On the Existence of a Progressive Variational Vademecum based in the Proper Generalized Decomposition for a Class of Elliptic Parametrised Problems

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Abstract

In this paper we introduce the mathematical analysis needed to explain the convergence of a Progressive Variational Vademecum based in the Proper Generalized Decomposition (PGD). The PGD is a novel technique appeared in the lately years for solving problems with high dimensions and also provides new tactics for obtaining the solution of elliptic and parabolic problems by means an abstract separation of variables method. In consequence this new scenario requires a mathematical framework in order to justify his usability to solve numerical problems. The PGD will help us in the change of paradigm.

The main goal of this paper is to give a mathematical environment to define the notion of Progressive Variational Vademecum. We will prove the convergence of this iterative procedure and we also provide the first order optimality conditions in order to construct the numerical approximations of the parametrised solutions. In particular, we illustrate this methodology by means a robot path planning problem. This is one of the common task for designing the trajectory or path of a mobile robot. The construction of a Progressive Variational Vademecum gives us a novel methodology for computing all the possible paths from any start and goal positions derived from a harmonic potential field in a predefined map.

Keywords: Variational Vademecum, Proper Generalized Decomposition, Tensor Hilbert Spaces, Robot Mobile Trajectories.

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1. Introduction

The human being throughout the history developed several facilities for giving fast responses to a variety of questions. Thus, *abaci* were used 2700 years B.C. in Mesopotamia. This abacus was a sort of counting frame primarily used for performing arithmetic calculations. We associate this abacus to a bamboo frame with beads sliding on wires, however, originally they were beans or stones moved in grooves in sand or on tablets of wood, stone, or metal. The abacus was in use centuries before the adoption of the written modern numeral system and is still widely used by readers. There are many variants, the Mesopotamian abacus, the Egyptian, Persian, Greek, Roman, Chinese, Indian, Japanese, Korean, native American, Russian, etc.

However, the initial arithmetic needs were rapidly complemented with more complex representations. Like for instance nomography. It is the graphical representation of mathematical relationships or laws. It is an area of practical and theoretical mathematics invented in 1880 by Philbert Maurice d'Ocagne and used extensively for many years to provide engineers with fast graphical calculations of complicated formulas to a practical precision.

Thus, a nomogram can be considered as a graphical calculating device. There are thousands of examples on the use of nomograms in all the fields of sciences and engineering. The former facilities allowed for fast calculations and data manipulations. Nomograms can be easily constructed when the mathematical relationships that they express are purely algebraic, eventually non-linear. In those cases it was easy to represent some outputs as a function of some inputs. The calculation of these data representations was performed off-line and then used on-line in many branches of engineering sciences for design and optimization. However, the former procedures fail when addressing more complex scenarios. Thus, sometimes engineers manipulate not properly understood physics and in that case the construction of nomograms based on a too coarse modelling could be dangerous. In that cases one could proceed by making several experiments from which defining a sort of experiment-based nomogram. In other cases the mathematical object to be manipulated consists of a system of complex coupled non-linear partial differential equations, whose solution for each possible combination of the values of the parameters that it involves is simply unimaginable for the nowadays computational availabilities. In these cases experiments or expensive computational solutions are performed for some possible states of the system, from which a simplified model linking the inputs to the outputs of interest is elaborated. These simplified models have different names: surrogate models, meta-models, response surface methodologies, ... Other associated tricky questions are the one that concerns the best sampling strategy (Latin hypercube,...) and also the one concerning the appropriate interpolation techniques for estimating the response at an unmeasured position from observed values at surrounding locations. However, we must accept a certain inevitable inaccuracy when estimating solutions from the available data. It is the price to pay if neither experimental measurements nor numerical solutions of the fine but expensive model are achievable for each possible scenario. Today many problems in science and engineering remain intractable, in spite of the impressive progresses attained in modelling, numerical analysis, discretization techniques and computer science during the last decade, because their numerical complexity, or the restrictions imposed by different requirements (real-time on deployed platforms, for instance) make them unaffordable for today's technologies.

Real time analysis of complex systems is suitable for speeding up engineering design, and compulsory for making possible a real time decision-making that needs the evaluation of many possible scenarios under the real time constraint. Decision-making is at the heart of material, processes and structural optimization and also of the incipient simulation-based control. Moreover, for democratizing the accessibility to efficient design technologies decision-making tools should require slight computing platforms. These apparently contradictory requirement: the real time evaluation of system responses involved in decision-making tools and the suitability of running these applications and tools in light computational devices, could be possible if we generate off-line a sort of *computational vademecum* containing the solution of the model under consideration for all the possible design scenarios and then we use it on-line for decision- making purposes. As this last step only involves post-processing of compressed data it could run in real time and even in very light computational devices, contributing as previously argued, to the democratization of these advanced decision-making tools.

Despite the improvement techniques in numerical problems, some challenging problems remain today intractable. Our computers and algorithms for addressing the models encountered in science and engineering are definitely suboptimal. It is faced that important limitations of today's computer capabilities. Nowadays the society needs fast and accurate solutions. For that reason there is a need for a new generation simulation techniques, beyond high-performance computing or nowadays approaches (proposed 40 years ago), to simply improve efficiency or to allow getting results when other alternatives fail in the above challenging scenarios, [7, 5].

Recently model order reduction opened new possibilities in order to construct a modern version of the ancient abacus and following [7] we will call it *computational vademecum*. It is due to a wide variety of applications lead to problems where the data or the desired solution can be represented by elements that we can choose from a dictionary of functions. In this context, tensor-based methods are receiving a growing interest in scientific computing for the numerical solution of problems defined in high dimensional tensor product spaces, such as partial differential equations, [11]. The last recently years a novel technique has been developed, called the Proper Generalized Decomposition (PGD), [7, 5]. The PGD is a methodology initially proposed for compute the variational solution of partial differential equations (PDE) defined in tensor product spaces. It consists in constructing a separated representation of the variational solution of a given PDE, [10].

The advantages of PGD can be divided in two: how to deal with high dimensional problems and new strategies for solving classical problems. For instance, parameters can be set as additional extra-coordinates of the model, [7, 5].

When we use the PGD, the resulting model is solved once for life, in order to obtain a general solution that includes all the solutions for every possible value of the parameters, that is, a sort of computational vademecum. The general PGD solution (the vademecum) is computed only once and off-line. The resulting multidimensional model we have access to the parametric solution that can be viewed as a sort of vademecum that can be then used on-line, [7, 5].

A real life problem, where is possible to apply the above strategy, is to compute the trajectory of a mobile robot free of collisions. It is a fundamental robotic task for guiding a robot from a star position to a goal position using a safe path. This task is called path planning. In many real-time situations path planning is hard and infeasible [14]. The complexity of the problem has motivated different publications in the field of robot path planning, for example: [14, 3, 17]

Even a lot of papers are devoted to present applications of the construction of a computational vademencum based in the PDG (or PGD-vademecum, in short) [6, 7, 5], under the authors knowledge, there are no papers than explain the mathematical analysis of this very promising engineering framework. The main goal of this paper is to give a mathematical environment to define the notion of Progressive Variational Vademecum. The numerical implementation of the Progressive Variational Vademecum gives us the computational vademecum introduced in [6]. It is based in the definition of Progressive PGD [11] previously introduced for some of the authors of the paper. This is our first contribution. Next, we will choose a convenient tensor Hilbert space that allows to prove the convergence to the Vademecum, that is, a solution that contains the whole set of parametrised solutions of the PDE under consideration. Our last contribution is to give a set of first order optimality conditions in order to perform a numerical approach to the construction of the Vademecum.

This paper is organized as follows. Section 2 introduces the notion of variational vademecum by using the potential flow theory to obtain a parametrized Laplace equation. In Section 3 is detailed some properties of Tensor Hilbert spaces that we will use to prove the results given in the paper. In Section 4 is shown a progressive construction of a variational vademecum based in the Proper Generalized Decomposition. The first order optimality conditions related with the above construction are explained in Section 5. In Section 6, some simulation examples are provided. Finally, Section 7 draws conclusions.

2. Motivation

In this section we introduce the notion of variational vademecum. To this end we will use the potential flow theory in robotics to illustrate the definition together a real life application.

2.1. Potential flow theory

Path planning based on the potential flow theory has been used in the literature during the last years, see [8]-[13], focused mainly in the resolution of the Laplace equation. First of all, let us outline the mathematical model describing the flow of an inviscid incompressible fluid. Assuming a steady state irrotational flow in the Eulerian framework, the velocity $\boldsymbol{v}$ obeys the relation

$$\bigtriangledown \times \boldsymbol{v} = \mathbf{0}, \tag{2.1}$$

and hence the velocity is the gradient of a scalar potential function, i.e. $\boldsymbol{v} = \nabla u$. Then the potential u appears as a solution of the Laplace equation:

$$\Delta u = 0. \tag{2.2}$$

By using a 2.5D mould filling model similar to [9] it is possible to introduce a localized fluid source (respectively, sink) modelled by a Dirac term δ_S (respectively, $-\delta_T$) added to the right hand side of (2.2). To this end we assume a unit amount of fluid injected at point S during a unit of time and the same unit withdrawn at point T, the velocity of the fluid is now the solution of the Poisson equation, that includes the source term $f = \delta_S - \delta_T$ as:

$$-\Delta u = \delta_S - \delta_T. \tag{2.3}$$

Equation (2.3) must be complemented by appropriate boundary conditions. In these sense, the fluid cannot flow through the boundaries, a condition expressed by $\boldsymbol{v} \cdot \mathbf{n}$ ($\mathbf{n}$ being a vector normal to the boundary Γ). The resolution of the Poisson equation under these conditions produces a potential field from the Starting point S (source) to the Target point T (sink), without deadlocks [13].

2.2. Source term definition and classical variational formulation

First, let's assume that the source term f is non-uniform, that is, $f = g_S \mathbf{1}_{\Omega_{\underline{X}}} - g_T \mathbf{1}_{\Omega_{\underline{X}}}$ where the function $\mathbf{1}_{\Omega_{\underline{X}}}(x,y) = 1$ when $(x,y) \in \Omega_{\underline{X}}$ and zero otherwise. The functions $g_S : \Omega_{\underline{X}} \times \Omega_{\underline{S}} \to \mathbb{R}$ and $g_T : \Omega_{\underline{X}} \times \Omega_{\underline{T}} \to \mathbb{R}$ are two-dimensional Gaussian density distributions centered in $\underline{S} = (s_1, s_2) \in \Omega_{\underline{X}}$ and $\underline{T} = (t_1, t_2) \in \Omega_{\underline{X}}$, respectively and both have equal variance given by a diagonal matrix $\Sigma = \mathrm{diag}(r,r)$ for some $r > 0$. More precisely, we can write $g_S = g_S((x,y);(s_1,s_2),r) = (2\pi r)^{-1} e^{-\frac{1}{2r}((x-s_1)^2+(y-s_2)^2)}$, $g_T = g_T((x,y);(t_1,t_2),r) = (2\pi r)^{-1} e^{-\frac{1}{2r}((x-t_1)^2+(y-t_2)^2)}$ and hence $\Omega_{\underline{X}} = \Omega_{\underline{x}} \times \Omega_{\underline{y}}$, $\Omega_{\underline{S}} = \Omega_{\underline{s}} \times \Omega_{\underline{r}}$ and $\Omega_{\underline{T}} = \Omega_{\underline{t}} \times \Omega_{\underline{r}}$. Here $\Omega_{\underline{X}} = \Omega_{\underline{s}} = \Omega_{\underline{t}} \subset \mathbb{R}^2$ and $\Omega_{\underline{r}} \subset (0,\infty)$. Then, the Poisson equation is now

$$-\Delta u(x,y) \quad = f((x,y);(s_1,s_2),(t_1,t_2),r) \tag{2.4}$$

where $f := g_S - g_T$, and the solution is in the form

$$u = u((x,y);(s_1,s_2),(t_1,t_2),r).$$

We recall that the Hilbert space $H_0^1(\Omega_{\underline{X}})$ is the closure of $C_c^\infty(\Omega_{\underline{X}})$ (functions in $C^\infty(\Omega_{\underline{X}})$ with compact support in Ω) in $H^1(\Omega_{\underline{X}})$ with respect to the norm in $H^1(\Omega_{\underline{X}})$. We equip $H_0^1(\Omega_{\underline{X}})$ with the norm

$$\|u\|_{H^1(\Omega_{\underline{X}})} := \left(\|\partial_x u\|^2_{L^2(\Omega_{\underline{X}})} + \|\partial_y u\|^2_{L^2(\Omega_{\underline{X}})} \right)^{1/2}$$

which is equivalent to the classical norm on $H^1(\Omega_{\underline{X}})$.

The classical variational formulation for (2.4) together $u|_{\partial\Omega_{\underline{X}}} = 0$ is: For each fixed $(s_1,s_2),(t_1,t_2)$ and r find $u \in H_0^1(\Omega_{\underline{X}})$ such that

$$\int_{\Omega_{\underline{X}}} \nabla_{\underline{X}} u \cdot \nabla_{\underline{X}} v = \int_{\Omega_{\underline{X}}} f v \tag{2.5}$$

holds for all $v \in H_0^1(\Omega_{\underline{X}})$. Here $\nabla_{\underline{X}}$ denotes the gradient in the coordinates $\underline{X} = (x,y)$.

2.3. A variational vademecum

From now one, we will assume that the common variance r take a fixed value and we will construct a cademecum by consider the solution of (2.4) for a fixed r and all values of

$$(\underline{X} = (x,y); \underline{S} = (s_1,s_2), \underline{T} = (t_1,t_2)) \in \Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}.$$

To this end we will consider in the next section a closed subspace, namely $\mathbf{H}_0$, of the tensor Hilbert space $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$, and then a variational vademecum $u \in \mathbf{H}_0$ can be introduced as follows. Find $u \in \mathbf{H}_0$ such that

$$\int_{\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}} \nabla_{\underline{X}} u \cdot \nabla_{\underline{X}} v = \int_{\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}} f v \tag{2.6}$$

holds for all $v \in \mathbf{H}_0$. In this case, u gives us the set of variational solutions of (2.4) for all possible parameter values $(s_1,s_2) \in \Omega_{\underline{S}}$ and $(t_1,t_2) \in \Omega_{\underline{T}}$. We remark that solving once (2.6) we solve variationally (2.4) for all possible parameter values $(s_1,s_2) \in \Omega_{\underline{S}}$ and $(t_1,t_2) \in \Omega_{\underline{T}}$.

2.4. A progresive variational vademecum

The mathematical analysis of a progressive PGD to solve (2.5) in the tensor Hilbert space $H_0^1(\Omega_{\underline{X}})$ for a fixed $(s_1,s_2)\in\Omega_{\underline{S}}$ and $(t_1,t_2)\in\Omega_{\underline{T}}$ has been introduced in [2] (see also [9, 11]). In the next sections we develop the mathematical analysis needed to justify the use of a progressive PGD to solve (2.6) and hence we give a constructive approach to obtain progresive variational vademecum.

3. Tensor Hilbert spaces

We first consider the definition of the algebraic tensor space ${}_a\bigotimes_{j=1}^d V_j$ generated from Hilbert spaces V_j $(1\le j\le d)$ equipped with norms $\|\cdot\|_j$. As underlying field we choose $\mathbb{R}$, but the results hold also for $\mathbb{C}$. The suffix 'a' in ${}_a\bigotimes_{j=1}^d V_j$ refers to the 'algebraic' nature. By definition, all elements of

$$\mathbf{V} := {}_a\bigotimes_{j=1}^d V_j$$

are *finite* linear combinations of elementary tensors $\mathbf{v}=\bigotimes_{j=1}^d v^{(j)}\ \left(v^{(j)}\in V_j\right)$.

A typical representation format is the Tucker or tensor subspace format

$$\mathbf{u}=\sum_{\mathbf{i}\in\mathbf{I}}\mathbf{a}_{\mathbf{i}}\bigotimes_{j=1}^d b_{i_j}^{(j)}, \tag{3.1}$$

where $\mathbf{I}=I_1\times\ldots\times I_d$ is a multi-index set with $I_j=\{1,\ldots,r_j\}$, $r_j\le\dim(V_j)$, $b_{i_j}^{(j)}\in V_j$ $\left(i_j\in I_j\right)$ are linearly independent (usually orthonormal) vectors, and $\mathbf{a}_{\mathbf{i}}\in\mathbb{R}$. Here, i_j are the components of $\mathbf{i}=(i_1,\ldots,i_d)$. The data size is determined by the numbers r_j collected in the tuple $\mathbf{r}:=(r_1,\ldots,r_d)$. The set of all tensors representable by (3.1) with fixed $\mathbf{r}$ is

$$\mathscr{M}_{\le\mathbf{r}}(\mathbf{V}) := \left\{\mathbf{v}\in\mathbf{V}:\ \begin{array}{l}\text{there are subspaces } U_j\subset V_j \text{ such that}\\ \dim(U_j)=r_j \text{ and } \mathbf{v}\in\mathbf{U}:={}_a\bigotimes_{j=1}^d U_j\,.\end{array}\right\} \tag{3.2}$$

To simplify the notations, the set of rank-one tensors (elementary tensors) will be denoted by

$$\mathscr{M}_{\le 1}(\mathbf{V}) := \mathscr{M}_{\le(1,\ldots,1)}(\mathbf{V}) = \left\{\otimes_{k=1}^d w^{(k)} : w^{(k)}\in V_k\right\}.$$

By definition, we then have $\mathbf{V}=\operatorname{span}\mathscr{M}_{\le 1}(\mathbf{V})$. We also introduce the set of rank-m ($m>1$) tensors defined by

$$\mathscr{R}_m(\mathbf{V}) := \left\{\sum_{i=1}^m \mathbf{z}_i : \mathbf{z}_i\in\mathscr{M}_{\le 1}(\mathbf{V})\setminus\{\mathbf{0}\}\right\}.$$

We say that $\mathbf{V}_{\|\cdot\|}$ is a *Hilbert tensor space* if there exists an algebraic tensor space $\mathbf{V}$ and a norm $\|\cdot\|$ on $\mathbf{V}$ such that $\mathbf{V}_{\|\cdot\|}$ is the completion of $\mathbf{V}$ with respect to the norm $\|\cdot\|$, i.e.

$$\mathbf{V}_{\|\cdot\|} := {}_{\|\cdot\|}\bigotimes_{j=1}^d V_j = \overline{{}_a\bigotimes_{j=1}^d V_j}^{\|\cdot\|}.$$

Observe that $\operatorname{span}\mathscr{M}_{\le 1}(\mathbf{V})$ is dense in $\mathbf{V}_{\|\cdot\|}$. Since $\mathscr{M}_{\le 1}(\mathbf{V})\subset\mathscr{M}_{\le\mathbf{r}}(\mathbf{V})$ for all $\mathbf{r}\ge(1,1,\ldots,1)$, then $\operatorname{span}\mathscr{M}_{\le\mathbf{r}}(\mathbf{V})$ is also dense in $\mathbf{V}_{\|\cdot\|}$.

3.1. Topological properties of Tensor Hilbert spaces

Any norm $\|\cdot\|$ on ${}_a\bigotimes_{j=1}^{d} V_j$ satisfying

$$\left\|\bigotimes_{j=1}^{d} v^{(j)}\right\| = \prod_{j=1}^{d} \|v^{(j)}\|_j \qquad \text{for all } v^{(j)} \in V_j \;\; (1 \leq j \leq d) \tag{3.3}$$

is called a *crossnorm.*

Remark 3.1. *Eq. (3.3) implies the inequality* $\|\bigotimes_{j=1}^{d} v^{(j)}\| \lesssim \prod_{j=1}^{d} \|v^{(j)}\|_j$ *which is equivalent to the continuity of the tensor product mapping*

$$\bigotimes : \bigtimes_{j=1}^{d} \left(V_j, \|\cdot\|_j\right) \longrightarrow \left({}_a\bigotimes_{j=1}^{d} V_j \,, \|\cdot\|\right), \tag{3.4}$$

given by $\otimes\left((v^{(1)},\ldots,v^{(d)})\right) = \otimes_{j=1}^{d} v^{(j)}$, *where* $(X, \|\cdot\|)$ *denotes a vector space* X *equipped with norm* $\|\cdot\|$.

As usual, the dual norm to $\|\cdot\|$ is denoted by $\|\cdot\|^*$. If $\|\cdot\|$ is a crossnorm and also $\|\cdot\|^*$ is a crossnorm on ${}_a\bigotimes_{j=1}^{d} V_j^*$, i.e.

$$\left\|\bigotimes_{j=1}^{d} \varphi^{(j)}\right\|^* = \prod_{j=1}^{d} \|\varphi^{(j)}\|_j^* \qquad \text{for all } \varphi^{(j)} \in V_j^* \;\; (1 \leq j \leq d), \tag{3.5}$$

$\|\cdot\|$ is called a *reasonable crossnorm.*

We recall that a sequence $v_m \in V$ is *weakly convergent* if $\lim_{m\to\infty}\langle \varphi, v_m\rangle$ exists for all $\varphi \in V^*$. We say that $(v_m)_{m\in\mathbb{N}}$ *converges weakly to* $v \in V$ if $\lim_{m\to\infty}\langle \varphi, v_m\rangle = \langle \varphi, v\rangle$ for all $\varphi \in V^*$. In this case, we write $v_m \rightharpoonup v$.

Definition 3.2. *A subset* $M \subset V$ *is called* weakly closed *if* $v_m \in M$ *and* $v_m \rightharpoonup v$ *implies* $v \in M$.

Note that 'weakly closed' is stronger than 'closed', i.e., M weakly closed $\Rightarrow$ M closed. The following proposition has been proved in [11].

Proposition 3.3. *Let* $\mathbf{V}_{\|\cdot\|}$ *be a Hilbert tensor space with a cross norm Then the set* $\mathscr{M}_{\leq \mathbf{r}}(\mathbf{V})$ *is weakly closed.*

4. A progressive construction of a variational vademecum

Let us consider $\Omega_{\underline{X}}, \Omega_{\underline{S}}, \Omega_{\underline{T}} \subset \mathbb{R}^2$ open and bounded domains and let us introduce the variables $\underline{X} = (x,y)$, $\underline{S} = (s_1,s_2)$ and $\underline{T} = (t_1,t_2)$. The aim of the paper is given $f(\underline{X};\underline{S}) \in L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}})$ and $g(\underline{X};\underline{T}) \in L^2(\Omega_{\underline{X}} \times \Omega_{\underline{T}})$ construct iteratively, by means a Greedy Rank-One Algorithm, a variational solution of the parametrised problem

$$-(\partial_x^2 + \partial_y^2)u(\underline{X};\underline{S},\underline{T}) = f(\underline{X};\underline{S}) - g(\underline{X};\underline{T}), \tag{4.1}$$

for $(\underline{X};\underline{S},\underline{T}) \in \Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}$ together the homogeneous boundary condition

$$u(\underline{X};\underline{S},\underline{T}) = 0 \text{ for all } (\underline{X};\underline{S},\underline{T}) \in \partial\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}. \tag{4.2}$$

Since $f(\underline{X};\underline{S}^{(0)}) \in L^2(\Omega_{\underline{X}})$ for each fixed $\underline{S}^{(0)} \in \Omega_{\underline{S}}$ and $g(\underline{X};\underline{T}^{(0)}) \in L^2(\Omega_{\underline{X}})$ for each fixed $\underline{T}^{(0)} \in \Omega_{\underline{T}}$, classical results give us, for each fixed $(\underline{S}^{(0)},\underline{T}^{(0)}) \in \Omega_{\underline{S}} \times \Omega_{\underline{T}}$, the existence and unicity of a weak solution for the PDE:

$$-(\partial_x^2+\partial_y^2)u(\underline{X};\underline{S}^{(0)},\underline{T}^{(0)}) = f(\underline{X};\underline{S}^{(0)}) - g(\underline{X};\underline{T}^{(0)}), \tag{4.3}$$

for $\underline{X} \in \Omega_{\underline{X}}$ together the homogeneous boundary condition

$$u(\underline{X};\underline{S}^{(0)},\underline{T}^{(0)}) = 0 \text{ for all } \underline{X} \in \partial\Omega_{\underline{X}}. \tag{4.4}$$

Thus, we have a map $u(\underline{X};\underline{S},\underline{T}) \in L^2(\Omega_{\underline{X}})$ that solves (4.1)-(4.2) for all $(\underline{S},\underline{T}) \in \Omega_{\underline{S}} \times \Omega_{\underline{T}}$. The idea of the Abacus introduced in [7] is to construct iteratively a global solution of the parametrised PDE (4.1)-(4.2) following the ideas that we will explain below.

Now, our main goal is construct iteratively the weak solution of (4.1)-(4.2) denoted by $u(\underline{X};\underline{S},\underline{T})$. To this end we introduce the following algebraic tensor product space

$$H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}) = \\ \operatorname{span}\{u_1(\underline{X})u_2(\underline{S})u_3(\underline{T}) : u_1(\underline{X}) \in H_0^1(\Omega_{\underline{X}}), u_2(\underline{S}) \in L^2(\Omega_{\underline{S}}) \text{ and } u_3(\underline{T}) \in L^2(\Omega_{\underline{T}})\}$$

endowed with the norm

$$\begin{aligned}\|u(\underline{X};\underline{S},\underline{T})\|_{(1,0,0)}^2 &:= \int_{\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}}} ((\partial_x u(\underline{X};\underline{S},\underline{T}))^2 + (\partial_y u(\underline{X};\underline{S},\underline{T}))^2) d\Omega_{\underline{X}}\, d\Omega_{\underline{S}}\, d\Omega_{\underline{T}} \\ &= \|\partial_x u(\underline{X};\underline{S},\underline{T})\|_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}^2 + \|\partial_y u(\underline{X};\underline{S},\underline{T})\|_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}^2.\end{aligned}$$

The norm $\|\cdot\|_{(1,0,0)}$ is indeed a cross-norm because

$$\|u_1(\underline{X})u_2(\underline{S})u_3(\underline{T})\|_{(1,0,0)} = \|u_1(\underline{X})\|_{H_0^1(\Omega_{\underline{X}})} \|u_2(\underline{S})\|_{L^2(\Omega_{\underline{S}})} \|u_3(\underline{T})\|_{L^2(\Omega_{\underline{T}})}$$

holds for all $u_1 \in H_0^1(\Omega_{\underline{X}})$, $u_2 \in L^2(\Omega_{\underline{S}})$ and $u_3 \in L^2(\Omega_{\underline{T}})$. By taking its completion over this norm we have the following Hilbert tensor space

$$\mathbf{H}_0 := \overline{H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})}^{\|\cdot\|_{(1,0,0)}} \subset L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}).$$

The inner product $\langle \cdot,\cdot \rangle_{(1,0,0)}$ is given by

$$\langle u,v \rangle_{(1,0,0)} = \int_{\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}}} (\partial_x u \partial_x v) + \partial_y u \partial_y v) d\Omega_{\underline{X}}\, d\Omega_{\underline{S}}\, d\Omega_{\underline{T}}.$$

In particular, we obtain for the rank-one tensors

$$\begin{aligned}&\langle u_1(\underline{X})u_2(\underline{S})u_3(\underline{T}), v_1(\underline{X})v_2(\underline{S})v_3(\underline{T})\rangle_{(1,0,0)} \\ &= \langle u_1(\underline{X}), v_1(\underline{X})\rangle_{H^1(\Omega_{\underline{X}})} \langle u_2(\underline{S}), v_2(\underline{S})\rangle_{L^2(\Omega_{\underline{S}})} \langle u_3(\underline{T}), v_3(\underline{T})\rangle_{L^2(\Omega_{\underline{T}})}.\end{aligned}$$

where $\nabla_{\underline{X}}$ denotes the gradient in the coordinates $\underline{X} = (x,y)$. Observe that $u(\underline{X};\underline{S},\underline{T}) = 0$ for $\underline{X} \in \partial\Omega_{\underline{X}}$ holds for all $u \in \mathbf{H}_0$ and that for each fixed $(\underline{S}^{(0)},\underline{T}^{(0)}) \in \Omega_{\underline{S}} \times \Omega_{\underline{T}}$, the set

$$\mathbf{F}_{(\underline{S}^{(0)},\underline{T}^{(0)})} := \{u \in \mathbf{H}_0 : u = u(\underline{X};\underline{S}^{(0)},\underline{T}^{(0)})\}$$

is a closed subspace of $\mathbf{H}_0$ linearly isomorphic to $H_0^1(\Omega_{\underline{X}})$. Next we introduce the set of tensors of bounded rank one:

$$\mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) := \{u_1(\underline{X})u_2(\underline{S})u_3(\underline{T}) : u_1(\underline{X}) \in H_0^1(\Omega_{\underline{X}}), u_2(\underline{S}) \in L^2(\Omega_{\underline{S}}) \text{ and } u_3(\underline{T}) \in L^2(\Omega_{\underline{T}})\}$$

The next lemma gives the main properties of $\mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$.

Lemma 4.1. *The set $\mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) \subset \mathbf{H}_0$ satisfy the following properties*

(a) $\operatorname{span} \mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ *is dense in* $\mathbf{H}_0$.

(b) It is a cone, that is, if $u \in \mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ then $\lambda u \in \mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ for all $\lambda \in \mathbb{R}$.

(c) It is a weakly closed set in $\mathbf{H}_0$.

Proof. The proofs of (a) and (b) are straightforward. (c) follows from Proposition 3.3 because the norm $\|\cdot\|_{(1,0,0)}$ is a cross-norm. □

Now, we consider the functional

$$J : \mathbf{H}_0 \longrightarrow \mathbb{R}$$

given by

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{(1,0,0)}^2 - \langle (f-g), u\rangle_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})} \\ &= \frac{1}{2}(\|\partial_x u\|_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}^2 + \|\partial_y u\|_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}^2) - \langle (f-g), u\rangle_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}. \end{aligned}$$

Then we can identify $J'(u) \in \mathbf{H}_0^*$ with an element in $\mathbf{H}_0$ denoted by $-\Delta_{\underline{X}}(u) - (f-g)$, here $-\Delta_{\underline{X}}(u) = -(\partial_x^2 + \partial_y^2)(u)$ denotes the Laplacian operator in coordinates $\overline{X} = (x,y)$. The following assumptions (A1)-(A3) on the functional are satisfied (see [4]).

(A1) J is Fréchet differentiable, with Fréchet differential $J' : \mathbf{H}_0 \to \mathbf{H}_0^*$.

(A2) J is elliptic and

(A3) $J' : \mathbf{H}_0 \longrightarrow \mathbf{H}_0^*$ is Lipschitz continuous on bounded sets.

Thanks to the Lemma 4.1 and that the functional J satisfies (A1)-(A2) we can introduce the following definition.

Definition 4.2 (Progressive Variational Vademecum)**.** *Since $J : \mathbf{H}_0 \longrightarrow \mathbb{R}$ satisfies (A1)-(A2) let $u \in \mathbf{H}_0$ be such that*

$$J(u) = \min_{\upsilon \in \mathbf{H}_0} J(\upsilon). \tag{4.5}$$

We define a Progressive Variational Vademecum $\{u_m\}_{m \geq 1}$ over the set of tensors of bounded rank-one $\mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ of u, as follows. We let $u_0 = 0$ and for $m \geq 1$, we *construct $u_m \in \mathbf{H}_0$ from $u_{m-1} \in \mathbf{H}_0$ as we show below. Since J satisfies (A3) and from Lemma 4.1 we can find an element*

$$\hat{z}_m \in \mathscr{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) \subset \mathbf{H}_0$$

such that

$$J(u_{m-1} + \hat{z}_m) = \min_{z \in \mathscr{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))} J(u_{m-1} + z) \quad (*).$$

Next before to update m to $m+1$, define $u_m = u_{m-1} + \hat{z}_m$, update m to $m+1$ and goto $(*)$.

The key point in the above procedure is the minimization problem $(*)$ because for each m we can consider that $J(u_{m-1} + \cdot)$ is a map

$$\begin{array}{rrcl} J(u_{m-1}+\cdot): & \mathscr{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) & \longrightarrow & \mathbb{R}, \\ & z & \mapsto & J(u_{m-1}+\cdot)(z) := J(u_{m-1}+z), \end{array}$$

where

$$J(u_{m-1}+z) = \frac{1}{2}\|u_{m-1}+z\|_{(1,0,0)}^2 - \langle (f-g), z \rangle_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})} - \langle (f-g), u_{m-1} \rangle_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}.$$

Then

$$J'(u_{m-1}+z) = -\Delta_{\underline{X}}(u_{m-1}+z) - (f-g) = -\Delta_{\underline{X}}(z) - (\Delta_{\underline{X}}(u_{m-1}) + (f-g)),$$

where $\Delta_{\underline{X}}(u_{m-1}) + (f-g)$ is the residual obtained in the previous step. Observe that at each update in the construction of a Progressive Variational Vademecum $\{u_m\}_{m\geq 1}$ over the set of tensors of bounded rank-one $\mathscr{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ of u, we obtain a rank-one function, namely

$$z_m(\underline{X};\underline{S},\underline{T}) = u_1^{(m)}(\underline{X})u_2^{(m)}(\underline{S})u_3^{(m)}(\underline{T}).$$

If $\hat{z}_m = 0$ then from Lemma 5 in [11] it follows that $u_m = u_{m-1} = u$ satisfy (4.5). In consequence,

$$u(\underline{X};\underline{S},\underline{T}) = \sum_{n=1}^{m-1} u_1^{(n)}(\underline{X})u_2^{(n)}(\underline{S})u_3^{(n)}(\underline{T}).$$

Otherwise, if $\hat{z}_m \neq 0$ we write

$$u_m(\underline{X};\underline{S},\underline{T}) = \sum_{n=1}^{m} u_1^{(n)}(\underline{X})u_2^{(n)}(\underline{S})u_3^{(n)}(\underline{T}) \in \mathbf{H}_0,$$

and continue. From Theorem 5 in [11] it follows the next result.

Theorem 4.3. *Let $u \in \mathbf{H}_0$ satisfy (4.5) and consider a Progressive Variational Vademecum $\{u_m\}_{m\geq 1}$ over $\mathscr{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ of u. Then $\{u_m\}_{m\geq 1}$, converges in $\mathbf{H}_0$ to u, that is,*

$$\lim_{m\to\infty} \|u - u_m\|_{(1,0,0)} = 0.$$

5. About the first order optimality conditions

In order to give the first order optimality condition for the minimization problem $(*)$, we will consider the set of tensors of fixed rank one:

$$\mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) := \mathcal{M}_{\leq 1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) \setminus \{0\}$$

and the restricted map

$$J : \mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) \subset L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}) \longrightarrow \mathbb{R},$$

given by

$$\begin{aligned} J(u_1u_2u_2) = {} & \frac{1}{2}\left(\|\partial_x u_1(\underline{X})u_2(\underline{S})u_3(\underline{T})\|^2_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})} + \|\partial_y u_1(\underline{X})u_2(\underline{S})u_3(\underline{T})\|^2_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}\right) \\ & - \langle (f-g), u_1(\underline{X})u_2(\underline{S})u_3(\underline{T})\rangle_{L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})}. \end{aligned}$$

It can be show that the set $\mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ is a Hilbert manifold modelled in a particular Hilbert space. In this paper we try to avoid the use of the differential geometry in infinite dimensions framework by means the use of an adequate parametrisation the set

$$\mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})),$$

considered as a sub-manifold of the Hilbert space $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$, in a local neighbourhood of the optimal point $\hat{z}$.

To this end we assume that $\hat{z} := \lambda u_1u_2u_3 \in \mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$, where $\lambda \in \mathbb{R} \setminus \{0\}$, $u_1 \in H_0^1(\Omega_{\underline{X}}) \setminus \{0\}$, $u_2 \in L^2(\Omega_{\underline{S}}) \setminus \{0\}$ and $u_3 \in L^2(\Omega_{\underline{T}}) \setminus \{0\}$ is the solution of $(*)$ when $u_{m-1} = 0$. Next, we will introduce a set, denoted by

$$\mathcal{U}(\lambda u_1u_2u_3) \subset \mathcal{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}})) \subset \mathbf{H}_0$$

containing $\lambda u_1u_2u_3$ and such that there exists an open set $\mathcal{U}$ in a Hilbert space $\mathscr{H}$ and a bijection

$$\varphi_{\lambda u_1u_2u_3} : \mathcal{U} \subset \mathscr{H} \longrightarrow \mathcal{U}(\lambda u_1u_2u_3).$$

In a second step we will prove that $\varphi_{\lambda u_1u_2u_3}$ considered as a map

$$\varphi_{\lambda u_1u_2u_3} : \mathcal{U} \subset \mathscr{H} \longrightarrow L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$$

is Fréchet differentiable and its derivative

$$\varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)) \in \mathcal{L}(\mathscr{H}, L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}}))$$

is injective. In consequence, the map

$$\varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)) : \mathscr{H} \longrightarrow \varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3))(\mathscr{H}) \subset L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$$

is a linear isomorphism between vector spaces. If $\varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3))(\mathscr{H})$ is a closed subspace of $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$ then we can identify this space with the Hilbert space $\mathscr{H}$, which is the tangent space of the open set $\mathscr{U}$ (that we can identify with $\mathscr{U}(\lambda u_1u_2u_3)$) at $\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)$. In consequence, the closed space

$$\mathrm{T}\mathscr{U}(\lambda u_1u_2u_3) := \varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3))(\mathscr{H}) \subset \mathbf{H}_0$$

is the tangent space of $\mathscr{U}(\lambda u_1u_2u_3)$ at $\lambda u_1u_2u_3$. Finally, we will consider the map

$$J \circ \varphi_{\lambda u_1u_2u_3} : \mathscr{U} \subset \mathscr{H} \longrightarrow \mathbb{R}.$$

Since

$$(J \circ \varphi_{\lambda u_1u_2u_3})'(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)) \in \mathscr{L}(\mathscr{H}, \mathbb{R}).$$

and by the chain rule

$$(J \circ \varphi_{\lambda u_1u_2u_3})'(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)) = J'(\lambda u_1u_2u_3)\left([\varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3))]\right).$$

Thus, if $\lambda u_1u_2u_3$ is a minimum of J in $\mathscr{U}(\lambda u_1u_2u_3)$ then it holds that

$$J'(\lambda u_1u_2u_3)\left(\varphi'_{\lambda u_1u_2u_3}(\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)\right)(h) = 0 \text{ for all } h \in \mathscr{H}.$$

Now, $J'(\lambda u_1u_2u_3) \in \mathbf{H}_0^*$, that we can identify with a vector in $\mathbf{H}_0$ also denoted by $J'(\lambda u_1u_2u_3)$, and $\left((\varphi^{-1}_{\lambda u_1u_2u_3}(\lambda u_1u_2u_3)\right)(h) \in \mathrm{T}\mathscr{U}(\lambda u_1u_2u_3)$ for all $h \in \mathscr{H}$. Thus, we obtain that the first order optimality condition means:

$$J'(\lambda u_1u_2u_3) \perp \mathrm{T}\mathscr{U}(\lambda u_1u_2u_3). \tag{5.1}$$

5.0.1. A parametrisation of $\mathscr{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$

Take $\lambda u_1u_2u_3 \in \mathscr{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ where $u_1 \in H_0^1(\Omega_{\underline{X}}), u_2 \in L^2(\Omega_{\underline{S}})$, $u_3 \in L^2(\Omega_{\underline{T}})$ and $\lambda \in \mathbb{R}_* := \mathbb{R} \setminus \{0\}$. Then let

$$U_{u_i} := \operatorname{span}\{u_i\}^{\perp},$$

be the orthogonal complement of the linear subspace $\operatorname{span}\{u_i\}$ for $1 \le i \le 3$ in $L^2(\Omega_{\underline{X}})$, $L^2(\Omega_{\underline{S}})$ and $L^2(\Omega_{\underline{T}})$, respectively. Observe that we take for the one-dimensional subspace $\operatorname{span}\{u_1\}$ its orthogonal complement in $L^2(\Omega_{\underline{X}})$ not in $H_0^1(\Omega_{\underline{X}})$ and hence

$$L^2(\Omega_{\underline{X}}) = \operatorname{span}\{u_i\} \oplus U_{u_1}.$$

Clearly,

$$H_0^1(\Omega_{\underline{X}}) = \operatorname{span}\{u_i\} \oplus (U_{u_1} \cap H_0^1(\Omega_{\underline{X}}))$$

holds. Now, we define an open neighbourhood of $\lambda u_1u_2u_3$ in the set $\mathscr{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ as follows. Let be introduce an open neighbourhood of $\lambda u_1u_2u_3$ in the set of tensor of fixed rank-one as follows:

$$\mathscr{U}(\lambda u_1u_2u_3) := \{\beta(u_1 + u_1^{\perp})(u_1 + u_2^{\perp})(u_1 + u_2^{\perp}) : (u_1^{\perp}, u_2^{\perp}, u_3^{\perp}, \beta) \in U_{u_1} \times U_{u_2} \times U_{u_3} \times \mathbb{R}_*\}$$

and then we introduce the map

$$\varphi_{\lambda u_1 u_2 u_3} : U_{u_1} \times U_{u_2} \times U_{u_3} \times \mathbb{R}_* \longrightarrow \mathscr{U}(\lambda u_1 u_2 u_3),$$

defined by

$$\varphi_{\lambda u_1 u_2 u_3}(u_1^\perp, u_2^\perp, u_3^\perp, \beta) := \beta(u_1 + u_1^\perp)(u_1 + u_2^\perp)(u_1 + u_2^\perp).$$

Clearly $\varphi_{\lambda u_1 u_2 u_3}(0,0,0,\lambda) = \lambda u_1 u_2 u_3$, and $\mathscr{U} := U_{u_1} \times U_{u_2} \times U_{u_3} \times \mathbb{R}_*$ is an open set of the product Hilbert space $\mathscr{H} := U_{u_1} \times U_{u_2} \times U_{u_3} \times \mathbb{R}$. Moreover, $\varphi_{\lambda u_1 u_2 u_3}$ considered as a map between $\mathscr{U}$ and $\mathbf{H}_0$ is Fréchet differentiable and its derivative at $(0,0,0,\lambda)$ is a linear map

$$\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda) : \mathscr{H} \to \mathbf{H}_0$$

given by

$$\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right)(\delta u_1, \delta u_2, \delta u_3, \delta\lambda) = (\delta\lambda) u_1 u_2 u_3 + \lambda\left((\delta u_1) u_2 u_3 + u_1(\delta u_2) u_3 + u_1 u_2 (\delta u_3)\right).$$

The next proposition give us the main properties of the map $\varphi_{\lambda u_1 u_2 u_3}$ and its derivative.

Proposition 5.1. *The map* $\varphi_{\lambda u_1 u_2 u_3} : \mathscr{U} \to \mathscr{U}(\lambda u_1 u_2 u_3)$ *is bijective and*

$$\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right) : \mathscr{H} \to \mathbf{H}_0$$

is a linear injective map and $\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right)(\mathscr{H})$ *is a closed subspace of* $\mathbf{H}_0$.

Proof. By definition $\varphi_{\lambda u_1 u_2 u_3}(\mathscr{U}) = \mathscr{U}(\lambda u_1 u_2 u_3)$ and hence the map $\varphi_{\lambda u_1 u_2 u_3}$ is sobrejective. Now, assume that $\varphi_{\lambda u_1 u_2 u_3}(\beta, u_1^\perp, u_2^\perp, u_3^\perp) = \varphi_{\lambda u_1 u_2 u_3}(\gamma, v_1^\perp, v_2^\perp, v_3^\perp), that is,$

$$\beta(u_1 + u_1^\perp)(u_2 + u_2^\perp)(u_3 + u_3^\perp) = \gamma(u_1 + v_1^\perp)(u_2 + v_2^\perp)(u_3 + v_3^\perp). \tag{5.2}$$

Since $\langle \cdot, \cdot \rangle_{(1,0,0)} = \langle \cdot, \cdot \rangle_{H_0^1(\Omega_{\underline{X}})} \langle \cdot, \cdot \rangle_{L^2(\Omega_{\underline{S}})} \langle \cdot, \cdot \rangle_{L^2(\Omega_{\underline{T}})}$ holds, we use first in both sides of (5.2) the linear map

$$\left\langle \cdot, \frac{u_1}{\|u_1\|_{H_0^1(\Omega_{\underline{X}})}} \frac{u_2}{\|u_2\|_{L^2(\Omega_{\underline{S}})}} \frac{u_3}{\|u_3\|_{L^2(\Omega_{\underline{T}})}} \right\rangle_{(1,0,0)}$$

obtaining that $\beta = \gamma$. Let P_{u_i} the orthogonal projection onto the linear space span $\{u_i\}$ for $1 \le i \le 3$. Then by using consecutively in (5.2) the linear maps $id_{H_0^1(\Omega_{\underline{X}})} \otimes P_{u_2} \otimes P_{u_3}$, $P_{u_1} \otimes id_{L^2(\Omega_{\underline{S}})} \otimes P_{u_3}$ and $P_{u_1} \otimes P_{u_2} \otimes id_{L^2(\Omega_{\underline{T}})}$ we obtain

$$(u_1 + u_1^\perp) u_2 u_3 = (u_1 + v_1^\perp) u_2 u_3 \tag{5.3}$$
$$u_1 (u_2 + u_2^\perp) u_3 = u_1 (u_2 + v_2^\perp) u_3 \tag{5.4}$$
$$u_1 u_2 (u_3 + u_3^\perp) = u_1 u_2 (u_3 + v_3^\perp) \tag{5.5}$$

Next we use the linear maps

$$id_{H_0^1(\Omega_{\underline{X}})} \otimes \langle \cdot, \frac{u_2}{\|u_2\|_{L^2(\Omega_{\underline{S}})}} \rangle_{L^2(\Omega_{\underline{S}})} \otimes \langle \cdot, \frac{u_3}{\|u_3\|_{L^2(\Omega_{\underline{T}})}} \rangle_{L^2(\Omega_{\underline{S}})}$$
$$\langle \cdot, \frac{u_1}{\|u_1\|_{H_0^1(\Omega_{\underline{X}})}} \rangle_{H_0^1(\Omega_{\underline{S}})} \otimes id_{L^2(\Omega_{\underline{S}})} \otimes \langle \cdot, \frac{u_3}{\|u_3\|_{L^2(\Omega_{\underline{T}})}} \rangle_{L^2(\Omega_{\underline{T}})}$$
$$\langle \cdot, \frac{u_1}{\|u_1\|_{H_0^1(\Omega_{\underline{X}})}} \rangle_{H_0^1(\Omega_{\underline{S}})} \otimes \langle \cdot, \frac{u_2}{\|u_2\|_{L^2(\Omega_{\underline{S}})}} \rangle_{L^2(\Omega_{\underline{S}})} \otimes id_{L^2(\Omega_{\underline{T}})},$$

(5.3),(5.4) and (5.5) respectively, obtaining

$$(u_1 + u_1^{\perp}) = (u_1 + v_1^{\perp})$$
$$(u_2 + u_2^{\perp}) = (u_2 + v_2^{\perp})$$
$$(u_3 + u_3^{\perp}) = (u_3 + v_3^{\perp}).$$

Thus, $u_i^{\perp} = v_i^{\perp}$ for $1 \leq i \leq 3$ and hence the map $\varphi_{\lambda u_1 u_2 u_3}$ is bijective. To end the proof of the proposition we show first that $\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right)$ is injective. To this end consider that

$$(\delta\lambda)u_1u_2u_3 + \lambda\left((\delta u_1)u_2u_3 + u_1(\delta u_2)u_3 + u_1u_2(\delta u_3)\right) = 0. \tag{5.6}$$

Now, we take into account the following continuous linear maps

$P_{u_1} \otimes P_{u_2} \otimes P_{u_2}$ the orthogonal projection onto $\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}$
$P_{U_{u_1}} \otimes P_{u_2} \otimes P_{u_2}$ the orthogonal projection onto $U_{u_1} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}$
$P_{u_1} \otimes P_{U_{u_2}} \otimes P_{u_2}$ the orthogonal projection onto $\mathrm{span}\{u_1\} \otimes_a U_{u_2} \otimes_a \mathrm{span}\{u_1\}$
$P_{u_1} \otimes P_{u_2} \otimes P_{U_{u_3}}$ the orthogonal projection onto $\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a U_{u_3}$.

By using consecutively the above four linear projections in (5.6) we obtain that $\delta\lambda = 0$ $\delta u_1 = 0$, $\delta u_2 = 0$ and $\delta u_3 = 0$, respectively. Thus $\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right)$ is injective and $\left(\varphi'_{\lambda u_1 u_2 u_3}(0,0,0,\lambda)\right)(\mathscr{H})$ is the closed subspace

$$(\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}) \oplus (U_{u_1} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}) \oplus$$
$$(\mathrm{span}\{u_1\} \otimes_a U_{u_2} \otimes_a \mathrm{span}\{u_3\}) \oplus (\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a U_{u_3})$$

in $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$. This concludes the proof. □

Remark 5.2. *From the proof of the above proposition we can identify the tangent space of $\mathscr{U}(\lambda u_1u_2u_3)$ at $\lambda u_1u_2u_3$ with the closed subspace*

$$\mathrm{T}\mathscr{U}(\lambda u_1u_2u_3) := (\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}) \oplus (U_{u_1} \otimes_a \mathrm{span}\{u_2\} \otimes_a \mathrm{span}\{u_3\}) \oplus$$
$$(\mathrm{span}\{u_1\} \otimes_a U_{u_2} \otimes_a \mathrm{span}\{u_3\}) \oplus (\mathrm{span}\{u_1\} \otimes_a \mathrm{span}\{u_2\} \otimes_a U_{u_3})$$

in $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$.

Thus the first order optimality condition (5.1) says us that if $\lambda u_1u_2u_3$ is a minimum in $\mathscr{U}(\lambda u_1u_2u_3)$ then

$$\left\langle J'(\lambda u_1u_2u_3),(\delta\lambda)u_1u_2u_3+\lambda\left((\delta u_1)u_2u_3+u_1(\delta u_2)u_3+u_1u_2(\delta u_3)\right)\right\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0 \quad (5.7)$$

holds for all $(\delta u_1,\delta u_2,\delta u_3,\delta\lambda)\in U_{u_1}\times U_{u_2}\times U_{u_3}\times\mathbb{R}$. We recall that in our case

$$J'(\lambda u_1u_2u_3)=\lambda(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g)\perp \mathrm{T}\mathscr{U}(\lambda u_1u_2u_3)$$

holds if $\lambda u_1u_2u_3$ is an stationary point of J at $\mathscr{U}(\lambda u_1u_2u_3)$. In order to construct an strategy to approximate an stationary point for the derivative J' we study the following four cases:

1. Take $\delta u_1=\delta u_2=\delta u_3=0$ then from (5.7) we obtain that
$$\langle\lambda(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g),(\delta\lambda)u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0 \quad (5.8)$$
holds for all $\delta\lambda\in\mathbb{R}$.
2. Take $\delta\lambda=0$, $\delta u_2=\delta u_3=0$ then from (5.7) we obtain that
$$\langle(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g),\lambda(\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0 \quad (5.9)$$
holds for all $\delta u_1\in U_{u_1}$.
3. Take $\delta\lambda=0$, $\delta u_1=\delta u_3=0$ then from (5.7) we obtain that
$$\langle\lambda(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g),\lambda u_1(\delta u_2)u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0 \quad (5.10)$$
holds for all $\delta u_2\in U_{u_2}$.
4. Take $\delta\lambda=0$, $\delta u_1=\delta u_2=0$ then from (5.7) we obtain that
$$\langle\lambda(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g),\lambda u_1u_2(\delta u_3)\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0 \quad (5.11)$$
holds for all $\delta u_3\in U_{u_3}$.

From (5.8) we obtain that

$$\langle\lambda(-\Delta_{\underline{X}}(u_1))u_2u_3-(f-g),u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=0$$

and hence

$$\lambda:=\frac{\langle(f-g),u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}}{\langle(-\Delta_{\underline{X}}(u_1))u_2u_3,u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}}=\frac{\langle(f-g),u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}}{\|u_1\|^2_{H^1_0(\Omega_{\underline{X}})}\|u_2\|^2_{L^2(\Omega_{\underline{S}})}\|u_3\|^2_{L^2(\Omega_{\underline{T}})}} \quad (5.12)$$

The equation (5.9) implies that

$$\langle-\Delta_{\underline{X}}(u_1)u_2u_3,(\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}=\langle f-g,(\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}, \quad (5.13)$$

holds for all $\delta u_1 \in U_{u_1}$. From (5.3) we can write

$$\langle(-\Delta_{\underline{X}}(u_1))u_2u_3, u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = \langle(f-g, u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} \tag{5.14}$$

Since $H_0^1(\Omega) = \operatorname{span}\{u_1\} \oplus (U_{u_1} \cap H_0^1(\Omega))$ combining (5.13) and (5.14) we obtain that

$$\langle-\Delta_{\underline{X}}(u_1)u_2u_3, (\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = \langle f-g, (\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}, \tag{5.15}$$

holds for all $\delta u_1 \in H_0^1(\Omega_{\underline{X}})$. Finally, (5.10) and (5.11) give us the conditions

$$\langle(f-g), u_1(\delta u_2)u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0 \text{ for all } \delta u_2 \in U_{u_2}, \tag{5.16}$$

$$\langle(f-g), u_1u_2(\delta u_3)\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0 \text{ for all } \delta u_3 \in U_{u_3}. \tag{5.17}$$

In consequence, we need to find $u_1u_2u_3 \in \mathscr{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ be such that (5.15)-(5.17) holds and then we take λ as in (5.12). From all said above we can state the following result.

Proposition 5.3. *Assume that $\lambda u_1u_2u_3 \in \mathscr{M}_{=1}(H_0^1(\Omega_{\underline{X}}) \otimes_a L^2(\Omega_{\underline{S}}) \otimes_a L^2(\Omega_{\underline{T}}))$ is a minimum of $(*)$ with $u_{m-1} = 0$. Then it holds*

$$\langle-\Delta_{\underline{X}}(u_1)u_2u_3, (\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = \langle f-g, (\delta u_1)u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} \textit{ for all } \delta u_1 \in H_0^1(\Omega_{\underline{X}})$$

$$\langle(f-g), u_1(\delta u_2)u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0 \textit{ for all } \delta u_2 \in \operatorname{span}\{u_2\}^\perp,$$

$$\langle(f-g), u_1u_2(\delta u_3)\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0 \textit{ for all } \delta u_3 \in \operatorname{span}\{u_3\}^\perp \textit{ and}$$

$$\lambda = \frac{\langle(f-g), u_1u_2u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}}{\|u_1\|^2_{H_0^1(\Omega_{\underline{X}})}\|u_2\|^2_{L^2(\Omega_{\underline{S}})}\|u_3\|^2_{L^2(\Omega_{\underline{T}})}}.$$

Proposition 5.3 allows us to use the following strategy in order implemented a Progressive Variational Vademecum.

1. Let be $u = 0$ and $r = f - g$.
2. Consider three finite dimensional subspaces $V_1 \subset H_0^1(\Omega_{\underline{X}})$, $V_2 \subset L^2(\Omega_{\underline{S}})$ and $V_3 \subset L^2(\Omega_{\underline{T}})$.
3. Take $\lambda_0 = 1$ and choose the functions $u_1^{(0)} \in V_1$, $u_2^{(0)} \in V_2$ and $u_3^{(0)} \in V_3$, randomly.
4. Let $U_2^{(0)} \subset V_2$ be a linear subspace such that $V_2 = \operatorname{span}\{u_2^{(0)}\} \oplus U_2^{(0)}$.
5. Find $u_1 \in V_1$ be such such that
$$\left\langle -\Delta_{\underline{X}}(u_1)u_2^{(0)}u_3^{(0)}, (\delta u_1)u_2^{(0)}u_3^{(0)} \right\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = \left\langle r, (\delta u_1)u_2^{(0)}u_3^{(0)} \right\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})}$$
holds for all $\delta u_1 \in V_1$.
6. Find $u_3 \in V_3$ be such that
$$\langle r, u_1(\delta u_2)u_3\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0$$
for all $\delta u_2 \in U_2^{(0)}$. Let $U_3 \subset V_3$ be a linear subspace such that $V_3 = \operatorname{span}\{u_3\} \oplus U_3$.

7. Find $u_2 \in V_2$ be such that

$$\langle r, u_1 u_2(\delta u_3)\rangle_{L^2(\Omega_{\underline{X}}\times\Omega_{\underline{S}}\times\Omega_{\underline{T}})} = 0$$

for all $\delta u_3 \in U_3$.

8. Compute λ from (5.12).

9. If $J(\lambda u_1 u_2 u_3) < J(\lambda_0 u_1^{(0)} u_2^{(0)} u_2^{(0)})$ we put $\lambda_0 = \lambda$ and $u_i^{(0)} = u_i$ for $1 \le i \le 3$ and goto 5, else put $r = r + \Delta_{\underline{X}}(\lambda_0 u_1^{(0)} u_2^{(0)} u_2^{(0)})$ and $u = u + \lambda_0 u_1^{(0)} u_2^{(0)} u_3^{(0)}$.

10. If $\|r\| <$ `tol` return u and STOP else goto 3.

6. An illustrative example

In order to illustrate the benefits of the PGD framework, we use the following example. Let us consider as domain $\Omega_{\underline{X}}$ a 5m $\times$ 5m square. We take a discretization of the domain by means $N_x = N_y = 50$ nodes on each side, that is, 2500 degrees of freedom and the variance r is set to 1.2. Figure 6.1 shows an example for $\underline{S} = (1,4), \underline{T} = (4,1)$. Left column shows the source term and the right column shows the resulting PGD reconstruction for $n = 200$ terms.

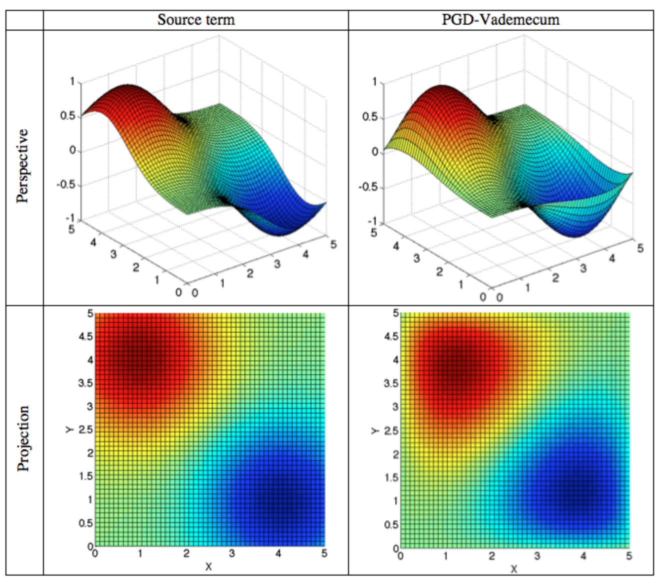

Figure 6.1: PGD reconstruction VS source term for $\underline{S} = (1,4), \underline{T} = (4,1)$

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The computational cost of the reconstruction is 0.0101s in a Mac with an Intel Core 2 Duo, 3.06 GHz and 4 GB RAM. It is worth comparing this value with the cost of a FEM approximation solving a standard linear system with where it rises to 4.7s.

6.1. Residual error

There are some techniques to measure the error approximation vs the number of PGD terms. One of the more appropriate error estimator is the $L^2(\Omega_{\underline{X}} \times \Omega_{\underline{S}} \times \Omega_{\underline{T}})$-residual $R(n)$ obtained by computing:

$$R(n) = \sqrt{R(n-1)^2 - \|u_1(\underline{X}) \cdot u_2(\underline{S}) \cdot u_3(\underline{T})\|^2_{L^2(\Omega_{\underline{X},\underline{S},\underline{T}})}} \tag{6.1}$$

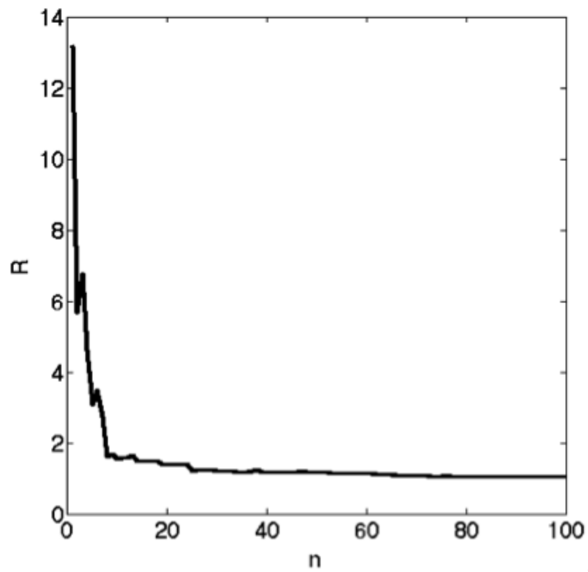

Figure 6.2: Residual error

Figure 6.2 shows one of the most important properties of the PGD: the first computed rank-one terms provide more energy to reduce the residual than the last ones. For a more detailed study of the behaviour of the residual error the interested reader can be see [1, Theorem 1].

6.2. Streamline computation

As explained in Section 1, the use of harmonic functions solve the problem of deadlocks present in APF-based techniques. Harmonic functions are based on flow dynamics, described by the Poisson equation, where the potential field is free of deadlocks and derives in a set of streamlines [8]-[13]. These streamlines are independent in time and describe the direction of a massless fluid element (particle) travelling from a start to a goal position, following the velocity field obtained from the gradient of the potential field as;

$$v_x = \frac{du}{dx}, v_y = -\frac{du}{dy} \tag{6.2}$$

The streamlines produced by the velocity field can be computed by means of any interpolation technique (linear, cubic, spline, etc). FIgure 6.3 shows examples of the streamlines resulting from a linear interpolation for the PGD reconstruction.

6.3. A Shortest Path Application

In order to test the advantages that PGD-Vademecum offers, a simulation with Matlab has been performed. An omnidirectional mobile robot navigates in a 5×5m square environment guided by a potential field (PGD) with $N_x \cdot N_y = 50 \times 50$ nodes, $N_{s_1} \cdot N_{s_2} = 5 \times 5$ nodes $N_{t_1} \cdot N_{t_2} = 5 \times 5$ nodes, $r = 0.7$ and $n = 200$. For a realistic implementation, we only reconstruct a Region Of Interest (ROI) in each algorithm execution. The ROI is composed by the surrounding nodes of the current robot position and its size depends on the maximum robot velocity. In the present example, for a specific Start and Goal configurations, the robot selects the shortest streamline, which is a straight line heading to the Goal. The next figure depicts different trajectories followed by the robot begining at the starting point $\underline{S} = (1,4)$ to subsequent target points $\underline{T} = (4,1),(3,4),(2,1),(4,3)$.

7. Conclusions and Future Work

The present paper provides the mathematical analysis needed to justify the use of the PGD-vademecum [7]. To this end we prove the convergence of a Progressive Variational Vademecum based in the PGD. From the point of view of the applications, the PGD-Vademecum is computed off-line and reconstructed on-line for any particular configuration. It is really fast because its formulation is a simple sum of products. In particular, in path planning robot applications, only the surrounding nodes of the robot position need to be reconstructed. As a consequence, the computational costs are nearly negligible. Moreover, the resulting paths are based on the Laplace/Poisson equation (harmonic functions) and, thus, are free of deadlocks. This property makes it a promising technique to solve likes the piano mover's. The only drawback noticed is the generation of a small offset in the start and goal positions due to the definition of the source term, as start and goal positions have a coupling effect. Solve this problem is part of our future research about the applications of the Progressive Variational Vademecum based in the PGD.

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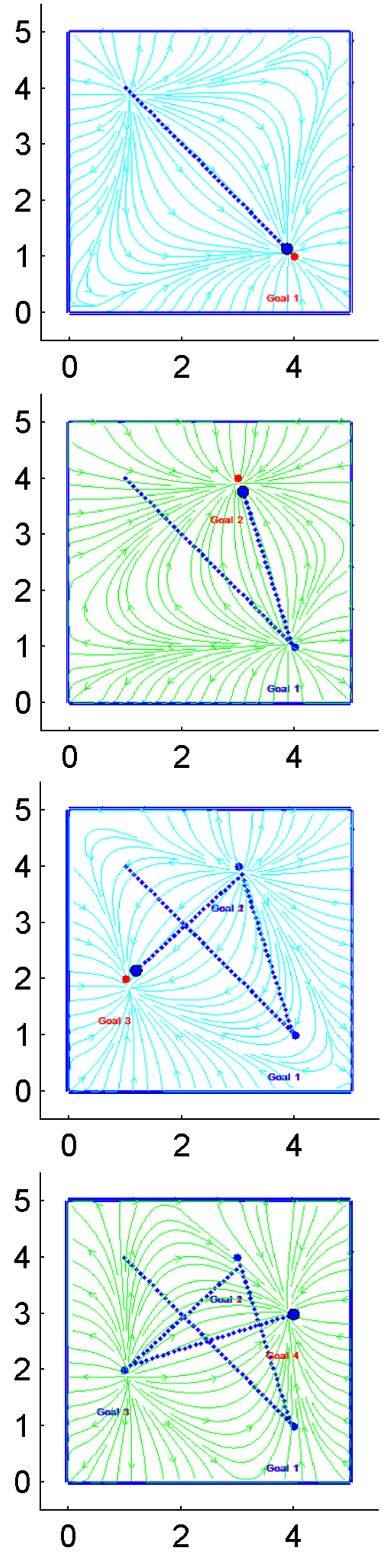

Figure 6.3: Simulation results for the goals $\underline{T} = (4,1),(3,4),(2,1),(4,3)$

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