HÖLDER REGULARITY FOR THE MOORE-GIBSON-THOMPSON EQUATION WITH INFINITE DELAY

LUCIANO ABADIAS, CARLOS LIZAMA, AND MARINA MURILLO-ARCILA

ABSTRACT. We characterize the well-posedness of a third order in time equation with infinite delay in Hölder spaces, solely in terms of spectral properties concerning the data of the problem. Our analysis includes the case of the linearized Kuznetzov and Westerwelt equations. We show in case of the Laplacian operator the new and surprising fact that for the standard memory kernel $g(t) = \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-at}$ the third order problem is ill-posed whenever $0 < \nu \leq 1$ and a is inversely proportional to the damping term of the given model.

1. INTRODUCTION

Our concern in this article is the study of well-posedness for the following abstract integro-differential equation of third order in time

$$(1.1) \qquad \tau u'''(t) + \kappa u''(t) - c^2 A u(t) - b A u'(t) + \int_{-\infty}^{t} g(t-s) A w(s) ds = f(t, u, u_t, u_{tt}),$$

where κ, b, c, τ are nonnegative real numbers, f is a vector-valued function, A is a closed linear operator with domain $D(A)$ defined on a Banach space X, where

$$(1.2) \qquad w(s) = u(s), \qquad\qquad\qquad \textbf{(Type I)}$$

$$(1.3) \qquad w(s) = u_t(s), \qquad\qquad\qquad \textbf{(Type II)}$$

$$(1.4) \qquad w(s) = \mu u(s) + u_t(s), \mu \neq 0 \qquad \textbf{(Type III)}$$

and the memory kernel g fulfills suitable assumptions. The well-posedness of higher order abstract Cauchy problems is a topic that has been studied for a long time, see e.g. [8, 6, 23, 32, 37, 40] and references therein. On the other hand, the equation (1.1) of type I, II and III were recently introduced by I. Lasiecka and X. Wang in an interesting series of articles [33, 34]. From then on, it began to attract the attention of an increasing number of researchers [17, 18].

In case $g \equiv 0$, the linear part of the third order equation (1.1) is referred to as the Moore-Gibson-Thomson equation (MGT-equation) [29]. This model represent powerful applications in different fields of practical interest such as high intensity ultrasound and vibrations of flexible materials, see e.g. [26, 28, 35]. In such cases, the operator A is usually the Laplacian and

$f(t, u, u_t, u_{tt}) = K((u_t)^2)_t$ for a suitable constant $K > 0$. In high intensity ultrasound, u denotes the potential velocity of the acoustic phenomenon described on some bounded $\mathbb{R}^3$-domains [29].

An operator-theoretical semigroup method to study the linear case $g \equiv 0$ was employed by Marchand-Devitt and Triggiani [35] and Kaltenbacher-Lasiecka and Marchand [28]. These authors rewrite the MGT-equation as a first order abstract Cauchy problem $u' = \mathcal{A}u + F$ in several state spaces. Then, well-posedness and exponential decay were obtained. It was proved (see [28, Theorem 1.1]) that if A is unbounded then, for $b = 0$ the operator matrix $\mathcal{A}$ cannot be the generator of a C_0-semigroup on the state space $\mathcal{H} := D(A^{1/2}) \times D(A^{1/2}) \times H$ and hence the problem is ill-posed, in the sense that the matrix operator $\mathcal{A}$ does not generate a strongly continuous semigroup on $\mathcal{H}$. On the other hand, a direct approach to well-posedness for the abstract MGT-equation without reduction of order was undertaken by Fernández, Lizama and Poblete [24]. However, this approach has been controversial [35]. More recently, in [25, Section 2] and [18] the MGT-equation has been studied from a different perspective, by means of the theory of integral equations. As a consequence, well-posedness and qualitative properties can be studied from a representation of the solution of the linear part of the MGT-equation by means of certain variation of constant formula. In the context of a Banach space X, this study seems to be possible using the theory of resolvent families deeply studied by Prüss [38]. It is worthwhile to mention that qualitative properties such as the existence of global attractors for the MGT-equation have also been studied, see for instance [11]. Results concerning the regularity of strong and mild solutions as well as compactness of trajectories have been provided in [25]. The study of the existence of asymptotically almost periodic and almost automorphic solutions can be found in [19] and [2], respectively. Existence of mild solutions with non local initial conditions by methods of Hausdorff measure of non compactness, has been established in [2] and concerning ill-posedness, the presence of chaos for the linear MGT-equation has been discovered in [14]. A variation of the MGT-equation in order to include non-degenerate equations was recently introduced by Cai and Bu [9].

In our case of study $g \neq 0$ and A is typically the Laplacian operator. Depending on the properties of the environment surrounding sound propagation, the memory kernel can exhibit several structures by selecting different values of g which yield to different possible configurations [17]. In the abstract case, $-A$ is a non-negative and self-adjoint operator defined on a Hilbert space H. There is little literature concerning the precise abstract model (1.1), and much work is left to be done in the qualitative analysis of (1.1) with memory term, specially concerning its treatment on general Banach spaces. In particular, the well-posedness of (1.1) in several classes of spaces of functions remains largely open. As it is well known, these results are necessary for the treatment of nonlinear problems [20].

In particular, the study of the well-posedness of the Moore-Gibson-Thomson equation (1.1) in Hölder spaces remains open.

The main objective of this article is to provide a complete answer to this open problem. We have succeeded in solving it by giving a complete *characterization* of C^α well-posedness, in the sense of *temporal* maximal regularity for the full model (1.1) in the vector-valued space of Hölder continuous functions $C^\alpha(\mathbb{R}, X)$, $0 < \alpha < 1$ and for all types I, II and III before mentioned. We note that C^α well-posedness of (1.1) without memory term has been characterized in [15] whereas L^p well-posedness has been analyzed in [24]. Hölder regularity has been also treated for other classes of abstract evolution equations [16].

More precisely, we prove in Theorem 3.12 that if $g \in L^1_{loc}(\mathbb{R}_+)$ satisfies certain conditions of regularity then, the following assertions are equivalent

(i) Equation (1.1) is C^α-well-posed;
(ii) $\{\frac{\tau(i\eta)^3 + \kappa(i\eta)^2}{c^2 + ib\eta - \tilde{g}(\eta)}\}_{\eta \in \mathbb{R}} \subseteq \rho(A)$ and

$$\sup_{\eta \in \mathbb{R}} \left\| \frac{(i\eta)^3}{c^2 + ib\eta - \tilde{g}(\eta)} \left(\frac{\tau(i\eta)^3 + \kappa(i\eta)^2}{c^2 + ib\eta - \tilde{g}(\eta)} - A \right)^{-1} \right\| < \infty.$$

Here $\rho(A)$ denotes the resolvent set of A and $\tilde{g}$ denotes the Fourier transform of g whenever it exists. An analogous result holds in the particular case $\tau = 0$. See Theorem 3.15 below. The main difference is that the term $(i\eta)^3$ must be replaced by $(i\eta)^2$ in front of the resolvent operator $(\cdot - A)^{-1}$, taking into account the higher order derivative term. This last model contains, as particular examples, the linear Westervelt or Kuznetsov equation, the viscoelastic membrane equation and the viscoelastic plate equation with memory in case $A = \Delta$, the Laplacian operator, see [30] and [36]. It is worthwhile to observe that, as an immediate consequence of our characterization, the following estimate

$$||u'''||_{C^\alpha(\mathbb{R},X)} + ||u''||_{C^\alpha(\mathbb{R},X)} + ||Au'||_{C^\alpha(\mathbb{R},X)} + ||Au||_{C^\alpha(\mathbb{R},X)} + ||g * Au||_{C^\alpha(\mathbb{R},X)} \leq C||f||_{C^\alpha(\mathbb{R},X)}$$

holds. Here C is a constant independent of f.

In order to achieve our main results, we use a method based on operator-valued Fourier multipliers. The main advantage of this method is that it does not require the representation of the solution by means of families of bounded linear operators as done, for instance, using the theory of C_0-semigroups of linear operators in [29] or other classes of resolvent families [25]. The key tool is a very general Fourier multiplier theorem due to Arendt, Batty and Bu [4, Theorem 5.3]. This method has been successfully used for the treatment of well-posedness of abstract evolution equations by S. Bu and collaborators [7, 9, 10] among other authors. See also [12, 21] and references therein.

Several examples are presented in the last section of this work. For instance, we consider the problem
(1.5)

$$\tau u_{ttt}(t,x) + \kappa u_{tt}(t,x) - c^2 \Delta u(t,x) - b\Delta u_t(t,x) + \int_{-\infty}^{t} \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} e^{-a(t-s)} \Delta u(s,x) ds = f(t,x)$$

with Dirichlet boundary conditions.

Suppose first that $\tau \neq 0$ in equation (1.5). Under the assumptions $0 < \nu \leq 1$, $c^2 > \frac{1}{a^\nu}$ and $-\tau c^2 + \kappa b \neq 0$ it is proved using our abstract characterization that the equation (1.5) is C^α-ill posed on the space $C^\alpha(\mathbb{R}, C_0(\Omega))$ for any $0 < \alpha < 1$. This is a new and surprising fact that has not been observed before.

On the other hand, if we suppose $\tau = 0$, then we prove that the equation (1.5) is C^α-well posed on the space $C^\alpha(\mathbb{R}, C_0(\Omega))$. This way the presence of the term of third order in time becomes crucial for C^α-well-posedness in case that $A = \Delta$, the Laplacian operator.

In contrast, an example of closed operator A is provided in the context of a Hilbert space showing that for $\tau \neq 0$, $g(t) = e^{-at}$ with $a > 1$ and $\kappa = b = c^2 = 1$ the corresponding MGTM equation is always C^α-well-posed.

It is worthwhile to mention that our characterization is very flexible with respect to the class of closed linear operators A that are admissible. Indeed, we provide an example that shows that A does not need to be the generator of any C_0-semigroup in order to obtain C^α well-posedness of (1.1).

We finish this paper with a criterion (see Theorem 5.4 below) that ensures the existence of a unique solution with C^α-temporal regularity for a nonlinear version of (1.1) given by

$$(1.6) \qquad \tau u'''(t) + \kappa u''(t) - c^2\phi_1(u)(t) - b\phi_2(u)(t) + \phi_3(u)(t) = f(t), \quad t \in \mathbb{R},$$

where $\phi_1, \phi_3 : C^\alpha(\mathbb{R}, D(A)) \to C^\alpha(\mathbb{R}, X)$ and $\phi_2 : C^{\alpha+1}(\mathbb{R}, D(A)) \to C^\alpha(\mathbb{R}, X)$.

2. Preliminaries

Let X, Y be Banach spaces and let $0 < \alpha < 1$ be fixed. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the space

$$\dot{C}^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : f(0) = 0, ||f||_\alpha < \infty\},$$

where

$$||f||_\alpha = \sup_{t \neq s} \frac{||f(t) - f(s)||}{|t - s|^\alpha}$$

and

$$C^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : ||f||_{C^\alpha} < \infty\}$$

with the norm

$$||f||_{C^\alpha} = ||f||_\alpha + ||f(0)||.$$

Let $k \in \mathbb{N}$, and $C^{\alpha+k}(\mathbb{R}, X)$ be the Banach space of all $u \in C^k(\mathbb{R}, X)$ such that $u^{(k)} \in C^\alpha(\mathbb{R}, X)$, equipped with the norm

$$||u||_{C^{\alpha+k}} = ||u^{(k)}||_{C^\alpha} + ||u(0)||.$$

Let $\Omega \subseteq \mathbb{R}$ be an open set. We denote by $C_c^\infty(\Omega)$ the space of all C^∞-functions in Ω having compact support in Ω. By $\mathcal{F}f$ or $\tilde{f}$ we denote the Fourier transform of a function $f \in L^1(\mathbb{R}, X)$, given by

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_\mathbb{R} e^{-ist} f(t)dt, \quad s \in \mathbb{R}.$$

The following notion of multiplier is originally due to Arendt, Batty and Bu.

Definition 2.1. [3] *Let $M : \mathbb{R}\backslash\{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha$-multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that*

$$(2.1) \qquad \int_\mathbb{R} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_\mathbb{R} (\mathcal{F}(\phi \cdot M))(s)f(s)ds$$

for all $f \in \dot{C}^\alpha(\mathbb{R}, X)$ and all $\phi \in C_c^\infty(\mathbb{R}\backslash\{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_\mathbb{R} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X, Y)$. Note that L is a bounded linear operator (cf. [3, Definition 5.2]). From Definition 2.1 and the relation

$$\int_\mathbb{R} (\mathcal{F}(\phi M))(s)ds = 2\pi(\phi M)(0) = 0,$$

we deduce that for $f \in C^\alpha(\mathbb{R}, X)$ we have $Lf \in C^\alpha(\mathbb{R}, X)$. Moreover, if $f \in C^\alpha(\mathbb{R}, X)$ is bounded then Lf is bounded as well (see [3, Remark 6.3]).

Remark 2.2. [31] The test function space $C_c^\infty(\Omega)$ in Definition 2.1 can be replaced by the space $C_c^1(\Omega)$ of all C^1-functions in Ω having compact support in Ω. It follows from the fact that if $\varphi \in C_c^1(\Omega)$ then $\rho_n * \varphi \in C_c^\infty(\Omega)$ where ρ_n denotes a sequence of mollifying functions, and $\rho_n * f \to f$ in $L^1(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$ (see e.g. [5, Théorème IV.22]).

The following operator-valued Fourier multiplier theorem, stated in [3, Theorem 5.3], which provides sufficient conditions to ensure when $M \in C^2(\mathbb{R} \backslash \{0\}, \mathcal{B}(X, Y))$ is a $\dot{C}^\alpha$-multiplier, plays a key role in the proof of the main results of the paper.

Theorem 2.3. *(Arendt-Batty-Bu) Let $M \in C^2(\mathbb{R} \backslash \{0\}, \mathcal{B}(X, Y))$ be such that*

$$(2.2) \qquad \sup_{s \neq 0} ||M(s)|| + \sup_{s \neq 0} ||sM'(s)|| + \sup_{s \neq 0} ||s^2 M''(s)|| < \infty.$$

Then M is a $\dot{C}^\alpha$-multiplier.

Remark 2.4. If X is B-convex, in particular if X is a UMD space, Theorem 2.3 also holds if (2.2) is replaced by the following weaker condition

$$(2.3) \qquad \sup_{s \neq 0} ||M(s)|| + \sup_{s \neq 0} ||sM'(s)|| < \infty,$$

where $M \in C^1(\mathbb{R} \backslash \{0\}, \mathcal{B}(X, Y))$ (cf. [3, Remark 5.5]).

Let $0 < \alpha < 1$. We denote by $L^1(\mathbb{R}_+, t^\alpha dt) \cap L^1_{loc}(\mathbb{R}_+)$ the set of all $a \in L^1_{loc}(\mathbb{R}_+)$ such that

$$(2.4) \qquad \int_0^\infty |a(t)| t^\alpha dt < \infty.$$

Observe that a function a satisfying the above conditions belongs to $L^1(\mathbb{R}_+)$. Now, given $v \in C^\alpha(\mathbb{R}, X)$ $(0 < \alpha < 1)$ and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$, we write

$$(2.5) \qquad (a * v)(t) = \int_0^\infty a(s) v(t-s) ds = \int_{-\infty}^t a(t-s) v(s) ds.$$

It follows from (2.4) that the above integral is well defined. Moreover,

$$(2.6) \qquad a * v \in C^\alpha(\mathbb{R}, X) \quad \text{and} \quad ||a * v||_\alpha \leq ||a||_1 ||v||_\alpha.$$

The Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ will be denoted by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt, \quad \mathfrak{Re}\, \lambda > \omega,$$

whenever the integral is absolutely convergent for $\mathfrak{Re}\, \lambda > \omega$. The relation between the Laplace transform of $f \in L^1(\mathbb{R}, X)$ satisfying $f(t) = 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

Let $u \in L^1_{loc}(\mathbb{R}, X)$ be a vector-valued function of subexponential growth, that is,

$$\int_{-\infty}^\infty e^{-\varepsilon|t|} ||u(t)|| dt < \infty, \quad \text{for each } \varepsilon > 0.$$

We denote by $\hat{u}$ the Carleman transform of u

$$\hat{u}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} u(t) dt & \mathfrak{Re}\, \lambda > 0, \\[2em] \displaystyle-\int_{-\infty}^0 e^{-\lambda t} u(t) dt & \mathfrak{Re}\, \lambda < 0. \end{cases}$$

By the above definition and the dominated convergence theorem we have

$$\lim_{\sigma \to 0^+} (\hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho)) = \tilde{u}(\rho),$$

see [38, p.19]. For more details about the Carleman transform see [4, Chapter 4] or [38]. For $\sigma > 0$, we also consider the operator L_σ defined by

$$(2.7) \qquad (L_\sigma u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho), \qquad \rho \in \mathbb{R},$$

see [31]. In [31, Proposition A.2 (ii)-(iii)] it is proved that

$$(2.8) \qquad (L_\sigma(v'))(\rho) = (\sigma + i\rho)(L_\sigma(v))(\rho) + 2\sigma\hat{v}(-\sigma + i\rho), \quad v \in C^{1+\alpha}(\mathbb{R}, X),$$

and

$$(2.9) \qquad (L_\sigma(a*v))(\rho) = \hat{a}(\sigma + i\rho)(L_\sigma v)(\rho) + G_a^v(\sigma, \rho), \quad v \in C^\alpha(\mathbb{R}, X),\, a \in L^1(\mathbb{R}_+, t^\alpha dt),$$

respectively, where

$$G_a^v(\sigma, \rho) \;\; := \;\; \int_{-\infty}^0 \left(\int_{-s}^\infty e^{-(\sigma+i\rho)(s+\tau)} a(\tau) d\tau + \int_0^{-s} e^{(\sigma-i\rho)(s+\tau)} a(\tau) d\tau \right.$$
$$\left. - e^{(\sigma-i\rho)s} \int_0^\infty e^{-(\sigma+i\rho)\tau} a(\tau) d\tau \right) v(s) \, ds.$$

The following result on properties of L_σ complement those given in [31, Appendix].

Proposition 2.5. *Let $\sigma > 0$ be given.*

(i) *If $v \in C^{\alpha+2}(\mathbb{R}, X)$ then*

$$(L_\sigma(v''))(\rho) = (\sigma + i\rho)^2(L_\sigma(v))(\rho) + 4\sigma\rho i\hat{v}(-\sigma + i\rho) - 2\sigma u(0), \quad \rho \in \mathbb{R}.$$

(ii) *If $v \in C^{\alpha+3}(\mathbb{R}, X)$ then*

$$(L_\sigma(v'''))(\rho) = (\sigma + i\rho)^3(L_\sigma(v))(\rho) - 2\sigma(\sigma^2 - 3\rho^2)\hat{v}(-\sigma + i\rho) - 4\sigma\rho iu(0) - 2\sigma u'(0), \quad \rho \in \mathbb{R}.$$

Proof. It is a consequence of (2.8) and the following well-known property: if $v' \in L^1_{loc}(\mathbb{R}, X)$ is of subexponential growth, then $\hat{v'}(\lambda) = \lambda\hat{v}(\lambda) - v(0)$, for $\mathfrak{Re}\,\lambda \neq 0$. $\qquad \square$

3. A CHARACTERIZATION OF C^α-WELL-POSEDNESS: TYPE I

In this section, we first state necessary conditions for the well-posedness of the Moore-Gibson-Thompson equation with memory (1.1) (MGTM-equation) of type I in vector-valued Hölder continuous spaces. Let $f \in C^\alpha(\mathbb{R}, X)$ and A be a closed linear operator in X. In the following, we consider the non-homogeneous MGTM-equation

$$(3.1) \qquad \tau u'''(t) + \kappa u''(t) - c^2 Au(t) - bAu'(t) + \int_{-\infty}^t g(t-s)Au(s) \, ds = f(t), \quad t \in \mathbb{R},$$

where $g \in L^1(\mathbb{R}_+, t^\alpha \, dt) \cap L^1_{loc}(\mathbb{R}_+)$ and τ, κ, c^2, b are positive real numbers. Recall that the domain of A, $D(A)$, is a Banach space when endowed with the graph norm.

Definition 3.1. *We say that the equation (3.1) is C^α-well-posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+3}(\mathbb{R}, X)$ satisfying (3.1).*

In [33, Theorem 1.4] the authors proved that

$$(3.2) \qquad c^2 > G(+\infty) = \int_0^\infty |g(s)|\, ds,$$

is a necessary condition to get exponentially stability of the standard energy for the weak solution of the homogeneous MGTM-equation of type I in a Hilbert space. This assumption implies that $c^2 + b\eta i - \tilde{g}(\eta) \neq 0$ for all $\eta \in \mathbb{R}$ and hence, by the Riemann-Lebesgue theorem, we get that the function

$$\gamma(\eta) := \frac{1}{c^2 + b\eta i - \tilde{g}(\eta)}$$

belongs to $C_0(\mathbb{R})$. The functions γ and β given by

$$\beta(\eta) := \frac{\tau(i\eta)^3 + \kappa(i\eta)^2}{c^2 + b\eta i - \tilde{g}(\eta)}, \quad \eta \in \mathbb{R},$$

will play a key role in the characterization of the well-posedness of equation (3.1).

Our first result in this section is the following theorem. Its proof follows some ideas provided in [31] and [3].

Theorem 3.2. *Let $g \in L^1(\mathbb{R}_+, t^\alpha\, dt) \cap L^1_{loc}(\mathbb{R}_+)$ be such that $c^2 + b\eta i - \tilde{g}(\eta) \neq 0$ for all $\eta \in \mathbb{R}$. If (3.1) is C^α-well-posed then the following statements hold*

(i) $\beta(\eta) \in \rho(A)$ *for all $\eta \in \mathbb{R}$,*

(ii) $\displaystyle\sup_{\eta \in \mathbb{R}} \|(i\eta)^3 \gamma(\eta)(\beta(\eta) - A)^{-1}\| < \infty.$

Proof. In order to prove (i), we first assume that there exist $\eta \in \mathbb{R}$ and $x \in D(A)$ such that $Ax = \beta(\eta)x$. It is easy to see that u is a solution of (3.1) for $f = 0$, and then $x = 0$ by uniqueness. Secondly, let $\eta \in \mathbb{R}$ and $y \in X$. Note that $f(t) = e^{i\eta t}y \in C^\alpha(\mathbb{R}, X)$, so $u = Lf$ is the unique solution of (3.1) associated to f, where $L : C^\alpha(\mathbb{R}, X) \to C^{\alpha+3}(\mathbb{R}, X) \cap C^{\alpha+1}(\mathbb{R}, D(A))$ is the bounded linear operator which associates to each $f \in C^\alpha(\mathbb{R}, X)$ the solution of (3.1). One can easily check that for a fixed $s \in \mathbb{R}$,

$$v_1(t) = u(t + s) \text{ and } v_2(t) = e^{i\eta s}u(t),$$

are both solutions of (3.1) with forcing term $e^{i\eta s}f(t)$. By uniqueness it follows that $u(t) = e^{i\eta t}u(0)$. Taking $x = u(0) \in D(A)$ and replacing u as above in (3.1) we get

$$u(t) = e^{i\eta t}\gamma(\eta)(\beta(\eta) - A)^{-1},$$

and for $t = 0$

$$x = \gamma(\eta)(\beta(\eta) - A)^{-1}y.$$

Therefore $(\beta(\eta) - A)$ is bijective.

In order to prove (ii), we observe that $\|e^{i\eta t}x\|_\alpha = K_\alpha|\eta|^\alpha\|x\|$, where $K_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$, see [3, Section 3]. Then we get

$$K_\alpha|\eta|^\alpha\|(i\eta)^3\gamma(\eta)(\beta(\eta) - A)^{-1}y\| = \|u'''\|_\alpha \leq \|u\|_{\alpha+3} \leq \|L\|\|f\|_\alpha$$

$$\leq \|L\|(\|f\|_\alpha + \|f(0)\|) \leq \|L\|(K_\alpha|\eta|^\alpha + 1) + \|y\|.$$

Therefore, for $\varepsilon > 0$ we obtain

$$\sup_{|\eta| > \varepsilon} \|(i\eta)^3 \gamma(\eta)(\beta(\eta) - A)^{-1}\| < \infty.$$

Also, using that $\gamma \in C_0(\mathbb{R})$ and the continuity of $\eta \to (i\eta)^3 \gamma(\eta)(\beta(\eta) - A)^{-1}$ at $\eta = 0$, we conclude (ii). $\qquad\square$

Remark 3.3. Note that condition (i) of the above theorem establishes that we need invertibility of the operator A to have C^α-well-posedness of (3.1).

Definition 3.4. [31] *We say that $h \in L^1_{loc}(\mathbb{R}_+)$ is n-regular on $\mathbb{R}$ if for all $0 \le k \le n$ the function h belongs to $L^1(\mathbb{R}_+, t^k \, dt)$ and*

$$\sup_{s \in \mathbb{R}} |s^k (\tilde{h}(s))^{(k)}| < \infty.$$

Remark 3.5. Let $h \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth and $n \in \mathbb{N}$. In [38, Definition 3.3], the following concept of n-regularity is considered:

$$|\lambda^k \hat{h}^{(k)}(\lambda)| \le C|\hat{h}(\lambda)|, \quad \text{for all } \mathfrak{Re}\, \lambda > 0, \ 0 \le k \le n,$$

where C is a positive constant. Observe that if we further assume that $h \in L^1(\mathbb{R}_+, t^n \, dt)$, this concept implies that

$$|s^k(\tilde{h}(s))^{(k)}| = |s^k \hat{h}^{(k)}(is)| \le C|\hat{h}(is)| \le C\|h\|_1, \quad 0 \le k \le n,$$

see [38, Lemma 8.1]. Therefore the function h is n-regular on $\mathbb{R}$ in the sense of Definition 3.4 too.

Remark 3.6. Observe that if h is 2-regular on $\mathbb{R}$, then $\tilde{h}$ is a $\dot{C}^\alpha$-multiplier.

Example 3.7. Let $a, \nu > 0$ be given. In the following, we will denote $g_{\nu,a}(t) := \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-at}$. This is the typical kernel used in viscoelasticity theory [39]. Observe that these functions are 2-regular on $\mathbb{R}$ since $\widetilde{g_{\nu,a}}(s) = \frac{1}{(is+a)^\nu}$ for all $s \in \mathbb{R}$.

The following Lemma provides several useful properties on the scalar function γ that will be crucial for the forthcoming section.

Lemma 3.8. *Let g be a 2-regular function on $\mathbb{R}$ such that $c^2 + bsi - \tilde{g}(s) \neq 0$ for all $s \in \mathbb{R}$. Then the following statements hold.*

   (i) *The functions $\gamma(s), s\gamma(s), s\gamma'(s), s^2\gamma'(s), s^2\gamma''(s), s^3\gamma''(s)$ are bounded on $\mathbb{R}$. In particular, $\gamma(s)$ and $s\gamma(s)$ are $\dot{C}^\alpha$-multipliers.*

   (ii) *The function γ satisfies*

$$\left|\frac{\gamma'(s)}{\gamma(s)}\right| \le C\gamma(s), \quad \left|\frac{s\gamma''(s)}{\gamma(s)}\right| \le C\gamma(s), \quad s \in \mathbb{R}.$$

Proof. By the Riemann-Lebesgue theorem we get that $\gamma(s) \in C_0(\mathbb{R})$ and $s\gamma(s) \in BC(\mathbb{R})$ since $c^2 + bsi - \tilde{g}(s) \neq 0$ for all $s \in \mathbb{R}$. In addition, using the 2-regularity of the function g and the identities

$$\gamma'(s) = ((\tilde{g}(s))' - bi)\gamma^2(s), \quad \gamma''(s) = (\tilde{g}(s))''\gamma^2(s) + 2\gamma(s)\gamma'(s)((\tilde{g}(s))' - bi),$$

for all $s \in \mathbb{R}$, we get the boundedness for the remaining functions. In order to prove (ii) it is enough to use (i) and $\sup_{s \in \mathbb{R}} |s(\tilde{g}(s))''| < \infty$. $\qquad\square$

In what follows, we will denote by $id(s) := is$ for $s \in \mathbb{R}$ and

$$M(s) := \gamma(s)(\beta(s) - A)^{-1}, \quad s \in \mathbb{R}.$$

The next Lemma shows that the particular structure of M gives immediate properties on additional operators related with M.

Lemma 3.9. *Let g be 2-regular on $\mathbb{R}$, and assume that $c^2 + bsi - \tilde{g}(s) \neq 0$ for all $s \in \mathbb{R}$, and $\{\beta(s)\}_{s \in \mathbb{R}} \subseteq \rho(A)$. If*

$$\sup_{s \in \mathbb{R}} \|s^3 M(s)\| < \infty,$$

then $(id)^3 M$ and $(id)^2 M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X)$, and idM, M and $\tilde{g}M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X, D(A))$.

Proof. First of all, we observe that $\sup_{s \in \mathbb{R}} \|(is)^j M(s)\| < \infty$ for $j = 0, 1, 2$, since $\sup_{s \in \mathbb{R}} \|s^3 M(s)\| < \infty$ and the continuity of $s \to M(s)$ in $s = 0$ holds. Also, we have the identities

$$M'(s) = \frac{\gamma'(s)}{\gamma(s)} M(s) - \frac{\beta'(s)}{\gamma(s)} M^2(s),$$

and

$$M''(s) = \frac{\gamma''(s)}{\gamma(s)} M(s) - \left(\frac{\gamma'(s)\beta''(s)}{\gamma^2(s)} + \frac{\gamma'(s)\beta'(s)}{\gamma^2(s)} + \frac{\beta''(s)}{\gamma(s)} \right) M^2(s) + 2 \left(\frac{\beta'(s)}{\gamma(s)} \right)^2 M^3(s),$$

where

$$\beta'(s) = (3\tau i(is)^2 + 2\kappa i(is))\gamma(s) + (\tau(is)^3 + \kappa(is)^2)\gamma'(s),$$

$$\beta''(s) = (-6\tau is - 2\kappa)\gamma(s) + (6\tau i(is)^2 + 4\kappa i(is))\gamma'(s) + (\tau(is)^3 + \kappa(is)^2)\gamma(s)''.$$

By Lemma 3.8, we get

$$(3.3) \qquad \left| \frac{\beta'(s)}{\gamma(s)} \right| \leq Cs^2, \quad \left| \frac{\beta''(s)}{\gamma(s)} \right| \leq Cs,$$

for $|s| > \varepsilon > 0$. We denote $M_j := (id)^j M$ for $j = 0, 1, 2, 3$. By the continuity of $s \to M(s)$ in $s = 0$, it is enough to prove that $\sup_{|s| > \varepsilon} \|sM_j'(s)\| < \infty$ and $\sup_{|s| > \varepsilon} \|s^2 M_j''(s)\| < \infty$ for each $j = 0, 1, 2, 3$. Let $j = 3$. First we have

$$
\begin{aligned}
\|sM_3'(s)\| &\leq C(\|s^3 M(s)\| + \|s^4 M'(s)\|) \\
&\leq C(\|M_3(s)\| + \|s^4 \gamma(s) M(s)\| + \|s^6 M^2(s)\|) \\
&\leq C(\|M_3(s)\| + \|M_3(s)\| + \|M_3^2(s)\|)),
\end{aligned}
$$

where we have applied Lemma 3.8 and (3.3). It shows that $\sup_{s \in \mathbb{R}} \|sM_3'(s)\| < \infty$. Secondly, we have the inequality

$$\|s^2 M_3''(s)\| \leq C(\|s^3 M(s)\| + \|s^4 M'(s)\| + \|s^5 M''(s)\|).$$

It is clear that the first two summands in the previous inequality are uniformly bounded. Using again Lemma 3.8 and (3.3) we get

$$
\begin{aligned}
\|s^5 M''(s)\| &\leq C(\|s^4 \gamma(s) M(s)\| + \|s^6 \gamma(s) M^2(s)\| + \|s^7 \gamma(s) M^2(s)\| \\
&\quad + \|s^6 M^2(s)\| + \|s^9 \gamma(s) M^3(s)\|) \\
&\leq C(\|M_3(s)\| + \|M_3^2(s)\| + \|M_3^2(s)\| + \|M_3^2(s)\| + \|M_3^3(s)\|).
\end{aligned}
$$

It implies that $\sup_{s\in\mathbb{R}}\|s^5M''(s)\| < \infty$. As a consequence, by Theorem 2.3, we obtain that M_3 is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X)$. The cases $j = 0, 1, 2$ are analogous. Observe that the 2-regularity of g implies that $\tilde{g}M$ is a $\dot{C}^\alpha$-multiplier. Finally, the identity

$$AM(s) = \beta(s)M(s) - \gamma(s)I = \gamma(s)(\tau(is)^3M(s) + \kappa(is)^2M(s) - I),$$

shows that M is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$. Also, $(id)M$ is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$, since $(id)\gamma$ is a $\dot{C}^\alpha$-multiplier (Lemma 3.8 (i)). It follows by the 2-regularity of g that $\tilde{g}M$ is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$, and we conclude the result. $\qquad\square$

Remark 3.10. By Lemma 3.8 (i), the condition $\sup_{s\in\mathbb{R}}\|s^3M(s)\| < \infty$ in the above theorem could be replaced by $\sup_{s\in\mathbb{R}}\|s^2(\beta(s) - A)^{-1}\| < \infty$.

The following theorem is one of the main results of this paper. It characterizes the C^α-well-posedness of equation (3.1) in Hölder continuous spaces, under the 2-regularity assumption on the function g and the following hypothesis: There exists $\varepsilon > 0$ such that

$$(3.4) \qquad\qquad \sup_{0\leq\sigma<\varepsilon,\,\rho\in\mathbb{R}} \left| \frac{1}{c^2 + b(\sigma + i\rho) - \hat{g}(\sigma + i\rho)} \right| < \infty.$$

Remark 3.11. It is interesting to observe that the assumption (3.2) i.e. $c^2 > G(+\infty)$, was used in [33] to get exponentially stability of the energy for weak solutions of the MGTM-equation of type I on Hilbert spaces. Note that it also implies condition (3.4). Therefore it seems natural to impose this condition in order to obtain C^α-well-posedness of (3.1).

We now present our main result.

Theorem 3.12. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is 2-regular on $\mathbb{R}$ and (3.4) holds. Then the following assertions are equivalent.*

  (i) *Equation (3.1) is C^α-well-posed.*
  (ii) *$\{\beta(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ and*

$$\sup_{s\in\mathbb{R}} \|(is)^3\gamma(s)(\beta(s) - A)^{-1}\| < \infty.$$

Proof. The implication $(i) \Rightarrow (ii)$ is a straightforward consequence of Theorem 3.2.

To prove $(ii) \Rightarrow (i)$, we take $f \in C^\alpha(\mathbb{R}, X)$. By Lemma 3.9, there exist $u_0, u_1, u_g \in C^\alpha(\mathbb{R}, D(A))$ and $u_2, u_3 \in C^\alpha(\mathbb{R}, X)$ such that

$$(3.5) \qquad\qquad \int_{\mathbb{R}} u_0(s)(\mathcal{F}\phi_0)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_0 \cdot M)(s)f(s)ds,$$

$$(3.6) \qquad\qquad \int_{\mathbb{R}} u_1(s)(\mathcal{F}\phi_1)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 \cdot id \cdot M)(s)f(s)ds,$$

$$(3.7) \qquad\qquad \int_{\mathbb{R}} u_g(s)(\mathcal{F}\phi_g)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_g \cdot \tilde{g} \cdot M)(s)f(s)ds,$$

$$(3.8) \qquad\qquad \int_{\mathbb{R}} u_2(s)(\mathcal{F}\phi_2)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot (id)^2 \cdot M)(s)f(s)ds,$$

and

$$(3.9) \qquad \int_{\mathbb{R}} u_3(s)(\mathcal{F}\phi_3)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_3 \cdot (id)^3 \cdot M)(s)f(s)ds,$$

for all $\phi_g, \phi_i \in C_c^1(\mathbb{R}\backslash\{0\})$ $(i = 0, 1, 2, 3)$ (cf. Remark 2.2), where $M(s) = \gamma(s)(\beta(s) - A)^{-1}$. Following the ideas of the proof of [15, Theorem 3.5], there exist $y_1, y_2, y_3 \in X$ such that

$$(3.10) \qquad u_0' = u_1 + y_1, \quad u_0'' = u_2 + y_2, \quad u_0''' = u_3 + y_3,$$

and $u_0 \in C^{\alpha+3}(\mathbb{R}, X)$. Also, note that if we take $\phi_0 = \phi_g \cdot \tilde{g}$ in (2.4) since $g \in L^1(\mathbb{R}_+, tdt)$, then using (3.7) one gets

$$\int_{\mathbb{R}} u_g(s)(\mathcal{F}\phi_g)(s) = \int_{\mathbb{R}} u_g(s)\mathcal{F}(\phi_g \cdot \tilde{g})(s)\, ds,$$

for all $\phi_g \in C_c^1(\mathbb{R}\backslash\{0\})$. Therefore by [31, Lemma 3.2, Remark 3.3], we obtain

$$(3.11) \qquad u_0 * g = u_g + y_g,$$

with $y_g \in X$.

Note that the operator $M(s)$ satisfies the following identity,

$$\tau(is)^3 M(s) + \kappa(is)^2 M(s) - c^2 AM(s) - b(is)AM(s) + \tilde{g}(s)AM(s) = I.$$

Therefore one gets

$$\tau \int_{\mathbb{R}} \mathcal{F}(\phi \cdot (id)^3 \cdot M)(s)f(s)ds + \kappa \int_{\mathbb{R}} \mathcal{F}(\phi \cdot (id)^2 \cdot M)(s)f(s)ds - c^2 \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot M)(s)f(s)ds$$
$$-b \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot (id) \cdot M)(s)f(s)ds + \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot \tilde{g} \cdot M)(s)f(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi)(s)f(s)ds,$$

for all $\phi \in C_c^1(\mathbb{R}\backslash\{0\})$. By (3.5)-(3.11) and the previous identity we have

$$\tau \int_{\mathbb{R}} u_0'''(s)(\mathcal{F}\phi)(s)ds + \kappa \int_{\mathbb{R}} u_0''(s)(\mathcal{F}\phi)(s)ds - c^2 \int_{\mathbb{R}} Au_0(s)(\mathcal{F}\phi)(s)ds$$
$$-b \int_{\mathbb{R}} Au_0'(s)(\mathcal{F}\phi)(s)ds + \int_{\mathbb{R}} A(g * u_0)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} f(s)\mathcal{F}(\phi)(s)f(s)ds,$$

for all $\phi \in C_c^1(\mathbb{R}\backslash\{0\})$. Then there existz $z \in X$ such that

$$\tau u_0'''(s) + \kappa u_0''(s) - c^2 Au_0(s) - bAu_0'(s) + (g * Au_0)(s) = f(s) + z \qquad s \in \mathbb{R}.$$

Since $\tilde{g}(0) \neq c^2$ and $0 \in \rho(A)$ we can take

$$u(t) := u_0(t) - (c^2 - \tilde{g}(0))^{-1}A^{-1}z.$$

It is easy to see that u solves (3.1).

In addition, note that $u, u' \in C^\alpha(\mathbb{R}, X)$, since $u \in C^{\alpha+3}(\mathbb{R}, X)$. Also, by Lemma 3.9 we have $AM, A(id)M$ are $\dot{C}^\alpha$-multipliers. Then there exist $u_4, u_5 \in C^\alpha(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} u_4(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot AM)(s)f(s)ds$$

and

$$\int_{\mathbb{R}} u_5(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot A(id)M)(s)f(s)ds,$$

for all $\phi \in C_c^1(\mathbb{R} \setminus \{0\})$. Using (3.5) and (3.6) and the closedness of A one gets

$$\int_{\mathbb{R}} Au_0(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot M)(s)f(s)ds = \int_{\mathbb{R}} u_4(s)(\mathcal{F}\phi)(s)ds$$

and

$$\int_{\mathbb{R}} Au_1(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot (id) \cdot M)(s)f(s)ds = \int_{\mathbb{R}} u_5(s)(\mathcal{F}\phi)(s)ds.$$

This implies that there exist $y_4, y_5 \in X$ such that $Au_0 = u_4 + y_4, Au_1 = u_5 + y_5$, and therefore $Au, Au' \in C^\alpha(\mathbb{R}, X)$. Finally note that $g * Au \in C^\alpha(\mathbb{R}, X)$ by (2.6).

Now, we prove uniqueness. Let $u \in C^{\alpha+3}(\mathbb{R}, X) \cap C^{\alpha+1}(\mathbb{R}, D(A))$ that solves the homogeneous MGTM-equation of Type I, that is,

$$(3.12) \qquad \tau u'''(t) + \kappa u''(t) - c^2 Au(t) - bAu'(t) + \int_{-\infty}^t g(t-s)Au(s)\,ds = 0, \quad t \in \mathbb{R}.$$

Then, $u''', u'', u', Au, Au', g * Au \in C^\alpha(\mathbb{R}, X)$.
Let $\sigma > 0$ and L_σ be the operator defined in (2.7). Applying L_σ to (3.12) we get

$$(c^2 + b(\sigma + i\rho) - \hat{g}(\sigma + i\rho))(\beta_\sigma(\rho) - A)(L_\sigma u)(\rho) = \sigma \left(4\tau i\rho u(0) + 2\kappa u(0) + 2\tau u'(0) \right)$$

$$-2\tau\sigma(\sigma^2 - 3\rho^2)\hat{u}(-\sigma + i\rho) - 4\kappa\sigma i\rho\hat{u}(-\sigma + i\rho) + 2\sigma A\hat{u}(-\sigma + i\rho) - G_g^{Au}(\sigma, \rho)$$

$$=: H_g(\sigma, \rho),$$

with

$$\beta_\sigma(\rho) := \frac{\tau(\sigma + i\rho)^3 + \kappa(\sigma + i\rho)^2}{c^2 + b(\sigma + i\rho) - \hat{g}(\sigma + i\rho)},$$

where we have applied (2.8), (2.9) and Proposition 2.5. Note that $\beta_0(\rho) = \beta(\rho) \in \rho(A)$ for all $\rho \in \mathbb{R}$, so we write

$$(\beta_\sigma(\rho) - \beta(\rho))(\beta(\rho) - A)^{-1}(L_\sigma u)(\rho) + (L_\sigma u)(\rho) = \frac{1}{c^2 + b(\sigma + i\rho) - \hat{g}(\sigma + i\rho)}(\beta(\rho) - A)^{-1}H_g(\sigma, \rho).$$

Let $\phi \in C_c^\infty(\mathbb{R})$. Multiplying by ϕ and integrating over $\mathbb{R}$ we obtain

$$(3.13) \qquad \int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho)d\rho = \int_{\mathbb{R}} N_\sigma(\rho)H_g(\sigma, \rho)d\rho - \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho)d\rho,$$

where

$$M_\sigma(\rho) := \phi(\rho)(\beta_\sigma(\rho) - \beta(\rho))(\beta(\rho) - A)^{-1}$$

and

$$N_\sigma(\rho) := \phi(\rho)\frac{1}{c^2 + b(\sigma + i\rho) - \hat{g}(\sigma + i\rho)}(\beta(\rho) - A)^{-1}.$$

By the 2-regularity of g and hypothesis (3.4), the operator families $\{M_\sigma(\rho)\}_{\rho \in \mathbb{R}}$ and $\{N_\sigma(\rho)\}_{\rho \in \mathbb{R}}$ are both in $C_c^2(\mathbb{R}, \mathcal{B}(X))$. Finally, we prove that both summands in the right hand side of (3.13) converge to zero as $\sigma \to 0$. Then

$$\int_{\mathbb{R}} u(\rho)(\mathcal{F}\phi)(\rho)d\rho = \lim_{\sigma \to 0} \int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho)d\rho = 0,$$

for all ϕ in the Schwartz space $\mathcal{S}(\mathbb{R})$, see [31, Proposition A.2 (i)], and therefore $u \equiv 0$.

In fact, using Lemmas [31, Lemma A.4] and [31, Lemma A.3] and taking into account that $\|M_\sigma\|_{L^1} + \|M_\sigma''\|_{L^1} \to 0$ as $\sigma \to 0$ (applying the Dominated Convergence Theorem since M_σ and M_σ'' have compact support and the identity $\beta_0^{(k)}(\rho) = \beta^{(k)}(\rho)$ holds for $k = 0, 1, 2$), we get that

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho) d\rho = 0.$$

In order to prove that

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) H_g(\sigma, \rho) d\rho = 0,$$

we observe, following the first part of [31, Lemma A.5], that $\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) G_g^{Au}(\sigma, \rho) d\rho = 0$. This follows from the 2-regularity of g, $\sup_{s \in \mathbb{R}} \|\gamma(s)(\beta(s) - A)^{-1}\| < \infty$, (3.4), the compact support of N_σ, N_σ'' and Lemma 3.8 which states that

$$\sup_{0 \leq \sigma < \varepsilon} (\|N_\sigma\|_{L^1} + \|N_\sigma''\|_{L^1}) < \infty.$$

Finally, the remaining terms of $H_g(\sigma, \rho)$ are of the form $h(\sigma, \rho)(1 + \hat{u}(-\sigma + i\rho))$, where h is continuous for all $\rho \in \mathbb{R}$, $|h(\sigma, \rho)| \leq C_{\varepsilon, \rho}$ for $0 \leq \sigma < \varepsilon$ and $h(\sigma, \rho) \to 0$ as $\sigma \to 0$. Then using similar arguments as in [31, Lemma A.5] we conclude the result. $\qquad\square$

Remark 3.13. *Under the hypothesis of Theorem 3.12 we say that equation* (3.1) *is* C^α-*ill-posed if condition* (ii) *fails.*

Remark 3.14. The proof of the above theorem states that $u''', u'', Au', Au, g * Au \in C^\alpha(\mathbb{R}, X)$. Moreover, using the closed graph theorem, we can conclude that there exists a positive constant C independent of $f \in C^\alpha(\mathbb{R}, X)$ such that

$$\|u'''\|_{C^\alpha(\mathbb{R},X)} + \|u''\|_{C^\alpha(\mathbb{R},X)} + \|Au'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)} + \|g * Au\|_{C^\alpha(\mathbb{R},X)} \leq C\|f\|_{C^\alpha(\mathbb{R},X)}.$$

In what follows we analyze the special case $\tau = 0$, namely,

$$(3.14) \qquad \kappa u''(t) - c^2 A u(t) - b A u'(t) + \int_{-\infty}^{t} g(t - s) A u(s)\, ds = f(t), \quad t \in \mathbb{R}.$$

Abstract integro-differential equations of second order with a memory term naturally appear in the theory of viscoelasticity. For instance, the model (3.14) with $b = 0$ and $A = \Delta$ corresponds to the viscoelastic membrane equation. In [1], decay estimates for the solutions of this abstract model defined on Hilbert spaces is studied. Moreover, for $A = -\Delta^2$ equation (3.14) models the viscoelastic plate equation with memory [36], [39].

We say that equation (3.14) is C^α-well-posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+2}(\mathbb{R}, X)$ satisfying (3.14). For this equation, the function γ defined previously

$$\gamma(\eta) = \frac{1}{c^2 + ib\eta - \tilde{g}(\eta)}, \quad \eta \in \mathbb{R},$$

does not change. However, the function β takes the form

$$\beta(\eta) = \frac{\kappa(i\eta)^2}{c^2 + b\eta i - \tilde{g}(\eta)} = \kappa(i\eta)^2 \gamma(\eta), \quad \eta \in \mathbb{R},$$

when $\tau = 0$. By the continuity of $s \to M(s) = \gamma(s)(\beta(s) - A)^{-1}$ at $s = 0$, we note that (ii) of Theorem 3.2, i.e.,

$$\sup_{s \in \mathbb{R}} \|(is)^3 \gamma(s)(\beta(s) - A)^{-1}\| < \infty,$$

implies (not only for $\tau = 0$) that

$$\sup_{s \in \mathbb{R}} \|(is)^2 \gamma(s)(\beta(s) - A)^{-1}\| < \infty.$$

Therefore, using the same ideas in the proof of Theorem 3.12, we obtain the following characterization of C^α-well-posedness of the MGTM equation of Type I with $\tau = 0$.

Theorem 3.15. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is 2-regular on $\mathbb{R}$ and (3.4) holds. Then the following assertions are equivalent.*

(i) *Equation (3.14) is C^α-well-posed.*
(ii) *$\{\beta(s)\}_{s \in \mathbb{R}} \subseteq \rho(A)$ and*

$$\sup_{s \in \mathbb{R}} \|(is)^2 \gamma(s)(\beta(s) - A)^{-1}\| < \infty.$$

4. Characterizations of C^α-well-posedness: Type II and Type III

In this section, our aim is to characterize the C^α-well-posedness of the MGTM-equation of Types II and III. Let $f \in C^\alpha(\mathbb{R}, X)$ and A be a closed linear operator in X. The non-homogeneous MGTM equations of Type II and III are given by

$$(4.1) \qquad \tau u'''(t) + \kappa u''(t) - c^2 A u(t) - b A u'(t) + \int_{-\infty}^t g(t - s) A u'(s) \, ds = f(t), \quad t \in \mathbb{R},$$

and

$$(4.2) \quad \tau u'''(t) + \kappa u''(t) - c^2 A u(t) - b A u'(t) + \int_{-\infty}^t g(t - s) A(\mu u(s) + u'(s)) \, ds = f(t), \, t, \mu \in \mathbb{R}, \mu \neq 0,$$

respectively, where $g \in L^1(\mathbb{R}_+, t^\alpha \, dt) \cap L^1_{loc}(\mathbb{R}_+)$. These versions of the MGTM-equation (3.1) were recently introduced by Lasiecka and Wang [33].

We introduce the following definition.

Definition 4.1. *We say that the equation (4.1)(resp. (4.2)) is C^α-well-posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+3}(\mathbb{R}, X)$ satisfying (4.1)(resp. (4.2)).*

We will use the same tools as the ones in the previous section in order to get C^α-well-posedness for the MGTM-equation of types II and III. We will omit unnecessary steps and we will write the main theorems directly. We first treat the MGTM equation of Type II.

In a similar way as in the above section, one can easily observe that the assumption

$$(4.3) \qquad |b| > G(+\infty) = \int_0^\infty |g(s)| \, ds,$$

implies that $c^2 + b\eta i - i\eta \tilde{g}(\eta) \neq 0$ for all $\eta \in \mathbb{R}$. Moreover, the function

$$\gamma_2(\eta) := \frac{1}{c^2 + b\eta i - i\eta \tilde{g}(\eta)},$$

belongs to $C_0(\mathbb{R})$. Observe that, instead of (4.3), we could suppose that the function g satisfies the condition $g' \in L^1(\mathbb{R}_+)$. This condition is verified for 2-regular derivable functions which are well defined at $t = 0$, for instance $g_{\nu,a}$ for $\nu \geq 1$. In this case, the following assumption

$$(4.4) \qquad\qquad c^2 > \int_0^\infty |g'(t)|\,dt,$$

is enough to ensure that

$$\gamma_2(\eta) = \frac{1}{c^2 + b\eta i - i\eta\tilde{g}(\eta)} = \frac{1}{c^2 + b\eta i - \widetilde{(g')}(\eta)},$$

belongs to $C_0(\mathbb{R})$. If any of the above assumptions for γ_2 hold, the function

$$\beta_2(\eta) := \frac{\tau(i\eta)^3 + \kappa(i\eta)^2}{c^2 + b\eta i - i\eta\tilde{g}(\eta)}, \quad \eta \in \mathbb{R},$$

is well-defined.

We introduce the following notion of regularity.

Definition 4.2. *We say that $h \in L^1_{loc}(\mathbb{R}_+)$ is strongly n-regular on $\mathbb{R}$ if for all $0 \leq k \leq n$ the function h belongs to $L^1(\mathbb{R}_+, t^k\,dt)$ and*

$$\sup_{s \in \mathbb{R}} |s^{k+1}(\tilde{h}(s))^{(k)}| < \infty.$$

Remark 4.3. Let h be a strongly 2-regular function. Then h is also 2-regular. Moreover, we have

$$\sup_{s \in \mathbb{R}} |s^k(s\tilde{h}(s))^{(k)}| < \infty, \quad k = 0, 1, 2.$$

In particular, $\tilde{h}$ and $s\tilde{h}(s)$ are $\dot{C}^\alpha$-multipliers. If we assume that h is differentiable, then h' is 2-regular as well.

Example 4.4. The functions $g_{\nu,a}(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}e^{-at}$ are strongly 2-regular on $\mathbb{R}$ for $a > 0$ and $\nu \geq 1$ since $\widetilde{g_{\nu,a}}(s) = \frac{1}{(is+a)^\nu}$ for all $s \in \mathbb{R}$.

The next result characterizes the C^α-well-posedness of equation (4.1) under the assumption of strong 2-regularity on the function g. Both, conditions (4.3) and (4.4), imply that there exists $\varepsilon > 0$ such that

$$(4.5) \qquad \sup_{0 \leq \sigma < \varepsilon,\, \rho \in \mathbb{R}} \left| \frac{1}{c^2 + b(\sigma + i\rho) - (\sigma + i\rho)\hat{g}(\sigma + i\rho)} \right| < \infty.$$

Theorem 4.5. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is strongly 2-regular on $\mathbb{R}$ and (4.5) holds. Then the following assertions are equivalent.*

(i) *Equation (4.1) is C^α-well-posed.*
(ii) *$\{\beta_2(s)\}_{s \in \mathbb{R}} \subseteq \rho(A)$ and*

$$\sup_{s \in \mathbb{R}} \|(is)^3 \gamma_2(s)(\beta_2(s) - A)^{-1}\| < \infty.$$

For the particular case $\tau = 0$,

$$(4.6) \qquad \kappa u''(t) - c^2 Au(t) - bAu'(t) + \int_{-\infty}^t g(t-s)Au'(s)\,ds = f(t), \quad t \in \mathbb{R},$$

the equation (4.6) is said C^α-well-posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+2}(\mathbb{R}, X)$ satisfying (4.6). For this equation, the function γ_2 does not change, and the function β_2 is

$$\beta_2(\eta) = \frac{\kappa(i\eta)^2}{c^2 + b\eta i - i\eta\tilde{g}(\eta)} = \kappa(i\eta)^2\gamma_2(\eta), \quad \eta \in \mathbb{R}.$$

It is not difficult to observe that we can obtain the following characterization of C^α-well-posedness of the MGTM equation of Type II with $\tau = 0$.

Theorem 4.6. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is strongly 2-regular on $\mathbb{R}$ and (4.5) holds. Then the following assertions are equivalent.*

(i) *Equation (4.6) is C^α-well-posed.*
(ii) *$\{\beta_2(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ and*

$$\sup_{s\in\mathbb{R}} \|(is)^2\gamma_2(s)(\beta_2(s) - A)^{-1}\| < \infty.$$

Finally, we study the MGTM-equation of Type III. Observe that if the following inequality holds

$$(4.7) \qquad\qquad |b| > \frac{c^2}{|\mu|} > G(+\infty),$$

then $c^2 + b\eta i - (\mu + i\eta)\tilde{g}(\eta) \neq 0$ for all $\eta \in \mathbb{R}$. Moreover, the function

$$\gamma_3(\eta) := \frac{1}{c^2 + b\eta i - (\mu + i\eta)\tilde{g}(\eta)},$$

belongs to $C_0(\mathbb{R})$. We note that condition (4.7) implies exponential stability of the standard energy for the weak solution of the homogeneous MGTM-equation of type III in a Hilbert space, see [33, Theorem 1.8]. On the other hand, if the function g satisfies that $g' \in L^1(\mathbb{R}_+)$, then the condition

$$(4.8) \qquad\qquad c^2 > |\mu|G(+\infty) + \int_0^\infty |g'(t)|\, dt,$$

also implies that

$$\gamma_3(\eta) = \frac{1}{c^2 + b\eta i - (\mu + i\eta)\tilde{g}(\eta)} = \frac{1}{c^2 + b\eta i - \mu\tilde{g}(\eta) - \widetilde{(g')}(\eta)} \in C_0(\mathbb{R}).$$

It is a straightforward consequence that

$$\beta_3(\eta) := \frac{\tau(i\eta)^3 + \kappa(i\eta)^2}{c^2 + b\eta i - (\mu + i\eta)\tilde{g}(\eta)}, \quad \eta \in \mathbb{R},$$

is well-defined if (4.7) or (4.8) holds. Also, the previous conditions imply that there exists $\varepsilon > 0$ such that

$$(4.9) \qquad \sup_{0\leq\sigma<\varepsilon,\, \rho\in\mathbb{R}} \left| \frac{1}{c^2 + b(\sigma + i\rho) - (\mu + \sigma + i\rho)\hat{g}(\sigma + i\rho)} \right| < \infty.$$

The characterization of C^α-well-posedness of MGTM equations of type III reads as follows.

Theorem 4.7. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is strongly 2-regular on $\mathbb{R}$ and (4.9) holds. Then the following assertions are equivalent.*

(i) *Equation (4.2) is C^α-well-posed.*

(ii) $\{\beta_3(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ and

$$\sup_{s\in\mathbb{R}} \|(is)^3\gamma_3(s)(\beta_3(s) - A)^{-1}\| < \infty.$$

Similarly to the previous cases, for $\tau = 0$,

$$(4.10) \qquad \kappa u''(t) - c^2 Au(t) - bAu'(t) + \int_{-\infty}^t g(t-s)A(\mu u(s) + u'(s))\, ds = f(t), \quad t \in \mathbb{R},$$

the equation (4.10) is said C^α-well-posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there exists a unique function $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+2}(\mathbb{R}, X)$ satisfying (4.10). In this case, the function γ_3 does not change, and the function β_3 is

$$\beta_3(\eta) = \frac{\kappa(i\eta)^2}{c^2 + b\eta i - (\mu + i\eta)\tilde{g}(\eta)} = \kappa(i\eta)^2\gamma_3(\eta), \quad \eta \in \mathbb{R}.$$

Then we get the following characterization of C^α-well-posedness of the MGTM equation of Type III with $\tau = 0$.

Theorem 4.8. *Let A be a closed linear operator defined on a Banach space X. Suppose $g \in L^1_{loc}(\mathbb{R}_+)$ is strongly 2-regular on $\mathbb{R}$ and (4.9) holds. Then the following assertions are equivalent.*
  (i) *Equation (4.10) is C^α-well-posed.*
  (ii) $\{\beta_3(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$ *and*

$$\sup_{s\in\mathbb{R}} \|(is)^2\gamma_3(s)(\beta_3(s) - A)^{-1}\| < \infty.$$

5. EXAMPLES

In this section we illustrate some of the main results in this paper with some examples of classes of operators A. Recall that we assume $\tau, \kappa, b, \mu > 0$.

Example 5.1. Let $A = \Delta$ be the Dirichlet Laplacian operator in $X = C_0(\Omega) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ with Ω Dirichlet regular, see [4, Theorem 6.1.9]. It is known that $0 \in \rho(A)$ and $-A$ is sectorial of angle 0, that is, $\sigma(A) \subseteq (-\infty, 0)$ and for all $\varphi \in (0, \pi)$

$$(5.1) \qquad \qquad \|\lambda(\lambda - A)^{-1}\| \le M_\varphi, \quad \lambda \in S_\varphi, M_\varphi > 0,$$

where $S_\varphi := \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |arg(\lambda)| < \varphi\}$, [27].

Also, we consider $g(t) = g_{\nu,a}(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}e^{-at}$ for $a > 0$ and $\nu > 0$. Note that $g_{\nu,a}$ is 2-regular on $\mathbb{R}$, see Example 3.7, and 2-strongly regular for $\nu \geq 1$, see Example 4.4.

First, we will prove that if $0 < \nu \le 1$ and $c^2 > \frac{1}{a^\nu} = G(+\infty)$, the corresponding MGTM equation of type I is C^α-ill-posed for $\tau \neq 0$. See Remark 3.13 for the precise meaning of ill-posedness. However, the Kuznetsov and Westervelt equations of Type I (MGTM equation for $\tau = 0$) are C^α-well-posed.

Indeed, recall that the assumption $c^2 > \frac{1}{a^\nu} = G(+\infty)$ implies the condition (3.4) by Remark 3.11. Next, note that

$$\begin{aligned}
\beta(\eta) &= \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2 - bi\eta - \overline{\tilde{g}(\eta)})}{|c^2 + bi\eta - \tilde{g}(\eta)|^2} \\
&= \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2(\eta^2 + a^2)^\nu - bi\eta(\eta^2 + a^2)^\nu - (\eta^2 + a^2)^{\nu/2}(\cos(\nu\theta) + i\sin(\nu\theta)))}{|c^2 + bi\eta - \tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu},
\end{aligned}$$

where $\theta = \arctan(\eta/a)$. Therefore the identity
(5.2)
$$\mathfrak{Re}\,\beta(\eta) = \frac{-\tau b\eta^4(\eta^2 + a^2)^\nu - \tau\eta^3(\eta^2 + a^2)^{\nu/2}\sin(\nu\theta) - \kappa\eta^2(\eta^2 + a^2)^{\nu/2}[c^2(\eta^2 + a^2)^{\nu/2} - \cos(\nu\theta)]}{|c^2 + bi\eta - \tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu}$$

shows that $\mathfrak{Re}\,\beta(\eta) < 0$ for $\eta \in \mathbb{R} \setminus \{0\}$ where we have used $c^2(\eta^2 + a^2)^{\nu/2} - \cos(\nu\theta) \geq c^2a^\nu - \cos(\nu\theta) > 0$, and $\sin(\nu\theta) = \sin(\nu\arctan(\eta/a)) > 0$ for $\eta > 0$ and $\sin(\nu\theta) = \sin(\nu\arctan(\eta/a)) < 0$ for $\eta < 0$ since $0 < \nu \leq 1$. If we assume $-\tau c^2 + \kappa b > 0$, which is the subcritical condition to get stability of the energy for the MGTM equation, see [33, Theorem 1.4], we have

(5.3) $$\mathfrak{Im}\,\beta(\eta) = \frac{(-\tau c^2 + \kappa b)\eta^3(\eta^2 + a^2)^\nu + (\tau\eta^3\sin(\nu\theta) + \kappa\eta^2\cos(\nu\theta))(\eta^2 + a^2)^{\nu/2}}{|c^2 + bi\eta - \tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu} \neq 0,$$

for $\eta \in \mathbb{R} \setminus \{0\}$. Then $\beta(\eta) \in \rho(A)$ for all $\eta \in \mathbb{R}$, and $\frac{\mathfrak{Im}\,\beta(\eta)}{\mathfrak{Re}\,\beta(\eta)} \to 0$ as $\eta \to \pm\infty$. This does not allow to use (5.1) to bound the resolvent.

Assuming $\tau \neq 0$, let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the spectrum of A with $0 > \lambda_1 \geq \ldots \geq \lambda_k \to \infty$ as $k \to \infty$. Then

$$\sup_{\eta\in\mathbb{R}} ||(i\eta)^3\gamma(\eta)(\beta(\eta) - \Delta)^{-1}|| \geq \sup_{\{\eta\in\mathbb{R}:\,\mathfrak{Re}\,\beta(\eta)=\lambda_k,\ k\in\mathbb{N}\}} ||(i\eta)^3\gamma(\eta)(\beta(\eta) - \Delta)^{-1}|||$$

$$\geq \sup_{\{\eta\in\mathbb{R}:\,\mathfrak{Re}\,\beta(\eta)=\lambda_k,\ k\in\mathbb{N}\}} \frac{|(i\eta)^3\gamma(\eta)|}{\text{dist}(\beta(\eta), \sigma(\Delta))} = \sup_{\{\eta\in\mathbb{R}:\,\mathfrak{Re}\,\beta(\eta)=\lambda_k,\ k\in\mathbb{N}\}} \left|\frac{(i\eta)^3\gamma(\eta)}{\mathfrak{Im}\,\beta(\eta)}\right| = \infty.$$

where we have used [22, Proposition 1.3, Chapter IV, p.240]. By Theorem 3.12, we conclude that equation (3.1) is C^α-ill-posed.

Assuming that $\tau = 0$, one gets $\mathfrak{Re}\,\beta(\eta) < 0$ and $\mathfrak{Im}\,\beta(\eta) \neq 0$ for $\eta \in \mathbb{R}\setminus\{0\}$, with $\left|\frac{\mathfrak{Im}\,\beta(\eta)}{\mathfrak{Re}\,\beta(\eta)}\right| \to \infty$ as $\eta \to \pm\infty$. Given $\varepsilon > 0$ we have

$$\sup_{|\eta|>\varepsilon} ||(i\eta)^2\gamma(\eta)(\beta(\eta) - \Delta)^{-1}|| \leq \sup_{|\eta|>\varepsilon} |\frac{(i\eta)^2}{\kappa(i\eta)^2}|\,||\beta(\eta)(\beta(\eta) - \Delta)^{-1}||$$

$$\leq M_{\varphi_0} \sup_{|\eta|>\varepsilon} |\frac{(i\eta)^2}{\kappa(i\eta)^2}| < \infty,$$

where $\pi > \varphi_0 := \pi - \min_{|\eta|>\varepsilon}\arctan\left(\left|\frac{\mathfrak{Im}\,\beta(\eta)}{\mathfrak{Re}\,\beta(\eta)}\right|\right)$. These facts and the continuity of $\eta \to (i\eta)^2\gamma(\eta)(\beta(\eta) - A)^{-1}$ for $\eta = 0$ imply that $\sup_{\eta\in\mathbb{R}} ||(i\eta)^2\gamma(\eta)(\beta(\eta) - A)^{-1}|| < \infty$. Therefore the equation (3.14) is C^α-well-posed, see Theorem 3.15.

Now, we study the C^α-well-posedness of the MGTM-equation of type II under the conditions $\nu \geq 1$, $b > \frac{1}{a^\nu}$ and $c^2 > \frac{1}{a^{\nu-1}}$ on the parameters. Indeed, observe that $b > \frac{1}{a^\nu} = G(+\infty)$ implies (4.5). Moreover, we have

$$\beta_2(\eta) = \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2 - bi\eta + i\eta\overline{\tilde{g}(\eta)})}{|c^2 + bi\eta - i\eta\tilde{g}(\eta)|^2}$$

$$= \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2(\eta^2 + a^2)^\nu - bi\eta(\eta^2 + a^2)^\nu + i\eta(\eta^2 + a^2)^{\nu/2}(\cos(\nu\theta) + i\sin(\nu\theta)))}{|c^2 + bi\eta - i\eta\tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu},$$

where $\theta = \arctan(\eta/a)$. Consequently,

$$\mathfrak{Re}\,\beta_2(\eta) = \frac{-\kappa\eta^2(\eta^2 + a^2)^{\nu/2}[c^2(\eta^2 + a^2)^{\nu/2} - \eta\sin(\nu\theta)] - \tau\eta^4(\eta^2 + a^2)^{\nu/2}[b(\eta^2 + a^2)^{\nu/2} - \cos(\nu\theta)]}{|c^2 + bi\eta - i\eta\tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu} < 0$$

for $\eta \in \mathbb{R} \setminus \{0\}$, since $b > \frac{1}{a^\nu} > \frac{\cos(\nu\theta)}{(\eta^2 + a^2)^{\nu/2}}$ and $c^2 > \frac{1}{a^{\nu-1}} > \frac{\eta\sin(\nu\theta)}{(\eta^2 + a^2)^{\nu/2}}$ for all $\eta \in \mathbb{R}$. In addition,

$$\mathfrak{Im}\,\beta_2(\eta) = \frac{-\tau c^2\eta^3(\eta^2 + a^2)^\nu - \kappa\eta^3(\eta^2 + a^2)^{\nu/2}\cos(\nu\theta) + \kappa b\eta^3(\eta^2 + a^2)^\nu + \tau\eta^4(\eta^2 + a^2)^{\nu/2}\sin(\nu\theta)}{|c^2 + bi\eta - i\eta\tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu}.$$

Now, we assume $\tau = 1, \kappa = 1, b = 16, c^2 = 2$ and $g(t) = g_{1,2}(t)$ (these parameters are under the hypothesis of Theorem 4.5) in order to ease the manipulation and draw the function β_2, see Figure 1. It shows that $\mathfrak{Im}\,\beta_2(\eta) \neq 0$, then $\beta_2(\eta) \in \rho(A)$ for all $\eta \in \mathbb{R}$, and $\frac{\mathfrak{Im}\,\beta_2(\eta)}{\mathfrak{Re}\,\beta_2(\eta)} \to 0$ as $\eta \to \pm\infty$. Using the same arguments that in the above case, the equation (4.1) is C^α-ill-posed (Theorem 4.5), however it is C^α-well-posed for $\tau = 0$ (Theorem 4.6).

FIGURE 1. Example of a parametric plot $(\mathfrak{Re}\,\beta_2(\eta), \mathfrak{Im}\,\beta_2(\eta))$
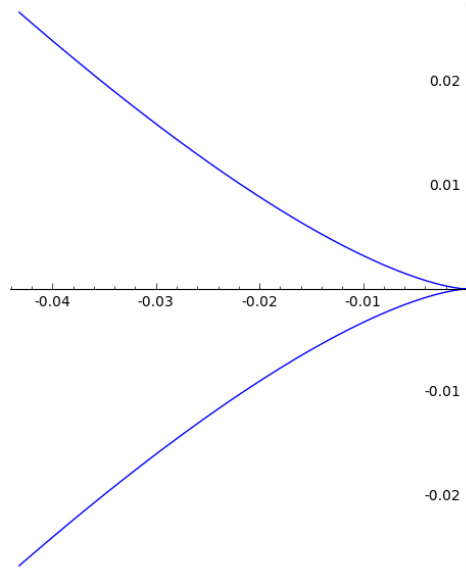
Finally, we consider the MGTM-equation of Type III for $\nu \geq 1$ and $b > \frac{c^2}{\mu} > \frac{1}{a^\nu} + \frac{1}{\mu a^{\nu-1}}$. We notice that $b > \frac{c^2}{\mu} > \frac{1}{a^\nu} + \frac{1}{\mu a^{\nu-1}}$ implies (4.9) and $0 < \mu < a\tan(\pi/2\nu)$. Note that the last assumption also implies $0 < \mu < \infty$ when $\nu = 1$ and $\mu \to 0$ as $\nu \to \infty$. We have

$$\begin{aligned}\beta_3(\eta) &= \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2 - bi\eta - (\mu - i\eta)\overline{\tilde{g}(\eta)})}{|c^2 + bi\eta - (\mu + i\eta)\tilde{g}(\eta)|^2} \\ &= \frac{(-\tau\eta^3 i - \kappa\eta^2)(c^2(\eta^2 + a^2)^\nu - bi\eta(\eta^2 + a^2)^\nu - (\mu - i\eta)(\eta^2 + a^2)^{\nu/2}(\cos(\nu\theta) + i\sin(\nu\theta)))}{|c^2 + bi\eta - (\mu + i\eta)\tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu},\end{aligned}$$

where $\theta = \arctan(\eta/a)$. Then

$$\mathfrak{Re}\,\beta_3(\eta) = \frac{-\kappa\eta^2(\eta^2 + a^2)^{\nu/2}[c^2(\eta^2 + a^2)^{\nu/2} - \eta\sin(\nu\theta) - \mu\cos(\nu\theta)]}{|c^2 + bi\eta - (\mu + i\eta)\tilde{g}(\eta)|^2(\eta^2 + a^2)^\nu}$$

$$-\frac{\tau\eta^4(\eta^2+a^2)^{\nu/2}[b(\eta^2+a^2)^{\nu/2}-\cos(\nu\theta)+(\mu/\eta)\sin(\nu\theta)]}{|c^2+bi\eta-(\mu+i\eta)\tilde{g}(\eta)|^2(\eta^2+a^2)^\nu}<0$$

for $\eta\in\mathbb{R}\setminus\{0\}$, by using $c^2>\frac{\mu}{a^\nu}+\frac{1}{a^{\nu-1}}>\frac{\mu\cos(\nu\theta)}{(\eta^2+a^2)^{\nu/2}}+\frac{\eta\sin(\nu\theta)}{(\eta^2+a^2)^{\nu/2}}$, and $b(\eta^2+a^2)^{\nu/2}-\cos(\nu\theta)+$ $(\mu/\eta)\sin(\nu\theta)>0$ for $\eta\in\mathbb{R}\setminus\{0\}$ (it is not difficult to prove the last inequality considering the cases $|\eta|>\mu$ and $|\eta|<\mu$, and using $\mu<a\tan(\pi/2\nu)$).

On the other hand,

$$\mathfrak{Im}\,\beta_3(\eta)=\frac{-\tau c^2\eta^3(\eta^2+a^2)^\nu-(\kappa-\tau\mu)\eta^3(\eta^2+a^2)^{\nu/2}\cos(\nu\theta)+\kappa b\eta^3(\eta^2+a^2)^\nu}{|c^2+bi\eta-(\mu+i\eta)\tilde{g}(\eta)|^2(\eta^2+a^2)^\nu}$$

$$+\frac{(\kappa\mu\eta^2+\tau\eta^4)(\eta^2+a^2)^{\nu/2}\sin(\nu\theta)}{|c^2+bi\eta-(\mu+i\eta)\tilde{g}(\eta)|^2(\eta^2+a^2)^\nu}.$$

Now, we consider $\tau=1,\kappa=1,b=16,c^2=4,\mu=1$ and $g(t)=g_{1,1}(t)$ (these parameters are under the hypothesis of Theorem 4.7). We observe in Figure 2 that $\mathfrak{Im}\,\beta_3(\eta)\neq0$. Then $\beta_3(\eta)\in\rho(A)$ for all $\eta\in\mathbb{R}$, and $\frac{\mathfrak{Im}\,\beta_3(\eta)}{\mathfrak{Re}\,\beta_3(\eta)}\to0$ as $\eta\to\pm\infty$. Therefore, equation (4.2) is C^α-ill-posed (Theorem 4.7), but it is C^α-well-posed for $\tau=0$ (Theorem 4.8).

FIGURE 2. Example of a parametric plot $(\mathfrak{Re}\,\beta_3(\eta),\mathfrak{Im}\,\beta_3(\eta))$
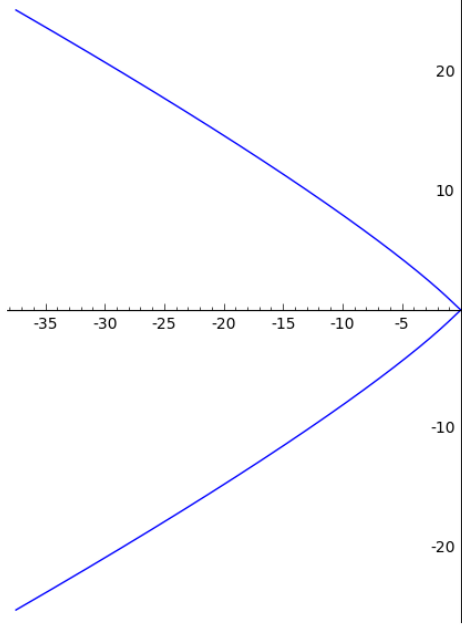
Example 5.2. Let A be the closed linear operator on $l^2(\mathbb{N})$ given by

$$(Au)_n=nu_n,\qquad D(A)=\{(u_n)\in l^2(\mathbb{N}):(n\cdot u_n)\in l^2(\mathbb{N})\}.$$

Let $b=\kappa=\tau=c^2=1$, and $g(t)=e^{-at}$ with $a>1$. Note A does not generate any C_0-semigroup on $l^2(\mathbb{N})$ because $\sigma(A)=\{n:n\in\mathbb{N}\}$. However, we will see that the MGTM-equation of Type I is still C^α-well posed.

Indeed, it is clear that $c^2 > G(+\infty)$ and g is 2-regular on $\mathbb{R}$. After some calculations we get

$$\beta(\eta) = \frac{\eta^2(\eta^2 - a)(a - \eta^2 - 1) - \eta^4(a+1)^2}{(a - \eta^2 - 1)^2 + (\eta + a\eta)^2} + i\frac{\eta^3(a+1)}{(a - \eta^2 - 1)^2 + (\eta + a\eta)^2}, \quad \eta \in \mathbb{R} \setminus \{0\}.$$

Moreover, $\beta(\eta) \in \rho(A)$ for all $\eta \in \mathbb{R}$ since $\mathfrak{Im}\,\beta(\eta) \neq 0$ for all $\eta \neq 0$.

Let $x = (x_n) \in l^2(\mathbb{N})$ and $\eta \neq 0$, then

$$||(i\eta)^3\gamma(\eta)(\beta(\eta) - A)^{-1}x||^2 = ||(i\eta)^3((i\eta)^2(i\eta + 1) - \frac{1}{\gamma(\eta)}A)^{-1}x||^2$$

$$= \sum_{n=1}^{\infty}\left|\frac{\eta^3}{(i\eta)^2(i\eta + 1) - \frac{n}{\gamma(\eta)}}x_n\right|^2 = \sum_{n=1}^{\infty}\frac{\eta^6(\eta^2 + a^2)}{(\eta^4 + \eta^2 + n(1-a))^2 + (a\eta^3 + \eta^3 + n\eta(a+1))^2}|x_n|^2$$

$$\leq \frac{a^2}{(a+1)^2}||x||^2 + \sum_{n=1}^{\infty}\frac{\eta^8}{(\eta^4 + \eta^2 + n(1-a))^2 + (a\eta^3 + \eta^3 + n\eta(a+1))^2}|x_n|^2.$$

Since $a > 0$, there exists $n_0 \in \mathbb{N}$ such that $n_0 - a \geq 0$ and therefore there exist constants $M_1, M_2 > 0$ such that

$$\sum_{n=1}^{\infty}\frac{\eta^8}{(\eta^4 + \eta^2 + n(1-a))^2 + (a\eta^3 + \eta^3 + n\eta(a+1))^2}|x_n|^2$$

$$\leq M_1 + \sum_{n=n_0}^{\infty}\frac{\eta^8}{(\eta^4 + \eta^2 + n(1-a))^2 + (a\eta^3 + \eta^3 + n\eta(a+1))^2}|x_n|^2$$

$$\leq M_1 + \sum_{n=n_0}^{\infty}\frac{\eta^8}{\eta^8 + 2\eta^6(n-a) + n\eta^2(2(n-a) + (2+n)a^2 + n)}|x_n|^2 \leq M_1 + M_2.$$

By the continuity of $\eta \to (i\eta)^3\gamma(\eta)(\beta(\eta) - A)^{-1}$ at $\eta = 0$ and the above estimates we get

$$\sup_{\eta \in \mathbb{R}}||(i\eta)^3\gamma(\eta)(b(\eta) - A)^{-1}|| < \infty.$$

Hence the equation (3.1) is C^α-well-posed by Theorem 3.12.

Our next example illustrates an application of C^α-well-posedness to obtain existence of solutions for a nonlinear version of the MGTM-equation.

Example 5.3. Let $f \in C^\alpha(\mathbb{R}, X)$ and A be a closed linear operator in X. We consider the following nonlinear version of (3.1),

$$(5.4) \qquad \tau u'''(t) + \kappa u''(t) - c^2\phi_1(u)(t) - b\phi_2(u)(t) + \phi_3(u)(t) = f(t), \quad t \in \mathbb{R},$$

where $\phi_1, \phi_3 : C^\alpha(\mathbb{R}, D(A)) \to C^\alpha(\mathbb{R}, X)$ and $\phi_2 : C^{\alpha+1}(\mathbb{R}, D(A)) \to C^\alpha(\mathbb{R}, X)$ are nonlinear functions, in general. In what follows the derivatives on $\phi_j(j = 1, 2, 3)$, are considered in the Fréchet sense.

We denote $Y := C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+3}(\mathbb{R}, X)$ and $Z := C^\alpha(\mathbb{R}, X)$. Inspired by a result of Clement and Da Prato [13, Theorem 4.1] we obtain the following result.

Theorem 5.4. *Let $f \in C^\alpha(\mathbb{R}, X)$ and $g \in L^1_{loc}(\mathbb{R}_+)$ be 2-regular on $\mathbb{R}$ and assume that f, g satisfy the condition (3.4). Let $\phi_j \in C^1(Y, Z)$ such that $\phi_j(0) = 0$ for $j = 1, 2, 3$ with $\phi'_1(0)v(t) = Av(t)$, $\phi'_2(0)v(t) = Av'(t)$ and $\phi'_3(0)v(t) = \int_{-\infty}^t g(t-s)Av(s)\,ds$ for all $v \in Y$ and $t \in \mathbb{R}$. If $\{\beta(s)\}_{s \in \mathbb{R}} \subseteq \rho(A)$ and*

$$\sup_{s \in \mathbb{R}}||(is)^3\gamma(s)(\beta(s) - A)^{-1}|| < \infty,$$

then there exist $r_0 > 0$ and $s_0 > 0$ such that for any $f \in C^\alpha(\mathbb{R}, X)$ with $\alpha \in (0,1)$ satisfying $\|f\|_{C^\alpha(\mathbb{R},X)} < r_0$, the problem (5.4) has a unique solution $u \in C^{\alpha+1}(\mathbb{R}, D(A)) \cap C^{\alpha+3}(\mathbb{R}, X)$ verifying the inequality

$$\|u\|_{C^{\alpha+1}(\mathbb{R},D(A))} + \|u\|_{C^{\alpha+1}(\mathbb{R},X)} < s_0.$$

Proof. We define the map $F : Y \to Z$ given by

$$F(u) := Lu - c^2\phi_1(u) - b\phi_2(u) + \phi_3(u),$$

where $Lu(t) := \tau u'''(t) + \kappa u''(t)$ for $u \in Y$ and $t \in \mathbb{R}$. Note that $F(0) = 0$, $F \in C^1(Y, Z)$ and $F'(u)v = Lv - c^2\phi_1'(u)v - b\phi_2'(u)v + \phi_3'(u)v$ for all $u, v \in Y$. Therefore $F'(0)v(t) = Lv(t) - c^2Av(t) - bAv'(t) + \int_{-\infty}^t g(t-s)Av(s)\, ds$, and by Theorem 3.12 $F'(0)$ is an isomorphism between Y and Z. Applying the Local Inversion Theorem we conclude the result. $\qquad\square$

References

[1] F. Alabau-Boussouira, P. Cannarsa and D. Sforza. *Decay estimates for second order evolution equations with memory.* J. Funct. Anal. **254** (2008), 1342–1372.

[2] D. Araya and C. Lizama. *Existence of asymptotically almost automorphic solutions for a third order differential equation.* E.J. Qualitative Theory of Diff. Eq. **53** (2012), 1–20.

[3] W. Arendt, C. J. Batty and S. Bu. *Fourier multipliers for Hölder continuous functions and maximal regularity.* Studia Math., **160** (2004), 23–51.

[4] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems.* Second edition, Monographs in Mathematics. **96**, Birkhäuser (2011).

[5] H. Brézis. *Analyse Fonctionnelle.* Masson, Paris, 1983.

[6] S. Bu. *Well-posedness of second order degenerate differential equations in vector-valued function spaces.* Studia Math. **214** (1) (2013), 1–16.

[7] S. Bu and Y. Fang. *Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces.* Studia Math. **184** (2) (2008), 103–119.

[8] S. Bu and G. Cai. *Well-posedness of second order degenerate differential equations in Hölder continuous function spaces.* Expo. Math. **34** (2) (2016), 223–236.

[9] G. Cai and S. Bu. *Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces.* Israel J. Math. **212** (1) (2016), 163–188.

[10] G. Cai and S. Bu. *Well-posedness of second order degenerate integrodifferential equations with infinite delay in vector-valued function spaces.* Math. Nachr. **289** (4) (2016), 436–451.

[11] A. H. Caixeta, I. Lasiecka and V. N. D. Cavalcanti. *Global attractors for a third order in time nonlinear dynamics.* J. Differential Equations, **261** (2016), 113–147.

[12] R. Chill and S. Srivastava. L_p-*maximal regularity for second order Cauchy problems.* Math. Z. **251** (4) (2005), 751–781.

[13] Ph. Clément and G. Da Prato. *Existence and regularity results for an integral equation with infinite delay in a Banach space.* Integral Equations and Operator Theory **11** (1988), 480–500.

[14] A. Conejero, C. Lizama and F. Rodenas. *Chaotic behaviour of the solutions of the Moore-Gibson-Thomson equation.* Appl. Math. Inf. Sci. **9** (5) (2015), 1–16.

[15] C. Cuevas and C. Lizama. *Well posedness for a class of flexible structure in Hölder spaces,* Math. Probl. Eng. vol. 2009, (2009), 1– 13.

[16] G. Da Prato and E. Sinestrari. *Hölder regularity for non autonomous abstract parabolic equations.* Israel J. Math. **42** 1-2 (1982), 1–19.

[17] F. Dell'oro, I. Lasiecka and V. Pata. *The Moore-Gibson-Thompson equation with memory in the critical case.* J. Differential Equations, to appear.

[18] F. Dell'oro and V. Pata. *On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity.* Appl. Math. Optim., to appear.

[19] B. De Andrade and C. Lizama. *Existence of asymptotically almost periodic solutions for damped wave equations.* J. Math. Anal. Appl., **382** (2011), 761–771.

[20] R. Denk, M. Hieber and J. Prüss. *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type.* Mem. Amer. Math. Soc. **166** (2003), no. 788.

[21] R. Denk, J. Prüss and R. Zacher. *Maximal Lp-regularity of parabolic problems with boundary dynamics of relaxation type.* J. Funct. Anal. **255** (11) (2008), 3149–3187.

[22] K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate texts in Mathematics. **194**, Springer, New York, 2000.

[23] H. O. Fattorini. *Second order linear differential equations in Banach spaces.* North-Holland Mathematics Studies, 108. Notas de Matemática [Mathematical Notes], 99. North-Holland Publishing Co., Amsterdam, 1985.

[24] C. Fernández, C. Lizama and V. Poblete. *Maximal regularity for flexible structural systems in Lebesgue spaces.* Math. Probl. Eng. vol 2010, (2010), 1–15.

[25] C. Fernández, C. Lizama and V. Poblete. *Regularity of solutions for a third order differential equation in Hilbert spaces.* Appl. Math. Comput.,**217** (21)(2011), 8522–8533.

[26] G. Gorain. *Stabilization for the vibrations modeled by the 'standard linear model' of viscoelasticity.* Proc. Indian Acad. Sci. (Math. Sci.) **120** (4) 2010, 495–506.

[27] M. Haase. *The functional calculus for sectorial operators.* Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel, 2006.

[28] B. Kaltenbacher, I. Lasiecka and R. Marchand. *Well-posedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound.* Control Cybernet. **40** (4) (2011), 971–988.

[29] B. Kaltenbacher, I. Lasiecka and M. K. Pospieszalska. *Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound.* Math. Models Methods Appl. Sci. **22**(11) (2012), 1–34.

[30] B. Kaltenbacher and I. Lasiecka. *Well-posedness of the Westervelt and the Kuznetsov equation with nonhomogeneous Neumann boundary conditions.* Discrete Contin. Dyn. Syst. 2011, Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. II, 763–773

[31] V. Keyantuo and C. Lizama. *Hölder continuous solutions for integro-differential equations and maximal regularity.* J. Differential Equations, **230** (2006), 634–660.

[32] N. T. Lan. *On the nonautonomous higher-order Cauchy problems.* Differential Integral Equations **14** (2) (2001), 241–256.

[33] I. Lasiecka and X. Wang. *Moore-Gibson-Thompson equation with memory, part I: Exponential decay of energy.* Z. Angew. Math. Phys., (2016) 67–17.

[34] I. Lasiecka and X. Wang. *Moore-Gibson-Thompson equation with memory, part II: General decay of energy.* J. Differential Equations. **259** (2015), 7610–7635.

[35] R. Marchand, T. McDevitt and R. Triggiani. *An abstract semigroup approach to the third-order Moore-Gibson-Thomson differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability,* Math. Methods Appl. Sci. **35** (15) (2012), 1896–1929.

[36] S. A. Messaoudi and S.E. Mukiana. *Existence and decay solutions to a viscoelastic plate equation.* Electr. J. Diff. Equ. **22** (2016), 1–14.

[37] F. Neubrander. *Well-posedness of higher order abstract Cauchy problems.* Trans. Amer. Math. Soc. **295** (1) (1986), 257–290.

[38] J. Prüss. Evolutionary Integral Equations and Applications. Monographs in Math. **87**, Birkhäuser, 1993.

[39] J. Prüss. *Decay properties for the solutions of a partial differential equation with memory.* Arch. Math. **92** (2009), 158–173.

[40] T.J. Xiao and J. Liang. *The Cauchy problem for higher-order abstract differential equations.* Lecture Notes in Mathematics, 1701. Springer-Verlag, Berlin, 1998.

Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile,
Las Sophoras 173, Estación Central, Santiago, Chile.
E-mail address: `luciano.abadias@gmail.com`

Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile,
Las Sophoras 173, Estación Central, Santiago, Chile.
E-mail address: `carlos.lizama@usach.cl`

BCAM-Basque Center for Applied Mathematics,
Mazarredo, 14, E48009 Bilbao, Basque Country, Spain.
E-mail address: `mmurillo@bcamath.org`