MULTI-REAL-LINEAR ISOMETRIES ON FUNCTION ALGEBRAS

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Abstract. Let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Y be a locally compact Hausdorff space. A k-real-linear map $T : A_1 \times ... \times A_k \longrightarrow C_0(Y)$ is called a *multi-real-linear (or k-real-linear) isometry* if

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k),$$

where $\|\cdot\|$ denotes the supremum norm. In this paper we study such maps and obtain generalizations of basically all known results concerning multilinear and real-linear isometries on function algebras.

1. Introduction

Let X be a locally compact Hausdorff space and let $C_0(X)$ (resp. $C(X)$ if X is compact) denote the Banach space of complex-valued continuous functions defined on X vanishing at infinity, endowed with the supremum norm $\|\cdot\|$. The classical Banach-Stone theorem gave the first characterization of surjective linear isometries between $C(X)$-spaces as weighted composition operators ([3, 1]). Several extensions of this theorem have been derived for different settings. Thus, Holsztyński ([6]) considered the non-surjective version of the Banach-Stone theorem and showed that if $T : C(X) \longrightarrow C(Y)$ is a linear isometry (not necessarily onto), then T can be represented as a weighted composition operator on a nonempty subset of Y.

In [12], the authors proved, based on the powerful Stone-Weierstrass theorem, the following bilinear version of Holsztyński's theorem:

Let $T : C(X) \times C(Y) \longrightarrow C(Z)$ be a bilinear (or 2-linear) isometry. Then there exist a closed subset Z_0 of Z, a surjective continuous mapping $\varphi : Z_0 \longrightarrow X \times Y$ and a unimodular function $a \in C(Z_0)$ such that $T(f, g)(z) = a(z)f(\pi_x(\varphi(z)))g(\pi_y(\varphi(z)))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$, where π_x and π_y are projection maps.

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More recently, in [8], the authors provided a weighed composition characterization of multilinear (k-linear) isometries on function algebras and extended the above results.

Another direction of extensions of the Banach-Stone theorem deals with its real-linear version, motivated by the fact that, thanks to the Mazur-Ulam theorem [9], every surjective isometry between two complex-linear function spaces is real-linear. Thus, in [4], Ellis considered two compact Hausdorff spaces, X_1 and X_2, a uniform algebra M_1 on X_1 and a unital closed separating subspace M_2 of $C(X_2)$ such that the Šilov boundaries of M_1 and M_2 are X_1 and X_2, respectively, and proved that if $T : M_1 \longrightarrow M_2$ is a surjective real-linear isometry, then there exist a clopen subset K of X_2 and a homeomorphism $\varphi : X_2 \longrightarrow X_1$ such that $T(f) = T(1) f \circ \varphi$ on K and $T(f) = T(1)\overline{f \circ \varphi}$ on $X_2 \setminus K$, where $\bar{\cdot}$ denotes the complex conjugate. In [11], Miura generalized this result to non-unital algebras and showed that if $T : A \longrightarrow B$ is a surjective real-linear isometry between two function algebras A and B, then there exist a homeomorphism $\varphi : Ch(B) \longrightarrow Ch(A)$, a continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that $T(f) = \omega f \circ \varphi$ on K and $T(f) = \omega \overline{f \circ \varphi}$ on $Ch(B) \setminus K$. More recently, in [10], the authors characterized surjective real-linear isometries between complex function spaces satisfying certain separating conditions and extended some previous results by a technique based on the extreme points. In [7], the non-surjective case is treated based on a different technique.

In this paper we combine both approaches by dealing with k-real-linear isometries. Let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Y be a locally compact Hausdorff space. Here we study a k-real-linear map $T : A_1 \times ... \times A_k \longrightarrow C_0(Y)$ satisfying

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k),$$

which we call a *multi-real-linear (or k-real-linear) isometry.*

We also check, based on an example, how different these isometries can be from the other so far studied cases.

2. Preliminaries

A *function algebra* A on a locally compact Hausdorff space X is a subalgebra of $C_0(X)$ which separates strongly the points of X in the sense that for each $x, x' \in X$ with $x \neq x'$, there exists an $f \in A$ with $f(x) \neq f(x')$ and for each $x \in X$, there exists an $h \in A$ with $h(x) \neq 0$. If X is a compact Hausdorff space, each unital uniformly closed function algebra on X is called a *uniform algebra* on X.

Let A be a function algebra on a locally compact Hausdorff space X, and let $\overline{A}$ stand for the uniform closure of A. The unique minimal closed subset of X with the property that every function in A assumes its maximum modulus on this set, which exists by [2], is called the *Šilov boundary* for A and is denoted by ∂A. The *Choquet boundary* $Ch(A)$ of A is the set of all $x \in X$ for which δ_x, the evaluation functional at the point x, is an extreme point of the unit ball of the dual space of $(A, \|\cdot\|)$. So it is apparent that $Ch(A) = Ch(\overline{A})$, and moreover, by [2, Theorem 1], $Ch(A)$ is dense in ∂A. It is said that $x \in X$ is a *strong boundary point* (or *weak peak point*) for A if for every neighborhood V of x, there exists a function $f \in A$ such that $\|f\| = 1 = |f(x)|$ and $|f| < 1$ on $X \setminus V$. It is known that for each uniformly closed function algebra A, then $Ch(A)$ coincides with the set of all strong boundary points (see [13]). Meantime, a function $f \in A$ is called a *peaking function* if $\|f\| = 1$ and for each $x \in X$, either $|f(x)| < 1$ or $f(x) = 1$. If we fix $x_0 \in X$, then $P_A(x_0)$ denotes the set of peaking functions f in A with $f(x_0) = 1$. Moreover, for an element $x_0 \in X$, we set $V_{x_0} := \{f \in A : f(x_0) = 1 = \|f\|\}$.

In the sequel, for each $f \in C_0(X)$, $M_f := \{x \in X : |f(x)| = \|f\|\}$ stands for the *maximum modulus set* of f.

It should be noted that in the proof of our results we shall apply the following versions of Bishop's Lemma (see [3, Theorem 2.4.1]) adapted to the context of uniformly closed function algebras, which can be obtained with exactly the same proofs as in [5, Lemma 2.3] and [14, Lemma 1].

Lemma 2.1. [5, Lemma 2.3] *Let A be a uniformly closed function algebra on a locally compact Hausdorff space X, $f \in A$ and $x_0 \in Ch(A)$. If $f(x_0) \neq 0$, then there exists a peaking function $h \in P_A(x_0)$ such that $\frac{fh}{f(x_0)} \in P_A(x_0)$.*

Lemma 2.2. [14, Lemma 1] *Assume that A is a uniformly closed function algebra on a locally compact Hausdorff space X and $f \in A$. Let $x_0 \in Ch(A)$ and arbitrary $r > 1$ (or $r \geq 1$ if $f(x_0) \neq 0$), then there exists a function $h \in r\|f\|P_A(x_0) = \{r\|f\|k : k \in P_A(x_0)\}$ such that*

$$|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)|$$

for every $x \notin M_h$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for all $x \in M_h$. Consequently, $\||f| + |h|\| = |f(x_0)| + |h(x_0)|$.

Let us remark that Lemma 2.1 is a version of the multiplicative Bishop's Lemma and Lemma 2.2 is the strong version of the additive Bishop's Lemma.

3. Previous lemmas

Let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively. In this section we shall prove some previous lemmas used in our main theorem (Theorem 4.1). First note that it is not difficult to extend a k-real-linear isometry $T : A_1 \times ... \times A_k \longrightarrow C_0(Y)$ to a k-real-linear isometry

$T : \overline{A_1} \times ... \times \overline{A_k} \longrightarrow C_0(Y)$, where $\overline{A_i}$ is the uniform closure of A_i $(i = 1, ..., k)$. So, without loss of generality, we can assume each A_i $(i = 1, ..., k)$ is a uniformly closed function algebra.

Lemma 3.1. *Let* $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$ *and* $(\alpha_1, ..., \alpha_k) \in \mathbb{T}^k$. *The set*

$$\mathcal{I}_{x_1,...,x_k}^{\alpha_1,...,\alpha_k} := \{y \in Y : y \in M_{T(f_1,...,f_k)} \text{ for all } (f_1, ..., f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}\}$$

is nonempty.

Proof. The proof is a modification of the proof of [7, Lemma 4.1]. Since for each $(f_1, ..., f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$, the maximum modulus set of $T(f_1, ..., f_k)$, $M_{T(f_1,...,f_k)}$, is a compact subset of the one point compactification Y_∞ of Y, it is enough to check that the family $\{M_{T(f_1,...,f_k)} : (f_1, ..., f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}\}$ has the finite intersection property. For this purpose, let $(f_1^1, ..., f_k^1), ..., (f_1^n, ..., f_k^n)$ be members in $\alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$. Define

$$f_i := \frac{1}{n} \sum_{j=1}^{n} f_i^j, \quad i \in \{1, ..., k\}.$$

Clearly, $(f_1, ..., f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$. Hence $\|T(f_1, ..., f_k)\| = \|f_1\| ... \|f_k\| = 1$. Then there is a point $y_0 \in Y$ such that

$$1 = |T(f_1, ..., f_k)(y_0)| = \frac{1}{n^k} \left| \sum_{1 \le i_1, ..., i_k \le n} T(f_1^{i_1}, ..., f_k^{i_k})(y_0) \right|.$$

Since for each $1 \le i_1, ..., i_k \le n$, $f_1^{i_1} \in \alpha_1 V_{x_1}$, ..., $f_k^{i_k} \in \alpha_k V_{x_k}$ and $\|T(f_1^{i_1}, ..., f_k^{i_k})\| = 1$, we conclude that $|T(f_1^{i_1}, ..., f_k^{i_k})(y_0)| = 1$. In particular, $y_0 \in \bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)}$. Therefore $\bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)} \neq \emptyset$, as was to be proved. □

Lemma 3.2. *Let* $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, $(\alpha_1, ..., \alpha_k) \in \mathbb{T}^k$ *and* $y \in \mathcal{I}_{x_1,...,x_k}^{\alpha_1,...,\alpha_k}$. *Let also* I *and* J *be two disjoint sets with* $I \neq \emptyset$ *and* $I \cup J = \{1, ..., k\}$. *If we assume that for each* $j \in J$, $h_j \in \alpha_j V_{x_j}$ *and for each* $i \in I$, $f_i \in A_i$ *with* $f_i(x_i) = 0$, *then* $T(F_1, ..., F_k)(y) = 0$, *where* $F_t = f_t$ *if* $t \in I$ *and* $F_t = h_t$ *if* $t \in J$.

Proof. Let us suppose, contrary to what we claim, that there exists $y_0 \in \mathcal{I}_{x_1,...,x_k}^{\alpha_1,...,\alpha_k}$ such that $T(F_1, ..., F_k)(y_0) \neq 0$. Without loss of generality, we may assume that $T(F_1, ..., F_k)(y_0) = e^{i\theta}$, where $-\pi < \theta \le \pi$. Fix a constant $r > 1$. For each $i \in I$, we can choose, by Lemma 2.2, a peaking function $h'_i \in V_{x_i}$ such that $\||f_i| + r_i|h'_i|\| = r_i$, where $r_i = r\|f_i\|$. In particular, putting $h_i = \alpha_i h'_i$ for each $i \in I$, we have $\| \pm f_i + r_i h_i\| = r_i$ and $T(h_1, ..., h_k)(y_0) = e^{i\theta'} \in \mathbb{T}$ for some $-\pi < \theta' \le \pi$.

We first assume that $card(I) = 1$. For simplicity, we can take $I = \{1\}$. We have

$$\begin{aligned} r &= \| \pm f_1 + r_1 h_1 \| \|h_2\| ... \|h_k\| = \|T(\pm f_1 + r_1 h_1, h_2, ..., h_k)\| \\ &\geq |T(\pm f_1 + r_1 h_1, h_2, ..., h_k)(y_0)| = | \pm T(f_1, h_2, ..., h_k)(y_0) + r_1 T(h_1, h_2, ..., h_k)(y_0)| \\ &= | \pm e^{i\theta} + r_1 e^{i\theta'}| = | \pm e^{i(\theta - \theta')} + r_1|, \end{aligned}$$

and consequently, $r_1 \geq \max\{|e^{i(\theta-\theta')} + r_1|, |-e^{i(\theta-\theta')} + r_1|\} > r_1$, which gives a contradiction. Thereby, $T(F_1, ..., F_k)(y) = 0$ for all $y \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$.

Now suppose that $I = \{1, 2\}$. Hence, from the previous part, we can conclude that

$$\begin{aligned} r_1 r_2 &= \| \pm f_1 + r_1 h_1 \| \|f_2 + r_2 h_2\| \|h_3\| ... \|h_k\| \\ &= \|T(\pm f_1 + r_1 h_1, f_2 + r_2 h_2, h_3, ..., h_k)\| \\ &\geq | \pm T(f_1, f_2, h_3, ..., h_k)(y_0) + r_2 T(f_1, h_2, h_3, ..., h_k)(y_0) \\ &+ r_1 T(h_1, f_2, h_3, ..., h_k)(y_0) + r_1 r_2 T(h_1, h_2, h_3, ..., h_k)(y_0)| \\ &= | \pm e^{i\theta} + r_1 r_2 e^{i\theta'}| = | \pm e^{i(\theta-\theta')} + r_1 r_2|, \end{aligned}$$

and so $r_1 r_2 \geq \max\{|e^{i(\theta-\theta')} + r_1 r_2|, |-e^{i(\theta-\theta')} + r_1 r_2|\} > r_1 r_2$, a contradiction which implies that the result is true when $I = \{1, 2\}$. Similarly, this result holds for all the other cases where $card(I) = 2$.

Now we can continue by induction: noting to the above explanation, let us assume that the result is true for $card(I) = l - 1$ and $3 \leq l \leq k$. We shall show that the result is held if $card(I) = l$. To this end, we suppose that $card(I) = l$ and $I = \{x_1, ..., x_l\}$, without loss of generality. If $l < k$, then we get

$$\begin{aligned} r_1 r_2 ... r_l &= \| \pm f_1 + r_1 h_1 \| \|f_2 + r_2 h_2\| ... \|f_l + r_l h_l\| \|h_{l+1}\| ... \|h_k\| \\ &= \|T(\pm f_1 + r_1 h_1, f_2 + r_2 h_2, ..., f_l + r_l h_l, h_{l+1}, ..., h_k)\| \\ &\geq |T(\pm f_1 + r_1 h_1, f_2 + r_2 h_2, ..., f_l + r_l h_l, h_{l+1}, ..., h_k)(z_0)| \\ &= | \pm T(f_1, ..., f_l, h_{l+1}, ..., h_k)(y_0) + r_1 r_2 ... r_l T(h_1, ..., h_k)(y_0)| \\ &= | \pm e^{i(\theta-\theta')} + r_1 ... r_l|, \end{aligned}$$

which is impossible as before. Therefore, $T(f_1, ..., f_l, h_{l+1}, ..., h_k)(y) = 0$ for all $y \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. Now if $l = k$, then $I = \{x_1, ..., x_k\}$ and similarly,

$$\begin{aligned} r_1 r_2 ... r_k &= \| \pm f_1 + r_1 h_1\| \|f_2 + r_2 h_2\| ... \|f_k + r_k h_k\| \\ &\geq |T(\pm f_1 + r_1 h_1, f_2 + r_2 h_2, ..., f_k + r_k h_k)(y_0)| \\ &= | \pm T(f_1, ..., f_k)(y_0) + r_1 r_2 ... r_k T(h_1, ..., h_k)(y_0)| \\ &= | \pm e^{i(\theta - \theta')} + r_1 r_2 ... r_k|, \end{aligned}$$

which again leads to a contradiction showing that $T(f_1, ..., f_k)(y) = 0$ for all $y \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. □

Lemma 3.3. *Let $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, $(\alpha_1, ..., \alpha_k) \in \mathbb{T}^k$, and $y \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. Then there exists a unique $\lambda \in \mathbb{T}$ such that $T(\alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}) \subseteq \lambda V_y$.*

Proof. Let $(f_1, ..., f_k), (g_1, ..., g_k) \in V_{x_1} \times ... \times V_{x_k}$. Then $(\alpha_1 f_1, ..., \alpha_k f_k), (\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$ and so $|T(\alpha_1 f_1, ..., \alpha_k f_k)(y)| = 1 = |T(\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)|$. It is also clear that

$$\frac{|T(\alpha_1 f_1, ..., \alpha_k f_k)(y) + T(\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)|}{2} = 1$$

because $\frac{\alpha_1 f_1 + \alpha_1 g_1}{2} \in \alpha_1 V_{x_1}$. Hence,

$$\frac{T(\alpha_1 f_1, ..., \alpha_k f_k)(y) + T(\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)}{2} = e^{i\theta}$$

for some $-\pi < \theta \leq \pi$. Then since $e^{i\theta}$ is an extreme point of the unit ball of $\mathbb{C}$, it follows that $T(\alpha_1 f_1, ..., \alpha_k f_k)(y) = T(\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)$. Continuing this process we get

$$\begin{aligned} T(\alpha_1 f_1, ..., \alpha_k f_k)(y) &= T(\alpha_1 g_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) \\ &= T(\alpha_1 g_1, \alpha_2 g_2, \alpha_3 f_3, ..., \alpha_k f_k)(y) \\ &= ... = T(\alpha_1 g_1, ..., \alpha_k g_k)(y). \end{aligned}$$

Therefore, $T(\alpha_1 f_1, ..., \alpha_k f_k)(y) = T(\alpha_1 g_1, ..., \alpha_k g_k)(y)$. Now, if we define $\lambda := T(\alpha_1 f_1, ..., \alpha_k f_k)(y)$ for some $(f_1, ..., f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$, then we conclude that $T(\alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}) \subseteq \lambda V_y$. □

Lemma 3.4. *Let $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ be distinct elements in $Ch(A_1) \times ... \times Ch(A_k)$, and $(\alpha_1, ..., \alpha_k) \in \mathbb{T}^k$. Then $\mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k} \cap \mathcal{I}^{\alpha_1,...,\alpha_k}_{x'_1,...,x'_k} = \emptyset$.*

Proof. Contrary to what we claim, assume that there exists $y_0 \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k} \cap \mathcal{I}^{\alpha_1,...,\alpha_k}_{x'_1,...,x'_k}$. Since $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ are distinct, the set $L = \{i : 1 \leq i \leq k, x_i \neq x'_i\}$ is nonempty. For each $i \in L$, we can choose a function $g_i \in A_i$ such that $g_i(x_i) = 1$ and $g_i(x'_i) = 0$, and then, by Lemma 2.1, a peaking function $h_i \in P_{A_i}(x_i)$ such that $g_i h_i \in P_{A_i}(x_i)$. Now if we let $f_i = g_i h_i$ for every $i \in L$, then $f_i \in V_{x_i}$ with $f_i(x_i) = 1$ and $f_i(x'_i) = 0$. Moreover, for each $j \in \{1, ..., k\} \setminus L$, we can also choose a peaking function $f_j \in V_{x_j}$. On one side, since $(\alpha_1 f_1, ..., \alpha_k f_k) \in \alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}$, $|T(\alpha_1 f_1, ..., \alpha_k f_k)(y_0)| = 1$. On the other side, by Lemma 3.2, $T(\alpha_1 f_1, ..., \alpha_k f_k)(y_0) = 0$, which is impossible. Therefore, $\mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k} \cap \mathcal{I}^{\alpha_1,...,\alpha_k}_{x'_1,...,x'_k} = \emptyset$. $\square$

Definition 3.5. *For each $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, let* $\mathcal{I}_{x_1,...,x_k} := \bigcap_{\alpha_1,...,\alpha_k \in \{1,i\}} \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$.

We should note that k-real-linear isometries behave differently from k-complex-linear isometries with respect to these sets. More precisely, as seen in [8] and in all previous papers dealing with 1-complex-linear (not necessarily surjective) isometries starting with Holsztyński's seminal paper ([6]), it is clear that $\mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k} = \mathcal{I}^{\alpha'_1,...,\alpha'_k}_{x_1,...,x_k}$ for each k-complex-linear isometry T, given any $\alpha_i, \alpha'_i \in \mathbb{T}$ $(1 \le i \le k)$. However, as the next example shows, this equality is no longer valid for k-real-linear isometries:

Example 3.6. *Let $T : C(\{x_1\}) \times C(\{x_2\}) \to C(\{y_1, y_2\})$ defined by $T(a + ib, c + id)(y_1) := ac$ and $T(a + ib, c + id)(y_2) := (a + ib)(c + id)$. It is apparent that T is a 2-real-linear isometry for which $\mathcal{I}^{1,1}_{x_1,x_2} = \{y_1, y_2\}$ and $\mathcal{I}^{1,i}_{x_1,x_2} = \{y_2\}$.*

In the complex-linear case, thanks to the above paragraph and Lemma 3.1, we infer that $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$ for each $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$. However, the authors are unaware whether each set $\mathcal{I}_{x_1,...,x_k}$ is nonempty for $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$ in the real-linear case. Hence we continue under the assumption that for each $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$. At the final remark of this paper, we provide several conditions which yield the nonemptiness of such sets.

Lemma 3.7. *If $y \in \mathcal{I}_{x_1,...,x_k}$, $\alpha_2, ..., \alpha_k \in \{1, i\}$ and $(f_1, ..., f_k) \in V_{x_1} \times ... \times V_{x_k}$, then we have either*

$$T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) = iT(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y),$$

or

$$T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) = -iT(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y).$$

A similar claim holds for the other indexes.

Proof. Let $y \in \mathcal{I}_{x_1,...,x_k}$, and put $\lambda_i := T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)$ and $\lambda_1 := T(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)$ for simplicity. We have

$$\begin{aligned} |\lambda_1 \pm \lambda_i| &= |T(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) \pm T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)| = |T(f_1 \pm if_1, f_2, ..., f_k)(y)| \\ &\le \|T(f_1 \pm if_1, , \alpha_2 f_2, ..., \alpha_k f_k)\| = \|f_1 \pm if_1\| \|f_2\| ... \|f_k\| \\ &= \|f_1\| |1 \pm i| = \sqrt{2}. \end{aligned}$$

Hence $|\lambda_1 \pm \lambda_i| \leq \sqrt{2}$, and since $|\lambda_1| = |\lambda_i| = 1$, it follows easily that $\lambda_i^2 = -\lambda_1^2$. Consequently, either $T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) = iT(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)$ or $T(if_1, \alpha_2 f_2, ..., \alpha_k f_k)(y) = -iT(f_1, \alpha_2 f_2, ..., \alpha_k f_k)(y)$. Analogously, a similar claim can be proved for the other indexes. □

Lemma 3.8. *Given* $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, *we have* $\mathcal{I}_{x_1,...,x_k} = \bigcap_{\alpha_1,...,\alpha_k \in \mathbb{T}} \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$.

Proof. Clearly, $\mathcal{I}_{x_1,...,x_k} \supseteq \bigcap_{\alpha_1,...,\alpha_k \in \mathbb{T}} \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. To see the converse inclusion, let $y \in \mathcal{I}_{x_1,...,x_k}$, $\beta_j \in \{1, i\}$ and put $\alpha_j = a_j + ib_j \in \mathbb{T}$, where $a_j, b_j \in \mathbb{R}$ and $j \in \{1, ..., k\}$. Given $(f_1, ..., f_k) \in V_{x_1} \times ... \times V_{x_k}$, from the previous lemma it follows that

$$\begin{aligned} T(\alpha_1 f_1, \beta_2 f_2, ..., \beta_k f_k)(y) &= a_1 T(f_1, \beta_2 f_2, ..., \beta_k f_k)(y) + b_1 T(if_1, \beta_2 f_2, ..., \beta_k f_k)(y) \\ &= (a_1 \pm ib_1) T(f_1, \beta_2 f_2, ..., \beta_k f_k)(y), \end{aligned}$$

and so $|T(\alpha_1 f_1, \beta_2 f_2, ..., \beta_k f_k)(y)| = 1$. Consequently,

$$y \in \bigcap \{\mathcal{I}^{\alpha_1,\beta_2,...,\beta_k}_{x_1,x_2,...,x_k} : \alpha_1 \in \mathbb{T}, \beta_2, ..., \beta_k \in \{1, i\}\}.$$

Now from the above argument and a discussion similar to the proof of the previous lemma we conclude that

$$\begin{aligned} T(\alpha_1 f_1, \alpha_2 f_2, \beta_3 f_3, ..., \beta_k f_k)(y) &= a_2 T(\alpha_1 f_1, f_2, \beta_3 f_3, ..., \beta_k f_k)(y) + b_2 T(\alpha_1 f_1, if_2, \beta_3 f_3, ..., \beta_k f_k)(y) \\ &= (a_2 \pm ib_2) T(\alpha_1 f_1, f_2, \beta_3 f_3, ..., \beta_k f_k)(y), \end{aligned}$$

which implies that $y \in \bigcap \{\mathcal{I}^{\alpha_1,\alpha_2,\beta_3,...,\beta_k}_{x_1,x_2,x_3,...,x_k} : \alpha_1, \alpha_2 \in \mathbb{T}, \beta_3, ..., \beta_k \in \{1, i\}\}$. Continuing this process, finally we deduce that $y \in \bigcap_{\alpha_1,...,\alpha_k \in \mathbb{T}} \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$, as claimed. □

Definition 3.9. *Let us define the set* $Y_0 := \{y \in Y : y \in \mathcal{I}_{x_1,...,x_k}$ *for some* $x_i \in Ch(A_i)$, $i = 1, ..., k\}$.

Y_0 is a non-empty set by our assumption after Example 3.6 and we can define a map $\varphi : Y_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$ by

$$\varphi(y) := (x_1, ..., x_k),$$

if $y \in \mathcal{I}_{x_1,...,x_k}$ for some $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$. From Lemma 3.4, for any distinct members $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$ it follows that $\mathcal{I}_{x_1,...,x_k} \cap \mathcal{I}_{x'_1,...,x'_k} = \emptyset$ and φ is well-defined. It is clear that φ is surjective by our assumption after Example 3.6.

As observed in Lemma 3.8, we have $\mathcal{I}_{x_1,...,x_k} = \bigcap_{\alpha_1,...,\alpha_k \in \mathbb{T}} \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. Now let us define a map $\Lambda : Y_0 \times \mathbb{T}^k \longrightarrow \mathbb{T}$ by

$$\Lambda(y, (\alpha_1, ..., \alpha_k)) := \lambda$$

such that $T(\alpha_1 V_{x_1} \times ... \times \alpha_k V_{x_k}) \subseteq \lambda V_y$, where $\varphi(y) = (x_1, ..., x_k)$. By Lemma 3.3, it is apparent that Λ is a well-defined map.

Definition 3.10. According to Lemma 3.7, $\Lambda(y,(i,1,...,1)) = \pm i\Lambda(y,(1,1,...,1))$ for all $y \in Y_0$. Set $K_1 := \{y \in Y_0 : \Lambda(y,(i,1,...,1)) = i\Lambda(y,(1,1,...,1))\}$ and, consequently, $Y_0 \setminus K_1 = \{y \in Y_0 : \Lambda(y,(i,1,...,1)) = -i\Lambda(y,(1,1,...,1))\}$. Analogously, for each $j \in \{2,...,k\}$, we can define a subset K_j of Y_0.

We remark that it is not difficult to see each K_j, $j \in \{1,...,k\}$, is a clopen subset of Y_0.

Lemma 3.11. *Let $y \in Y_0$, $\varphi(y) = (x_1,...,x_k)$, $h_j \in V_{x_j}$ $(1 \le j \le k)$, and let also I be a non-empty subset of $\{1,...,k\}$. Assume that for each $t \in I$, $f_t = ih_t$ and for each $t \notin I$, $f_t = h_t$. Then*

$$T(f_1,...,f_k)(y) = i_1...i_k T(h_1,...,h_k)(y),$$

where

$$i_t = \begin{cases} i & y \in K_t, \\ -i & y \in Y_0 \setminus K_t, \end{cases}$$

when $t \in I$ and $i_t = 1$ when $t \notin I$.

Proof. Put $n = card(I)$. For $n = 1$, the result follows from Lemma 3.7.

Step 1. Suppose that $n = 2$. We may assume, without loss of generality, that $I = \{1,2\}$. Lemma 3.7 shows that $T(f_1,...,f_k)(y) = \pm iT(h_1,f_2,...,f_k)(y)$. Then $T(f_1,...,f_k)(y) = \mp T(h_1,h_2,f_3,...,f_k)(y)$. We claim that

$$T(f_1,...,f_k)(y) = \begin{cases} -T(h_1,...,h_k)(y) & y \in (K_1 \cap K_2) \cup (K_1^c \cap K_2^c), \\ T(h_1,...,h_k)(y) & y \in (K_1 \cup K_2) \setminus (K_1 \cap K_2). \end{cases}$$

Suppose, on the contrary, that $y \in K_1 \cap K_2$ and $T(f_1,...,f_k)(y) = T(h_1,...,h_k)(y)$. Then taking into account the k-real-linearity of T we have

$$\begin{aligned} T(ih_1,(i+1)h_2,h_3,...,h_k)(y) &= T(ih_1,ih_2,h_3...,h_k)(y) + T(ih_1,h_2,h_3...,h_k)(y) \\ &= T(h_1,...,h_k)(y) + iT(h_1,...,h_k)(y) \\ &= (1+i)T(h_1,...,h_k)(y) \\ &= T(h_1,(i+1)h_2,h_3,...,h_k)(y), \end{aligned}$$

which implies that $T((i-1)h_1,(i+1)h_2,h_3,...,h_k)(y) = 0$ and it is a contradiction since it is not difficult to see that on $\mathcal{I}_{x_1,...,x_k}$, $|T((i-1)h_1,(i+1)h_2,h_3,...,h_k)(y)| = |(i-1)(i+1)| = 2$, by Lemma 3.8. Hence this argument shows that $T(f_1,...,f_k)(y) = -T(h_1,...,h_k)(y)$ for each $y \in K_1 \cap K_2$. The other cases can be derived similarly and so the result holds for all the cases where $card(I) = 2$.

Step 2. Next, assume that the result is true for $card(I) = l-1$ and $3 \le l < k$, and we prove the result for the case where $card(I) = l$. We suppose, with no loss of generality, that $I = \{1,...,l\}$. Again from Lemma 3.7, we conclude that $T(ih_1,...,ih_l,h_{l+1},...,h_k)(y) = \pm iT(ih_1,...,ih_{l-1},h_l,...,h_k)(y)$. Then we have $T(ih_1,...,ih_l,h_{l+1},...,h_k)(y) = \pm ii_1...i_{l-1}T(h_1,...,h_k)(y)$. We claim that

$$T(f_1,...,f_k)(y) = \begin{cases} ii_1...i_{l-1}T(h_1,...,h_k)(y) & y \in K_l, \\ -ii_1...i_{l-1}T(h_1,...,h_k)(y) & y \in Y_0 \setminus K_l. \end{cases}$$

Suppose, on the contrary, that $y \in Y_0 \backslash K_l$ and $T(ih_1,...,ih_l,h_{l+1},...,h_k)(y) = ii_1...i_{l-1}T(h_1,...,h_k)(y)$. Then, we deduce that

$$\begin{aligned} T(ih_1,...,ih_{l-1},(i+1)h_l,h_{l+1},...,h_k)(y) &= T(ih_1,...,ih_{l-1},ih_l,h_{l+1},...,h_k)(y) \\ &\quad + T(ih_1,...,ih_{l-1},h_l,h_{l+1},...,h_k)(y) \\ &= ii_1...i_{l-1}T(h_1,...,h_k)(y) + i_1...i_{l-1}T(h_1,...,h_k)(y) \\ &= i_1...i_{l-1}(i+1)T(h_1,...,h_k)(y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} T(h_1,ih_2,...,ih_{l-1},(i+1)h_l,h_{l+1},...,h_k)(y) &= T(h_1,ih_2,...,ih_l,h_{l+1},...,h_k)(y) \\ &\quad + T(h_1,ih_2,...,ih_{l-1},h_l,h_{l+1},...,h_k)(y) \\ &= -ii_2...i_{l-1}T(h_1,...,h_k)(y) + i_2...i_{l-1}T(h_1,...,h_k)(y) \\ &= i_2...i_{l-1}(-i+1)T(h_1,...,h_k)(y). \end{aligned}$$

Therefore, adding the above two expressions,

$$T((i+1)h_1,ih_2,...,ih_{l-1},(i+1)h_l,h_{l+1},...,h_k)(y) = i_2...i_{l-1}(i_1 i + i_1 - i + 1)T(h_1,...,h_k)(y),$$

and so

$$T((i+1)h_1,ih_2,...,ih_{l-1},(i+1)h_l,h_{l+1},...,h_k)(y) = \begin{cases} 0 & y \in K_1, \\ (2-2i)T(h_1,...,h_k)(y) & y \in Y_0 \setminus K_1, \end{cases}$$

which is impossible because $|T((i+1)h_1,ih_2,...,ih_{l-1},(i+1)h_l,h_{l+1},...,h_k)(y)| = 2$, by Lemma 3.8. Thus from this argument we conclude that $T(ih_1,...,ih_l,h_{l+1},...,h_k)(y) = ii_1...i_{l-1}T(h_1,...,h_k)(y)$ for each $y \in Y_0 \setminus K_l$. The other cases can be obtained in a similar way. So the result holds for all cases where $n = l$.

Step 3. Finally suppose that the result is true when $card(I) = k-1$. We shall show the validity of the result for the case where $card(I) = k$. By Lemma 3.7, we can see that $T(ih_1,...,ih_k)(y) = \pm iT(ih_1,...,ih_{k-1},h_k)(y)$, and so $T(ih_1,...,ih_k)(y) = \pm ii_1...i_{k-1}T(h_1,...,h_k)(y)$. We claim that

$$T(ih_1,...,ih_k)(y) = \begin{cases} ii_1...i_{k-1}T(h_1,...,h_k)(y) & y \in K_k, \\ -ii_1...i_{k-1}T(h_1,...,h_k)(y) & y \in Y_0 \setminus K_k. \end{cases}$$

Suppose, on the contrary, that $y \in K_k$ and $T(ih_1, ..., ih_k)(y) = -ii_1...i_{k-1}T(h_1, ..., h_k)(y)$. Then

$$\begin{aligned} T(ih_1, ih_2, ..., ih_{k-1}, (i+1)h_k)(y) &= -ii_1...i_{k-1}T(h_1, ..., h_k)(y) + T(ih_1, , ..., ih_{k-1}, h_k)(y) \\ &= (-ii_1...i_{k-1} + i_1...i_{k-1})T(h_1, ..., h_k)(y), \end{aligned}$$

and

$$\begin{aligned} T(h_1, ih_2, ..., ih_{k-1}, (i+1)h_k)(y) &= i_2...i_kT(h_1, ..., h_k)(y) + i_2...i_{k-1}T(h_1, ..., h_k)(y) \\ &= (i_2...i_k + i_2...i_{k-1})T(h_1, ..., h_k)(y), \end{aligned}$$

thus adding the above two relations we have

$$T((i+1)h_1, ih_2, ..., ih_{k-1}, (i+1)h_k)(y) = i_2...i_{k-1}(-ii_1 + i_1 + i + 1)T(h_1, ..., h_k)(y),$$

and consequently,

$$T((i+1)h_1, ih_2, ..., ih_{k-1}, (i+1)h_k)(y) = \begin{cases} i_2...i_{k-1}(2+2i)T(h_1, ..., h_k)(y) & y \in K_1, \\ 0 & y \in Y_0 \setminus K_1, \end{cases}$$

which is impossible since $|T((i+1)h_1, ih_2, ..., ih_{k-1}, (i+1)h_k)(y)| = |(i+1)^2| = 2$, by Lemma 3.8. Therefore, $T(ih_1, ..., ih_k)(y) = ii_1...i_{k-1}T(h_1, ..., h_k)(y)$ for all $y \in K_k$, as asserted. Similarly, for every $y \in Y_0 \setminus K_k$ we have $T(ih_1, ..., ih_k)(y) = -ii_1...i_{k-1}T(h_1, ..., h_k)(y)$. □

Lemma 3.12. *Let $y \in Y_0$ and $(\alpha_1, ..., \alpha_k) \in \mathbb{C}^k$. Then*

$$\Lambda(y, (\alpha_1, ..., \alpha_k)) = \alpha_1^*...\alpha_k^*\Lambda(y, (1, ..., 1)),$$

where, for each $j \in \{1, ..., k\}$, $\alpha_j^* = \alpha_j$ if $y \in K_j$ and $\alpha_j^* = \overline{\alpha_j}$ if $y \in Y_0 \setminus K_j$.

Proof. For each $j \in \{1, ..., k\}$, choose $f_j \in V_{x_j}$. Let $\alpha_j = a_j + ib_j$, where $a_j, b_j \in \mathbb{R}$. Since T is k-real-linear, if $y \in \cap_{j=1}^k K_j$, then, from the preceeding lemma, it follows that

$$\begin{aligned} T(a_1f_1 + ib_1f_1, ..., a_kf_k + ib_kf_k)(y) &= \sum_{c_{i_j} \in \{a_j, ib_j\}, (1 \le j \le k)} c_{i_1}...c_{i_k}T(f_1, ..., f_k)(y) \\ &= \alpha_1...\alpha_kT(f_1, ..., f_k)(y) \\ &= \alpha_1...\alpha_k\Lambda(y, (1, ..., 1)), \end{aligned}$$

and if $y \in \cap_{j=1}^k (Y_0 \setminus K_j)$, similarly we have

$$\begin{aligned} T(a_1f_1 + ib_1f_1, ..., a_kf_k + ib_kf_k)(y) &= \sum_{c_{i_j} \in \{a_j, -ib_j\}, (1 \le j \le k)} c_{i_1}...c_{i_k}T(f_1, ..., f_k)(y) \\ &= \overline{\alpha_1}...\overline{\alpha_k}T(f_1, ..., f_k)(y) \\ &= \overline{\alpha_1}...\overline{\alpha_k}\Lambda(y, (1, ..., 1)). \end{aligned}$$

The other cases can be obtained similarly. □

Remark 3.13. We define the map $\omega : Y_0 \longrightarrow \mathbb{T}$ by

$$\omega(y) := \Lambda(y, (1, ..., 1))$$

for all $y \in Y_0$. Indeed, if $(x_1, ..., x_k) = \varphi(y)$, then $\omega(y) = T(f_1, ..., f_k)$, where $(f_1, ..., f_k) \in V_{x_1} \times ... \times V_{x_k}$. Moreover, by the above lemma, for all $(\alpha_1, ..., \alpha_k) \in \mathbb{C}^k$ we have

$$\Lambda(y, (\alpha_1, ..., \alpha_k)) = \alpha_1^* ... \alpha_k^* \omega(y),$$

where, for each $j \in \{1, ..., k\}$, $\alpha_j^* = \alpha_j$ if $y \in K_j$ and $\alpha_j^* = \overline{\alpha_j}$ if $y \in Y_0 \setminus K_j$.

Lemma 3.14. *Let $y \in Y_0$ with $\varphi(y) = (x_1, ..., x_k)$, and $(f_1, ..., f_k) \in A_1 \times V_{x_2} \times ... \times V_{x_k}$. Then*

$$T(f_1, ..., f_k)(y) = \omega(y) \begin{cases} f_1(x_1) & y \in K_1, \\ \overline{f_1(x_1)} & y \in Y_0 \setminus K_1. \end{cases}$$

A similar assertion holds for the other indexes.

Proof. If $f_1(x_1) = 0$, then from Lemma 3.2, $T(f_1, ..., f_k)(y) = 0$. Now assume that $f_1(x_1) \neq 0$. Hence choosing h_1 as a function in V_{x_1}, again by Lemma 3.2, we have $T(f_1 - f_1(x_1)h_1, f_2, ..., f_k)(y) = 0$, and so

$$T(f_1, ..., f_k)(y) = T(f_1(x_1)h_1, f_2, ..., f_k)(y).$$

Now, from the previous lemma, we infer that

$$T(f_1, ..., f_k)(y) = \begin{cases} f_1(x_1)T(h_1, f_2, ..., f_k)(y) & y \in K_1, \\ \overline{f_1(x_1)}T(h_1, f_2, ..., f_k)(y) & y \in Y_0 \setminus K_1, \end{cases}$$

as claimed. Similarly, the other cases can be concluded. □

4. Main result

Let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively. Let also recall here our assumption after Example 3.6 that for each $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$.

Theorem 4.1. *Suppose that $T : A_1 \times ... \times A_k \longrightarrow C_0(Y)$ is a k-real-linear isometry. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$, (possibly empty) clopen subsets $K_1, ..., K_k$ of Y_0 and a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ such that for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $y \in Y_0$,*

$$T(f_1, ..., f_k)(y) = \omega(y) \prod_{j=1}^{k} f_j(\pi_j(\varphi(y)))^*,$$

where π_j *is the* j*th projection map and for each* $j \in \{1, ..., k\}$, $f_j(\pi_j(\varphi(y)))^* = f_j(\pi_j(\varphi(y)))$ *if* $y \in K_j$ *and* $f_j(\pi_j(\varphi(y)))^* = \overline{f_j(\pi_j(\varphi(y)))}$ *if* $y \in Y_0 \setminus K_j$.

Proof. Let Y_0 be the set introduced in Definition 3.9. Fix $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$ and $h_j \in V_{x_j}$ for each j, $j = 1, ..., k$. Then for each j, $j = 1, ..., k$, we can define a real-linear isometry as follows:

$$\begin{cases} T_j : A_j \longrightarrow C_0(Y) \\ T_j(f) := T(h_1, ..., h_{j-1}, f, h_{j+1}, ..., h_k). \end{cases}$$

According to [7], there exist a nonempty subset Y_j of Y, a subset $\mathcal{K}_j$ of Y_j, a continuous surjective map $\varphi_j : Y_j \longrightarrow Ch(A_j)$ such that, for each $f_j \in A_j$,

$$T_j(f_j)(y) = T(h_1, ..., h_k)(y) \begin{cases} f_j(\varphi_j(y)) & y \in \mathcal{K}_j, \\ \overline{f_j(\varphi_j(y))} & y \in Y_j \setminus \mathcal{K}_j. \end{cases}$$

Namely, $Y_j \supseteq \bigcup_{x'_j \in Ch(A_j)} \mathcal{I}_{x_1,...,x'_j,...,x_k}$ and if $y \in \mathcal{I}_{x_1,...,x'_j,...,x_k}$, then $\varphi_j(y) = x'_j$.

Let $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $y \in \mathcal{I}_{x_1,...,x_k}$. From the description of T_j, it easily follows that $y \in \mathcal{K}_j$ if $y \in K_j$, and $y \notin \mathcal{K}_j$ if $y \notin K_j$, where K_j is the clopen subset of Y_0 introduced in Definition 3.10. We now claim that

$$T(f_1(x_1)h_1, f_2, h_3, ..., h_k)(y) = f_1(x_1)^* T_2(f_2)(y),$$

where $f_1(x_1)^* = f_1(x_1)$ if $y \in K_1$, and $f_1(x_1)^* = \overline{f_1(x_1)}$ if $y \in Y_0 \setminus K_1$. First note that the k-real-linearity of T yields

$$T(f_1(x_1)h_1, f_2, h_3, ..., h_k)(y) = \mathrm{Re}f_1(x_1)T(h_1, f_2, h_3, ..., h_k)(y) + \mathrm{Im}f_1(x_1)T(ih_1, f_2, h_3, ..., h_k)(y).$$

On the other hand, by Lemma 3.2, $T(ih_1, f_2 - f_2(x_2)h_2, h_3, ..., h_k)(y) = 0$ and so, using the preceding remark, we deduce that

$$\begin{aligned} T(ih_1, f_2, h_3, ..., h_k)(y) &= T(ih_1, f_2(x_2)h_2, h_3, ..., h_k)(y) \\ &= \mathrm{Re}f_2(x_2)T(ih_1, h_2, ..., h_k)(y) + \mathrm{Im}f_2(x_2)T(ih_1, ih_2, h_3, ..., h_k)(y) \end{aligned}$$

$$= \omega(y) \begin{cases} i\mathrm{Re}f_2(x_2) - \mathrm{Im}f_2(x_2) = iT_2(f_2)(y) & y \in K_1 \cap K_2, \\ i\mathrm{Re}f_2(x_2) + \mathrm{Im}f_2(x_2) = iT_2(f_2)(y) & y \in K_1 \setminus K_2, \\ -i\mathrm{Re}f_2(x_2) + \mathrm{Im}f_2(x_2) = -iT_2(f_2)(y) & y \in K_2 \setminus K_1, \\ -i\mathrm{Re}f_2(x_2) - \mathrm{Im}f_2(x_2) = -iT_2(f_2)(y) & y \in Y_0 \setminus (K_1 \cup K_2). \end{cases}$$

Now combining the latter relations implies that

$$T(f_1(x_1)h_1, f_2, h_3, ..., h_k)(y) = \begin{cases} T_2(f_2)(y)(\mathrm{Re}f_1(x_1) + i\mathrm{Im}f_1(x_1)) = T_2(f_2)(y)f_1(x_1) & y \in K_1, \\ T_2(f_2)(y)(\mathrm{Re}f_1(x_1) - i\mathrm{Im}f_1(x_1)) = T_2(f_2)(y)\overline{f_1(x_1)} & y \in Y_0 \setminus K_1, \end{cases}$$

as claimed.

Similarly, $T(f_1, f_2(x_2)h_2, h_3, ..., h_k)(y) = f_2(x_2)^* T_1(f_1)(y)$, where $f_2(x_2)^* = f_2(x_2)$ if $y \in K_2$, and $f_2(x_2)^* = \overline{f_2(x_2)}$ if $y \in Y_0 \setminus K_2$. Now using again Lemmas 3.2 and 3.14, Remark 3.13 and the above two equations it follows that

$$\begin{aligned}
0 &= T(f_1 - f_1(x_1)h_1, f_2 - f_2(x_2)h_2, h_3, ..., h_k)(y) \\
&= T(f_1, f_2, h_3, ..., h_k)(y) - T(f_1(x_1)h_1, f_2, h_3, ..., h_k)(y) \\
&- T(f_1, f_2(x_2)h_2, h_3, ..., h_k)(y) + f_1(x_1)^* f_2(x_2)^* T(h_1, ..., h_k)(y) \\
&= T(f_1, f_2, h_3, ..., h_k)(y) - f_1(x_1)^* T_2(f_2)(y) - f_2(x_2)^* T_1(f_1)(y) + f_1(x_1)^* f_2(x_2)^* T(h_1, ..., h_k)(y) \\
&= T(f_1, f_2, h_3, ..., h_k)(y) - f_1(x_1)^* T(h_1, ..., h_k)(y) f_2(x_2)^* \\
&- f_2(x_2)^* T(h_1, ..., h_k)(y) f_1(x_1)^* + f_1(x_1)^* f_2(x_2)^* T(h_1, ..., h_k)(y) \\
&= T(f_1, f_2, h_3, ..., h_k)(y) - f_1(x_1)^* f_2(x_2)^* T(h_1, ..., h_k)(y),
\end{aligned}$$

where, as above, $f_j(x_j)^* = f_j(x_j)$ if $y \in K_j$, and $f_j(x_j)^* = \overline{f_j(x_j)}$ if $y \in Y_0 \setminus K_j$.

Thus $T(f_1, f_2, h_3, ..., h_k)(y) = T(h_1, ..., h_k)(y) f_1(x_1)^* f_2(x_2)^*$. By continuing this process and applying Lemma 3.2, we finally see that

$$\begin{aligned}
0 &= T(f_1 - f_1(x_1)h_1, ..., f_k - f_k(x_k)h_k)(y) \\
&= T(f_1, ..., f_k)(y) - T(h_1, ..., h_k)(y) f_1(x_1)^* ... f_k(x_k)^*,
\end{aligned}$$

thereby, $T(f_1, ..., f_k)(y) = T(h_1, ..., h_k)(y) f_1(x_1)^* ... f_k(x_k)^*$, where $f_j(x_j)^* = f_j(x_j)$ if $y \in K_j$, and $f_j(x_j)^* = \overline{f_j(x_j)}$ if $y \in Y_0 \setminus K_j$.

Consider φ as introduced after Definition 3.9. Now let us recall the unimodular function $\omega : Y_0 \longrightarrow \mathbb{T}$ defined in Remark 3.13; that is, if $y \in Y_0$ then $\omega(y) := T(h_1, ..., h_k)(y)$, where $h_j \in P_{A_j}(\pi_j(\varphi(y)))$. Besides, from the above argument, it follows that if $y \in Y_0$ with $\varphi(y) = (x_1, ..., x_k)$ and $(f_1, ..., f_k) \in A_1 \times ... \times A_k$, then

$$T(f_1, ..., f_k)(y) = \omega(y) \prod_{j=1}^{k} f_j(x_j)^* = \omega(y) \prod_{j=1}^{k} f_j(\pi_j(\varphi(y)))^*,$$

that is,

$$T(f_1, ..., f_k)(y) = \omega(y) \prod_{j=1}^{k} f_j(\pi_j(\varphi(y)))^*$$

where $f_j(\pi_j(\varphi(y)))^* = f_j(\pi_j(\varphi(y)))$ if $y \in K_j$ and $f_j(\pi_j(\varphi(y)))^* = \overline{f_j(\pi_j(\varphi(y)))}$ if $y \in Y_0 \setminus K_j$.

Next we prove that φ is continuous. Suppose that $y_0 \in Y_0$, $\varphi(y_0) = (x_1, ..., x_k)$ and $U_1 \times ... \times U_k$ is a neighborhood of $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$. For each j, $j = 1, ..., k$, there is a neighborhood U'_j of x_j in X_j with $U_j = U'_j \cap Ch(A_j)$. Choose a peaking function $f_j \in V_{x_j}$ such that

$|f_j| < \frac{1}{2}$ on $X_j \setminus U'_j$ $(j = 1, ..., k)$. Then $|T(f_1, ..., f_k)(y_0)| = 1$. Set

$$V := \{z \in Y_0 : |T(f_1, ..., f_k)(z)| > \frac{1}{2}\}.$$

Clearly, V is a neighborhood of y_0 such that $\varphi(V) \subseteq U_1 \times ... \times U_k$ because if $z \in V$ and $\varphi(z) = (x'_1, ..., x'_k)$, then

$$\frac{1}{2} < |T(f_1, ..., f_k)(z)| = \prod_{j=1}^{k} |f_j(x'_j)| \leq |f_j(x'_j)| \quad (j = 1, ..., k).$$

Hence $x'_j \in U_j$ and so $(x'_1, ..., x'_k) \in U_1 \times ... \times U_k$. Therefore, φ is continuous.

To complete the proof, it suffices to check the continuity of ω. Let $y_0 \in Y_0$. Then $y_0 \in \mathcal{I}_{x_1,...,x_k}$ for a unique $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$. For each j, $j = 1, ..., k$, choose a peaking function $f_j \in P_{A_j}(x_j)$ and take

$$U_j := \{x \in Ch(A_j) : f_j(x) \neq 0\}.$$

Then $U = U_1 \times ... \times U_k$ is a neighborhood of $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$ and consequently $\varphi^{-1}(U)$ is a neighborhood of y_0. We have

$$\omega(y) = \frac{T(f_1, ..., f_k)(y)}{\prod_{j=1}^{k} f_j(\pi_j(\varphi(y)))^*} \quad (y \in \varphi^{-1}(U)),$$

where $f_j(\pi_j(\varphi(y)))^* = f_j(\pi_j(\varphi(y)))$ if $y \in K_j$ and $f_i(\pi_i(\varphi(y)))^* = \overline{f_j(\pi_j(\varphi(y)))}$ if $y \in Y_0 \setminus K_j$. So taking into account that K_j is a clopen subset of Y_0, from the continuity of the functions $T(f_1, ..., f_k)$, $f_j \circ \pi_j \circ \varphi$ and $\overline{f_j \circ \pi_j \circ \varphi}$ we conclude that ω is continuous at y_0. □

It should be noted that if T is a k-linear-isometry, then, as mentioned before Example 3.6, we have $\mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k} = \mathcal{I}^{\alpha'_1,...,\alpha'_k}_{x_1,...,x_k}$ for all $(\alpha_1, ..., \alpha_k), (\alpha'_1, ..., \alpha'_k) \in \mathbb{T}^k$ and $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, and furthermore, $K_j = Y_0$ for all $j \in \{1, ..., k\}$. So we can obtain immediately the main result in [8] as follows:

Corollary 4.2. *Suppose that $T : A_1 \times ... \times A_k \longrightarrow C_0(Y)$ is a k-linear isometry. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$, and a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ such that*

$$T(f_1, ..., f_k)(y) = \omega(y) \prod_{j=1}^{k} f_j(\pi_j(\varphi(y)))$$

for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $y \in Y_0$, where π_j is the jth projection map.

Remark 4.3. As announced after Example 3.6, we provide several conditions each of which implies the nonemptiness of the sets $\mathcal{I}_{x_1,...,x_k}$:

- Given $(x_1,...,x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, there exists $(f_1,...,f_k) \in V_{x_1} \times ... \times V_{x_k}$ such that $\bigcap_{(\alpha_1,...,\alpha_k)\in\{1,i\}^k} M_{T(\alpha_1 f_1,...,\alpha_k f_k)} \neq \emptyset$.

  Let us give an explanation to see $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$ in this case. Consider y in the above non-empty intersection. Given $(\alpha_1,...,\alpha_k) \in \{1,i\}^k$ and $(g_1,...,g_k) \in V_{x_1} \times ... \times V_{x_k}$, then from Lemma 3.2 we have $T(\alpha_1 f_1 - \alpha_1 g_1, f_2, ..., f_k)(y) = 0$, and so $|T(\alpha_1 g_1, f_2, ..., f_k)(y)| = |T(\alpha_1 f_1, f_2, ..., f_k)(y)| = 1$. This argument yields $y \in \mathcal{I}^{\alpha_1,1,...,1}_{x_1,x_2,...,x_k}$. Then again by using Lemma 3.2 (twice) we get

$$|T(\alpha_1 g_1, \alpha_2 g_2, f_3, ..., f_k)(y)| = |T(\alpha_1 g_1, \alpha_2 f_2, f_3, ..., f_k)(y)|$$
$$= |T(\alpha_1 f_1, \alpha_2 f_2, f_3, ..., f_k)(y)| = 1,$$

  and consequently, $y \in \mathcal{I}^{\alpha_1,\alpha_2,1,...,1}_{x_1,x_2,x_3,...,x_k}$. By continuing this process, finally it is concluded that $y \in \mathcal{I}^{\alpha_1,...,\alpha_k}_{x_1,...,x_k}$. Therefore, $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$.
- Given $(x_1,...,x_k) \in Ch(A_1) \times ... \times Ch(A_k)$, there exists $(f_1,...,f_k) \in V_{x_1} \times ... \times V_{x_k}$ such that all the functions $|T(\alpha_1 f_1,...,\alpha_k f_k)|$, $(\alpha_1,...,\alpha_k) \in \{1,i\}^k$, peak at the same points.
- In the unital case, $\bigcap_{(\alpha_1,...,\alpha_k)\in\{1,i\}^k} M_{T(\alpha_1,...,\alpha_k)} \neq \emptyset$.
- In the surjective case when $k = 1$ (see [7, Corollary 3.11]).

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