [x^-(\lambda), x^+(\lambda)]$. Since for any $\lambda \in [0, 1]$ $u^-(\lambda) + x^-(\lambda) = 0$ and $u^+(\lambda) + v^+(\lambda) = 0$, we have $x^-(\lambda) = -u^-(\lambda)$ and $x^+(\lambda) = -u^+(\lambda)$ ($\lambda \in [0, 1]$). Being the functions u^- and $x^- = -u^-$ non-decreasing and the functions u^+ and $v^+ = -u^+$ non-increasing, we obtain that u^-, u^+, x^- and x^+ are constant in $[0, 1]$ so that $u^- = u^+$. Therefore $u = [u^-(\lambda), u^+(\lambda)] \in \mathbb{R}$. $\square$

We finish this section with several useful properties of $(\mathbb{E}^1, d_\infty)$.

Proposition 2.2. *The Souslin number of* $(\mathbb{E}^1, d_\infty)$ *is* $\mathfrak{c}$.

Proof. For each $a \in (0,1]$ define the fuzzy number $u_a = [u_a^-, u_a^+]$, where $u_a^-(\lambda) = 0$ and

$$u_a^+(\lambda) = \begin{cases} 1 \text{ if } \lambda \in [0,a], \\ 0 \text{ if } \lambda \in (a,1], \end{cases}$$

for any $\lambda \in [0,1]$. It is easy to see that $B_{1/2}(u_a) \cap B_{1/2}(u_b) = \emptyset$ for any $a, b \in (0,1]$, $a \neq b$. Hence $\{B_{1/2}(u_a) : a \in (0,1]\}$ is a pairwise disjoint family of nonempty open sets of $(\mathbb{E}^1, d_\infty)$. Since the cardinality of $(\mathbb{E}^1, d_\infty)$ is $\mathfrak{c}$, so is the Souslin number of $(\mathbb{E}^1, d_\infty)$. □

Proposition 2.3. *The space* $(\mathbb{E}^1, d_\infty)$ *is not locally compact at any fuzzy number* $u \in \mathbb{E}^1$.

Proof. It is easy to see that the sequence $\{v_n : n \in \mathbb{N}\} \subset (\mathbb{E}^1, d_\infty)$ where $v_n^-(\lambda) = 0$ and $v_n^+(\lambda) = (1-\lambda)^n$ for any $\lambda \in [0,1]$ does not have any convergent subsequence.

Fix $u \in \mathbb{E}^1$. For a given number $\epsilon > 0$, define the sequence of fuzzy numbers $\{w_n : n \in \mathbb{N}\}$ by the rule $w_n^-(\lambda) = u^-(\lambda)$ and $w_n^+(\lambda) = u^+(\lambda) + \epsilon(1-\lambda)^n$ for any $\lambda \in [0,1]$. Notice that $\{w_n : n \in \mathbb{N}\}$ has no convergent subsequences and it is contained in $B_\epsilon(u)$. Therefore u does not have any compact neighborhood. This completes the proof. □

The interested reader is referred to [13] for a characterization of compact sets in $(\mathbb{E}^1, d_\infty)$.

Recall that a space X is called cofinally Čech-complete if there exists a locally compact space Z and an embedding $e\colon X \to Z$ of X into Z satisfying $\chi(e(X), Z) \leq \aleph_0$. Among other reasons, cofinally Čech complete spaces are interesting because a metrizable space admits a cofinally complete metric if and only if it is cofinally Čech complete (see [21]).

Corollary 2.4. *The space* $(\mathbb{E}^1, d_\infty)$ *is not cofinally Čech-complete.*

Proof. It follows from Proposition 2.3 that the set of points of $(\mathbb{E}^1, d_\infty)$ that have no compact neighborhood is not compact. By [14, 19], $(\mathbb{E}^1, d_\infty)$ is not cofinally Čech-complete. □

A space X is hemicompact if in the family of all compact subspaces of X ordered by inclusion there exists a countable cofinal subfamily. The completion of $\mathbb{E}^1$ with the *pointwise uniformity* is a hemicompact space (see [12] for details). However, since any hemicompact first countable space is locally compact ([1]), we have

Proposition 2.5. *The space* $(\mathbb{E}^1, d_\infty)$ *is not hemicompact.*

3. The Space $C_p(X, \mathbb{E}^1)$

From now on, if no confusion is possible, we will denote the metric space $(\mathbb{E}^1, d_\infty)$ by $\mathbb{E}^1$. We begin by showing a basic but helpful property of the fuzzy-valued functions. As usual, given two functions f, g from a space X into $\mathbb{E}^1$, by $f+g$ (respectively, fg) it is understood the pointwise addition (respectively, the pointwise multiplication).

Proposition 3.1. *Let X be a space. If $f, g\colon X \to \mathbb{E}^1$ are two continuous functions, then $f+g$ and fg are continuous.*

Proof. Take a point $x_0 \in X$ and $\epsilon > 0$. We can choose open sets U_1, U_2 containing x_0 such that $f(U_1) \subset B_{\epsilon/2}(f(x_0))$ and $f(U_2) \subset B_{\epsilon/2}(f(x_0))$. Consider the open set $U = U_1 \cap U_2$. If $y \in U$, then

$$\begin{aligned} d_\infty(f(y)+g(y), f(x_0)+g(x_0)) &\leq d_\infty(f(y)+g(y), f(y)+g(x_0)) + \\ &+ d_\infty(g(y), g(x_0)) + d_\infty(f(y), f(x_0)) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $f+g$ is continuous at x_0.

Now we prove that fg is continuous. For this, take as above a point $x_0 \in X$ and choose an open set W_1 such that $x_0 \in W_1$ and $f(W_1) \subset B_1(f(x_0))$. If $y \in W_1$, then $d_\infty(f(y), f(x_0)) < 1$. Hence, for any $\lambda \in [0,1]$,

$$\max\{|(f(y))^-(\lambda) - (f(x_0))^-(\lambda)|, |(f(y))^+(\lambda) - (f(x_0))^+(\lambda)|\} < 1.$$

Thus, for any $\lambda \in [0,1]$, we have

$$|(f(y))^-(\lambda)| \leq 1 + |(f(y))^-(x_0)|, \quad |(f(y))^+(\lambda)| \leq 1 + |(f(y))^+(x_0)|$$

and

$$\max\{|(f(y))^-(0)|, |(f(y))^+(0)|\} \leq 1 + \max\{|(f(x_0))^-(0)|, |(f(x_0))^+(0)|\}.$$

Let r (respectively, s) denote $\max\{|(f(x_0))^-(0)|, |(f(x_0))^+(0)|\}$ (respectively, $\max\{|(g(x_0))^-(0)|, |(g(x_0))^+(0)|\}$).

Choose open sets W_2 and W_3 in X containing x_0 such that $f(W_2) \subset B_{\epsilon_1}(f(x_0))$ and $f(W_3) \subset B_{\epsilon_1}(g(x_0))$ with $\epsilon_1 = \dfrac{\epsilon}{1+r+s}$.

Consider the open set $W = W_1 \cap W_2 \cap W_3$. It is clear that $x_0 \in W$. Now, if $y \in W$, then

$$\begin{aligned} d(f(y)g(y), f(x_0)g(x_0)) &\leq d(f(y)g(y), f(y)g(x_0)) + d(f(y)g(x_0), f(x_0)g(x_0)) \\ &< (1+r)\frac{\epsilon}{1+r+s} + \frac{s\epsilon}{1+r+s} = \frac{(1+r+s)\epsilon}{1+r+s} = \epsilon. \end{aligned}$$

Therefore fg is continuous at the point x_0. $\square$

We look now at three properties which are interesting in themselves and for future applications.

Proposition 3.2. *Let X be a space. If $x_1, x_2, \ldots, x_n$ are different points of X and $u_1, u_2, \ldots, u_n \in \mathbb{E}^1$, then there exists a continuous function $f\colon X \to \mathbb{E}^1$ such that $f(x_i) = u_i$ for any $i = 1, 2, \ldots, n$.*

Proof. Since X is Tychonoff, there exists,for all $i = 1, 2, \ldots, n$, a continuous function $f_i : X \to \mathbb{R}$ such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ whenever $i \neq j$. The function $f\colon X \to \mathbb{E}^1$ defined by the rule $f(x) = f_1(x)u_1 + f_2(x)u_2 + \cdots + f_n(x)u_n$ is well defined and continuous by Proposition 3.1. Notice that

$f(x_i) = f_1(x_i)u_1 + f_2(x_i)u_2 + \cdots + f_i(x_i)u_i + \cdots + f_n(x_i)u_n$

which implies that $f(x_i) = u_i$ $(i = 1, 2, \ldots, n)$. □

From now on, $(\mathbb{E}^1)^X$ stands for the product space $\prod_{x \in X} (\mathbb{E}^1)_x$ where each $(\mathbb{E}^1)_x$ coincides with $\mathbb{E}^1$. In other words, $(\mathbb{E}^1)^X$ is the space of all functions from X into $\mathbb{E}^1$ equipped with the pointwise topology.

Proposition 3.3. *For any space X, the function space $C_p(X, \mathbb{E}^1)$ is dense in $(\mathbb{E}^1)^X$.*

Proof. Take an arbitrary function $f \in (\mathbb{E}^1)^X$ and the open set in $(\mathbb{E}^1)^X$ defined as $U = \langle f; x_1, x_2, \ldots, x_n; \epsilon \rangle$ with $\epsilon > 0$. The previous Proposition 3.2 tells us that there exists a function $g \in C_p(X, \mathbb{E}^1)$ such that $g(x_i) = f(x_i)$ for any $i = 1, 2, \ldots, n$. It is evident that $g \in U$. This proves that $C_p(X, \mathbb{E}^1)$ is a dense subset of $(\mathbb{E}^1)^X$. □

Another interesting property of the space $C_p(X, \mathbb{E}^1)$ is

Proposition 3.4. *For any space X, the space $C_p(X, \mathbb{E}^1)$ is homeomorphic to $C_p(X, B_1(0))$.*

Proof. Consider the homeomorphism $\alpha = \frac{2 \arctan}{\pi} : \mathbb{R} \to (-1, 1)$. Now, if $f \in C_p(X, \mathbb{E}^1)$, let $\varphi(f) : C_p(X, \mathbb{E}^1) \to C_p(X, B_1(0))$ be the function defined by the rule $\varphi(f)(x) = [\alpha(f(x)^-), \alpha(f(x)^+)]$ for all $x \in X$. Since α is non-decreasing and $f(x)^-(\lambda) \leq f(x)^+(\lambda)$, we have that $\alpha(f(x)^-(\lambda)) \leq \alpha(f(x)^+(\lambda))$. Moreover, since the functions $f(x)^-$ and α are non-decreasing so is $\alpha(f(x))^-$. In a similar way, we have that the function $\alpha(f(x))^+$ is non-increasing. It follows from the continuity of α and the properties of $f(x)^-$ and $f(x)^+$ that $\varphi(f)(x) \in B_1(0)$. Take now $f \in C_p(X, \mathbb{E}^1)$ and an open set $W = \langle h; x_1, \ldots, x_n; \epsilon \rangle$ (in $C_p(X, B_1(0))$) containing $h = \varphi(f)$.

Now, since the function α is uniformly continuous, there exists $\delta > 0$ such that $|\alpha(z_1) - \alpha(z_2)| < \frac{\epsilon}{2}$ whenever $|z_1 - z_2| < \delta$. Define now $U = \langle f; x_1, \ldots, x_n; \delta \rangle$ and take $g \in U$. For all $\lambda \in [0, 1]$, it follows from $d_\infty(f(x_k), g(x_k)) < \delta$ that $|f(x_k)^-(\lambda) - g(x_k)^-(\lambda)| < \delta$ and $|f(x_k)^+(\lambda) - g(x_k)^+(\lambda)| < \delta$ for any $k = 1, \ldots, n$. Thus,

$$|\alpha(f(x_k)^-(\lambda)) - \alpha(g(x_k)^-(\lambda))| < \frac{\epsilon}{2} \text{ and } |\alpha(f(x_k)^+(\lambda)) - \alpha(g(x_k)^+(\lambda))| < \frac{\epsilon}{2}$$

for all $\lambda \in [0, 1]$. Then $d_\infty(\varphi(f)(x_k), \varphi(g)(x_k)) \leq \frac{\epsilon}{2} < \epsilon$ $(k = 1, 2, \ldots, n)$ which implies that $\varphi(g) \in W$. We have just proved that the function φ is continuous.

We now prove that φ is a bijection. Injectivity is an easy consequence of the fact of being α injective. Moreover, if $f \in C_p(X, B_1(0))$, then, for any $x \in X$, define $h(x) = [\frac{\pi \tan(f(x)^-)}{2}, \frac{\pi \tan(f(x)^+)}{2}]$. It is easy to see that $h \in C_p(X, \mathbb{E}^1)$ and $\varphi(h) = f$. Thus, φ is surjective.

We will finish the proof by showing that φ^{-1} is continuous. We will use the fact that, for any $0 < r < 1$, the function $\beta(x) = \frac{\pi tan x}{2}$ is a uniformly continuous function from the real interval $[-r, r]$ into $\mathbb{R}$. Take $f \in C_p(X, B_1(0))$ and $V = \langle \varphi^{-1}(f); x_1, \ldots, x_n; \epsilon \rangle$. If $k \in \{1, \ldots, n\}$, then $-1 < f(x_k)^-(0) \leq f(x_k)^-(\lambda) \leq f(x_k)^+(\lambda) \leq f(x_k)^+(0) < 1$ for all $\lambda \in [0, 1]$. Now choose $r \in \mathbb{R}$ and $\delta > 0$ such that

$-1 < -r < f(x_k)^-(0) \leq f(x_k)^+(0) < r < 1$, $-r < f(x_k)^-(0) - \delta \leq f(x_k)^+(0) < f(x_k)^+(0) + \delta < r$ and $|\beta(z_1) - \beta(z_2)| < \frac{\epsilon}{2}$ for $z_1, z_2 \in [-r, r]$ with $|z_1 - z_2| < \delta$. Define $Z = \langle f; x_1, \ldots, x_n; \delta\rangle$. If $g \in Z$ and $k \in \{1, \ldots, n\}$, then $|g(x_k)^-(\lambda) - f(x_k)^-(\lambda)| < \delta$ and $|g(x_k)^+(\lambda) - f(x_k)^+(\lambda)| < \delta$ for all $\lambda \in [0, 1]$. It is easy to see that $g(x_k)^-(\lambda), g(x_k)^+(\lambda), f(x_k)^-(\lambda), f(x_k)^+(\lambda) \in [-r, r]$ for all $\lambda \in [0, 1]$. Hence $|\beta(g(x_k)^-(\lambda)) - \beta(f(x_k)^-(\lambda))| < \frac{\epsilon}{2}$ and $|\beta(g(x_k)^+(\lambda)) - \beta(f(x_k)^+(\lambda))| < \frac{\epsilon}{2}$ for all $\lambda \in [0, 1]$ which implies that $d_\infty(\varphi^{-1}(f), \varphi^{-1}(g)) \leq \frac{\epsilon}{2} < \epsilon$. Therefore $\varphi^{-1}(Z) \subset V$ and, consequently, φ^{-1} is continuous. Hence $C_p(X, \mathbb{E}^1)$ and $C_p(X, B_1(0))$ are homeomorphic. □

A helpful property is

Proposition 3.5. *Let A be a closed susbet of a space X. If $x_1, x_2, \ldots, x_n \in X \setminus A$ and $u_1, u_2, \ldots, u_n \in \mathbb{E}^1$, then there exists a continuous function $g\colon X \to \mathbb{E}^1$ such that $g|_A = f$ and $g(x_i) = u_i$ for any $i = 1, 2, \ldots, n$.*

Proof. For any $i = 1, 2, \ldots, n$ there exists a continuous function $g_i\colon X \to \mathbb{R}$ such that $g_i(A \cup \{x_1, x_2, \ldots, x_n\} \setminus \{x_i\}) = \{0\}$ and $g_i(x_i) = 1$. Consider now a function $h\colon X \to \mathbb{R}$ such that $h(A) = \{1\}$ and $h(\{x_1, x_2, \ldots, x_n\}) = \{0\}$ and let g be the function from X into $\mathbb{E}^1$ defined as $g(x) = h(x)f(x) + \Sigma_{i=1}^n u_i g_i(x)$. If $a \in A$, then

$$g(a) = h(a)f(a) + \Sigma_{i=1}^n u_i g_i(a) = f(a) + 0 = f(a).$$

Observe that, for $1 \leq k \leq n$, we have

$$g(x_k) = h(x_k)f(x_k) + \Sigma_{i=1}^n u_i g_i(x_k) = u_k g_k(x_k) = u_k.$$

Therefore the function g satisfies all the desired properties. □

Two functions which play an important role in C_p–theory are the so-called *restriction function* and *dual function*.

Definition 3.6. Let Y be a subset of a space X. The function $\pi_Y\colon C_p(X, \mathbb{E}^1) \to C_p(Y, \mathbb{E}^1)$ defined by $\pi_Y(f) = f|_Y$ is called the restriction function.

The following proposition follows from Propositions 3.2, 3.3 and 3.5. Recall that a function $f\colon X \to Z$ is called *open* if $f(V)$ is an open set of Z whenever V is open in X.

Proposition 3.7. *If Y is a subset of a space X, then the restriction function π_Y on Y enjoys the following properties:*

(i) *π_Y is a continuous function and $\overline{\pi_Y(C_p(X, \mathbb{E}^1))} = C_p(Y, \mathbb{E}^1)$;*
(ii) *π_Y is injective if and only if Y is dense in X;*
(iii) *if the function $\pi|_Y$ is a homeomorphism, then $Y = X$;*
(iv) *if Y is closed in X, then $\pi|_Y$ is an open function.*

Definition 3.8. Let $\varphi\colon X \to Y$ be a continuous function. The function

$$\varphi^*\colon C_p(Y, \mathbb{E}^1) \to C_p(X, \mathbb{E}^1)$$

defined by the rule $\varphi^*(f) = f \circ \varphi$ for all $f \in C_p(Y, \mathbb{E}^1)$ is called the dual function of φ.

The following result follows from Propositions 3.2, 3.3, 3.5 and 3.7. A function $\varphi\colon X \to Y$ is said to be a closed function if $\varphi(A)$ is closed in Y whenever A is a closed set of X.

Proposition 3.9. *If $\varphi\colon X \to Y$ is a continuous function, then the following conditions hold:*

(i) *The dual function φ^* is continuous;*
(ii) *if $\varphi(X)$ is surjective, then $\varphi^*\colon C_p(Y,\mathbb{E}^1) \to \varphi^*(C_p(Y,\mathbb{E}^1) \subset C_p(X,\mathbb{E}^1)$ is a homeomorphism;*
(iii) *if $\varphi(X)$ is surjective, then $\varphi^*(C_p(Y,\mathbb{E}^1)$ is dense in $C_p(X,\mathbb{E}^1)$ if and only if φ is a condensation;*
(iv) *if $\varphi(X)$ is surjective and φ is a closed function, then $\varphi^*(C_p(Y,\mathbb{E}^1)$ is closed in $C_p(X,\mathbb{E}^1)$.*

The following result establishes an important difference between $C_p(X)$ and $C_p(X,\mathbb{E}^1)$. It is well known that for any $f \in C_p(X)$ the function $\varphi_f : C_p(X) \to C_p(X)$ defined by $\varphi_f(g) = f + g$ is a homeomorphism. However, we have

Proposition 3.10. *If $f \in C_p(X,\mathbb{E}^1)$, then the function $\varphi_f\colon C_p(X,\mathbb{E}^1) \to C_p(X,\mathbb{E}^1)$ is continuous and injective. Moreover, the following assertions hold:*

(i) *If $f \in C_p(X)$, then φ_f is a homeomorphism;*
(ii) *if $f \in C_p(X,\mathbb{E}^1) \setminus C_p(X)$, then φ_f is not surjective.*

Proof. Take $g_0 \in C_p(X,\mathbb{E}^1)$ and an open set $W = \langle f + g_0; x_1,\dots,x_n;\epsilon\rangle$. Consider the open set $U = \langle g_0; x_1,\dots,x_n;\epsilon\rangle$ and choose $h \in U$. It follows from Proposition 2.1 that $d_\infty(f(x_k)+h(x_k), f(x_k)+g_0(x_k)) = d_\infty(h(x_k), g_0(x_k)) < \epsilon$ for any $k = 1,\dots,n$. Then $\varphi_f(U) \subset W$ which implies that φ_f is continuous. Now, if $g, h \in C_p(X,\mathbb{E}^1)$ with $f \neq g$, choose a point $x_0 \in X$ for which $g(x_0) \neq h(x_0)$. Since $d_\infty(f(x_0)+g(x_0), f(x_0)+h(x_0)) = d_\infty(g(x_0), h(x_0)) \neq 0$, we have $(f+g)(x_0) \neq (f+h)(x_0)$. Hence φ_f is injective.

Suppose now that $f \in C_p(X,\mathbb{R})$ and take $g \in C_p(X,\mathbb{E}^1)$. The function f has an opposite with respect to addition $h = -f \in C_p(X,\mathbb{R})$. Hence $\varphi_f(g+h) = f+g+h = g$. Thus, φ_f is surjective. Moreover, the function $\varphi_h : C_p(X,\mathbb{E}^1) \to C_p(X,\mathbb{E}^1)$ is continuous and $\varphi_f^{-1} = \varphi_h$ because $\varphi_h(\varphi_f(g)) = \varphi_h(f+g) = f+g+h = f$. Therefore φ_f is a homeomorphism. This shows (i).

To conclude the proof, let $f \in C_p(X,\mathbb{E}^1)\setminus C_p(X)$ and choose a point $x_0 \in X$ such that $f(x_0) \in \mathbb{E}^1 \setminus \mathbb{R}$. If $g \in C_p(X,\mathbb{E}^1)$, then the equation $f(x_0) + g(x_0) = 0$ does not have a solution in $\mathbb{E}^1$. This fact shows that φ_f is not surjective: indeed, there is no function $g \in C_p(X,\mathbb{E}^1)$ such that $\varphi_f(g) = f + g = 0$. This proves (ii). □

A subset A of $\mathbb{E}^1$ is called *support bounded* if there exists a positive real number L such that $\max\{|u^-(0)|, |u^+(0)|\} \leq L$ for all $u \in A$. By using the metric d_∞, this is equivalent to saying that there is a ball $B_r(0)$ containing A. It is a well-known fact that every compact subset of $\mathbb{E}^1$ is support bounded.

We will prove two properties of $C_p(X,\mathbb{E}^1)$ which imply that X is countable. These results are motivated by the well-known fact that $C_p(X,Y)$ is metrizable if and only if X is countable and Y is metrizable. Note that this result implies that $C_p(X, \mathbb{E}^1)$ is metrizable if and only if X is countable.

Proposition 3.11. *If there exists a compact subspace $K \subset C_p(X, \mathbb{E}^1)$ such that $\chi(K, C_p(X, \mathbb{E}^1)) \leq \omega$, then X is countable.*

Proof. Let $\mathcal{B} = \{W_n : n \in \mathbb{N}\}$ be a countable base of K in $C_p(X, \mathbb{E}^1)$. Given a natural number $n \in \mathbb{N}$, choose, for any $f \in K$, an open neighborhood $U_f^n = \langle f; x_1^f, \ldots, x_{k_f}^f; \epsilon_f \rangle$ such that $U_f^n \subset W_n$. The family $\{U_f^n : f \in K\}$ is an open cover of K. Take a finite subcover $\{U_{f_1}^n, \ldots, U_{f_m}^n\}$ of K and consider the finite set

$$A_n = \{x_1^{f_1}, \ldots, x_{k_{f_1}}^{f_1}, \ldots, x_1^{f_m}, \ldots, x_{k_{f_m}}^{f_m}\}.$$

Now define the countable set $A = \bigcup\{A_n : n \in \mathbb{N}\}$. We will prove that $A = X$. To proceed by contradiction, suppose that there exists $x \in X \setminus A$ and consider the function $e_x : C_p(X, \mathbb{E}^1) \to \mathbb{E}^1$ defined by $e_x(f) = f(x)$. Since the set $e_x(K) \subset \mathbb{E}^1$ is compact, there exists a real number r_x such that $e_x(K) \subset B_{r_x}(0)$. It is easy to see that $K \subset \langle 0; x; r_x \rangle = [x; B_{r_x}(0)]$ and, consequently, there exists $W_n \in \mathcal{B}$ such that $K \subset W_n \subset \langle 0; x; r_x \rangle$. Then $K \subset \bigcup\{U_{f_i}^n : i = 1, \ldots, m\} \subset \langle 0; x; r_x \rangle$. Thus, $U_{f_1}^n = \langle f_1; x_1^{f_1}, \ldots, x_{k_{f_1}}^{f_1}; \epsilon_{f_1} \rangle \subset \langle 0; x; r_x \rangle$. It follows from Proposition 3.2 that there is $h \in C_p(X, \mathbb{E}^1)$ such that $h(x_i^{f_1}) = f(x_i^{f_1})$ for any $i = 1, \ldots, k_{f_1}$ and $h(x) = r_x + 1$. Hence $h \in U_{f_1}^n$ and $h \notin \langle 0; x; r_x \rangle$ which is a contradiction because $U_{f_1}^n \subset \langle 0; x; r_x \rangle$. Therefore $X = A$ and X is countable. □

A space X is called Čech-complete if the remainder $\beta X \setminus X$ is the union of countably many closed sets of βX where, as usual, βX denotes the Stone-Čech compactification of X.

Proposition 3.12. *The space $C_p(X, \mathbb{E}^1)$ contains a dense Čech-complete subspace if and only if X is countable and discrete.*

Proof. Let $Z \subset C_p(X, \mathbb{E}^1)$ be a dense and Čech-complete subset. Suppose that X is not discrete and choose a non-open singleton subset A of X. Consider now the function $f \in \mathbb{R}^X \setminus C_p(X, \mathbb{R})$ defined by the rule $f(A) = 0$ and $f(X \setminus A) = 1$. Since f has an opposite with respect to addition in $(\mathbb{E}^1)^X$, it follows from Proposition 3.10 that the function $\varphi_f : (\mathbb{E}^1)^X \to (\mathbb{E}^1)^X$ defined as $\varphi_f(g) = f + g$ is a homeomorphism. Then the space $f + Z = \{f + g : g \in Z\}$ is a dense Čech-complete subspace of $(\mathbb{E}^1)^X$ and $f + Z \subset (\mathbb{E}^1)^X \setminus C_p(X, \mathbb{E}^1)$. Consequently, $Z \cap (f + Z) = \emptyset$. This fact leads us to a contradiction because the intersection of two dense Čech-complete subspaces of a Tychonoff space cannot be empty. Therefore X is discrete.

Moreover, since Z is Čech-complete, we can find a compact set $K \subset Z$ with $\chi(K, Z) \leq \omega$. Being Z dense in $C_p(X, \mathbb{E}^1)$, we have $\chi(K, C_p(X, \mathbb{E}^1)) = \chi(K, Z) \leq \omega$. By Proposition 3.11, X is countable.

To see the converse, assume that X is countable and discrete. Then $C_p(X, \mathbb{E}^1) = (\mathbb{E}^1)^\omega$ and the result follows from the fact that $(\mathbb{E}^1)^\omega$ is a Čech-complete space (because it is a complete metric space). □

An interesting question in C_p–theory is under what conditions the algebraic and/or topological structure of $C_p(X, \mathbb{E}^1)$ determines the space X. We turn now to the study of this problem. A function $\varphi\colon C_p(X, \mathbb{E}^1) \to \mathbb{E}^1$ is said to be *an additive functional* if $\varphi(f+g) = \varphi(f) + \varphi(g)$. Notice that if φ is an additive functional, then $\varphi(0) = 0$. Indeed, $\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$ so that, for all $\lambda \in [0,1]$, $\varphi(0)^-(\lambda_1) = (\varphi(0)+\varphi(0))^-(\lambda) = \varphi(0)^-(\lambda_1)+\varphi(0)^-(\lambda_1)$ and $\varphi(0)^+(\lambda_1) = (\varphi(0)+\varphi(0))^+(\lambda) = \varphi(0)^+(\lambda_1)+\varphi(0)^+(\lambda_1)$. Thus, $\varphi(0) = 0$. Our first result states an important property of additive functionals on $C_p(X, \mathbb{E}^1)$.

Proposition 3.13. *If $\varphi\colon C_p(X, \mathbb{E}^1) \to \mathbb{E}^1$ is an additive functional, then $\varphi(C_p(X))$ is a subset of $\mathbb{R}$.*

Proof. By (v) of Proposition 2.1, a function $f \in C_p(X, \mathbb{E}^1)$ has an opposite with respect to addition if and only if $f \in C_p(X)$. Thus, if $f \in C_p(X)$ and $g = -f$, then $\varphi(f+g) = \varphi(f) + \varphi(g) = 0$ which implies $\varphi(f) \in \mathbb{R}$. □

An additive functional $\varphi\colon C_p(X, \mathbb{E}^1) \to \mathbb{E}^1$ is called *a linear functional* if $\varphi(uf) = u\varphi(f)$ for all $f \in C_p(X, \mathbb{E}^1)$ and all $u \in \mathbb{E}^1$, and it is said to be *a linear multiplicative functional* if $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \mathbb{E}^1$. It is easy to prove that $\varphi(1) = 1$ whenever φ is a linear multiplicative functional. Note that every linear multiplicative functional is a linear functional. A homeomorphism $\xi\colon C_p(X, \mathbb{E}^1) \to C_p(Y, \mathbb{E}^1)$ is called a *topological isomorphism* if $\xi(f+g) = \xi(f) + \xi(f)$ and $\xi(fg) = \xi(f)\xi(g)$ for all $f, g \in C_p(X, \mathbb{E}^1)$. In this case, we say that $C_p(X, \mathbb{E}^1)$ and $C_p(Y, \mathbb{E}^1)$ are topologically isomorphic. The corresponding definitions for $C_p(X)$ and $C_p(Y)$ are self-explanatory.

Proposition 3.14. *Given two spaces X and Y, the following conditions are equivalent:*

(i) *$C_p(X, \mathbb{E}^1)$ and $C_p(Y, \mathbb{E}^1)$ are topologically isomorphic,*
(ii) *$C_p(X)$ and $C_p(Y)$ are topologically isomorphic,*
(iii) *X and Y are homeomorphic.*

Proof. (i)$\Longrightarrow$(ii) Suppose that ξ is a topological isomorphism from $C_p(X, \mathbb{E}^1)$ into $C_p(Y, \mathbb{E}^1)$. Let $f_0 \in C_p(X, \mathbb{E}^1)$ denote the constant function $f_0(x) = 0$ for any $x \in X$. Since $\xi(f_0) = \xi(f_0+f_0) = \xi(f_0)+\xi(f_0)$, we have that $\xi(f_0) = h_0$ where $h_0(y) = 0$ for any $y \in Y$. Thus, if $f \in C_p(X, \mathbb{R})$ and $g = -f \in C_p(X, \mathbb{R})$, then $\xi(f)+\xi(g) = h_0$. Consequently, if $y \in Y$, then the equation $\xi(f)(y) + \xi(g)(y) = h_0(y) = 0$ has a solution if and only if $\xi(f)(y) \in \mathbb{R}$. Hence $\xi(f) \in C_p(Y, \mathbb{R})$. In a similar way we can show that $\xi^{-1}(C_p(Y, \mathbb{R}) \subset C_p(X, \mathbb{R})$ so that $\xi|_{C_p(X,\mathbb{R})} : C_p(X, \mathbb{R}) \to C_p(Y, \mathbb{R})$ is a topological isomorphism.
(ii)$\Longrightarrow$(iii) It suffices to apply Nagata's theorem ([16]).
(iii)$\Longrightarrow$(i) If $r\colon X \to Y$ is a homeomorphism, then Proposition 3.9 tells us that $r^*\colon C_p(Y, \mathbb{E}^1) \to r^*(C_p(Y, \mathbb{E}^1)) \subset C_p(X, \mathbb{E}^1)$ is a homeomorphism. Now, for any $f \in C_p(X, \mathbb{E}^1)$, consider the function $g = f \circ r^{-1}$. Notice that $r^*(g) = g \circ r = f \circ r^{-1} \circ r = f$ which implies that $r^*(C_p(Y, \mathbb{E}^1)) = C_p(X, \mathbb{E}^1)$. Therefore $C_p(X, \mathbb{E}^1)$ and $C_p(Y, \mathbb{E}^1)$ are homeomorphic. If $f, g \in C_p(Y, \mathbb{E}^1)$, then $r^*(f+g)(x) = (f+g) \circ r(x) = f(r(x)) + g(r(x)) = r^*(f)(x) + r^*(g)(x)$ and $r^*(fg)(x) =$

$(fg) \circ r(x) = [f(r(x))][g(r(x))] = [r^*(f)(x)][r^*(g)(x)]$ for any $x \in X$. Hence $r^*(f+g) = r^*(f) + r^*(g)$, $r^*(fg) = r^*(f)r^*(g)$ and r^* is an isomorphism. Thus, $C_p(X, \mathbb{E}^1)$ and $C_p(Y, \mathbb{E}^1)$ are topologically isomorphic. □

Let $L(X)$ denote the set of all continuous linear functionals on the function space $C_p(C_p(X, \mathbb{E}^1), \mathbb{E}^1)$.

Proposition 3.15. *Let X be a space. The set $L(X)$ is closed in $C_p(C_p(X, \mathbb{E}^1), \mathbb{E}^1)$.*

Proof. Take $\varphi \in \overline{L(X)}$, $f, g \in C_p(X, \mathbb{E}^1)$ and $u \in \mathbb{E}^1$. First we prove that $\varphi(f+g) = \varphi(f) + \varphi(g)$. To do so, let $\epsilon > 0$ and choose $\psi \in \langle \varphi; f, g, f+g; \frac{\epsilon}{3} \rangle \cap L(X)$. Since $d_\infty(\psi(f), \varphi(f)) < \frac{\epsilon}{3}$, $d_\infty(\psi(g), \varphi(g)) < \frac{\epsilon}{3}$ and $d_\infty(\psi(f+g), \varphi(f+g)) < \frac{\epsilon}{3}$, we have

$$\begin{aligned} d_\infty(\varphi(f+g), \varphi(f)+\varphi(g)) &\leq d_\infty(\varphi(f+g), \psi(f+g)) \\ &+ d_\infty(\psi(f+g), \varphi(f)+\varphi(g)) \\ &< \frac{\epsilon}{3} + d_\infty(\psi(f)+\psi(g), \varphi(f)+\varphi(g)). \end{aligned}$$

Notice that

$$\begin{aligned} d_\infty(\psi(f)+\psi(g), \varphi(f)+\varphi(g)) &\leq d_\infty(\psi(f)+\psi(g), \psi(f)+\varphi(g)) \\ &+ d_\infty(\psi(f)+\varphi(g), \varphi(f)+\varphi(g)) \\ &= d_\infty(\psi(g), \varphi(g)) + d_\infty(\psi(f), \varphi(f)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \end{aligned}$$

and, consequently, $d_\infty(\varphi(f+g), \varphi(f)+\varphi(g)) < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$. Thus, $d_\infty(\varphi(f+g), \varphi(f)+\varphi(g)) = 0$. We have just shown that $\varphi(f+g) = \varphi(f)+\varphi(g)$.

Now we prove that $\varphi(uf) = u\varphi(f)$. Let r denote $\max\{|u^-(0)|, |u^+(0)|\}$. Given $\epsilon > 0$ and a continuous linear functional $\rho \in \langle \varphi; f, uf; \frac{\epsilon}{r+1} \rangle \cap L(X)$, we have

$$d_\infty(\varphi(f), \rho(f)) < \frac{\epsilon}{r+1},\ d_\infty(\varphi(uf), \rho(uf)) < \frac{\epsilon}{r+1},$$

and

$$d_\infty(\varphi(uf), u\varphi(f)) \leq d_\infty(\varphi(uf), \rho(uf)) + d_\infty(\rho(uf), u\varphi(f)).$$

Since $d_\infty(\varphi(uf), u\varphi(f)) < \frac{\epsilon}{r+1} + d_\infty(u\rho(f), u\varphi(f)) \leq \frac{\epsilon}{r+1} + \frac{r\epsilon}{r+1} = \epsilon$, we have $\varphi(uf) = u\varphi(f)$. Therefore $\varphi \in L(X)$ and, consequently, $L(X)$ is a closed subspace of the function space $C_p(C_p(X, \mathbb{E}^1), \mathbb{E}^1)$. □

Let A be a subset of $C_p(X, \mathbb{E}^1)$. For each $x \in X$ define the function $e_x^A \colon A \to \mathbb{E}^1$ by $e_x^A(f) = f(x)$ for all $f \in A$. The function e_x^A belongs to $C_p(A, \mathbb{E}^1)$ because it is the restriction to A of the projection of $(\mathbb{E}^1)^X$ onto the x-coordinate. The function $e^A \colon X \to C_p(A, \mathbb{E}^1)$ defined by the rule $e^A(x) = e_x^A$ is called the evaluation function. We now take a look at some properties involving the evaluation function.

Let X, Y be two spaces. A family of functions $A \subset C_p(X, Y)$ is said to separate points and closed sets of X if for every $x \in X$ and every closed set G of X such that $x \notin G$ there exists $f \in A$ for which $f(x) \notin \overline{f(G)}$. The family A separates the points of X if $f(x) \neq f(y)$ whenever $x \neq y$ for all $x, y \in X$.

Proposition 3.16. *For each space X, the following properties hold:*

(i) *The evaluation function* $e^A \colon X \to C_p(A, \mathbb{E}^1)$ *is continuous;*

(ii) *the evaluation function* $e^A \colon X \to C_p(A, \mathbb{E}^1)$ *is injective if and only if* A *separates the points of* X*;*

(iii) *the evaluation function* $e^A \colon X \to e^A(X) \subset C_p(A, \mathbb{E}^1)$ *is a homeomorphism if and only if the family* $U_A = \{f^{-1}(U) : f \in A$ *and* U *is an open set of* $\mathbb{E}^1\}$ *is a subbase of* X*;*

(iv) *if* A *separates points and closed sets of* X*, then the evaluation function*

$$e^A \colon X \to e^A(X) \subset C_p(A, \mathbb{E}^1)$$

is a homeomorphism.

Proof. (i) For a given $x \in X$ take an open set $U = \langle e^A(x); f_1, \ldots, f_n; \epsilon \rangle$ of $C_p(A, \mathbb{E}^1)$. For $k = 1, \ldots, n$, the set $V_k = f_k^{-1}(B_\epsilon(e_x^A(f_k))) = f_k^{-1}(B_\epsilon(f_k(x)))$ is an open set containing x and, consequently, x belongs to the open set $V = \cap\{V_k : k = 1, \ldots, n\}$. It is easy to see that $e^A(V) \subset U$ and hence e^A is continuous at the point x. Being x an arbitrary point of X, we have proved that e^A is continuous on X.
(ii) If e^A is injective, then for any pair of different points $x, y \in X$ we have that $e^A(x) \neq e^A(y)$. Hence there exists $f \in A$ such that $f(x) = e_x^A(f) \neq e_y^A(f) = f(y)$. Conversely, if A separates the points of X, given $x, y \in X$ with $x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$. Thus, $e_x^A(f) \neq e_y^A(f)$ and e^A is injective.
(iii) Assume that e^A is a homeomorphism. Since $\mathcal{D} = \{[f; U] \cap e^A(X) : f \in A$ and U is an open set of $\mathbb{E}^1\}$ is a subbase of $e^A(X) \subset C_p(A, \mathbb{E}^1)$ with $[f; U] = \{\varphi \in C_p(A, \mathbb{E}^1) : \varphi(f) \in U\}$, the family $\mathcal{B} = \{(e^A)^{-1}(W) : W \in \mathcal{D}\}$ is a subbase of the topology of X. Now, if $W \in \mathcal{D}$, it is easy to see that $(e^A)^{-1}(W) = f^{-1}(U)$ and hence $\mathcal{B} = \{f^{-1}(U) : f \in A$ and U is an open set of $\mathbb{E}^1\}$.

To see the converse, choose two different points $x, y \in X$ and assume that $\mathcal{E} = \{f^{-1}(U) : f \in A$ and U is an open set of $\mathbb{E}^1\}$ is a subbase of the topology of X. Then there exist $n \in \mathbb{N}$ and $f_1^{-1}(U_1), \ldots, f_n^{-1}(U_n) \in \mathcal{E}$ such that $x \in \bigcap\{f_i^{-1}(U_i) : i = 1, \ldots, n\}$ and $y \notin \bigcap\{f_i^{-1}(U_i) : i = 1, \ldots, n\}$. It is easy to see that there is i for which $y \notin f_i^{-1}(U_i)$. This implies that $\mathcal{E}$ separates the points of X. Moreover, it follows from (i) and (ii) that e^A is continuous and injective.

Next take a function $f \in e^A(X)$ and a basic open set (of X) W containing $(e^A)^{-1}(f)$. Then there exists $f_1^{-1}(U_1), \ldots, f_n^{-1}(U_n) \in \mathcal{E}$ such that $(e^A)^{-1}(f) \in \bigcap\{f_i^{-1}(U_i) : i = 1, \ldots, n\} \subset W$. Consider now the set $Z = \bigcap\{[f_i; U_i] \cap e^A(X) : i = 1, \ldots, n\}$. Z is an open set of $e^A(X)$ containing f and it is easy to see that $(e^A)^{-1}(Z) \subset W$. This proves the continuity of $(e^A)^{-1}$. Therefore e^A is a homeomorphism from X into $e^A(X)$.

(iv) Suppose that A separates points and closed sets of X. Take a point $x \in X$ and an open set (of X) U such that $x \in U$. There exists $f \in A$ such that $f(x) \notin \overline{f(X \setminus U)}$ and hence the set $\mathbb{E}^1 \setminus \overline{f(X \setminus U)}$ is open and contains $f(x)$. Therefore $x \in f^{-1}(\mathbb{E}^1 \setminus \overline{f(X \setminus U)})$ which implies that

$$U_A = \{f^{-1}(U) : f \in A \text{ and } U \text{ is an open set of } \mathbb{E}^1\}$$

is a subbase of the topology of X. It follows from (iii) that $e^A : X \to e^A(X)$ is a homeomorphism. $\square$

Proposition 3.17. *Let X be an arbitrary space. If $e\colon X \to C_p(C_p(X,\mathbb{E}^1),\mathbb{E}^1)$ is the evaluation function, then $e\colon X \to e(X) \subset C_p(C_p(X,\mathbb{E}^1),\mathbb{E}^1)$ is a homeomorphism.*

Proof. For any $x \in X$ and any closed set G of X with $x \notin G$, there exists a continuous function $f\colon X \to \mathbb{E}^1$ such that $f(x) = 1$ and $f(G) = 0$. Observe that $f(x) \notin \overline{f(G)}$, hence $C_p(X,\mathbb{E}^1)$ separates points and closed sets of X. Thus, Proposition 3.16 (iv) applies. $\square$

We now turn to some results related to cardinal functions. In our first result we use the well-known fact that $|X| = \chi(C_p(X))$.

Proposition 3.18. *Given a space X, the equality*

$$w(C_p(X,\mathbb{E}^1)) = |X|\mathfrak{c} = \chi(C_p(X,\mathbb{E}^1))\mathfrak{c}$$

holds.

Proof. It is easy to see that $w(C_p(X,\mathbb{E}^1)) \leq |X|\mathfrak{c}$. On the other hand,

$$|X| = \chi(C_p(X)) \leq \chi(C_p(X,\mathbb{E}^1)).$$

Then

$$w(C_p(X,\mathbb{E}^1)) \leq |X|\mathfrak{c} \leq \chi(C_p(X,\mathbb{E}^1))\mathfrak{c} \leq w(C_p(X,\mathbb{E}^1))\mathfrak{c}.$$

Since $w(C_p(X,\mathbb{E}^1)) \geq \mathfrak{c}$ for any space X, we have

$$w(C_p(X,\mathbb{E}^1)) = \chi(C_p(X,\mathbb{E}^1))\mathfrak{c} = |X|\mathfrak{c}.$$

$\square$

Corollary 3.19. *Let X be a space. If $|X| \geq \mathfrak{c}$, then $w(C_p(X,\mathbb{E}^1)) = |X|$.*

Proposition 3.20. *Given a space X, the equality $nw(C_p(X,\mathbb{E}^1)) = nw(X)\mathfrak{c}$ holds.*

Proof. It is easy to see that $nw(C_p(X,\mathbb{E}^1)) \geq \mathfrak{c}$ for any space X. Then, since $nw(X) = nw(C_p(X))$ for any space X (see, for example, [23, S.172]), we have

$$nw(X) = nw(C_p(X)) \leq nw(C_p(X,\mathbb{E}^1)) \leq nw(X)\mathfrak{c}$$

which implies

$$nw(C_p(X,\mathbb{E}^1)) = nw(C_p(X,\mathbb{E}^1))\mathfrak{c} = nw(X)\mathfrak{c}.$$

$\square$

Proposition 3.21. *Given a space X, the following assertions hold:*

(i) $d(X) \leq iw(C_p(X,\mathbb{E}^1)) \leq d(X)\mathfrak{c}$;
(ii) $d(C_p(X,\mathbb{E}^1)) \leq iw(X)\mathfrak{c}$;
(iii) $iw(X)\mathfrak{c} = d(C_p(X,\mathbb{E}^1))\mathfrak{c}$.

Proof. (i) $d(X) = iw(C_p(X)) \leq iw(C_p(X,\mathbb{E}^1))$. Take a dense subset $Y \subset X$ such that $|Y| = d(X)$. Since the function $\pi|_Y : C_p(X,\mathbb{E}^1) \to \pi|_Y(C_p(X,\mathbb{E}^1)) \subset C_p(Y,\mathbb{E}^1)$ is a condensation, it follows from Proposition 3.18 that

$$w(\pi|_Y(C_p(X,\mathbb{E}^1))) \leq w(C_p(Y,\mathbb{E}^1)) = |Y|\mathfrak{c} = d(X)\mathfrak{c}.$$

Hence $iw(C_p(X,\mathbb{E}^1)) \leq d(X)\mathfrak{c}$.

(ii) If $\varphi\colon X \to Y$ is a condensation, then Proposition 3.9 implies that $\varphi^* : C_p(Y,\mathbb{E}^1) \to \varphi^*(C_p(Y,\mathbb{E}^1)) \subset C_p(X,\mathbb{E}^1)$ is a homeomorphism and $\varphi^*(C_p(Y,\mathbb{E}^1))$ is a dense subset of $C_p(X,\mathbb{E}^1)$. Then

$$d(\varphi^*(C_p(Y,\mathbb{E}^1))) \leq nw(\varphi^*(C_p(Y,\mathbb{E}^1))) = nw(C_p(Y,\mathbb{E}^1)) = nw(Y)\mathfrak{c} \leq w(Y)\mathfrak{c}$$

so that $d(C_p(X,\mathbb{E}^1)) \leq d(\varphi^*(C_p(Y,\mathbb{E}^1))) \leq iw(X)\mathfrak{c}$.

(iii) It follows from Proposition 3.17 and (i) that $iw(X) \leq iw(C_p(C_p(X,\mathbb{E}^1),\mathbb{E}^1))$ and $iw(C_p(C_p(X,\mathbb{E}^1),\mathbb{E}^1)) \leq d(C_p(X,\mathbb{E}^1))\mathfrak{c}$. Moreover, (ii) tells us that

$$d(C_p(X,\mathbb{E}^1))\mathfrak{c} \leq iw(X)\mathfrak{c}$$

which implies that $iw(X)\mathfrak{c} = d(C_p(X,\mathbb{E}^1))\mathfrak{c}$. □

Proposition 3.22. *If X is a Lindelöf space and $X \subset C_p(Y,\mathbb{E}^1)$ for some space Y, then $w(K) \leq |Y|$.*

Proof. For any $y \in Y$ consider the set $K_y = \pi_y(K) \subset \mathbb{E}^1$ where π_y is the projection function onto the y-coordinate. Note that K_y is a Lindelöf metrizable space for any $y \in Y$. Thus, the weight of each K_y is countable and, consequently, so is the weight of $\prod_{y\in Y} K_y$ is $\leq |Y|$. The inclusion $K \subset \prod_{y\in Y} K_y$ implies that $w(K) \leq |Y|$. □

Proposition 3.23. *The space X^n is Lindelöf for any $n \in \mathbb{N}$ if and only if the tightness of $C_p(X,\mathbb{E}^1)$ is countable.*

Proof. Choose $f \in C_p(X,\mathbb{E}^1)$ and $A \subset C_p(X,\mathbb{E}^1)$ such that $f \in \overline{A}$. Select $g_z \in A \cap \langle f; x_1,\ldots,x_n; \frac{1}{n}\rangle$ for each $n \in \mathbb{N}$ and $z = (x_1,\ldots,x_n) \in X^n$. Then, for any $i = 1,\ldots,n$, we have $d_\infty(g_z(x_i), f(x_i)) < \frac{1}{n}$, and there exists an open set U_i^z of X such that $x_i \in U_i^z$ with $d_\infty(g_z(x), f(x)) < \frac{1}{n}$ for any $x \in U_i^z$.

Next, for any $z \in X^n$, take the open set $U_1^z \times \cdots \times U_n^z$ and consider a countable subcover $\mathcal{W}_n$ of the cover $\mathcal{U}_n = \{U_1^z \times \cdots \times U_n^z : z \in X^n\}$ of X^n. Define $B_n = \{g_z : U_1^z \times \cdots \times U_n^z \in \mathcal{W}_n\}$ and consider the countable subset B of A defined as $B = \bigcup\{B_n : n \in \mathbb{N}\}$. Now take an open set $\langle f; r_1,\ldots,r_n;\epsilon\rangle$ with $0 < \epsilon < \frac{1}{n}$ and choose $z \in X^n$ such that $z = (r_1,\ldots,r_n) \in U_1^z \times \cdots \times U_n^z$. Then $d_\infty(g_z(r_i), f(r_i)) < \frac{1}{n} < \epsilon$ for any $i = 1,\ldots,n$. Hence $g_z \in B \cap \langle f; r_1,\ldots,r_n;\epsilon\rangle$ which implies that $f \in \overline{B}$. Therefore the tightness of $C_p(X,\mathbb{E}^1)$ is countable.

To see the converse, it suffices to observe that $C_p(X) \subset C_p(X,\mathbb{E}^1)$ which implies that $t(Cp(X)) = \omega$. Therefore the space X^n is Lindelöf for any $n \in \mathbb{N}$ (see [23, S.149]). □

For a given space X, we say that $A \subset X$ is support-bounded if for any $f \in C_p(X,\mathbb{E}^1)$ there exists $r \in \mathbb{R}$ such that $f(A) \subset B_r(0)$. A Baire space is a topological space in which the intersection of every countable collection of dense open sets is an open set. Complete metric spaces and locally compact Hausdorff spaces are examples of Baire spaces according to the well-known Baire category theorem. Baire spaces have many important applications in several branches of functional analysis, topological algebra, etc. By the way of illustration, we can comment that Fréchet spaces are Baire spaces and the Baire category theorem can be applied to obtain the Banach-Steinhaus theorem and the open-mapping theorem (see, for example, [17]).

Proposition 3.24. *If $C_p(X, \mathbb{E}^1)$ is a Baire space, then any support-bounded subset of X is finite.*

Proof. If $A \subset X$ is an infinite support-bounded set, then for each $n \in \mathbb{N}$ the set $D_n = \{f \in C_p(X, \mathbb{E}^1) :$ there exists $x \in A$ with $f(x)^+(0) > n\}$ is open and dense in $C_p(X, \mathbb{E}^1)$. Take now an open set $\langle g; x_1, \dots, x_n; \epsilon \rangle$ and a point $x_0 \in A \setminus \{x_1, \dots, x_n\}$. It follows from Proposition 3.2 that there is a function $f \in C_p(X, \mathbb{E}^1)$ such that $f(x_0) = n + 1$ and $f(x_k) = g(x_k)$ for any $k = 1, \dots, n$. Since $f(x_0)^+(0) = n + 1$, we have $f \in D_n \cap \langle g; x_1, \dots, x_n; \epsilon \rangle$.

Given $f \in D_n$, find $z \in A$ such that $f(z)^+(0) > n$ and define $\delta = f(z)^+(0) - n > 0$. The set $\langle f; z; \delta \rangle$ is open and is contained in D_n. If $h \in \langle f; z; \delta \rangle$, then $|h(z)^+(0) - f(z)^+(0)| \le d_\infty(h(z), f(z)) < \delta$ which implies that $h(z)^+(0) > n$ and that $\langle f; z; \delta \rangle \subset D_n$. Therefore D_n is open and dense in $C_p(X, \mathbb{E}^1)$ for any $n \in \mathbb{N}$. We will finish the proof by showing that $\bigcap\{D_n : n \in \mathbb{N}\} = \emptyset$ which contradicts that $C_p(X, \mathbb{E}^1)$ is a Baire space. To proceed by contradiction, assume that there exists $f \in \bigcap\{D_n : n \in \mathbb{N}\}$. Then, for any $n \in \mathbb{N}$, there is $x_n \in A$ such that $f(x_n)^+(0) > n$ so that $f(x_n) \notin B_n(0)$ because $d_\infty(f(x_n), 0) \ge f(x_n)^+(0) > n$. Hence $f(A)$ is not support-bounded. This contradiction concludes the proof. □

4. Compactness and $C_p(X, \mathbb{E}^1)$-theory

In real analysis, compactness of subsets of $C_p(X)$ plays an important role in functional analysis, general topology and its applications. In this framework, one of the most celebrated results is Grothendieck's theorem which states that if X is a countably compact space and $A \subset C_p(X)$ is a countably compact set in $C_p(X)$ (i.e., for any infinite set $B \subset A$, the space $C_p(X)$ contains a limit point of B), then the closure of A in $C_p(X)$ is compact. For $C_p(X, \mathbb{E}^1)$ we have the following version of Grothendieck's theorem. It may be worth reminding the reader that a space X is said to be pseudocompact if every real-valued continuous function on X is bounded.

Proposition 4.1. *If X is a countably compact space and Y is a closed pseudocompact subspace of $C_p(X, \mathbb{E}^1)$, then Y is compact.*

Proof. For any $x \in X$ we know that the evaluation function $e^x : C_p(X, \mathbb{E}^1) \to \mathbb{E}^1$ defined by $e^x(f) = f(x)$ for all $f \in C_p(X, \mathbb{E}^1)$ is continuous. Consequently, for any $x \in X$, the set $K_x = e^x(Y) \subset \mathbb{E}^1$ is compact because it is pseudocompact and metrizable. Thus, Y is a subset of the compact space $\prod_{x \in X} K_x \subset (\mathbb{E}^1)^X$. Suppose, to derive a contradiction, that Y is not compact. Since Y is closed in $C_p(X, \mathbb{E}^1)$, there exists a discontinuous function f with $f \in \overline{Y} \setminus C_p(X, \mathbb{E}^1) \subset (\mathbb{E}^1)^X$. Thus, there is $a \in X$ and $A \subset X$ such that $a \in \overline{A}$ and $f(a) \notin \overline{f(A)}$. Hence we can find open sets $U, V \subset \mathbb{E}^1$ such that $f(a) \in U$, $f(A) \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$.

By induction on n we now define sequences $\{f_n : n \in \mathbb{N}\} \subset Y$, $\{U_n : n \in \mathbb{N}\}$, where U_n is open in X and $a \in U_n$ for all $n \in \mathbb{N}$, and $\{a_n : n \in \mathbb{N}\} \subset A$ with the properties

(i) $\overline{U_{n+1}} \subset U_n$ and $a_n \in U_n$ for any $n \in \mathbb{N}$,

(ii) $f_n(U_n) \subset U$ for any $n \in \mathbb{N}$,

(iii) $f_{n+1}(a_i) \in V$ for any $n \in \mathbb{N}$ and $i \leq n$.

To do so, notice that there exists $f_0 \in Y$ such that $f_0(a) \in U$ because $f \in \overline{Y}$. Since the function f_0 is continuous, there is an open set (of X) U_0 such that $a \in U_0$ and $f(U_0) \subset U$. The point a belongs to the closure of A and, consequently, there exists $a_0 \in A \cap U_0$. It is straightforward to see that the triple (f_0, U_0, a_0) satisfies the properties (i), (ii) and (iii).

Suppose that f_i, U_i, a_i are defined satisfying properties (i)-(iii) for any $i \leq n$. It is evident that $f(a_i) \in V$ for any $i \leq n$. The set $Y \cap [a, a_0, \ldots, a_n; U, V, \ldots, V]$ is nonempty because $f \in [a, a_0, \ldots, a_n; U, V, \ldots, V]$ and $f \in \overline{Y}$. Hence there exists $f_{n+1} \in Y \subset C_p(X, \mathbb{E}^1)$ for which $f_{n+1}(a) \in U$ and $f_{n+1}(a_i) \in V$ for any $i \leq n$. Then there exists an open set U_{n+1} in X such that $a \in U_{n+1}$, $\overline{U_{n+1}} \subset U_n$ and $f_{n+1}(U_{n+1}) \subset U$. Take any point $a_{n+1} \in A \cap U_{n+1}$. Then $\{a_i, f_i, U_i : i \leq n+1\}$ satisfies (i)-(iii). This completes the induction step.

Take now a cluster point b of the sequence $S = \{a_n : n \in \mathbb{N}\}$. It is easy to see that $b \in \bigcap\{U_n : n \in \mathbb{N}\} = \bigcap\{\overline{U_n} : n \in \mathbb{N}\}$ because, for a given $n \in \mathbb{N}$, we have $x_i \in U_n$ for any $i > n$. Notice that $f_n(b) \in f_n(U_n) \subset U$. Define the countable set $D = \{y\} \cup \{a_n : n \in \mathbb{N}\}$ ant take the restriction function $\pi|_D : C_p(X, \mathbb{E}^1) \to C_p(D, \mathbb{E}^1)$. The function $\pi|_D$ sends Y to a pseudocompact subspace of the metrizable space $C_p(D, \mathbb{E}^1)$. Since the closure in $C_p(D, \mathbb{E}^1)$ of $\{g_n = \pi|_D(f_n) : n \in \mathbb{N}\}$ is a metrizable compact space, we can consider that g_n converges to some $g \in \pi|_D(Y)$. Moreover, $\langle g; b; \epsilon\rangle \cap \{g_n : n \in \mathbb{N}\} \neq \emptyset$ for any $\epsilon > 0$ and $g(b) \in \overline{\{g_n(b) : n \in \mathbb{N}\}} = \overline{\{f_n(b) : n \in \mathbb{N}\}}$. For any $n \in \mathbb{N}$ we have $f_n(b) \in f_n(\bigcap\{U_k : k \in \mathbb{N}\}) \subset f_n(U_n) \subset U$ which implies that $g(b) \in \overline{U}$. If follows from continuity of g that $g(b) \in \overline{\{g(a_n) : n \in \mathbb{N}\}}$. If $n \in \mathbb{N}$, then $f_k(a_n) \in V$ for any $k > n$. Hence $g(a_n) \in \overline{V}$ for any $n \in \mathbb{N}$ which implies that $g(b) \in \overline{V}$. Thus, $g(b) \in \overline{U} \cap \overline{V}$, which leads us to a contradiction. Therefore Y is a compact space. □

As a corollary we obtain a version of Grothendiek's theorem in the realm of fuzzy analysis.

Corollary 4.2. *Let X be a countably compact space. If A is a countably compact set in $C_p(X)$, then the closure of $C_p(X)$ is compact.*

Recall that a space X is said to be σ-compact if it is the union of countably many compact sets.

Proposition 4.3. *If X is a σ-compact space, then there exists a compact space K such that $C_p(X, \mathbb{E}^1)$ is homeomorphic to a subspace of $C_p(K, \mathbb{E}^1)$.*

Proof. Suppose that $X = \bigcup\{X_n : n \in \mathbb{N}\}$ where each X_n is compact and put $Y = C_p(X, \mathbb{E}^1)$. The evaluation function $e^Y : X \to C_p(Y, \mathbb{E}^1)$ sends homeomorphically X to $Z = e(X) \subset C_p(Y, \mathbb{E}^1)$. Observe that Z is σ-compact and that the family $U_A = \{\varphi^{-1}(U) : \varphi \in Z$ and U is an open set of $\mathbb{E}^1\}$ is a subbase of Y because for any $x \in X$ and any open set $U \subset \mathbb{E}^1$ we have $\varphi_x^{-1}(U) = e_x^{-1}(U) = \{h \in Y =$

$C_p(X, \mathbb{E}^1) : h(x) \in U\} = [x; U]$. Then it follows from Proposition 3.4 that there exists a homeomorphism $\varphi : C_p(Y, \mathbb{E}^1) \to C_p(Y, B_1(0))$. Therefore, if $Z = \bigcup\{Z_n : n \in \mathbb{N}\}$ where each Z_n is compact, then for any $n \in \mathbb{N}$ the space $A_n = \varphi(Z_n) \subset C_p(Y, B_1(0))$ is also compact. Consider now, for each $n \in \mathbb{N}$, the compact space $K_n = \{\frac{f}{n} : f \in A_n\}$ and define $K = \bigcup\{K_n : n \in \mathbb{N}\} \cup \{h\}$ where $h : Y \to B_1(0)$ is the constant function $h \equiv 0$. Let $\mathcal{U}$ be an open cover of A by open sets of $C_p(Y, B_1(0))$. Then there is $W \in \mathcal{U}$ such that $h \in W$. We can assume that $W = \langle h; y_1, \dots, y_n; \epsilon\rangle$ for some $\{y_1, \dots, y_n\} \subset Y$ and $\epsilon > 0$. If we choose $m \in \mathbb{N}$ with $\frac{1}{m} < \epsilon$, then it is easy to see that $K_n \subset W$ for any $n \geq m$ (notice that $K_q \subset B_{\frac{1}{q}}(0)$ for any $q \in \mathbb{N}$). Hence there exists a finite subcover $\mathcal{U}$ covering K. We have just proved that K is compact. It is straightforward to show that K separates points and closed sets of Y and, consequently, the evaluation function $e^K : Y \to C_p(K, \mathbb{E}^1)$ sends Y homeomorphically to a subspace of $C_p(K, \mathbb{E}^1)$. This completes the proof. □

A space X is said to be scattered if every nonempty subset $A \subseteq X$ has an isolated point relative to A. Recall that a space X is called Fréchet-Urysohn if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence from A converging to x. For compact spaces, we have the following relationship between both properties in C_p–theory.

Proposition 4.4. *If X is a compact space, then $C_p(X, \mathbb{E}^1)$ is Fréchet-Urysohn if and only if X is scattered.*

Proof. Suppose that X is scattered. Choose $f \in C_p(X, \mathbb{E}^1)$ and $A \subset C_p(X, \mathbb{E}^1)$ such that $f \in \overline{A}$ where the closure is taking in $C_p(X, \mathbb{E}^1)$. Proposition 3.23 tells us that there exists a countable set $B \subset A$ such that $f \in \overline{B}$. Let e^B be the evaluation function $e^B : X \to C_p(B, \mathbb{E}^1)$ and $Y = e^B(X) \subset C_p(B, \mathbb{E}^1)$. The product space $(\mathbb{E}^1)^B$ is metrizable; hence $C_p(B, \mathbb{E}^1)$ is also metrizable, which implies that Y is a scattered compact with a countable base. Thus, Y is countable. Take now the dual function $e^{B^*} : C_p(Y, \mathbb{E}^1) \to C_p(X, \mathbb{E}^1)$. By Proposition 3.9 we have that $e^{B^*} : C_p(Y, \mathbb{E}^1) \to e^{B^*}(C_p(Y, \mathbb{E}^1))$ is a homeomorphism. Since e^B is a closed function, Proposition 3.9 tells us that $e^{B^*}(C_p(Y, \mathbb{E}^1))$ is a closed set of $C_p(X, \mathbb{E}^1)$. It is easy to see that $B \subset e^{B^*}(C_p(Y, \mathbb{E}^1))$ and, consequently, $\overline{B} \subset e^{B^*}(C_p(Y, \mathbb{E}^1))$ which implies that $\overline{B}$ is a metrizable compact space. Thus, there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset B \subset A$ such that $\lim_{n\to\infty} f_n = f$. Therefore $C_p(X, \mathbb{E}^1)$ is Fréchet-Urysohn.

To see the converse, notice that if $C_p(X, \mathbb{E}^1)$ is Fréchet-Urysohn, then $C_p(X) \subset C_p(X, \mathbb{E}^1)$ is also Fréchet-Urysohn. Therefore X is scattered. □

A space X is called ω-monolithic if, for every $Y \subset X$ with $|Y| \leq \aleph_0$, we have $nm(\overline{Y}) \leq \aleph_0$. In the spirit of the previous result, we can prove the following

Proposition 4.5. *If K and X are compact spaces with $X \subset C_p(K, \mathbb{E}^1)$, then X is Fréchet-Urysohn and ω-monolithic.*

Proof. Take a countable set $A \subset X \subset C_p(K, \mathbb{E}^1)$ and consider the evaluation function $e^A : K \to C_p(A, \mathbb{E}^1)$. Notice that $C_p(A, \mathbb{E}^1)$ is a subspace of a countable product of metrizable spaces so that it is metrizable as well.

Then compactness of K implies that the space $Y = e^A(K) \subset C_p(A, \mathbb{E}^1)$ is compact and metrizable. Observe that $(e^A)^* : C_p(Y, \mathbb{E}^1) \to (e^A)^*(C_p(Y, \mathbb{E}^1)) \subset C_p(K, \mathbb{E}^1)$ is a homeomorphism. Thus, $(e^A)^*(C_p(Y, \mathbb{E}^1))$ is a closed subspace of $C_p(K, \mathbb{E}^1)$ with $A \subset (e^A)^*(C_p(Y, \mathbb{E}^1))$. Therefore $\overline{A}$ is homeomorphic to a compact subspace of $C_p(Y, \mathbb{E}^1)$. Now let Z be a dense countable subset of Y. Since the function $\pi|_Z : C_p(Y, \mathbb{E}^1) \to C_p(Z, \mathbb{E}^1)$ is injective and the space $C_p(Z, \mathbb{E}^1)$ is metrizable, we have that A condenses onto the metrizable compact space $\pi|_Z(A)$. Therefore A is metrizable and compact. Thus, $nw(\overline{A}) = w(\overline{A}) = \aleph_0$ which implies that X is ω-monolithic.

To see that X is Fréchet-Urysohn, choose $x \in X$ and let $B \subset X$ such that $x \in \overline{B}$. Proposition 3.23 tells us that the tightness of $C_p(K, \mathbb{E}^1)$ is countable. Hence the tightness of X is countable. Take now a sequence $S = \{x_n : n \in \mathbb{N}\} \subset B$ such that $x \in \overline{S}$. Since $\overline{S}$ is ω-monolithic, it is metrizable and compact. Therefore there exists a sequence $\{y_n : n \in \mathbb{N}\} \subset S \subset B$ which converges to x. This completes the proof. □

5. **Conclusion**

We establish the basic properties of the space $C_p(X, \mathbb{E}^1)$ of all continuous fuzzy-valued functions on a space X endowed with the pointwise topology. The restriction, dual and evaluation function (useful tools in the theory) are also studied. We introduce several relationships between the notion of compactness and $C_p(X, \mathbb{E}^1)$. In our research we state some properties of the space of fuzzy numbers equipped with the topology induced by the metric d_∞.

This paper was conceived as an introduction to C_p-theory in fuzzy analysis. As far as the authors know it is the first attempt to set the foundations of this theory. As in the case of the pointwise topology in the realm of real analysis, it can be used for further development of the theory of function spaces in fuzzy analysis and its applications.

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