CHAOTIC SEMIGROUPS FROM SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We give general conditions on given parameters to ensure Devaney and distributional chaos for the solution C_0-semigroup corresponding to a class of second-order partial differential equations. We also provide a critical parameter that led us to distinguish between stability and chaos for these semigroups. In the case of chaos, we prove that the C_0-semigroup admits a strongly mixing measure with full support. We also give concrete examples of partial differential equations, such as the telegraph equation, whose solutions satisfy these properties.

1. Introduction

The phenomenon of chaos is usually identified with nonlinear phenomena, but chaos also appears in linear dynamical systems provided that the underlying space is infinite-dimensional. The theory of chaos in finite-dimensional dynamical systems has been well-developed and includes both discrete maps and systems of ordinary differential equations. This theory has led to important applications in physics, chemistry, biology, and engineering. However, for a long period of time, there was no theory of chaos for partial differential equations (PDE's). In terms of applications, most of the important natural phenomena are described by PDE's: nonlinear wave equations, Maxwell equations, Navier-Stokes equations, and so on. These equations model a wide variety of phenomena in cell proliferation, electrostatics, electrodynamics, elasticity, fluid flow, heat conduction, sound propagation, or traffic modelling.

The study of C_0-semigroups has been widely identified with partial of parabolic and hyperbolic type differential equations. It is now well known that the solutions of these equations can be represented in terms of C_0-semigroups [27]. They permit the solution to the corresponding abstract Cauchy problem to be described in a broader setting, for instance, including non-differentiable integrable functions as initial conditions.

In this paper, we provide a new insight into the chaotic behavior of any C_0-semigroup that is solution of a certain class of second order partial differential equations, considering both Devaney and distributional chaos. The study will be carried out on Herzog type spaces [32]. Herzog's result was later improved in [24].

These spaces consist of analytic functions regulated by a parameter, or a tuner, that allows their growth at infinity to be controlled. They were initially introduced in order to study the universality of the solution operators of the heat equation. In [16], Chan & Shapiro studied the dynamics of the translation operator on spaces of analytic functions of slow growth and characterized when the derivative operator was bounded on these spaces. Then, since the derivative operator is the infinitesimal generator of the translation semigroup, we can conclude that the translation semigroup is uniformly continuous and all its operators can be obtained via the exponential formula. See for instance [27, Th. 3.7]. Interesting constructions and counterexamples have been given in the framework of certain subspaces of analytic functions, see for instance [10, 40, 14]. Godefroy & Shapiro also considered Hardy and Bergman spaces for studying the dynamics of shift operators, see [28].

Date: July 12, 2017.

The first and third authors are supported in part by MINECO and FEDER, grant MTM2016-75963-P. The second author is partially supported by CONICYT, under Fondecyt Grant number 1140258 and CONICYT-PIA, Anillo ACT1416.

Our results show an interesting duality between chaos and stability, which is distinguished by means of a critical parameter depending on the given equation. We obtain sharp conditions involving the coefficients of the equation and the tuner. This is a remarkable phenomenon that appears to be linked to the dependence of the tuning parameter of the underlying Herzog space. A similar analysis of the existence of chaos versus stability can also be found in [6, 15].

We point out that our results refer to the dynamics on the whole space. Special cases in which the chaotic behaviour is analyzed in certain subspaces can be found in [5], where the notion of subchaos is described. See also [6, 9] for the study of the stability on a dense subspace of the phase space.

In the last section, we analyze the linear telegraph equation on an infinite transmission lossless line. This equation is obtained from a system of two coupled differential equations that represent the voltage and current on an electrical transmission line with distance and time. These equations can be simplified into a single second order differential equation, either for the voltage or the current. This model shows that electromagnetic waves can travel with a speed close to the speed of light, although electrons travel with a drift velocity of only a few centimeters per second, which makes a wave propagation phenomenon of the electric field appear.

This permits performing an analysis of the linear dynamics, using a similar approach to [22] for the solution of the wave equation, see also [20, 21]. Though the chaos for the solutions of this equation has not been obtained, some results on the dynamics of the solutions have already been obtained. Bereanu [11] has studied the existence, non existence and multiplicity of the periodic solutions of the nonlinear telegraph equation with bounded non linearities and in [34] the authors introduced a maximum principle for the bounded and periodic solutions of this equation. Abdusalam has given asymptotic techniques in order to find traveling waves solution in [1]. Finite difference schemes have been widely used for solving this equation, see for example [18].

2. Preliminaries

Let X be a separable infinite-dimensional Banach space. We recall that $\{T_t\}_{t\geq 0}$, with $T_t : X \to X$ a continuous and linear map on X for each $t \geq 0$, is a C_0*-semigroup* if $T_0 = I$, $T_{t+s} = T_t \circ T_s$, and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$, which is the convergence of the operatos of the semigroup to the identity when t tends to 0 with respect to the strong operator topology. If the limit holds uniformly over bounded subsets of X we say that the C_0-semigroup is *uniformly continuous.*

Let us consider the following abstract Cauchy problem on X:

$$\begin{cases} u_t(t,x) = Au(t,x), \\ u(0,x) = \varphi(x) \text{ with } \varphi(x) \in X. \end{cases} \tag{1}$$

The solution to (1) can be represented by a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X whose infinitesimal generator is A. Provided $A \in L(X)$ is a generator, then the operators in the C_0-semigroup can be represented as $T_t = e^{tA} = \sum_{k=0}^{\infty}(tA)^n/n!$ for all $t \geq 0$, see for instance [27, Chapter I, Proposition 3.5].

A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is called a *periodic point* for the semigroup $\{T_t\}_{t\geq 0}$ if there exists some $t > 0$ such that $T_t x = x$. $\{T_t\}_{t\geq 0}$ is *transitive* if for any pair U, V of nonempty open sets of X, there exists some $t_0 \geq 0$ such that $T_{t_0}(U) \cap V \neq \emptyset$, and it is *topologically mixing* if $T_t(U) \cap V \neq \emptyset$ for all $t \geq t_0$. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in X. We point out that these two requirements also yield the sensitive dependence on the initial conditions, as seen by Banks et al for the discrete case [7, 31], and [4] for the case of C_0-semigroups.

It is well-known that, topologically mixing property for C_0-semigroups is strictly stronger than hypercyclicity, see for instance [12]. Further information on the dynamics of C_0-semigroups can be found in [31, Chapter 7].

Another variation of the definition of chaos is the notion of *distributional chaos* introduced by Schweizer & Smítal [39], see also [33, 37] for its presentation in the infinite-dimensional linear setting. A C_0-semigroup

$\{T_t\}_{t\geq 0}$ on X is said to be *distributionally chaotic* if there exists an uncountable subset $S \subset X$ and $\delta > 0$ such that, for each pair of distinct points $x, y \in S$ and for every $\varepsilon > 0$, we have $\overline{\text{Dens}}(\{s \geq 0; ||T_s x - T_s y|| > \delta\}) = 1$ and $\overline{\text{Dens}}(\{s \geq 0; ||T_s x - T_s y|| < \varepsilon\}) = 1$, where $\overline{\text{Dens}}$ stands for the upper density of a set of real positive numbers, that is, given $M \subset \mathbb{R}_+$,

$$\overline{\text{Dens}}(M) := \limsup_{N\to\infty} \frac{\lambda(M \bigcap [0, N])}{N},$$

where λ is the Lebesgue measure on $\mathbb{R}_+$. The set S is called the scrambled set and the C_0-semigroup is said to be *densely distributionally chaotic* if S is dense on X.

Remark 2.1. *Upper denstiy can be easily illustrated with the next two examples: For $0 < \varepsilon < 1$ and $k \in \mathbb{N}$, the upper density of the set $\cup_{n=1}^{\infty}[kn, kn + \varepsilon]$ is ε/k; however, the upper density of the set $\cup_{n=1}^{\infty}[n^2, n^2 + \varepsilon]$ is* 0.

Given a pair of elements in the scrambled set S, the notion of distributional chaos indicates that we can find intervals, as long as we want, such that the upper density of times in which the orbits of both elements are close enough is positive, and long intervals such that the set of times in which both orbits differ by at least a certain positive amount, and also have positive upper density.

Now, we present a criterion that ensures Devaney chaos for C_0-semigroups, that is a variation of the Desch-Schappacher-Webb (DSW) criterion [25], see [31, Th. 7.30].

Theorem 2.2. *Let X be a complex separable Banach space, and $\{T_t\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator $(A, D(A))$. Assume that there exists an open connected subset U and a weakly holomorphic function $f : U \to X$, such that*

(i) $U \cap i\mathbb{R} \neq \emptyset$,
(ii) $f(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U$,
(iii) *for any* $x^* \in X^*$, *if* $\langle f(\lambda), x^*\rangle = 0$ *for all* $\lambda \in U$, *then* $x^* = 0$.

Then the semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic and topologically mixing.

Finally, we recall the definition of the space of analytic functions of Herzog type [32]. Given $\rho > 0$, let:

$$X_\rho = \left\{ f : \mathbb{R} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\} \tag{2}$$

endowed with the norm $||f|| = \sup_{n\geq 0} |a_n|$. This space is isometrically isomorphic to $c_0(\mathbb{N}_0) := \{a_n : \mathbb{N}_0 \to \mathbb{C} : \lim_{n\to\infty} |a_n| = 0\}$. To finish this section, we recall that a function $f : U \to X$, defined on an open subset $U \subset \mathbb{C}$ is weakly holomorphic if for all $x^* \in X^*$, $x^* \circ f$ is holomorphic.

3. Chaotic behavior for a class of partial differential equations

In this section, we will study the chaotic behavior of second-order partial differential equations with respect to the time and space such as the following:

$$\frac{\partial^2 u}{\partial t^2}(t, x) + \gamma \frac{\partial u}{\partial t}(t, s) + \theta u(t, x) = \alpha \frac{\partial^2 u}{\partial x^2}(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{3}$$

where γ, θ and $\alpha \in \mathbb{R}$. This equation can be reduced to a first order system on the phase space that is the product of a certain space of Herzog type with itself. Setting $u_1 = u$ and $u_2 = \frac{\partial u}{\partial t}$ we have

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \alpha \frac{\partial^2}{\partial x^2} - \theta I & -\gamma I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \\ \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{cases} \tag{4}$$

Since the second order differential operator $\frac{\partial^2}{\partial x^2}$ turns out to be a bounded operator on X_ρ, then the operator-valued matrix

$$A := \begin{pmatrix} 0 & I \\ \alpha\frac{\partial^2}{\partial x^2} - \theta I & -\gamma I \end{pmatrix}, \tag{5}$$

defines a bounded operator on $X := X_\rho \oplus X_\rho$ for every $\rho > 0$ and, consequently, we have that $(e^{tA})_{t\geq 0}$ is the uniformly continuous semigroup associated with the Cauchy problem (4). The following theorem is one of the main results of the present study.

Theorem 3.1. *Let γ, θ, α be real positive numbers. Suppose that*

$$\gamma^2 = 4\theta. \tag{6}$$

Then A generates a uniformly continuous semigroup which is Devaney and distributionally chaotic, topologically mixing and admits a strongly mixing measure with full support on $X_\rho \times X_\rho$, for each $\rho > \frac{\gamma}{2\sqrt{\alpha}}$.

Proof. Our purpose is to apply Theorem 2.2. Since $\rho > \frac{\gamma}{2\sqrt{\alpha}}$ there exists $\epsilon > 0$ such that $\epsilon < 2\rho\sqrt{\alpha} - \gamma$. We define

$$U := \{z \in \mathbb{C} \, : \, |z + \frac{\gamma}{2}| < \frac{\gamma}{2} + \frac{\epsilon}{2} \, \}. \tag{7}$$

Note that $U \cap i\mathbb{R} \neq \emptyset$ because $\epsilon > 0$. It satisfies condition (i) in Theorem 2.2. For each $\lambda \in \mathbb{C}$ we define $R_\lambda := \frac{\lambda^2+\gamma\lambda+\theta}{\alpha} = \frac{1}{\alpha}(\lambda + \frac{\gamma}{2})^2$, where the last identity follows from (6).

We now solve $Af_{z_0,z_1}(\lambda) = \lambda f_{z_0,z_1}(\lambda)$. We then obtain weakly analytic functions $f_{z_0,z_1}(\lambda)$ on the set U given by

$$f_{z_0,z_1}(\lambda) = \begin{pmatrix} \varphi_\lambda \\ \lambda\varphi_\lambda \end{pmatrix}, \tag{8}$$

where $\varphi_\lambda(x) := \cosh\left(\frac{1}{\sqrt{\alpha}}(\lambda + \frac{\gamma}{2})x\right) z_0 + \sinh\left(\frac{1}{\sqrt{\alpha}}(\lambda + \frac{\gamma}{2})x\right) z_1, \; z_0, z_1 \in \mathbb{C}$. It is easy to see that

$$\alpha\varphi_\lambda^{''}(x) = (\lambda^2 + \gamma\lambda + \theta)\varphi_\lambda(x), \text{ for every } x \in \mathbb{R}. \tag{9}$$

We will show that $f_{z_0,z_1}(\lambda) \in X_\rho \times X_\rho$ for all $\lambda \in U$.

Indeed, first note that we can rewrite

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty} a_n(\lambda)\frac{(\rho x)^n}{n!}, \, x \in \mathbb{R}, \tag{10}$$

where $a_n(\lambda) = \frac{1}{\alpha^{n/2}}\frac{(\lambda+\gamma/2)^n}{\rho^n}z_0, \; n = 0, 2, 4, \ldots$ and $a_n(\lambda) = \frac{1}{\alpha^{n/2}}\frac{(\lambda+\gamma/2)^n}{\rho^n}z_1, \; n = 1, 3, 5, \ldots$ See for instance [19, 21]. Therefore, by definition, it is enough to prove that $\left|\frac{1}{\rho\sqrt{\alpha}}(\lambda + \frac{\gamma}{2})\right| < 1$. Indeed, by the choice of $\epsilon > 0$, we have for all $\lambda \in U$ that

$$\left|\frac{1}{\rho\sqrt{\alpha}}(\lambda + \frac{\gamma}{2})\right| < \frac{\gamma + \epsilon}{2\rho\sqrt{\alpha}} < 1, \tag{11}$$

which proves the claim and gives condition (ii) in Theorem 2.2.

It only remains to show, that for any $x^* \in X_\rho^* \oplus X_\rho^*$, the functions $\lambda \to \langle f_{z_0,z_1}(\lambda), x^*\rangle$, z_0, $z_1 \in \mathbb{C}$, are holomorphic on U, and if they all vanish on U, then $x^* = 0$. Since X_ρ is isometrically isomorphic to c_0, in what follows, we identify the dual space X_ρ^* with ℓ^1.

Let $x^* \in X_\rho^* \oplus X_\rho^*$. It can be represented in a canonical way by $(x_1^*, x_2^*) = ((x_{1,n}^*)_{n\geq 0}, (x_{2,n}^*)_{n\geq 0}) \in \ell^1 \oplus \ell^1$. Then, we have

$$0 = \langle f_{z_0,z_1}(\lambda), x^*\rangle = \langle \varphi_\lambda, x_1^*\rangle + \langle \lambda\varphi_\lambda, x_2^*\rangle, \tag{12}$$

for all $\lambda \in U, z_0, z_1 \in \mathbb{C}$. This last equation can be reformulated in the following way:

$$0 = \sum_{n=0}^{\infty} a_n(\lambda) x^*_{1,n} + \lambda \sum_{n=0}^{\infty} a_n(\lambda) x^*_{2,n}. \tag{13}$$

Let $\lambda_0 := -\frac{\gamma}{2}$. It is clear that $\lambda_0 \in U$ and evaluating (13) in λ_0 we get :

$$z_0 x^*_{1,0} + \lambda_0 z_0 x^*_{2,0} = 0, \tag{14}$$

for all $z_0 \in \mathbb{C}$. Now, differentiating (13) with respect to λ we obtain:

$$\begin{aligned} &z_0 \sum_{n=2,4,\ldots}^{\infty} \frac{n(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{1,n} + z_1 \sum_{n=1,3,\ldots}^{\infty} \frac{n(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{1,n} + \\ &+ z_0 \sum_{n=0,2,4,\ldots}^{\infty} \frac{(\lambda+\gamma/2)^{n}}{\alpha^{n/2}\rho^n} x^*_{2,n} + z_1 \sum_{n=1,3,\ldots}^{\infty} \frac{(\lambda+\gamma/2)^{n}}{\alpha^{n/2}\rho^n} x^*_{2,n} \\ &+ \lambda \left(z_0 \sum_{n=2,4,\ldots}^{\infty} \frac{n(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{2,n} + z_1 \sum_{n=1,3,\ldots}^{\infty} \frac{n(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{2,n} \right) = 0 \end{aligned} \tag{15}$$

Evaluating (15) in $\lambda = \lambda_0$ we obtain:

$$\frac{z_1}{\sqrt{\alpha}\rho} x^*_{1,1} + z_0 x^*_{2,0} + \frac{\lambda_0}{\sqrt{\alpha}\rho} z_1 x^*_{2,1} = 0, \tag{16}$$

for all $z_0, z_1 \in \mathbb{C}$. Therefore, solving (14) and (16),we have $x^*_{1,0} = 0$, $x^*_{2,0} = 0$ and

$$\frac{z_1}{\sqrt{\alpha}\rho} x^*_{1,1} + \frac{\lambda_0}{\sqrt{\alpha}\rho} z_1 x^*_{2,1} = 0. \tag{17}$$

Now, we divide (13) by $(\lambda + \frac{\gamma}{2})$ and we differentiate with respect to λ obtaining:

$$\begin{aligned} &z_0 \sum_{n=2,4,\ldots}^{\infty} \frac{(n-1)(\lambda+\gamma/2)^{n-2}}{\alpha^{n/2}\rho^n} x^*_{1,n} + z_1 \sum_{n=3,5,\ldots}^{\infty} \frac{(n-1)(\lambda+\gamma/2)^{n-2}}{\alpha^{n/2}\rho^n} x^*_{1,n} + \\ &+ z_0 \sum_{n=2,4,\ldots}^{\infty} \frac{(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{2,n} + z_1 \sum_{n=1,3,\ldots}^{\infty} \frac{(\lambda+\gamma/2)^{n-1}}{\alpha^{n/2}\rho^n} x^*_{2,n} \\ &+ \lambda \left(z_0 \sum_{n=2,4,\ldots}^{\infty} \frac{(n-1)(\lambda+\gamma/2)^{n-2}}{\alpha^{n/2}\rho^n} x^*_{2,n} + z_1 \sum_{n=3,5,\ldots}^{\infty} \frac{(n-1)(\lambda+\gamma/2)^{n-2}}{\alpha^{n/2}\rho^n} x^*_{2,n} \right) = 0. \end{aligned} \tag{18}$$

Evaluating (18) in λ_0, we get:

$$\frac{z_0}{\alpha\rho^2} x^*_{1,2} + \frac{1}{\sqrt{\alpha}\rho} z_1 x^*_{2,1} + \frac{\lambda_0 z_0}{\alpha\rho^2} x^*_{2,2} = 0, \tag{19}$$

for all $z_0 \in \mathbb{C}$. Therefore, solving (19) and (17), we obtain $x^*_{1,1} = 0$ and $x^*_{2,1} = 0$. Proceeding inductively we will get that $x^*_{i,n} = 0$ for $i = 1, 2$ and $n \in \mathbb{N}$. We finally have $x^* = 0$ and we conclude the result by applying Theorem 2.2. Finally, it is well known that distributional chaos holds whenever the DSW criterion can be applied [8, 13]. Moreover, when DSW holds, the C_0-semigroup admits a strongly mixing measure with full support on $X_\rho \times X_\rho$, [35]. □

Roughly speaking, the existence of distributional chaos means that we can pick two initial vectors from an uncountable set such that there will be intervals of arbitrary time length in which the trajectories of the solutions are very similar and intervals have arbitrary time length in which there exists at least some positive difference between them. The mixing properties indicate that given any pair of open sets U and V on the space X, where we take the initial conditions, there will exist an instant of time t_0 such that for every time $t_1 \geq t_0$ we can find the orbit of an elements that at $t = 0$ lies in U and later, at time t_1, visits V.

Remark 3.2. *One can also consider the equation* (3) *including a source term* $g(t,x)$

$$\frac{\partial^2 u}{\partial t^2}(t,x) + \gamma\frac{\partial u}{\partial t}(t,s) + \theta u(t,x) = \alpha\frac{\partial^2 u}{\partial x^2}(t,x) + g(t,x), \quad t \geq 0, \quad x \in \mathbb{R}. \tag{20}$$

Then the abstract Cauchy problem can be reformulated as follows in the same way as in [23]

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \alpha\frac{\partial^2}{\partial x^2} - \theta I & -\gamma I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t,x) \end{pmatrix}; \\ \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{cases} \tag{21}$$

Taking the operator A *as in* (5)*, the solution family of operators to this abstract Cauchy problem is given by*

$$u(t,x) = e^{tA}\Phi(x) + \int_0^t e^{(t-s)A}\Psi(s,x)ds, \qquad \text{for every } x \in \mathbb{R}, t \geq 0. \tag{22}$$

where we have used the following notation

$$u(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}, \quad \Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad \Psi(t,x) = \begin{pmatrix} 0 \\ g(t,x) \end{pmatrix}. \tag{23}$$

Given U *defined as in* (7)*, for any arbitrary value* $\lambda \in U$ *with* $\Re(\lambda) < 0$ *we also obtain the topologically mixing property for the solution family* (22)*, which is not a semigroup, where*

$$\Psi(t,x) = \begin{pmatrix} \varphi_\lambda(x) \\ \lambda\varphi_\lambda(x) \end{pmatrix}, \tag{24}$$

by following exactly the technique shown in Theorem 2.2 and Remarks 1,2, and 3 in [23].

Recall that a uniformly continuous semigroup is said to be stable, if $\lim_{t\to\infty} \|e^{At}\| = 0$. For more information related to stability of C_0-semigroups, we refer to [27, Chapter V] and [26]. We can prove the following result.

Theorem 3.3. *Let* $\theta \in \mathbb{R}, \gamma > 0, \alpha > 0$. *Suppose that*

$$\gamma^2 - 4\theta < 0. \tag{25}$$

Then A *generates a uniformly continuous and stable semigroup on* $X_\rho \times X_\rho$ *for each* $\rho < \frac{\gamma}{2\sqrt{\alpha}}$.

Proof. It remains to prove stability. Since A is bounded, it is enough to show that the spectrum of A is contained in the negative real axis. Then the result is a consequence of the spectral mapping theorem [27, Chapter I, Lemma 3.13]. Indeed, we shall consider the eigenvalue problem for A

$$Af = \lambda f, \quad \lambda \in \mathbb{C}, \qquad \text{for every } f \in X_\rho \times X_\rho. \tag{26}$$

As in the proof of the above theorem, this leads to the equation

$$\varphi''(x) = R_\lambda\varphi(x) \qquad \text{for every } x \in \mathbb{R}, \varphi \in X_\rho, \tag{27}$$

where $R_\lambda := \frac{\lambda^2+\gamma\lambda+\theta}{\alpha}$. Since $\varphi \in X_\rho$, we have $\varphi(x) = \sum_{n=0}^{\infty} \frac{a_n\rho^n}{n!}x^n$, where $a_n \to 0$. From (27) we obtain that the sequence $(a_n)_n$ satisfies $\rho^2 a_{n+2} = R_\lambda a_n$, $n \in \mathbb{N}_0$. Therefore the sequence $(a_n)_n$ must be defined as powers of $\dfrac{R_\lambda}{\rho^2}$. This, together with the condition $a_n \to 0$, implies that we have $|R_\lambda| < \rho^2$. Hence

$$\left|\lambda + \frac{\gamma}{2} - ic\right| \left|\lambda + \frac{\gamma}{2} + ic\right| = \left|\lambda^2 + \gamma\lambda + \theta\right| < \alpha\rho^2, \tag{28}$$

where $c := \sqrt{\theta - \frac{\gamma^2}{4}}$. Let $\lambda = a + ib, a, b \in \mathbb{R}$. Then the above inequality gives

$$\left(\left(a + \frac{\gamma}{2}\right)^2 + (c-b)^2\right)\left(\left(a + \frac{\gamma}{2}\right)^2 + (c+b)^2\right) < \alpha^2\rho^4. \tag{29}$$

In particular, it implies $|a + \frac{\gamma}{2}| < \sqrt{\alpha}\rho < \frac{\gamma}{2}$ and therefore $\Re(\lambda) < 0$, which proves that claim and the theorem. □

Remark 3.4. *Comparing Theorems 3.1 and 3.3 we observe that concerning Herzog's spaces, a critical point is* $\rho = \frac{\gamma}{2\sqrt{\alpha}}$ *from where stability and chaos are divided. Observe that for* $\gamma = 0$ *there is no possibility to have stability according to Theorem 3.3. It is interesting to note that* $\gamma = 0$ *indicates that the damping effect in the equation* (3) *disappears, which is consistent with the stabilization of the solution on time. Therefore, in this sense, both theorems are dual and the conditions are sharp.*

4. Application to the telegraph equation

The telegraph equation models a piece of telegraph wire as an electrical circuit which consists of a resistor of resistance R and a coil of inductance L. The function $u(x,t)$ represents the voltage at position x and time t. We also suppose that current can escape from the wire to the ground, either through a resistor of conductance G or through a capacitor of capacitance C. This equation is given by

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + (a+b)\frac{\partial u}{\partial t} + abu = 0, \tag{30}$$

where $a = \frac{G}{C}, b = \frac{R}{L}$, and $c^2 = \frac{1}{LC}$. We identify the parameters in (3) as $\alpha = c^2, \gamma = (a+b)$, and $\theta = ab$.

The usual frame in which the telegraph equation is applied is the design of circuits on wires of finite length. The abstract model of an infinite transmission line corresponds to the situation in which the energy supplied at the source is transmitted without dissipation along the line, and no reflections are considered. This situation is equivalent to the finite case in which the impedance at one extreme is equal to the characteristic impedance (ratio of voltage and current) at the other extreme (source), see for instance [38, 42]. This is known as a *matched* line [2]. Lossless lines are transmission lines with no resistance and no dielectric loss. Maxwell equations for an infinite lossless transmission line can be transformed into telegraph equations [36]. The connection of the telegraph equation with the wave equation was observed by Griego and Hersh [30] in the Banach setting, see also [29]. The asymptotic analysis of the solutions of the telegraph equation on $L^2(\mathbb{R})$ was considered in [3].

By Theorem 3.1, if $a = b$ we have that the associated uniformly continuous semigroup of (30) is Devaney and distributionally chaotic, and topologically mixing on $X_\rho \oplus X_\rho$ for all $\rho > \frac{a}{c}$. In particular, if $a = 0$ then the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{31}$$

is always chaotic on $X_\rho \cong c_0(\mathbb{N}_0)$ for all $\rho > 0$. We remark that the presence of chaos for the wave equation has been previously observed in the literature by other authors. See [17, 41] and references therein.

Remark 4.1. *As mentioned in* [22, Sec. 3] *it follows from the existence of chaos in the sense of Devaney that for every* $\varepsilon > 0, \ell_0 > 0$ *and* $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ *real-valued continuous functions defined on the whole line,* *there exist a solution* $\bar{u}(t,x)$ *of* (30) *and* $t_0, t_1 > 0$ *such that* $\bar{u}(t,x) = \bar{u}(t+t_0,x)$ *for all* $t > 0, x \in \mathbb{R}$ *and*

$$\sup_{x\in[-\ell_0,\ell_0]} |\bar{u}(0,x) - \varphi_1(x)| < \varepsilon, \qquad \sup_{x\in[-\ell_0,\ell_0]} |\bar{u}_t(0,x) - \varphi_2(x)| < \varepsilon,$$
$$\sup_{x\in[-\ell_0,\ell_0]} |\bar{u}(t_1,x) - \varphi_3(x)| < \varepsilon, \qquad \sup_{x\in[-\ell_0,\ell_0]} |\bar{u}_t(t_1,x) - \varphi_4(x)| < \varepsilon.$$

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