Equicontinuity criteria for metric-valued sets of continuous functions[☆]

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Abstract

Combining ideas of Troallic [1] and Cascales, Namioka, and Vera [2], we prove several characterizations of *almost equicontinuity* and *hereditarily almost equicontinuity* for subsets of metric-valued continuous functions when they are defined on a Čech-complete space. We also obtain some applications of these results to topological groups and dynamical systems.

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1. Introduction

Let X and (M, d) be a Hausdorff, completely regular space and a metric space, respectively, and let $C(X, M)$ denote the set of all continuous functions from X to M. A subset $G \subseteq C(X, M)$ is said to be *almost equicontinuous* if G is equicontinuous on a dense subset of X. If G is almost equicontinuous for every closed nonempty subset of X, then it is said that G is *hereditarily almost equicontinuous*. The main goal of this paper is to extend to arbitrary topological spaces these two important notions, which were introduced in the setting of topological dynamics studying the enveloping semigroup of a flow [3, 4, 5].

In addition to their intrinsic academic interest, it turns out that these two concepts have found application in other different settings as it will be made clear in the sequel. First, we shall provide some basic notions and terminology.

Given $F \subseteq X$, the symbol $t_p(F)$ (resp. $t_\infty(F)$) will denote the topology, on $C(X, M)$, of pointwise convergence (resp. uniform convergence) on F. For a set G of functions from X to M and $Z \subseteq X$, the symbol $G|_Z$ will denote the set $\{g|_Z : g \in G\}$. We denote by $\overline{G}^{M^X}$ the closure of G in the Tychonoff product space M^X.

The symbolism $(F, t_p(\overline{G}^{M^X}))$ will denote the set F equipped with the weak topology generated by the functions in $\overline{G}^{M^X}|_F$. In like manner, the symbol $[A]^{\leq \omega}$ will denote the set of all countable subsets of A. A topological space X is said to be *Čech-complete* if it is a G_δ-subset of its Stone-Čech compatification. The family of Čech-complete spaces includes all complete metric spaces and all locally compact spaces. Several quotient spaces are used along the paper. For the reader's sake, a detailed description of them is presented at the Appendix.

We now formulate our main results.

***Theorem* A**. Let X and (M, d) be a Čech-complete space and a separable metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:

(a) G is almost equicontinuous.

(b) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.

Then $(b) \Rightarrow (c) \Rightarrow (a)$. If X is also a hereditarily Lindelöf space, then all conditions are equivalent.

Next result characterizes hereditarily almost equicontinuous families of functions defined on a Čech-complete space (this question has been studied in detail in [6] for compact spaces).

***Theorem* B**. Let X and (M, d) be a Čech-complete space and a metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:

(a) G is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

Remark 1.3. If G is a subset of $C(X, M)$ such that $K \stackrel{\text{def}}{=} \overline{G}^{M^X}$ is contained in $C(X, M)$, then the implication $(c) \Rightarrow (a)$ in Theorem **B** provides a different proof of the celebrated Namioka Theorem [7, Theorem 2.3]. Indeed, given any $L \in [G]^{\leq \omega}$ and any separable compact subset F of X, since $K \subseteq C(X, M)$ and F is separable, it follows that $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable. Thus G (and therefore K) is hereditarily almost equicontinuous.

Corollary 1.4. *With the same hypothesis of Theorem **B**, consider the following three properties:*

(a) G is hereditarily almost equicontinuous.

(b) G is hereditarily almost equicontinuous on F, for all F a separable and compact subset of X.

(c) $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf, for all F a separable and compact subset of X.

Then $(a) \Leftrightarrow (b) \Leftarrow (c)$.

2. Applications

The results formulated in the previous section have consequences in different settings. First, we consider an application to fragmentability.

A topological space X is said to be *fragmented by a pseudometric ρ* if for each nonempty subset A of X and for each $\epsilon > 0$ there exists a nonempty open subset U of X such that $U \cap A \neq \emptyset$ and $\rho\text{-}diam(U \cap A) \leq \epsilon$. This notion was introduced by Jayne and Rogers in [8]. Further work has been done by many workers. It will suffice to mention here the contribution by Namioka [9] and Ribarska [10].

Let X be a topological space, (M, d) a metric space and $G \subseteq M^X$ a family of functions. Whenever feasible, for example if $\overline{G}^{M^X}$ is compact, we will consider the pseudometric $\rho_{G,d}$, defined as follows:

$$\rho_{G,d}(x, y) \stackrel{\text{def}}{=} \sup_{g \in G} d(g(x), g(y)), \quad \forall x, y \in X.$$

Therefore, taking into account Definition 3.1 and Lemma 3.2, we have the following proposition.

Proposition 2.1. *Let X and (M, d) be a topological space and a metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following two properties:*

(a) G is hereditarily almost equicontinuous.

(b) X is fragmented by $\rho_{G,d}$.

Then (a) implies (b). If X is a hereditarily Baire space, then (a) and (b) are equivalent.

As a consequence, we have the following corollary of Theorem B.

Corollary 2.2. *Let X and (M,d) be a Čech-complete space and a metric space, respectively, and let $G \subseteq C(X,M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) *X is fragmented by $\rho_{G,d}$.*

(b) *F is fragmented by $\rho_{L,d}$, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.*

(c) *$((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.*

(d) *$(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.*

It is easy to check that, in the context of topological groups, the notion of almost equicontinuity is equivalent to equicontinuity. This fact allows us to characterize equicontinuous subsets of group homomorphisms using Theorem A.

From here on, if X and M are topological groups, the symbol $CHom(X,M)$ will denote the set of continuous homomorphisms of X into M. Recall that a topological group G is said to be *ω-narrow* if for every neighborhood V of the neutral element, there exists a countable subset E of G such that $G = EV$.

Corollary 2.3. *Let X and (M,d) be a Čech-complete topological group and a metric separable group, respectively, and let G be a subset of $CHom(X,M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:*

(a) *G is equicontinuous.*

(b) *G is relatively compact in $CHom(X,M)$ with respect to the compact open topology.*

(c) *There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.*

(d) *There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.*

Then (c) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Leftrightarrow$ (b). If X is also ω-narrow, then all conditions are equivalent. Furthermore (c) and (d) are also true for $F = X$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from Ascoli Theorem. So, after Theorem A, it will suffice to show the implication (a) $\Rightarrow$ (c) for an ω-narrow X. Now, assuming that G is equicontinuous, it follows that $K \overset{\text{def}}{=} \overline{G}^{M^X} \subseteq CHom(X,M)$.

Thus K is an equicontinuous compact subset of continuous group homomorphisms. As a consequence, it is known that K is metrizable. (see [11, Cor. 3.5]). $\square$

Extending a result given by Troallic in [1, Corollary 3.2], we can reduce the verification of hereditarily almost equicontinuity to countable subsets. The equivalence $(a) \Leftrightarrow (b)$ bellow is due to Troallic (op. cit.).

Corollary 2.4. *Let X and (M, d) be a Čech-complete topological group and a metric group, respectively, and let G be a subset of $CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) G is equicontinuous.

(b) L is equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

For a function $f : X \times Y \to M$ let $f_x : Y \to M$ ($f^y : X \to M$) be $f(x, \cdot)$ for a fixed $x \in X$ ($f(\cdot, y)$ for a fixed $y \in Y$, resp.).

A variation of the celebrated Namioka Theorem [7] is also obtained as a corollary of Theorems **A** and **B** (cf. [12, 13, 14]).

Corollary 2.5. *Let X, H, and (M, d) be a Čech-complete space, a compact space, and a metric space, respectively, and let $f : X \times H \to M$ be a map satisfying that $f_x \in C(H, M)$ for every $x \in X$ and there is a dense subset G of H such that $f^g \in C(X, M)$ for every $g \in G$. Suppose that any of the two following equivalent conditions holds.*

(a) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.

(b) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.

Then there exists a G_δ and dense subset F in X such that f is jointly continuous at each point of $F \times H$.

Finally, we obtain some applications to dynamical systems [4, 15, 5]. Recall that a dynamical system, or a G-space, is a Hausdorff space X on which a topological group G acts continuously. We denote such a system by (G, X). For each $g \in G$ we have the self-homeomorphism $x \mapsto gx$ of X that we call g-translation.

Corollary 2.6. *Let X be a Polish G-space such that $\overline{G}^{X^X}$ is compact. The following properties are equivalent:*

(a) G is almost equicontinuous.

(b) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{X^X})|_F$ is metrizable.

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{X^X}))$ is Lindelöf.

Corollary 2.7. *Let X be a completely metrizable G-space such that $\overline{G}^{X^X}$ is compact. Then the following conditions are equivalent:*

(a) G is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [G]^{\leq \omega}$ and F a compact subset of X.

(c) $((\overline{L}^{X^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq \omega}$ and F a compact subset of X.

(d) $(F, t_p(\overline{L}^{X^X}))$ is Lindelöf, for all $L \in [G]^{\leq \omega}$ and F a compact subset of X.

In [16, Problem 28], Arkhangel'skii raises the following question: Let X be a Lindelöf space and let K be a compact subset of $(C(X), t_p(X))$. Is it true that the tightness of K is countable? As far as we know, this question is still open in ZFC. Here we provide a partial answer to Arkhangel'skii's question.

Corollary 2.8. *Let X be a Lindelöf space and let K be a compact subspace of $(C(X), t_p(X))$. If there is a a dense subset $G \subseteq K$ such that $(X, t_p(G))$ is Čech-complete and hereditarily Lindelöf, then K is metrizable.*

Proof. The proof of this result is consequence of Theorem B. Indeed, remark that, if F is a subset of X that is closed in the $t_p(G)$-topology, then F will be Čech-complete and hereditarily Lindelöf as well. Moreover, since $G \subseteq K$, it follows that F is also closed in the $t_p(K)$-topology and, as a consequence, Lindelöf. Applying Corollary 1.4 to the (compact) space K, which is equipped with the $t_p(X)$-topology, it follows that G is hereditarily almost equicontinuous on X. Since $(X, t_p(G))$ is hereditarily Lindelöf, Proposition 4.6 yields the metrizability of $K = \overline{G}^{\mathbb{R}^X}$. □

3. Basic results

Within the setting of dynamical systems, the following definitions appear in [3].

Definition 3.1. Let X and (M, d) be a topological space and a metric space respectively, and let $G \subseteq C(X, M)$. According to [3], we say that a point $x \in X$ is an *equicontinuity point* of G when for every $\epsilon > 0$ there is a neighborhood U of x such that $diam(g(U)) < \epsilon$ for all $g \in G$. We say that G is *almost equicontinuous* when the subset of equicontinuity points of G is dense in X. Furthermore, it is said that G is *hereditarily almost equicontinuous* if $G|_A$ is almost equicontinuous for every nonempty closed subset A of X.

The proof of the following lemma is known. However it is very useful in order to obtain subsets of continuous functions that are not almost equicontinuous. We include its proof here for completeness sake.

Lemma 3.2. *Let X and (M, d) be a topological space and a metric space respectively, and let $G \subseteq C(X, M)$. Consider the following two properties:*

(a) G is almost equicontinuous.

(b) For every nonempty open subset U of X and $\epsilon > 0$, there exists a nonempty open subset $V \subseteq U$ such that $diam(g(V)) < \epsilon$ for all $g \in G$.

Then (a) implies (b). If X is a Baire space, then (a) and (b) are equivalent. Furthermore, in this case, the subset of equicontinuity points of G is a dense G_δ-set in X.

Proof. That (a) implies (b) is obvious. Assume that X is a Baire space and (b) holds. Given $\epsilon > 0$ arbitrary, we consider the open set $O_\epsilon \stackrel{\text{def}}{=} \bigcup \{U \subseteq X : \quad U$ is a nonempty open subset $\wedge\ diam(g(U)) < \epsilon \quad \forall g \in G\}$. By (b), we have that O_ϵ is nonempty and dense in X. Since X is Baire, taking $W \stackrel{\text{def}}{=} \bigcap_{n < \omega} O_{\frac{1}{n}}$, we obtain a dense G_δ subset which is the subset of equicontinuity points of G. $\square$

Remark 3.3. As a consequence of assertion (b) in Lemma 3.2, it follows that, when X is a Baire space, a subset of functions G is hereditarily almost equicontinuous if, and only if, $G|_A$ is almost equicontinuous for every nonempty (non necessarily closed) subset A of X. Since we mostly work with Baire spaces here, we will make use of this fact in some parts along the paper.

Note that the set of equicontinuity points of a subset of functions G is a G_δ-set. Next corollary is a straightforward consequence of Lemma 3.2.

Corollary 3.4. *Let X and (M, d) be a topological space and a metric space respectively, and let $G \subseteq C(X, M)$. Suppose there is an open basis $\mathcal{V}$ in X and $\epsilon > 0$ such that for every $V \in \mathcal{V}$, there is $g_V \in G$ with $diam(g_V(V)) \geq \epsilon$. Then G is not almost equicontinuous.*

Let 2^ω be the Cantor space and let $2^{(\omega)}$ denote the set of finite sequences of 0's and 1's. For a $t \in 2^{(\omega)}$, we designate by $|t|$ the length of t. For $\sigma \in 2^\omega$ and $n > 0$ we write $\sigma|n$ to denote $(\sigma(0), \ldots, \sigma(n-1)) \in 2^{(\omega)}$. If $n = 0$ then $\sigma|0 \stackrel{\text{def}}{=} \emptyset$.

Applying Corollary 3.4, it is easy to obtain subsets of continuous functions that are not almost equicontinuous.

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Example 3.5. Let $X = 2^\omega$ be the Cantor space and let $G = \{\pi_n\}_{n<\omega}$ be the set of all projections of X onto $\{0,1\}$. Then G is not almost equicontinuous.

Proof. Let $U \neq \emptyset$ be an open subset in X. Then, for some index $n < \omega$ we have $\pi_n(U) = \{0,1\}$, which implies $diam(\pi_n(U)) > 1/2$. Therefore G is not almost equicontinuous by Corollary 3.4. $\square$

The precedent result can be generalized in order to obtain a more general example of non-almost equicontinuous set of functions. It turns out that this example is universal in a sense that will become clear along the paper.

Example 3.6. Let $X = 2^\omega$ be the Cantor space and let (M, d) be a metric space. Let $\{U_t : t \in 2^{(\omega)}\}$ be the canonical open basis of X. If $G = \{g_t\}_{t\in 2^{(\omega)}}$ is a set of continuous functions on X into M satisfying that $diam(g_t(U_t)) \geq \epsilon$ for some fixed $\epsilon > 0$ and all $t \in 2^{(\omega)}$, then G is not almost equicontinuous.
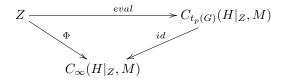
Next result gives a sufficient condition for the equicontinuity of a family of functions. It extends a well known result by Corson and Glicksberg [17]. However, we remark that the subset F found in the lemma below can become the empty set if Z is a first category subset of X.

Lemma 3.7. *Let X and (M, d) be a topological space and a separable metric space, respectively. If $G \subseteq C(X, M)$ and $(\overline{G}^{M^X})|_Z$ is metrizable and compact for some dense subset Z of X, then there is a residual subset F in Z such that G is equicontinuous at every point in F. In case Z is of second category in X, it follows that F will be necessarily nonempty.*

Proof. Set $H \stackrel{\text{def}}{=} \overline{G}^{M^X}$ and consider the map $eval : X \to C(H, M)$, $x \mapsto eval_x$; defined by $eval_x(f) \stackrel{\text{def}}{=} f(x)$ for all $x \in X$ and $f \in H$.

For simplicity's sake, the symbols $C_{t_p(G)}(H|_Z, M)$ and $C_\infty(H|_Z, M)$ will denote the space $C(H|_Z, M)$ equipped with the pointwise convergence $t_p(G)$ and the uniform convergence topology, respectively.

Now set Φ such that the following diagram commutes

$$
\begin{array}{ccc}
Z & \xrightarrow{\quad eval \quad} & C_{t_p(G)}(H|_Z, M) \\
& {\scriptstyle \Phi} \searrow \quad \quad {\scriptstyle id} \swarrow & \\
& C_\infty(H|_Z, M) &
\end{array}
$$

Remark that the evaluation map, $eval$, is continuous because $G \subseteq C(X, M)$. Since $H|_Z$ is $t_p(Z)$-compact and metrizable and Z is dense in X, it follows that $C_\infty(H|_Z, M)$ is separable and metrizable (see [18, Cor. 4.2.18]). Therefore, for every $n < \omega$, there is a sequence of closed balls $\{\overline{B}(u_i^{(n)}, 1/n) : i < \omega\}$ that covers $C_\infty(H|_Z, M)$. Furthermore, since G is dense in H, we have that each $\overline{B}(u_i^{(n)}, 1/n)$ is also closed in $C_{t_p(G)}(H|_Z, M)$. As a consequence $K_{(i,n)} \stackrel{\text{def}}{=}$

$\Phi^{-1}(\overline{B}(u_i^{(n)}, 1/n)) = eval^{-1}(\overline{B}(u_i^{(n)}, 1/n))$ is closed in Z for all $i, n < \omega$, because $eval$ is continuous.

We have that $Z \subseteq \bigcup_{i<\omega} K_{(i,n)}$ for every $n < \omega$, so $Z \subseteq \bigcap_{n<\omega} \bigcup_{i<\omega} K_{(i,n)}$. Observe that $\bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$ is a set of first category in Z. As a consequence

$$F \overset{\text{def}}{=} Z \setminus \bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$$

is a residual set in Z.

We now verify that G is equicontinuous at each point $z \in F$. Let $z \in F$ and $\epsilon > 0$ arbitrary. Take $n_0 < \omega$ such that $2/n_0 < \epsilon$. Since $z \in F \subseteq \bigcap_{n<\omega} \bigcup_{i<\omega} K_{(i,n)} \subseteq \bigcup_{i<\omega} K_{(i,n_0)}$ there is $i_0 < \omega$ such that $z \in K_{(i_0,n_0)}$. We claim that $z \in int_Z(K_{(i_0,n_0)})$. Indeed, if we assume that $z \notin int_Z(K_{(i_0,n_0)})$, then $z \in K_{(i_0,n_0)} \setminus int_Z(K_{(i_0,n_0)})$. Therefore, $z \in \bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$ and $z \notin F$, which is a contradiction.

Since $z \in int_Z(K_{(i_0,n_0)})$ there is a nonempty open set A in X such that $int_Z(K_{(i_0,n_0)}) = A \cap Z$. Note that $A \cap Z$ is dense on A because Z is dense in X. So, $z \in A = \overline{A \cap Z}^A \subseteq \overline{A \cap Z}^X$.

Let $a, b \in A \cap Z$. Then $\Phi(a) = eval_a, \Phi(b) = eval_b \in \overline{B}(u_{i_0}^{(n_0)}, 1/n_0)$. Consequently, $d(g(a), g(b)) \le 2/n_0$ for every $g \in G$. So, given $x, y \in A \subseteq \overline{A \cap Z}^X$ we have that $d(g(x), g(y)) \le 2/n_0$ for every $g \in G$. Then $diam(g(A)) \le 2/n_0 < \epsilon$ for all $g \in G$. $\square$

Remark 3.8. Let X be a topological space, (M, d) be a metric space and G be a subset of $C(X, M)$ that we consider equipped with the pointwise convergence topology $t_p(X)$ in the sequel, unless otherwise stated.

Set

$$K \overset{\text{def}}{=} \{\alpha : M \to [-1, 1] : |\alpha(m_1) - \alpha(m_2)| \le d(m_1, m_2), \quad \forall m_1, m_2 \in M\}.$$

It is readily seen that K is a compact subspace of $[-1, 1]^M$.

Consider the evaluation map $\varphi : X \times G \to M$ defined by $\varphi(x, g) \overset{\text{def}}{=} g(x)$ for all $(x, g) \in X \times G$, which is clearly separately continuous. The map φ has associated a separately continuous map $f : X \times (G \times K) \to [-1, 1]$ defined by $f(x, (g, \alpha)) \overset{\text{def}}{=} \alpha(g(x))$ for all $(x, (g, \alpha)) \in X \times (G \times K)$.

Set

$$\nu : \overline{G}^{M^X} \times K \to [-1, 1]^X$$

defined by

$$\nu(h, \alpha) \overset{\text{def}}{=} \alpha \circ h \text{ for all } h \in \overline{G}^{M^X} \text{ and } \alpha \in K.$$

We claim that ν is continuous. Indeed, let $\{(h_\delta, \alpha_\delta)\}_{\delta \in \Delta} \subseteq \overline{G}^{M^X} \times K$ be a net that converges to $(h, \alpha) \in \overline{G}^{M^X} \times K$. Given $\epsilon > 0$ and $x \in X$, then there

exists $\delta_0 \in \Delta$ such that $d(h_\delta(x), h_0(x)) < \epsilon/2$ and $|\alpha_\delta(h_0(x)) - \alpha_0(h_0(x))| < \epsilon/2$ for all $\delta > \delta_0$. Therefore, we have that $|\nu(h_0, \alpha_0)(x) - \nu(h_\delta, \alpha_\delta)(x)| = |\alpha_0(h_0(x)) - \alpha_\delta(h_\delta(x))| \leq |\alpha_0(h_0(x)) - \alpha_\delta(h_0(x))| + |\alpha_\delta(h_0(x)) - \alpha_\delta(h_\delta(x))| \leq |\alpha_0(h_0(x)) - \alpha_\delta(h_0(x))| + d(h_0(x), h_\delta(x)) < \epsilon$ for all $\delta > \delta_0$. Since $G \subseteq C(X, M)$, we have that $\nu(G \times K) \subseteq C(X, [-1, 1])$.

For $m_0 \in M$, define $\alpha_{m_0} \in [-1, 1]^M$ by $\alpha_{m_0}(m) \stackrel{\text{def}}{=} d(m, m_0)$ for all $m \in M$. It is easy to check that $\alpha_{m_0} \in K$.

Lemma 3.9. *Let X be a topological space, (M, d) a metric space and G a subset of $C(X, M)$. Let K and ν be the space and the map defined in Remark 3.8. Then, for every subset F of X, the identity map $\text{id} : (F, t_p(\overline{G}^{M^X})) \to (F, t_p(\nu(\overline{G}^{M^X} \times K)))$ is a homeomorphism.*

Proof. Let $\{x_\delta\}_{\delta \in \Delta} \subseteq F$ be a net that $t_p(\overline{G}^{M^X})$-converges to x. Since α is continuous, for any $(h, \alpha) \in \overline{G}^{M^X} \times K$, we have $\lim_{\delta \in \Delta} \nu(h, \alpha)(x_\delta) = \lim_{\delta \in \Delta} \alpha(h(x_\delta)) = \alpha(h(x)) = \nu(h, \alpha)(x)$. So, id is continuous. Conversely, let $\{x_\delta\}_{\delta \in \Delta} \subseteq F$ be a net that $t_p(\nu(\overline{G}^{M^X} \times K))$-converges to $x_0 \in F$. Given $h \in \overline{G}^{M^X}$ arbitrary, take $\alpha_{h(x_0)} \in K$. So, fixed $\epsilon > 0$, there is $\delta_0 \in \Delta$ such that $\epsilon > |\nu(h, \alpha_{h(x_0)})(x_\delta) - \nu(h, \alpha_{h(x_0)})(x_0)| = |d(h(x_\delta), h(x_0)) - d(h(x_0), h(x_0))| = d(h(x_\delta), h(x_0))$ for every $\delta > \delta_0$. That is, the net $\{x_\delta\}_{\delta \in \Delta}$ converges to x_0 in $t_p(\overline{G}^{M^X})$, which completes the proof. $\square$

It is well known that the metric $\bar{d} : M \times M \to \mathbb{R}$ defined by $\bar{d}(m_1, m_2) \stackrel{\text{def}}{=} \min\{d(m_1, m_2), 1\}$ for all $m_1, m_2 \in M$ induces the same topology as d. So, without loss of generality, we work with this metric from here on.

The following lemma reduces many questions related to a general metric space M to the interval $[-1, 1]$ (cf. [19]).

Lemma 3.10. *Let X and (M, d) be a topological and a metric space, respectively. If G is a subset of $C(X, M)$, then G is equicontinuous at a point $x_0 \in X$ if and only if $\nu(G \times K)$ is equicontinuous at it.*

Proof. Assume that G is equicontinuous at x_0. Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $d(g(x_0), g(x)) < \epsilon$ for all $x \in U$ and $g \in G$. Let $\alpha \in K$, $x \in U$ and $g \in G$, then we have

$$|\nu(g, \alpha)(x_0) - \nu(g, \alpha)(x)| = |\alpha(g(x_0)) - \alpha(g(x))| \leq d(g(x_0), g(x)) < \epsilon.$$

Conversely, assume that $\nu(G \times K)$ is equicontinuous in x_0. Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $|\nu(g, \alpha)(x_0) - \nu(g, \alpha)(x)| < \epsilon$ for all $x \in U$, $g \in G$ and $\alpha \in K$.

For $g \in G$, consider the map $\alpha_{g(x_0)} \in K$. In order to finish the proof, it will suffice to observe that

$$|\alpha_{g(x_0)}(g(x_0)) - \alpha_{g(x_0)}(g(x))| = d(g(x), g(x_0))$$
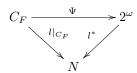
for all $x \in U$ and $g \in G$. $\square$

Corollary 3.11. *Let X and (M, d) be a topological and a metric space, respectively, and let G be an arbitrary subset of $C(X, M)$. Then G is (hereditarily) almost equicontinuous if and only if $\nu(G \times K)$ is (hereditarily) almost equicontinuous.*

4. Proof of main results

The following technical lemma is essential in most results along this paper. The construction of the proof is based on an idea that appears in [13] and [2]. We recall that a topological space is *hemicompact* if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence. Every compact space or every locally compact and Lindelöf space is hemicompact.

Lemma 4.1. *Let X and (M, d) be a Čech-complete space and a hemicompact metric space, respectively, and let G be a subset of $C(X, M)$ such that $\overline{G}^{M^X}$ is compact. If G is not almost equicontinuous, then for every G_δ and dense subset F of X there exists a countable subset L in G, a compact separable subset $C_F \subseteq F$, a compact subset $N \subseteq M$ and a continuous and surjective map Ψ of C_F onto the Cantor set 2^ω such that for every $l \in L$ there exists a continuous map $l^* : 2^\omega \to N$ satisfying that the following diagram is commutative*
Diagram 1:

$$
\begin{array}{ccc}
C_F & \xrightarrow{\ \ \Psi\ \ } & 2^\omega \\
& {\scriptstyle l|_{C_F}} \searrow \quad \swarrow {\scriptstyle l^*} & \\
& N &
\end{array}
$$

Furthermore, the subset $L^* \overset{\text{def}}{=} \{l^* : l \in L\} \subseteq C(2^\omega, N)$ separate points in 2^ω and is not almost equicontinuous on 2^ω.

Proof. Let F be a G_δ and dense subset of X. Then there is a sequence $\{W_n\}_{n=1}^\infty$ of open dense subsets of X such that $W_s \subseteq W_r$ if $r < s$ and $F = \bigcap\limits_{n=1}^\infty W_n$.

Let $\{M_n\}_{n<\omega}$ be a sequence of compact subsets, that we obtain by hemicompactness such that $M = \bigcup\limits_{n<\omega} M_n$ and for every compact subset $K \subseteq M$ there is $n < \omega$ such that $K \subseteq M_n$.

For each $n < \omega$ we consider the closed subset $X_n = \{x \in X : g(x) \in M_n \quad \forall g \in G\}$. We claim that $X = \bigcup\limits_{n<\omega} X_n$. Indeed, let $x \in X$. Since $\overline{G}^{M^X} \subseteq M^X$ is compact and the xth projection π_x is continuous, then $\pi_x(\overline{G}^{M^X}) \subseteq M$ is compact. So, there is $n_x < \omega$ such that $\pi_x(\overline{G}^{M^X}) \subseteq M_{n_x}$ by hemicompactness. Therefore $x \in X_{n_x}$.

Since G is not almost equicontinuous there exists a nonempty open subset U of X and $\epsilon > 0$ such that for all nonempty open subset $V \subseteq U$ there exists a function $g_V \in G$ such that $diam(g_V(V)) \geq 2\epsilon > \epsilon$ by Lemma 3.2.

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Note that U is Čech-complete. If we express $U = \bigcup_{n \in \omega} (U \cap X_n)$, by Baire's theorem, there is $n_0 < \omega$ such that $\tilde{U} \overset{\text{def}}{=} int_U(U \cap X_{n_0}) \neq \emptyset$ and open in X.

Set $C = \overline{\tilde{U}}^{X_{n_0}}$, which is closed in X, and $O_n = W_n \cap \tilde{U} = W_n \cap \tilde{U} \cap C$ that is open and dense in C for each $n < \omega$. Then $O_s \subseteq O_r$ if $r < s$ and $H = \bigcap_{n=1}^{\infty} O_n \subseteq F$ is a dense G_δ subset of C, which is a Baire space. Remark further that $g(x) \in M_{n_0}$ for all $x \in C$ and $g \in G$. Since M_{n_0} is compact, every function $f \in C(C, M_{n_0})$ can be extended to a continuous function $f^\beta \in C(\beta C, M_{n_0})$. Set $G^\beta = \{g^\beta : g \in G\} \subseteq C(\beta C, M_{n_0})$.

The space C, being Čech-complete, is a dense G_δ subset of its Stone-Čech compactification βC. Therefore, since H is a G_δ subset of C, it follows that H also is a dense G_δ subset of βC. Consider a sequence $\{E_n\}_{n=1}^{\infty}$ of open dense subsets of βC such that $E_s \subseteq E_r$ if $r < s$ and $H = \bigcap_{n=1}^{\infty} E_n$. We have that $H = \bigcap_{n=1}^{\infty} (E_n \cap O_n^\beta)$, where $O_n^\beta = \beta C \setminus \overline{(C \setminus O_n)}^{\beta C}$ is open in βC and $O_n^\beta \cap C = O_n$.

By induction on $n = |t|$ with $t \in 2^{(\omega)}$, we construct a family $\{U_t : t \in 2^{(\omega)}\}$ of nonempty open subsets of βC and a family of countable functions $L \overset{\text{def}}{=} \{g_t : t \in 2^{(\omega)}\} \subseteq G$, satisfying the following conditions for all $t \in 2^{(\omega)}$:

(i) $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta C} \subseteq O_0^\beta \overset{\text{def}}{=} \beta C \setminus \overline{(C \setminus \tilde{U})}^{\beta C}$ (remark that $O_0^\beta \cap C = \tilde{U}$);

(ii) $U_{ti} \subseteq \overline{U_{ti}}^{\beta C} \subseteq E_{|t|} \cap O_{|t|}^\beta \cap U_t$ for $i = 0, 1$ (where $E_0 \overset{\text{def}}{=} \beta C$);

(iii) $U_{t0} \cap U_{t1} = \emptyset$;

(iv) $d(g_t(x), g_t(y)) > \epsilon$, $\forall x \in U_{t0} \cap C$ and $\forall y \in U_{t1} \cap C$;

(v) whenever $s, t \in 2^{(\omega)}$ and $|s| < |t|$, $diam(g_s(U_{tj} \cap C)) < \frac{1}{|t|}$ for $j = 0, 1$.

Indeed, if $n = 0$, by regularity we can find $U_\emptyset$ an open set in βC such that $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta C} \subseteq E_0 \cap O_0^\beta$. For $n \geq 0$, suppose $\{U_t : |t| \leq n\}$ and $\{g_t : |t| < n\}$ have been constructed satisfying $(i) - (v)$. Fix a $t \in 2^{(\omega)}$ with $|t| = n$. Since U_t is open in βC, there is an open set A_t in X such that $U_t \cap C = A_t \cap C$. Therefore

$$U_t \cap C = (A_t \cap C) \cap O_0^\beta = A_t \cap (O_0^\beta \cap C) = A_t \cap \tilde{U}$$

is open in X and included in U.

By assumption there exist $g_t \in G$ such that $diam(g_t(U_t \cap C)) > \epsilon$. Consequently, we can find $x_t, y_t \in V_t \cap C$ such that $d(g_t(x_t), g_t(y_t)) > \epsilon$. By continuity, we can select two open disjoint neighbourhoods in βC, S_{t0} and S_{t1} of x_t and y_t, respectively, satisfying conditions (iii) and (iv).

If $i \in \{0, 1\}$, observe that $U_t \cap S_{ti} \cap O_0^\beta$ is open in βC and nonempty. Since $E_{|t|} \cap O_{|t|}^\beta$ is dense in βC then $U_t \cap S_{ti} \cap E_{|t|} \cap O_{|t|}^\beta$ is a nonempty open subset

of βC. By regularity there exists a nonempty open subset U_{ti} of βC such that $U_{ti} \subseteq \overline{U_{ti}}^{\beta C} \subseteq U_t \bigcap S_{ti} \bigcap E_{|t|} \cap O^\beta_{|t|}$. Therefore, U_{t0} and U_{t1} satisfies conditions $(ii), (iii)$ and (iv) and, by continuity, we can adjust the open sets to satisfy (v).

Set $K \overset{\text{def}}{=} \bigcap_{n=0}^\infty \bigcup_{|t|=n} \overline{U_t}^{\beta C}$, which is closed in βC and, as a consequence, also compact. Remark that we can express $K = \bigcup_{\sigma \in 2^\omega} \bigcap_{n=0}^\infty \overline{U_{\sigma|n}}^{\beta C}$. Therefore, for each $\sigma \in 2^\omega$, we have $\bigcap_{n=0}^\infty \overline{U_{\sigma|n}}^{\beta C} \neq \emptyset$ by the compactness of βC, which implies $K \neq \emptyset$. Furthermore, since $K \subseteq \bigcap_{n=0}^\infty (E_n \cap O^\beta_n) = H \subseteq F$, it follows that K is contained in F.

Let $\Psi : K \to 2^\omega$ be the canonical map defined such that $\Psi^{-1}(\sigma) = \bigcap_{n=0}^\infty \overline{U_{\sigma|n}}^{\beta C}$ for all $\sigma \in 2^\omega$. Clearly Ψ is onto and continuous. Observe that for each $t \in 2^{(\omega)}$ and $\sigma \in 2^\omega$, $g_t(\Psi^{-1}(\sigma))$ is a singleton by (iv). Therefore, g_t lifts to a continuous function g_t^* on 2^ω such that $g_t(x) = g_t^*(\Psi(x))$ for all $x \in K$.

Take a countable subset D of K such that $\Psi(D) = 2^{(\omega)}$ and makes $\Psi|_D$ injective. Set $C_F \overset{\text{def}}{=} \overline{D}^K$. Note that $2^{(\omega)}$ is a countable dense subset of 2^ω.

We have that $\Psi_{|C_F} : C_F \to 2^\omega$ is an onto and continuous map. We consider the set $L^* \subseteq C(2^\omega, M_{n_0})$ defined by $L^* = \{l^* : l \in L|_{C_F}\}$ that makes the diagram 1 commutative. We claim that L^* separates points in 2^ω and, as a consequence, defines its topology. Indeed, let $\sigma, \sigma' \in 2^\omega$ be two arbitrary points such that $\sigma \neq \sigma'$. Since Ψ is an onto map there exist $x, y \in C_F$ such that $\sigma = \Psi(x)$ and $\sigma' = \Psi(y)$. Therefore, $x \in \bigcap_{n=0}^\infty \overline{U_{\sigma|n}}^{\beta C}$ and $y \in \bigcap_{n=0}^\infty \overline{U_{\sigma'|n}}^{\beta C}$. Since $\sigma \neq \sigma'$, there is $n_0 \in \omega$ such that $\sigma|n_0 = \sigma'|n_0$ and $\sigma(n_0 + 1) \neq \sigma'(n_0 + 1)$. Taking $t = \sigma|n_0$, then by (iv) we know that $d(g_t(x), g_t(y)) > \epsilon$. So, $g_t^*(\sigma) \neq g_t^*(\sigma')$.

On the other hand, by the commutativity of Diagram 1, and taking into account how L and L^* have been defined, it is easily seen that L^* is not almost equicontinuous on 2^ω using Example 3.6. $\qquad\square$

Applying Corollary D of [2] by Cascales, Namioka and Vera and Facts 5.1, 5.2, 5.3 and 5.4, next result follows easily.

Proposition 4.2. *Let X be a compact space, (M, d) be a compact metric space and let G be a subset of $C(X, M)$. If $(X, t_p(\overline{G}^{M^X}))$ is Lindelöf, then G is hereditarily almost equicontinuous.*

Using Lemma 4.1, the constraints in Proposition 4.2 can be relaxed as the following result shows.

Proposition 4.3. *Let X be a Čech-complete space, (M, d) be a compact metric space and let G be a subset of $C(X, M)$. If there exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf, then G is almost equicontinuous.*
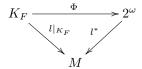
Proof. Reasoning by contradiction, suppose that G is not almost equicontinuous. By Lemma 4.1 there exists a compact separable subset C_F of F, a continuous onto map $\Psi : C_F \to 2^\omega$, and a countable subset L of G such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in C_F$ separate points in 2^ω and is not almost equicontinuous.

Let K_F be the closure of C_F in F with respect to the initial topology generated by the maps in L. Using a compactness argument, it follows that if $p \in K_F$ then there is $x_p \in C_F$ such that $l(p) = l(x_p)$ for all $l \in L$. Indeed, let $p \in K_F$. Then there is a net $\{x_\delta\}_{\delta \in \Delta} \subseteq C_F$ that $t_p(L)$-converges to p. Since C_F is compact there is a subnet $\{x_\gamma\}_{\gamma \in \Gamma}$ such that converges to $x_0 \in C_F$. Given $l \in L$, we know that $\lim_{\gamma \in \Gamma} l(x_\gamma) = l(x_0)$ because l is continuous. Therefore, $l(x_0) = \lim_{\gamma \in \Gamma} l(x_\gamma) = l(p)$. Consequently, we can extend Ψ to a map $\Phi : K_F \to 2^\omega$ by $\Phi(p) = \Psi(x_p)$ for all $p \in K_F$.

Let's see that Φ is well-defined. Let $p \in K_F$, suppose that there are $x_p, \tilde{x}_p \in C_F$ such that $x_p \neq \tilde{x}_p$ and $l(p) = l(x_p) = l(\tilde{x}_p)$ for all $l \in L$. Since the Diagram 1 commutes, we know that $l^*(\Psi(x_p)) = l^*(\Psi(\tilde{x}_p))$ for all $l^* \in L^*$. So, $\Psi(x_p) = \Psi(\tilde{x}_p)$ because L^* separates points in 2^ω.
Observe that the following diagram is commutative
Diagram 2:

$$
\begin{array}{ccc}
K_F & \xrightarrow{\quad \Phi \quad} & 2^\omega \\
& \searrow{\scriptstyle l|_{K_F}} \quad {\scriptstyle l^*}\swarrow & \\
& M &
\end{array}
$$

Certainly, let $p \in K_F$, then there is $x_p \in C_F$ such that $\Phi(p) = \Psi(x_p)$. Given $l \in L$, we have that $l(p) = l(x_p) = l^*(\Psi(x_p)) = l^*(\Phi(p))$.
We claim that $\Phi : (K_F, t_p(L)) \to (2^\omega, t_p(L^*))$ is also continuous. Indeed, let $\{h_\delta\}_{\delta \in \Delta} \subseteq K_F$ a net that $t_p(L)$-converges to $h_0 \in K_F$. For each $\delta \in \Delta$ there is $x_\delta \in C_F$ such that $\Phi(h_\delta) = \Psi(x_\delta)$ and $l(h_\delta) = l(x_\delta)$ for all $l \in L$. Analogously, there is $x_0 \in C_F$ such that $\Phi(h_0) = \Psi(x_0)$ and $l(h_0) = l(x_0)$ for all $l \in L$.
Since C_F is compact there is a subnet $\{x_\gamma\}_{\gamma \in \Gamma}$ such that converges to $\tilde{x} \in C_F$. Given $l \in L$, we know that $\lim_{\gamma \in \Gamma} l(x_\gamma) = l(\tilde{x})$ because l is continuous. On the other hand, we also have that $\lim_{\gamma \in \Gamma} l(x_\gamma) = \lim_{\gamma \in \Gamma} l(h_\gamma) = l(h_0) = l(x_0)$. Therefore, $l(\tilde{x}) = l(x_0)$ for all $l \in L$. So, $\Psi(\tilde{x}) = \Psi(x_0)$ because L^* separates points in 2^ω. The continuity follows because $\lim_{\gamma \in \Gamma} \Phi(h_\gamma) = \lim_{\gamma \in \Gamma} \Psi(x_\gamma) = \Psi(\tilde{x}) = \Psi(x_0) = \Phi(h_0)$.

Now, since K_F is $t_p(L)$-closed in F, it follows that it is also $t_p(\overline{G}^{M^X})$-closed in F.

By our initial assumption, we have that F is $t_p(\overline{G}^{M^X})$-Lindelöf, which implies that also K_F is $t_p(\overline{G}^{M^X})$-Lindelöf.

We claim that $(2^\omega, t_p(\overline{L^*}^{M^{2^\omega}}))$ is also Lindelöf. Indeed, it is enough to prove that Φ is continuous on K_F when it is equipped with the $t_p(\overline{G}^{M^X})$-topology

and 2^ω is equipped with the $t_p(\overline{L^*}^{M^{2^\omega}})$-topology.

Take a map $k \in \overline{L^*}^{M^{2^\omega}}$ and let $\{l^*_\gamma\}_{\gamma \in \Gamma} \subseteq L^*$ be a net converging to k pointwise on 2^ω. Since $\overline{G}^{M^X}$ is compact, we may assume wlog that $\{l_\gamma\}_{\gamma \in \Gamma} \subseteq L\ t_p(X)$-converges to $h \in \overline{G}^{M^X}$. Therefore, for each $x \in K_F$ we have that $k(\Phi(x)) = \lim_{\gamma \in \Gamma} l^*_\gamma(\Phi(x)) = \lim_{\gamma \in \Gamma} l_\gamma(x) = h(x)$. That is $k \circ \Phi = h$. Since h is continuous on K_F, the continuity of Φ follows.

By Proposition 4.2, this implies that L^* is a hereditarily almost equicontinuous family on 2^ω, which is a contradiction. $\square$

Proposition 4.4. *Let X be a Čech-complete space, (M, d) be a metric space and let G be a subset of $C(X, M)$ such that $\overline{G}^{M^X}$ is compact. If there exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf, then G is almost equicontinuous.*

Proof. Let K and ν defined as in Remark 3.8. Since $\nu(\overline{G}^{M^X} \times K)$ is a compact subset of $[-1,1]^X$, it follows that $\overline{\nu(G \times K)}^{[-1,1]^X} = \nu(\overline{G}^{M^X} \times K)$.

By Lemma 3.9 we know that $(F, t_p(\nu(\overline{G}^{M^X} \times K)))$ is Lindelöf . Now, applying Proposition 4.3 to the subset $\nu(G \times K) \subseteq C(X, [-1, 1])$, it follows that $\nu(G \times K)$ is almost equicontinuous. Therefore, G is almost equicontinuous by Corollary 3.11. $\square$

The following lemma is known. We refer to [11, Cor. 3.5] for its proof.

Lemma 4.5. *Let X be a Lindelöf space, (M, d) be a metric space. If G is an equicontinuous subset of $C(X, M)$, then $\overline{G}^{M^X}$ is metrizable.*

We are now in position of proving Theorem **A**.

Proof of Theorem A. $(b) \Rightarrow (c)$ Since $(\overline{G}^{M^X})|_F$ is compact metric, it follows by Lemma 3.7 that there is a dense subset E such that G is equicontinuous at the points in E with respect to X. Since E is dense in F, which is dense in X, it follows that E is also be dense in X. Moreover, if Y denotes the G_δ subset of equicontinuity points of G in X, since $E \subseteq Y$, it follows that Y, the set of equicontinuity points of G is a dense G_δ-set in X. Set $K \overset{\text{def}}{=} (\overline{G}^{M^X})$. The equicontinuity of G at the points in Y combined with the density of $E \subseteq F$ in Y, implies that the map $\Theta : K|_F \longrightarrow K|_Y$ defined by $\Theta(f|_F) \overset{\text{def}}{=} f|_Y$ is a homeomorphism of $K|_F$ onto $K|_Y$.

By our initial assumption we have that $K|_F$ is compact and metrizable, which yields the metrizability of $K|_Y$. Thus, the evaluation map $Eval : Y \longrightarrow C_\infty(K|_Y, M)$ is a well defined and continuous map. We know that $C_\infty(K|_Y, M)$ is a separable space by [18, Cor. 4.2.18]. Therefore $(Eval(Y), t_\infty(K|_Y))$ and $(Y, t_\infty(K|_Y))$ are Lindelöf spaces. As a consequence $(Y, t_p(K|_Y))$ must be also Lindelöf and we are done.

$(c) \Rightarrow (a)$ This implication is Proposition 4.4

$(a) \Rightarrow (b)$ Suppose that X is Čech-complete and hereditarily Lindelöf. By Lemma 3.2, the subset, F, of equicontinuity points of G is a dense G_δ-set in X, which is a Lindelöf space by our initial assumption. Since G is equicontinuous on F, Lemma 4.5 implies that $(\overline{G}^{M^X})|_F$ must be metrizable. $\square$

The following result can be found in [5, Prop. 2.5 and Section 5] in the setting of compact metric spaces. Notwithstanding this, the proof given there can be adapted easily for Čech-complete and hereditarily Lindelöf spaces, as it is formulated in the next proposition. A sketch of the proof is included here for completeness sake.

Proposition 4.6. *Let X be a hereditarily Lindelöf space, (M, d) is a metric space and $G \subseteq C(X, M)$. If $H \stackrel{\text{def}}{=} \overline{G}^{M^X}$ is compact and hereditarily almost equicontinuous, then H is metrizable.*

Proof. The symbol $C_\infty(H, M)$ denote the space $C(H, M)$ equipped with the uniform convergence topology. Consider the map $eval : X \to C_\infty(H, M)$ defined by $eval(x)[h] \stackrel{\text{def}}{=} h(x)$ for all $x \in X$ and $h \in H$.

By Proposition 2.1 X is fragmented by $\rho_{G,d}$. Thus, for each nonempty subset A of X and for each $\epsilon > 0$ there exists a nonempty open subset U of X such that $U \cap A \neq \emptyset$ and $diam(h(U \cap A)) \leq \epsilon$ for all $h \in H$. Thus, d_∞-$diam(eval(U \cap A)) \leq \epsilon$.

We claim that $eval(X)$ is separable. Indeed, pick $\epsilon > 0$. Let $\mathcal{A}$ be the collection of all open subsets O of X such that $eval(O)$ can be covered by countably many sets of diameter less than ϵ. Since X is hereditarily Lindelöf there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B$. Take $V \stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} A$. Observe that V is the largest element of $\mathcal{A}$. Let's see that $A \stackrel{\text{def}}{=} X \setminus V$ is empty. Assume that $A \neq \emptyset$. Then there is a nonempty set U of X such that $U \cap A \neq \emptyset$ and d_∞-$diam(eval(U \cap A)) \leq \epsilon$. Since $eval(U \cup V) = eval(U \cap A) \cup eval(V)$ we know that $eval(U \cup V)$ can be covered by countably many sets of diameter less than ϵ. So, $U \cup V \in \mathcal{A}$ and we arrive to a contradiction because $U \cap (X \setminus V) \neq \emptyset$. Since $X = V \in \mathcal{A}$ and ϵ was arbitrary $eval(X)$ is separable.

There is a dense and countable subset D of $eval(X)$. We know that D separates points of H because $eval(X)$ also separates points. Let $\Delta D : H \to M^D$ be the diagonal product. Since ΔD is an embedding and M^D is metrizable we conclude that H is metrizable. $\square$

Next result is due basically to Namioka [9, Lemma 2.1]. It can also be found in [20, Lemma 6.4.], where the reference to Namioka is acknowledged. Again, we include a sketch of the proof here for completeness sake.

Lemma 4.7. *Let X, Y and (M, d) be two arbitrary compact spaces and a metric space, respectively, and let G be a subset of $C(Y, M)$. Suppose that $p : X \longrightarrow Y$*

is a continuous onto map. Then $G \circ p \stackrel{\text{def}}{=} \{g \circ p : g \in G\} \subseteq C(X, M)$ is hereditarily almost equicontinuous if and only if G is also hereditarily almost equicontinuous.

Proof. In order to prove this result, we will apply Lemma 3.2. Assume that $G \circ p$ is hereditarily almost equicontinuous. Let A be a closed (and compact) subset of Y, U be a nonempty relatively open set in A and $\epsilon > 0$. By Zorn's Lemma, there exists a minimal compact subset Z of X such that $p(Z) = A$. Since $\tilde{U} \stackrel{\text{def}}{=} p^{-1}(U) \cap Z$ is a nonempty relatively open set in Z and $(G \circ p)|_Z$ is almost equicontinuous there is a nonempty relatively open set $\tilde{V} \subseteq \tilde{U}$ in Z such that $diam((g \circ p)(\tilde{V})) < \epsilon$ for all $g \in G$. Let $V \stackrel{\text{def}}{=} A \setminus p(Z \setminus \tilde{V})$, that is relatively open set in A. We claim that $V \neq \emptyset$. Indeed, assume that $V = \emptyset$. Then $A = p(Z \setminus \tilde{V})$ and this contradicts the minimality of Z. Since $V \subseteq p(\tilde{V})$ we have that $diam(g(V)) < \epsilon$ for all $g \in G$.

Conversely, let Z be a closed subset of X, $\tilde{U}$ be a nonempty relatively open set in Z and $\epsilon > 0$. Consider the closed subset $W_0 \stackrel{\text{def}}{=} \overline{p(\tilde{U})}$ of Y. Since $G|_{W_0}$ is almost equicontinuous there is a nonempty relatively open set V_0 in Y such that $V_0 \cap W_0 \neq \emptyset$ and $diam(g(V_0 \cap W_0)) < \epsilon$ for all $g \in G$. Take $\tilde{V} \stackrel{\text{def}}{=} p^{-1}(V_0) \cap \tilde{U}$. Since $\tilde{V}$ is a nonempty relatively open set in Z and $p(\tilde{V}) \subseteq V_0 \cap W_0$ we conclude that $diam(g(p(\tilde{V}))) < \epsilon$ for all $g \in G$. $\square$

Remark 4.8. If the map p of the previous lemma is open or quasi-open we obtain the same result for almost equicontinuity. Recall that a map $f : X \to Y$ between two topological spaces is *quasi-open* if for any nonempty open set $U \subseteq X$ the interior of $f(U)$ in Y is nonempty.

Proof. Let U be a nonempty open set of Y and $\epsilon > 0$. Since $G \circ p$ is almost equicontinuous and $\tilde{U} = p^{-1}(U)$ is an open subset of X there is a nonempty open subset $\tilde{V} \subseteq \tilde{U}$ of X such that $diam((g \circ p)(\tilde{V})) < \epsilon$ for all $g \in G$. Since the nonempty open set $V \stackrel{\text{def}}{=} int(p(\tilde{V}))$ is included in $p(\tilde{V})$ we have that $diam(g(V)) < \epsilon$ for all $g \in G$.

Conversely, let $\tilde{U}$ be a nonempty open set of X and $\epsilon > 0$. Take $U \stackrel{\text{def}}{=} int(p(\tilde{U})) \neq \emptyset$. Since G is almost equicontinuous there is a nonempty open subset $V \subseteq U$ of Y such that $diam(g(V)) < \epsilon$ for all $g \in G$. So, taking the open subset $\tilde{V} \stackrel{\text{def}}{=} p^{-1}(V) \cap \tilde{U}$, we conclude that $diam((g \circ p)(\tilde{V})) < \epsilon$ for all $g \in G$. $\square$

Proposition 4.9. *Let X be a Čech-complete space, (M, d) be a hemicompact metric space and $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) G is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

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Proof. (a) implies (b) is trivial. To see the other implication, assume, reasoning by contradiction, that (a) does not hold. Then there must be some closed subset $A \subseteq X$ such that $G|_A$ is not almost equicontinuous. By Lemma 4.1 there exists a compact and separable subset F of X, an onto and continuous map $\Psi : F \to 2^\omega$, and a countable subset L of G such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in F$ is not almost equicontinuous. Therefore, L is not hereditarily almost equicontinuous on F by Lemma 4.7 and we arrive to a contradiction. $\square$

We can now prove Theorem B.

Proof of Theorem B. $(b) \Rightarrow (a)$ is a direct consequence of Proposition 4.9 and Corollary 3.11.

$(a) \Rightarrow (c)$ Let $L \in [G]^{\leq \omega}$ and let F be a separable and compact subset of X. L defines an equivalence relation on F by $x \sim y$ if and only if $l(x) = l(y)$ for all $l \in L$. If $\widetilde{F} = F/\sim$ is the compact quotient space and $p : F \to \widetilde{F}$ denotes the canonical quotient map, each $l \in L$ has associated a map $\tilde{l} \in C(\widetilde{F}, M)$ defined as $\tilde{l}(\tilde{x}) \overset{\text{def}}{=} l(x)$ for any $x \in F$ with $p(x) = \tilde{x}$. Furthermore, if $\tilde{L} \overset{\text{def}}{=} \{\tilde{l} : l \in L\}$, we can extend this definition to the closure of $\tilde{L}$ in $M^{\widetilde{F}}$. Thus, each $l \in \overline{L}^{M^F}$ has associated a map $\tilde{l} \in \overline{\tilde{L}}^{M^{\widetilde{F}}}$ such that $\tilde{l} \circ p = l$. By construction, we have that $\tilde{L}$ separates the points in $\widetilde{F}$. Since $\tilde{L}$ is countable it follows that $(\widetilde{F}, t_p(\tilde{L}))$ is a compact metric space. On the other hand, G is hereditarily almost equicontinuous on X. Applying Lemma 4.7 to F and $\widetilde{F}$, it follows that $\tilde{L}$ is hereditarily almost equicontinuous on $\widetilde{F}$. Therefore, the space $\overline{\tilde{L}}^{M^{\widetilde{F}}}$ is metrizable by Proposition 4.6. In order to finish the proof, it suffices to remark that $\overline{L}^{M^F}$ is canonically homeomorphic to $\overline{\tilde{L}}^{M^{\widetilde{F}}}$ (see Fact 5.7).

$(c) \Rightarrow (d)$ Let $L \in [G]^{\leq \omega}$ and let F be a separable and compact subset of X. We know that $H \overset{\text{def}}{=} ((\overline{L}^{M^X})|_F, t_p(F))$ is compact metric. Since F is separable, we have that $l(F)$ is a separable for every $l \in L$. Hence $N \overset{\text{def}}{=} \overline{\bigcup_{l \in L} l(F)}^M$ is a separable subset of M. Now, remark that M can be replaced by N without loss of generality. On the other hand, since $F \subseteq C(H, M)$ and H is compact metric, it follows that $(F, t_\infty(H))$ is separable and metrizable by [18, Cor. 4.2.18], which implies that $(F, t_\infty(H))$ is Lindelöf. Since the the topology $t_p(H)$ is weaker than $t_\infty(H)$, we deduce that $(F, t_p(H))$ must be Lindelöf.

$(d) \Rightarrow (b)$ By Lemma 3.9, for all $L \in [G]^{\leq \omega}$ and F a separable compact subset of X, we have that $(F, t_p(\nu(\overline{L}^{M^X} \times K)))$ is Lindelöf . Applying [2, Corollary D], it follows that $\nu(\overline{L}^{M^X} \times K)$ is hereditarily almost equicontinuous for all $L \in [G]^{\leq \omega}$ and F a separable compact subset of X. Thus, Corollary 3.11 yields (b). $\square$

5. Appendix

It is well known that for every compact metric space (M, d), there is a canonical continuous one-to-one mapping $\mathcal{E}_M : M \longrightarrow [0, 1]^\omega$ that embeds M into $[0, 1]^\omega$ as a closed subspace. Let $\rho_n : [-1, 1] \longrightarrow [0, 1]$ the map defined by $\rho_n(r) = \frac{|r|}{2^n}$ for every $n < \omega$. Along this paper, we will consider that $[0, 1]^\omega$ is equipped with the metric ρ defined by

$$\rho((x_n), (y_n)) = \sum_{n < \omega} \rho_n(x_n - y_n)$$

The proof of the following lemma is obtained by a standard argument of compactness, using the continuity of $\mathcal{E}_M^{-1}$ and that every continuous map defined on a compact space is uniformly continuous. We omit its proof here.

Fact 5.1. *Let (M, d) be a compact metric space. Let $\mathcal{E}_M : M \longrightarrow [0, 1]^\omega$ denote its attached embedding into $[0, 1]^\omega$, and let $\pi_n : [0, 1]^\omega \to [0, 1]$ denote the nth canonical projection. Then, for every $\epsilon > 0$, there is $\delta > 0$ and $n_0 < \omega$ such that if $(x, y) \in M \times M$ and $\rho_n(\pi_n(\mathcal{E}_M(x)) - \pi_n(\mathcal{E}_M(y))) < \delta/2n_0$ for $n \leq n_0$ then $d(x, y) < \epsilon$.*

We know recall some simple remarks that will be used along the paper.

Fact 5.2. *Let X be a topological space and (M, d) a compact metric space. If π_n is the nth projection mapping defined above, then the following map is continuous if we consider that the two spaces have the topology of pointwise convergence.*

$$\pi_n^* : M^X \to [0, 1]^X$$

defined by $\pi_n^*(f) \stackrel{\text{def}}{=} \pi_n \circ \mathcal{E}_M \circ f$, $f \in M^X$, for each $n < \omega$.

For each $S \subseteq M^X$ and each $n < \omega$ we define $S_n \stackrel{\text{def}}{=} \pi_n^*(S)$.

Fact 5.3. *Let X be a Baire space, (M, d) be a compact metric space, $G \subseteq C(X, M)$ and $H \stackrel{\text{def}}{=} \overline{G}^{M^X}$. Then $H_n = \overline{G_n}^{[0,1]^X}$.*

Proof. Indeed, since π_n^* is continuous we have that $H_n = \pi_n^*(H) = \pi_n^*(\overline{G}^{M^X}) \subseteq \overline{\pi_n^*(G)}^{[0,1]^X} = \overline{G_n}^{[0,1]^X}$. For the reverse inclusion, remark that $\overline{G_n}^{[0,1]^X}$ is the smallest closed subset that contains G_n and $G_n \subseteq H_n$. $\square$

Fact 5.4. *Let X be a Baire space, (M, d) be a compact metric space and $G \subseteq C(X, M)$. If G_n is almost equicontinuous for every $n < \omega$, then G is almost equicontinuous.*

Proof. For each $n \in \omega$ there exists a dense G_δ subset D_n of X such that G_n is equicontinuous on D_n. Since X is a Baire space, the $D = \bigcap_{n < \omega} D_n$ is dense in X. We claim that G is equicontinuous in D. Indeed, let $x_0 \in D$ and

$\epsilon > 0$. By Fact 5.1 we get $\delta > 0$ and $n_0 < \omega$. Take $\epsilon_0 = \frac{\delta}{2n_0}$. For each $n < n_0$, being G_n equicontinuous in x_0, there is an open neighbourhood U_n of x_0 such that $|g_n(x_0) - g_n(x)| < \epsilon_0$ for all $x \in U_n$ and $g_n \in G_n$. Consider the open neighbourhood $U = \bigcap_{n<n_0} U_n$ of x_0. So, let an arbitrary $g \in G$ and $x \in U$, then $\rho_n(\pi_n(\mathcal{E}_M(g(x_0))) - \pi_n(\mathcal{E}_M(g(x)))) = \rho_n(\pi_n^*(g)(x_0) - \pi_n^*(g)(x)) = \frac{|\pi_n^*(g)(x_0) - \pi_n^*(g)(x)|}{2^n} < \frac{\epsilon_0}{2^n} \leq \frac{\delta}{2n_0}$. Consequently, $d(g(x_0), g(x)) < \epsilon$ by Fact 5.1. $\qquad\square$

Fact 5.5. *The diagonal map $\Delta : H \to \prod_{n<\omega} H_n$ defined by $\Delta(h) = (\pi_n \circ \mathcal{E}_M \circ h)_{n<\omega}$ for each $h \in H$, is a homeomorphism of H onto its image.*

Fact 5.6. *Given a subset $L \subseteq G$, it defines an equivalence relation on X by $x \sim y$ if and only if $l(x) = l(y)$ for all $l \in L$. Let $\widetilde{X} = X/\sim$ be the quotient space and let $p : X \to \widetilde{X}$ denote the canonical quotient map, then each $l \in L$ has associated a map $\tilde{l} \in C(\widetilde{X}, M)$ defined as $\tilde{l}(\tilde{x}) \overset{\text{def}}{=} l(x)$ for any $x \in X$ with $p(x) = \tilde{x}$. Furthermore, if $\tilde{L} \overset{\text{def}}{=} \{\tilde{l} : l \in L\}$, we can extend this definition to the closure of $\tilde{L}$. Thus, each $l \in \overline{L}^{M^X}$ has associated a map $\tilde{l} \in \overline{\tilde{L}}^{M^{\widetilde{X}}}$ such that $\tilde{l} \circ p = l$.*

Fact 5.7. *Let L be a countable subset of $G \subseteq C(X, M)$. We denote by X_L the topological space $(\widetilde{X}, t_p(\tilde{L}))$, which is metrizable because $\tilde{L}$ is countable. Consider the map $p^* : (M^{\widetilde{X}}, t_p(\widetilde{X})) \to (M^X, t_p(X))$ defined by $p^*(\tilde{f}) = \tilde{f} \circ p$, for each $\tilde{f} \in M^{\widetilde{X}}$. Then p^* is a homeomorphism of $\overline{\tilde{L}}^{M^{\widetilde{X}}}$ onto $\overline{L}^{M^X}$.*

Proof. We observe that p^* is continuous, since a net $\{\tilde{f}_\alpha\}_{\alpha \in A}$ $t_p(\widetilde{X})$-converges to $\tilde{f}$ in $\overline{\tilde{L}}^{M^{\widetilde{X}}}$ if and only if $\{\tilde{f}_\alpha \circ p\}_{\alpha \in A}$ $t_p(X)$-converges to $\tilde{f} \circ p$ in $\overline{L}^{M^X}$.

Let's see that $p^*(\overline{\tilde{L}}^{M^{\widetilde{X}}}) = \overline{L}^{M^X}$. Indeed, since p^* is continuous we have that $p^*(\overline{\tilde{L}}^{M^{\widetilde{X}}}) \subseteq \overline{p^*(\tilde{L})}^{M^X} = \overline{L}^{M^X}$. We have the other inclusion because $\overline{L}^{M^X}$ is the smaller closed set that contains L and $L \subseteq p^*(\overline{\tilde{L}}^{M^{\widetilde{X}}})$.

Let $\tilde{f}, \tilde{g} \in \overline{\tilde{L}}^{M^{\widetilde{X}}}$ such that $\tilde{f} \neq \tilde{g}$. Then there exists $\tilde{x} \in \widetilde{X}$ such that $\tilde{f}(\tilde{x}) \neq \tilde{g}(\tilde{x})$. Let $x \in X$ an element such that $\tilde{x} = p(x)$. Thus $(\tilde{f} \circ p)(x) \neq (\tilde{g} \circ p)(x)$. So, p^* is injective because $\tilde{f} \circ p \neq \tilde{g} \circ p$.

Finally, we arrive to the conclusion that $p^*|_{\overline{\tilde{L}}^{M^{\widetilde{X}}}}$ is a homeomorphism because it is defined between compact spaces. $\qquad\square$

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