

# SHARP VALUES FOR THE CONSTANTS IN THE POLYNOMIAL BOHNENBLUST-HILLE INEQUALITY

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ABSTRACT. In this paper we prove that the complex polynomial Bohnenblust-Hille constant for 2-homogeneous polynomials in  $\mathbb{C}^2$  is exactly  $\sqrt[4]{\frac{3}{2}}$ . We also give the exact value of the real polynomial Bohnenblust-Hille constant for 2-homogeneous polynomials in  $\mathbb{R}^2$ . Finally, we provide lower estimates for the real polynomial Bohnenblust-Hille constant for polynomials in  $\mathbb{R}^2$  of higher degrees.

## 1. PRELIMINARIES: WHAT YOU NEED TO KNOW

Any homogeneous polynomial in  $\mathbb{K}^n$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) of degree  $m \in \mathbb{N}$  can be written as

$$(1.1) \quad P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha,$$

where  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  and  $a_\alpha \in \mathbb{K}$ . Here,  $\mathcal{P}(^m\mathbb{K}^n)$  stands for the finite dimension linear space of all the homogeneous polynomials of degree  $m$  on  $\mathbb{K}^n$ .

If  $\|\cdot\|$  is a norm on  $\mathbb{K}^n$ , then the formula

$$\|P\| := \sup\{|P(x)| : x \in \mathbf{B}_X\},$$

for all  $P \in \mathcal{P}(^m\mathbb{K}^n)$ , where  $\mathbf{B}_X$  is the unit ball of the Banach space  $X = (\mathbb{K}^n, \|\cdot\|)$ , defines a norm in  $\mathcal{P}(^m\mathbb{K}^n)$  usually called polynomial norm. The space  $\mathcal{P}(^m\mathbb{K}^n)$  endowed with the polynomial norm induced by  $X$  is denoted by  $\mathcal{P}(^mX)$ .

Other norms customarily used in  $\mathcal{P}(^m\mathbb{K}^n)$  besides the polynomial norm are the  $\ell_p$  norms of the coefficients. Namely, if  $P$  is as in (1.1) and  $p \geq 1$ , then

$$|P|_p := \left( \sum_{|\alpha|=m} |a_\alpha|^p \right)^{\frac{1}{p}},$$

defines another norm in  $\mathcal{P}(^m\mathbb{K}^n)$ . It is important to point out that although the polynomial norm is, most of the times, very difficult to compute, the  $\ell_p$  norm of the coefficients is fairly easy to obtain. Since  $\mathcal{P}(^m\mathbb{K}^n)$  is finite dimensional, the polynomial norm  $\|\cdot\|$  and the  $\ell_p$  norm  $|\cdot|_p$  ( $p \geq 1$ ) are equivalent, and therefore there exist constants  $k(m, n)$ ,  $K(m, n) > 0$  such that

$$(1.2) \quad k(m, n)|P|_p \leq \|P\| \leq K(m, n)|P|_p,$$

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for all  $P \in \mathcal{P}(^m\mathbb{K}^n)$ . The latter inequalities may provide a good estimate on  $\|P\|$  as long as we know the exact value of the best possible constants  $k(m, n)$  and  $K(m, n)$  appearing in (1.2).

The problem presented above is an extension of the the well known polynomial Bohnenblust-Hille inequality (polynomial BH inequality for short). It was proved in [5] that there exists a constant  $D_m > 0$  such that for every  $P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$  we have

$$(1.3) \quad |P|_{\frac{2m}{m+1}} \leq D_m \|P\|,$$

where  $\ell_\infty^n(\mathbb{R})$  and  $\ell_\infty^n(\mathbb{C})$  are, respectively, the real and complex versions of  $\ell_\infty^n$ . Observe that (1.3) coincides with the first inequality in (1.2) for  $p = \frac{2m}{m+1}$  except for the fact that  $D_m$  in (1.3) can be chosen in such a way that it is independent from the dimension  $n$ . Actually Bohnenblust and Hille showed that  $\frac{2m}{m+1}$  is optimal in (1.3) in the sense that for  $p < \frac{2m}{m+1}$ , any constant  $D$  fitting in the inequality

$$|P|_p \leq D \|P\|,$$

for all  $P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$  depends necessarily on  $n$ .

The best constants in (1.3) depend considerably on whether we consider the real or the complex version of  $\ell_\infty^n$ , which motivates the following definition:

**Definition 1.1.** *The polynomial Bohnenblust-Hille constant for polynomials of degree  $m$  is defined as*

$$D_{\mathbb{K},m} := \inf \left\{ D > 0 : |P|_{\frac{2m}{m+1}} \leq D \|P\|, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K})) \right\}.$$

If we restrict attention to a certain subset  $E$  of  $\mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$  for some  $n \in \mathbb{N}$ , then we define

$$D_{\mathbb{K},m}(E) := \inf \left\{ D > 0 : |P|_{\frac{2m}{m+1}} \leq D \|P\| \text{ for all } P \in E \right\}.$$

For simplicity we will often use the notation  $D_{\mathbb{K},m}(n)$  instead of  $D_{\mathbb{K},m}(\mathcal{P}(^m\ell_\infty^n(\mathbb{K})))$ . Note that

$$1 \leq D_{\mathbb{K},m}(n) \leq D_{\mathbb{K},m},$$

for all  $m, n \in \mathbb{N}$ .

A good idea of the asymptotic growth of the constants  $D_{\mathbb{K},m}$  and  $D_{\mathbb{K},m}(n)$  is provided by the following definition:

**Definition 1.2.** *The asymptotic hypercontractivity constant of the polynomial BH inequality is*

$$H_{\mathbb{K},\infty} := \limsup_m \sqrt[m]{D_{\mathbb{K},m}}.$$

Similarly, if we restrict attention to polynomials in  $n$  variables then we define

$$H_{\mathbb{K},\infty}(n) := \limsup_m \sqrt[m]{D_{\mathbb{K},m}(n)}.$$

Of course  $1 \leq H_{\mathbb{K},\infty}(n) \leq H_{\mathbb{K},\infty}$ , for all  $n \in \mathbb{N}$ .

It was shown in [10] that the complex polynomial Bohnenblust–Hille inequality is, at most, hypercontractive. In [2] the estimate on  $D_{\mathbb{C},m}$  was improved. In fact the authors show that for every  $B > 1$  there exists  $A > 0$  such that  $D_{\mathbb{C},m} \leq Ae^{B\sqrt{m \log m}}$ , from which it follows that  $H_{\mathbb{C},\infty}(n) = H_{\mathbb{C},\infty} = 1$ , for all  $n \in \mathbb{N}$ . For the real case, it has been recently proved in [7] that  $H_{\mathbb{R},\infty} = 2$ . However, not many exact values of  $D_{\mathbb{K},m}(n)$  are known so far. This paper is devoted to calculate, explicitly or numerically some values of these constants.

This paper is arranged in two main sections. In Section 2, we employ some results on the geometry of spaces of polynomials in order to provide the exact value of  $D_{\mathbb{C},2}(2)$ . In Section 3 we use a similar technique to find the exact value of  $D_{\mathbb{R},2}(2)$ . We also provide lower estimates for  $D_{\mathbb{R},m}(2)$  and  $H_{\mathbb{R},\infty}(2)$  by means of numerical calculus.

The polynomial Bohnenblust–Hille inequality has important applications in different fields of Mathematics and Physics and has been studied in depth by many authors since a multilinear version of the Bohnenblust–Hille inequality was proved in 1931 (see [2–6, 9, 11–17, 20–27]) and the references therein.

## 2. THE EXACT VALUE OF $D_{\mathbb{C},2}(2)$

Throughout this section we will often identify any two-variable polynomial  $az^2 + bwz + cw^2$  or any one-variable polynomial  $a\lambda^2 + b\lambda + c$ , for  $a, b, c \in \mathbb{K}$ , with the vector  $(a, b, c) \in \mathbb{K}^3$ . Also, we use the standard notation  $\|az^2 + bwz + cw^2\|_{\mathbb{D}}$  for the supremum of  $|az^2 + bwz + cw^2|$  for  $z, w$  in the unit disk  $\mathbb{D}$  of  $\mathbb{C}$ . Similarly,  $\|a\lambda^2 + b\lambda + c\|_{\mathbb{D}}$  stands for the supremum of  $|a\lambda^2 + b\lambda + c|$  for  $\lambda \in \mathbb{D}$ . Observe that

$$\|az^2 + bwz + cw^2\|_{\mathbb{D}} = \|a\lambda^2 + b\lambda + c\|_{\mathbb{D}} = \max_{|\lambda|=1} |a\lambda^2 + b\lambda + c|,$$

being the last of the latter equalities due to the Maximum Modulus Principle.

The main result of this section depends upon the following lemma, which is of independent interest.

**Lemma 2.1.** *Let  $a, b, c \in \mathbb{C}$ . There exist  $a', b', c' \in \mathbb{R}$  such that*

$$\|az^2 + bwz + cw^2\|_{\mathbb{D}} \geq \|a'z^2 + b'zw + c'w^2\|_{\mathbb{D}} \quad \text{and} \quad \|(a, b, c)\|_{\frac{4}{3}} = \|(a', b', c')\|_{\frac{4}{3}}.$$

*Proof.* If we perform the change of variables

$$z \mapsto ze^{-\frac{i \arg(a)}{2}} \quad \text{and} \quad w \mapsto we^{-\frac{i \arg(c)}{2}},$$

in  $\|az^2 + bwz + cw^2\|_{\mathbb{D}}$ , we can assume (without loss of generality) that  $a, c \geq 0$ . We can also assume that  $a \geq c$  by swapping  $z$  and  $w$ . We have:

$$\begin{aligned} |a\lambda^2 + b\lambda + c|^2 &= (a\lambda^2 + b\lambda + c) \left( a\bar{\lambda}^2 + \bar{b}\bar{\lambda} + c \right) \\ &= a^2 + ac\lambda^2 + a\bar{b}\lambda + ac\bar{\lambda}^2 + c^2 + c\bar{b}\bar{\lambda} + ab\bar{\lambda} + bc\lambda + |b|^2 \\ &= a^2 + c^2 + |b|^2 + 2 [ac\operatorname{Re}(\lambda^2) + a\operatorname{Re}(\bar{b}\lambda) + c\operatorname{Re}(b\lambda)] \\ &= a^2 + c^2 + |b|^2 + 2 [ac\operatorname{Re}(\lambda^2) + (a+c)\operatorname{Re}(b)\operatorname{Re}(\lambda) + (a-c)\operatorname{Im}(b)\operatorname{Im}(\lambda)]. \end{aligned}$$

Similarly, if  $a', b', c'$  are real numbers, then:

$$|a'\lambda^2 + b'\lambda + c'|^2 = a'^2 + c'^2 + |b'|^2 + 2 [a'c'\operatorname{Re}(\lambda^2) + (a' + c')b'\operatorname{Re}(\lambda)].$$

(1) Assume first  $a \geq c \geq |b|$ . Then, choose

$$a' = \frac{(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{4}}}{2^{\frac{3}{4}}}, \quad c' = -a', \quad b' = a.$$

Then,  $\|(a, b, c)\|_{\frac{4}{3}} = \|(a', b', c')\|_{\frac{4}{3}}$ . On the other hand,

$$a'^2 + c'^2 + |b'|^2 + 2 [a'c'\operatorname{Re}(\lambda^2) + (a' + c')b'\operatorname{Re}(\lambda)] = 2a'^2 + a^2 - 2a'^2\operatorname{Re}(\lambda^2),$$

so that it is easy to see

$$\|(a', b', c')\|_{\mathbb{D}}^2 = 4a'^2 + a^2 = \sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}} + a^2.$$

Also, giving the value  $\lambda = 1$  (if  $\operatorname{Re}(b) \geq 0$ ) or  $\lambda = -1$  (if  $\operatorname{Re}(b) \leq 0$ ), we can see that

$$\|az^2 + cw^2 + b zw\|_{\mathbb{D}}^2 \geq a^2 + c^2 + |b|^2 + 2ac.$$

Now, we want

$$\sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}} + a^2 \leq a^2 + c^2 + |b|^2 + 2ac,$$

that is,

$$\frac{c^2 + |b|^2 + 2ac}{\sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}}} \geq 1.$$

Divide both, numerator and denominator, by  $a^2$  in order to convert the problem in having to achieve

$$\frac{x^2 + y^2 + 2x}{\sqrt{2}(x^{\frac{4}{3}} + y^{\frac{4}{3}})^{\frac{3}{2}}} \geq 1,$$

for  $0 \leq y \leq x \leq 1$ .

- (2) Assume next  $a \geq |b| \geq c$ . In this second part of the proof we shall need to employ a couple of real valued functions that will come in handy to achieve our purpose. Let us first focus our attention on the choice of the constants  $a'$ ,  $b'$ ,  $c'$ , as before,

$$a' = \frac{\left(|c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{2^{\frac{3}{4}}}, \quad c' = -a', \quad b' = a.$$

In this case, choose

$$\lambda = \operatorname{sign}(\operatorname{Re}(b))\sqrt{\frac{1}{2}} + i\operatorname{sign}(\operatorname{Im}(b))\sqrt{\frac{1}{2}}.$$

Then,

$$\begin{aligned} \|(a, b, c)\|_{\mathbb{D}}^2 &\geq a^2 + c^2 + |b|^2 + 2 \left[ \sqrt{\frac{1}{2}}(a+c)|\operatorname{Re}(b)| + \sqrt{\frac{1}{2}}(a-c)|\operatorname{Im}(b)| \right] \\ &\geq a^2 + c^2 + |b|^2 + \sqrt{2}|b|(a-c). \end{aligned}$$

Hence, we will achieve the desired result if we can guarantee

$$\sqrt{2} \left( |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}} \right)^{\frac{3}{2}} + a^2 \leq a^2 + c^2 + |b|^2 + \sqrt{2}(a-c)|b|,$$

in other words,

$$1 \leq \Phi_1(x, y) := \frac{x^2 + y^2 + \sqrt{2}(1-x)y}{\sqrt{2} \left( x^{\frac{4}{3}} + y^{\frac{4}{3}} \right)^{\frac{3}{2}}},$$

where  $0 \leq x \leq y \leq 1$ .

Let us focus now in another choice of constants  $a'$ ,  $b'$ ,  $c'$ :

$$a' = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{(2 + k^{4/3})^{\frac{3}{4}}}, \quad c' = -a', \quad b' = ka,$$

where  $k$  has been chosen so that  $\|(a, b, c)\|_{\frac{4}{3}} = \|(a', b', c')\|_{\frac{4}{3}}$ . It can be proved that  $k \approx 2.828$ . Still giving the value  $\lambda = \text{sign}(\text{Re}(b))$

$$\|(a, b, c)\|_{\mathbb{D}}^2 \geq a^2 + c^2 + |b|^2 + 2ac,$$

and again we guarantee that we achieve what we are searching for if we get

$$4a'^2 + b'^2 = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{2}}}{(2 + k^{4/3})^{\frac{3}{2}}}(4 + k^2) \leq a^2 + c^2 + |b|^2 + 2ac,$$

in other words,

$$1 \leq \Psi_1(x, y) := \frac{(1 + x^2 + y^2 + 2yx)(2 + k^{4/3})^{\frac{3}{2}}}{(4 + k^2)\left(y^{\frac{4}{3}} + x^{\frac{4}{3}} + 1\right)^{\frac{3}{2}}},$$

with  $0 \leq x \leq y \leq 1$ .

The reader can check using elementary calculus that, if

$$H(x, y) := \max\{\Phi_1(x, y), \Psi_1(x, y)\},$$

then

$$1 \leq H(x, y) \text{ for every } 0 \leq x \leq y \leq 1.$$

(3) Assume finally  $|b| \geq a \geq c$ . Then, we may choose

$$a' = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{(2 + k^{4/3})^{\frac{3}{4}}}, \quad c' = -a', \quad b' = ka,$$

where  $k$  is chosen as in the previous case. For  $\lambda = \text{sign}(\text{Re}(b))$ , we still need to make sure that

$$1 \leq \Phi_2(x, y) := \frac{(1 + x^2 + y^2 + 2xy)(2 + k^{4/3})^{\frac{3}{2}}}{(4 + k^2)(x^{4/3} + y^{4/3} + 1)^{\frac{3}{2}}}.$$

For  $\lambda = \text{sign}(\text{Re}(b))\sqrt{\frac{1}{2}} + i\text{sign}(\text{Im}(b))\sqrt{\frac{1}{2}}$ , we need to make sure that

$$1 \leq \Psi_2(x, y) := \frac{(1 + x^2 + y^2 + \sqrt{2}(y - x))(2 + k^{4/3})^{\frac{3}{2}}}{(4 + k^2)(x^{4/3} + y^{4/3} + 1)^{\frac{3}{2}}}.$$

Next, choose

$$a' = \frac{(|a|^{4/3} + |c|^{4/3})^{3/4}}{2^{3/4}}, \quad c' = -a' \quad \text{and} \quad b' = |b|,$$

such that

$$\|(a', b', c')\|_{\mathbb{D}}^2 = 4 \frac{(|a|^{4/3} + |c|^{4/3})^{3/2}}{2^{3/2}} + |b|^2,$$

and

$$\lambda = \text{sign}(\text{Re}(b))\sqrt{1 - \left(\frac{a - c}{|b|}\right)^2} + i\text{sign}(\text{Im}(b))\frac{a - c}{|b|}.$$

In that case,

$$\begin{aligned} \|(a, b, c)\|_{\mathbb{D}}^2 &\geq a^2 + c^2 + |b|^2 + 2 [ac\operatorname{Re}(\lambda^2) + (a+c)\operatorname{Re}(b)\operatorname{Re}(\lambda) + (a-c)\operatorname{Im}(b)\operatorname{Im}(\lambda)] \\ &= a^2 + c^2 + |b|^2 + 2 \left[ ac \left( 1 - 2 \left( \frac{a-c}{|b|} \right)^2 \right) + (a+c)|\operatorname{Re}(b)|\sqrt{1 - \left( \frac{a-c}{|b|} \right)^2} \right. \\ &\quad \left. + (a-c)|\operatorname{Im}(b)|\frac{a-c}{|b|} \right]. \end{aligned}$$

Assume first  $|\operatorname{Im}(b)| \geq \frac{\sqrt{2}}{2}$ . Then,

$$\begin{aligned} \|(a, b, c)\|_{\mathbb{D}}^2 &\geq a^2 + c^2 + |b|^2 \\ &\quad + 2 \left[ ac \left( 1 - 2 \left( \frac{a-c}{|b|} \right)^2 \right) + (a-c)\frac{\sqrt{2}}{2}|b|\frac{a-c}{|b|} \right]. \end{aligned}$$

Hence, we will achieve what we are searching for if we can assure that

$$1 \leq \Omega_2^{(1)}(x, y) := \frac{x^2 + y^2 + 2xy(1 - 2(y-x)^2) + \sqrt{2}(y-x)^2}{\sqrt{2}(x^{4/3} + y^{4/3})^{3/2}}.$$

On in on, we need to prove

$$1 \leq \max\{\Phi_2(x, y), \Psi_2(x, y), \Omega_2^{(1)}(x, y)\},$$

for  $0 \leq x \leq y \leq 1$ . This can be done by means of elementary calculus, and we leave it as an exercise to the reader.

On the other hand if, instead, we have  $|\operatorname{Re}(b)| \geq \frac{\sqrt{2}}{2}$ , then

$$\begin{aligned} \|(a, b, c)\|_{\mathbb{D}}^2 &\geq a^2 + c^2 + |b|^2 \\ &\quad + 2 \left[ ac \left( 1 - 2 \left( \frac{a-c}{|b|} \right)^2 \right) + (a+c)\frac{\sqrt{2}}{2}|b|\sqrt{1 - \left( \frac{a-c}{|b|} \right)^2} \right], \end{aligned}$$

and (in this case) we will be working with the condition

$$1 \leq \Omega_2^{(2)}(x, y) := \frac{x^2 + y^2 + 2xy(1 - 2(y-x)^2) + \sqrt{2}(y+x)\sqrt{1 - (y-x)^2}}{\sqrt{2}(x^{4/3} + y^{4/3})^{3/2}},$$

and, in conclusion, we shall need to guarantee that

$$1 \leq \max\{\Phi_2(x, y), \Psi_2(x, y), \Omega_2^{(2)}(x, y)\},$$

for  $0 \leq x \leq y \leq 1$ , which we also leave as an exercise to the reader.

And, with this last case, the proof is complete.  $\square$

In order to prove that  $D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}}$  we will also need the following description of the extreme points of the unit ball of  $\mathbb{R}^3$  endowed with the norm

$$\|(a, b, c)\|_{\mathbb{D}} := \sup\{|az^2 + bz + c| : |z| \leq 1\}$$

for  $a, b, c \in \mathbb{R}$ . This norm has been studied by Aron and Klimek in [1], where they denote it by  $\|\cdot\|_{\mathbb{C}}$ . Observe, once again that

$$\|(a, b, c)\|_{\mathbb{D}} = \|az^2 + bwz + cz^2\|_{\mathbb{D}}.$$

**Theorem 2.2** (Aron and Klimek, [1]). *Let  $E_{\mathbb{R}}$  be the real subspace of  $\mathcal{P}({}^2\ell_{\infty}^2(\mathbb{C}))$  given by  $\{az^2 + bwz + cw^2 : (a, b, c) \in \mathbb{R}^3\}$ . Then*

$$\text{ext}(\mathbf{B}_{E_{\mathbb{R}}}) = \left\{ \left( s, \sqrt{4|s||t| \left( \frac{1}{(|s| + |t|)^2} - 1 \right)}, t \right) : (s, t) \in G \right\},$$

where  $\text{ext}(\mathbf{B}_{E_{\mathbb{R}}})$  is the set of extreme points of the unit ball of  $E_{\mathbb{R}}$ , namely  $\mathbf{B}_{E_{\mathbb{R}}}$  and  $G = \{(s, t) \in \mathbb{R}^2 : |s| + |t| < 1 \text{ and } |s + t| \leq (s + t)^2\} \cup \{\pm(1, 0), \pm(0, 1)\}$ .

**Theorem 2.3.** *The optimal complex polynomial Bohnenblust-Hille constant for polynomials in  $E_{\mathbb{R}}$ , which we denote by  $D_{\mathbb{C},2}(E_{\mathbb{R}})$ , is given by  $D_{\mathbb{C},2}(E_{\mathbb{R}}) = \sqrt[4]{\frac{3}{2}}$ . Moreover,*

$$D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}} \approx 1.1066.$$

*Proof.* Using convexity we have

$$\begin{aligned} D_{\mathbb{C},2}(E_{\mathbb{R}}) &= \sup\{\|(a, b, c)\|_{\frac{4}{3}} : \|az_1^2 + bz_1z_2 + cz_2^2\| \leq 1\} \\ &= \sup\{\|(a, b, c)\|_{\frac{4}{3}} : \|(a, b, c)\|_{\mathbb{C}} \leq 1\} \\ &= \sup\{\|(a, b, c)\|_{\frac{4}{3}} : (a, b, c) \in \text{ext}(\mathbf{B}_{E_{\mathbb{R}}})\}, \end{aligned}$$

Hence

$$D_{\mathbb{C},2}(E_{\mathbb{R}}) = \sup \left\{ \left( |s|^{\frac{4}{3}} + |t|^{\frac{4}{3}} + \left[ 4|s||t| \left( \frac{1}{(|s| + |t|)^2} - 1 \right) \right]^{\frac{2}{3}} \right)^{\frac{3}{4}} : (s, t) \in G \right\}.$$

If  $\Phi(s, t) = \left( |s|^{\frac{4}{3}} + |t|^{\frac{4}{3}} + \left[ 4|s||t| \left( \frac{1}{(|s| + |t|)^2} - 1 \right) \right]^{\frac{2}{3}} \right)^{\frac{3}{4}}$  for  $(s, t) \in G$ , one can prove using elementary calculus that  $\Phi$  attains its maximum on  $G$  at  $\pm \left( \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6} \right)$  and  $\Phi \left( \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6} \right) = \sqrt[4]{\frac{3}{2}}$ . Finally, from Lemma 2.1 we also obtain that  $D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}} \approx 1.1066$ .  $\square$

Observe that the fact that  $D_{\mathbb{C},2}(2) \geq \sqrt[4]{\frac{3}{2}}$  was already proved in [22]. The reader can find a sketch of the graph of  $\Phi$  on the part of  $G$  contained in the second quadrant in Figure 1.

### 3. THE EXACT VALUE OF $D_{\mathbb{R},2}(2)$ AND LOWER BOUNDS FOR $D_{\mathbb{R},m}(2)$

In [7] it is proved that the asymptotic hypercontractivity constant of the real polynomial BH inequality is exactly 2. Is it true that  $H_{\mathbb{R},\infty}(2) = 2$ ? The results presented here suggest that, perhaps  $H_{\mathbb{R},\infty}(2) < 2$ . In this section, as we did in the previous one, we will also identify polynomials with the vector of its coefficients.

**Remark 3.1.** *Throughout this section we will compute several times norms of polynomials on the real line numerically. This is done by using Matlab. In particular, if  $P(x)$  is a real polynomial on  $\mathbb{R}$ , we apply the predefined Matlab function `roots.m` to  $P'$  in order to obtain an approximation of all the critical points of  $P$ . If  $x_1, \dots, x_k$  are all the roots of  $P'$  in  $[-1, 1]$ , then we approach the norm of  $P$  as*

$$\|P\| := \max\{|P(x)| : x \in [-1, 1]\} = \max\{|P(x_i)|, |P(\pm 1)| : i = 1, \dots, k\}.$$

Another Matlab predefined function, namely `conv.m`, is used in order to multiply polynomials. This is done to obtain Figure 4.

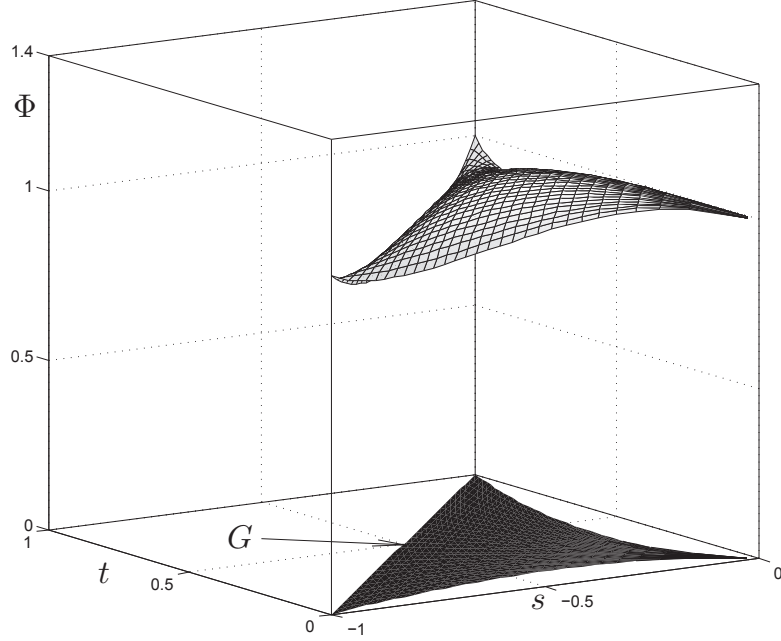


FIGURE 1. Graph of  $\Phi$  on  $G$  intersected with the second quadrant.

**3.1. The exact calculation of  $D_{\mathbb{R},2}(2)$ .** The value of the constant  $D_{\mathbb{R},2}(2)$  can be obtained using the geometry of the unit ball of  $\mathcal{P}(\ell_\infty^2(\mathbb{R}))$  described in [8]. We state the result we need for completeness:

**Theorem 3.2.** [Choi, Kim [8]] *The set  $\text{ext}(\mathbf{B}_{\mathcal{P}(\ell_\infty^2(\mathbb{R}))})$  of extreme points of the unit ball of  $\mathcal{P}(\ell_\infty^2(\mathbb{R}))$  is given by*

$$\text{ext}(\mathbf{B}_{\mathcal{P}(\ell_\infty^2(\mathbb{R}))}) = \{\pm x^2, \pm y^2, \pm(tx^2 - ty^2 \pm 2\sqrt{t(1-t)}xy) : t \in [1/2, 1]\}.$$

As a consequence of the previous result, we obtain the following:

**Theorem 3.3.** *Let  $f$  be the real valued function given by*

$$f(t) = \left[ 2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}} \right]^{\frac{3}{4}}.$$

We have that  $D_{\mathbb{R},2}(2) = f(t_0) \approx 1.837373$ , where

$$t_0 = \frac{1}{36} \left( 2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16 \right) \approx 0.867835.$$

The exact value of  $f(t_0)$  is given by

$$\left( \frac{\left( 2\sqrt[3]{107+9\sqrt{129}} + \sqrt[3]{856-72\sqrt{129}} + 16 \right)^{4/3}}{18 \cdot 6^{2/3}} + \frac{1}{9 \left( -\frac{2\sqrt[3]{107+9\sqrt{129}} + (107+9\sqrt{129})^{2/3} - 2\sqrt[3]{107-9\sqrt{129}} + (107-9\sqrt{129})^{2/3} - 60}{-2} \right)^{2/3}} \right)^{3/4},$$

Moreover, the following normalized polynomials are extreme for this problem:

$$P_2(x, y) = \pm(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1-t_0)}xy).$$



*Proof.* Let

$$f(t) = \left[ 2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}} \right]^{\frac{3}{4}}.$$

We just have to notice that due to the convexity of the  $\ell_p$ -norms and Theorem 3.2 we have

$$\begin{aligned} D_{\mathbb{R},2}(2) &= \sup\{|\mathbf{a}|_{\frac{4}{3}} : \mathbf{a} \in \mathbf{B}_{\mathcal{P}(2\ell_{\infty}^2\mathbb{R})}\} \\ &= \sup\{|\mathbf{a}|_{\frac{4}{3}} : \mathbf{a} \in \text{ext}(\mathbf{B}_{\mathcal{P}(2\ell_{\infty}^2\mathbb{R})})\} = \sup_{t \in [1/2,1]} f(t). \end{aligned}$$

Some calculations will show that the last supremum is attained at  $t = t_0$ , concluding the proof.  $\square$

Now, if  $\mathbf{a}_n$  is the vector of the coefficients of  $P_n^2$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.1) \quad D_{\mathbb{R},2n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{4n}{2n+1}}}{\|P_2\|^n}.$$

Since  $\|P_2\| = 1$ , then (3.1) with  $n = 300$  (see also Figure 4) proves that

$$D_{\mathbb{R},600}(2) \geq (1.36117)^{600},$$

providing numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.36117.$$

**3.2. Educated guess for the exact calculation of  $D_{\mathbb{R},3}(2)$ .** To the authors' knowledge the calculation of  $\|P\|$  is, in general, far from being easy. However there is a way to compute  $\|P\|$  for specific cases. For instance Greu, Muñoz and Seoane prove in [18, Lemma 3.12] the following formula:

**Lemma 3.4.** *If for every  $a, b \in \mathbb{R}$  we define  $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$  then*

$$\|P_{a,b}\| = \begin{cases} \left| a - \frac{b^2}{3a} + \frac{2b^3}{27a^2} + \frac{2a}{27} \left( -\frac{3b}{a} + \frac{b^2}{a^2} \right)^{\frac{3}{2}} \right| & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3 - 2\sqrt{3}, \\ |2a + 2b| & \text{otherwise,} \end{cases}$$

where

$$b_1 = \frac{3}{7} \left( 3 - \frac{2\sqrt[3]{9}}{\sqrt[3]{-12 + 7\sqrt{3}}} + 2\sqrt[3]{-36 + 21\sqrt{3}} \right) \approx -1.6692.$$

From Lemma 3.4 we have the following sharp polynomial Bohnenblust-Hille type constant:

**Theorem 3.5.** *Let  $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$  for  $a, b \in \mathbb{R}$  and consider the subset of  $\mathcal{P}(^3\ell_{\infty}^2(\mathbb{R}))$  given by  $E = \{P_{a,b} : a, b \in \mathbb{R}\}$ . Then*

$$\frac{|(a, b, b, a)|_{\frac{3}{2}}}{\|P_{a,b}\|} = \begin{cases} \frac{27a^2(2|a|^{\frac{3}{2}} + 2|b|^{\frac{3}{2}})^{\frac{2}{3}}}{|27a^3 - 9ab^2 + 2b^3 + 2\text{sign}(a)(-3ab + b^2)|^{\frac{3}{2}}}, & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3 - 2\sqrt{3}, \\ \frac{(2|a|^{\frac{3}{2}} + 2|b|^{\frac{3}{2}})^{\frac{2}{3}}}{2|a+b|}, & \text{otherwise} \end{cases}$$

where  $b_1$  was defined in Lemma 3.4. Moreover, the above function attains its maximum when  $\frac{b}{a} = b_1$ , which implies that

$$D_{\mathbb{R},3}(E) = \frac{\left( 2 + 2|b_1|^{\frac{3}{2}} \right)^{\frac{2}{3}}}{2|1 + b_1|} \approx 2.5525$$

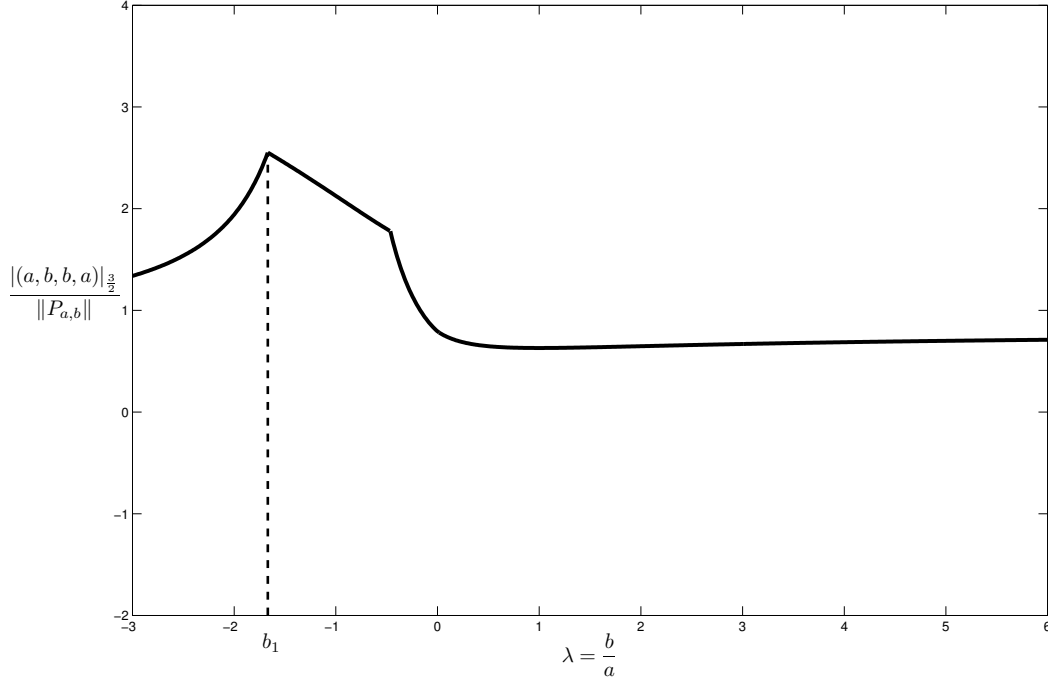


FIGURE 2. Graph of the quotient  $\frac{|(a,b,b,a)|_{\frac{3}{2}}}{\|P_{a,b}\|}$  as a function of  $\lambda = \frac{a}{b}$ .

The authors have numerical evidence to state that

$$D_{\mathbb{R},3}(2) = D_{\mathbb{R},3}(E).$$

Moreover, one polynomial for which  $D_{\mathbb{R},3}(2)$  would be attained is

$$P_3(x, y) = x^3 + b_1x^2y + b_1xy^2 + y^3,$$

where  $b_1 \approx -1.6692$  is as in Lemma 3.4. It can be proved from Lemma 3.4 that

$$\|P_3\| \approx 1.33848,$$

up to 5 decimal places. If  $\mathbf{a}_n$  is the vector of the coefficients of  $P_3(x, y)^n$  and we use the fact that

$$(3.2) \quad D_{\mathbb{R},3n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{6n}{3n+1}}}{\|P_3\|^n},$$

then putting  $n = 200$  in (3.2) we obtain, for instance,

$$D_{\mathbb{R},600}(2) \geq (1.42234)^{600},$$

which provides numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.42234.$$

**3.3. Numerical calculation of  $D_{\mathbb{R},5}(2)$ .** Let us define the polynomial

$$P_5(x, y) = ax^5 - bx^4y - cx^3y^2 + cx^2y^3 + bxy^4 - ay^5,$$

with

$$\begin{aligned} a &= 0.19462, \\ b &= 0.66008, \\ c &= 0.97833. \end{aligned}$$

The norm of  $P_5$  can be calculated numerically (using Remark 3.1), and it turns out to be

$$\|P_5\| = 0.28617,$$

up to 5 decimal places. The authors have numerical evidence showing that

$$D_{\mathbb{R},5}(2) \approx 6.83591.$$

In any case we have

$$D_{\mathbb{R},5}(2) \geq \frac{|(a, -b, -c, c, b, -a)|_{\frac{5}{3}}}{\|P_5\|} \approx 6.83591.$$

It is interesting to observe that we can improve numerically the estimate  $H_{\infty, \mathbb{R}}(2) \geq \sqrt[8]{27} \approx 1.50980$  (see [7, Theorem 4.2]) by considering polynomials of the form  $P_5^n$ . Indeed, if  $\mathbf{a}_n$  is the vector of the coefficients of  $P_5^n$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.3) \quad D_{\mathbb{R},5n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{10n}{5n+1}}}{\|P_5\|^n},$$

Using (3.3) with  $n = 120$  we obtain, in particular (see also Figure 4)

$$D_{\mathbb{R},600}(2) \geq (1.54987)^{600},$$

providing numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.54987.$$

**3.4. Educated guess for the exact calculation of  $D_{\mathbb{R},6}(2)$ .** The authors have numerical evidence pointing to the fact that an extreme polynomial in the Bohnenblust-Hille inequality for polynomials in  $\mathcal{P}({}^6\ell_{\infty}^2(\mathbb{R}))$  may be of the form

$$Q_{a,b}(x, y) = ax^5y + bx^3y^3 + axy^5.$$

This motivates a deeper study of this type of polynomials, which we do in the following result.

**Theorem 3.6.** *Let  $Q_{a,b}(x, y) = ax^5y + bx^3y^3 + axy^5$  for  $a, b \in \mathbb{R}$  and consider the subspace of  $\mathcal{P}({}^6\ell_{\infty}^2(\mathbb{R}))$  given by  $F = \{Q_{a,b} : a, b \in \mathbb{R}\}$ . Suppose  $\lambda_0 < \lambda_1$  are the only two roots of the equation*

$$\frac{|3\lambda^2 - 20 + \lambda\sqrt{9\lambda^2 - 20}|}{25} \sqrt{\frac{-3\lambda - \sqrt{9\lambda^2 - 20}}{10}} = |2 + \lambda|.$$

Then if  $\lambda = \frac{b}{a}$  we have

$$\frac{|(0, a, 0, b, 0, a, 0)|_{\frac{12}{7}}}{\|Q_{a,b}\|} = \begin{cases} \frac{25\sqrt{10}(2+|\lambda|^{\frac{12}{7}})^{\frac{12}{7}}}{|3\lambda^2 - 20 + \lambda\sqrt{9\lambda^2 - 20}| \sqrt{-3\lambda - \sqrt{9\lambda^2 - 20}}}, & \text{if } a \neq 0 \text{ and } \lambda_0 < \frac{b}{a} < \lambda_1, \\ \frac{(2+|\lambda|^{\frac{12}{7}})^{\frac{7}{12}}}{|2+\lambda|}, & \text{otherwise.} \end{cases}$$

Observe that  $\lambda_0 \approx -2.2654$ ,  $\lambda_1 \approx -1.6779$  and the above function attains its maximum when  $\frac{b}{a} = \lambda_0$  (see Figure 3), which implies that

$$D_{\mathbb{R},6}(F) = \frac{\left(2 + |\lambda_0|^{\frac{12}{7}}\right)^{\frac{12}{7}}}{|2 + \lambda_0|} \approx 10.7809.$$

*Proof.* We do not lose generality by considering only polynomials of the form  $Q_{1,\lambda}$ , in which case

$$\|Q_{1,\lambda}\| = \sup\{|x^5 + \lambda x^3 + x| : x \in [0, 1]\}.$$

The polynomial  $q_\lambda(x) := x^5 + \lambda x^3 + x$  has no critical points if  $\lambda > -\frac{2\sqrt{5}}{3}$ , otherwise it has the following critical points in  $[0, 1]$ :

$$x_0 := \sqrt{\frac{-3\lambda - \sqrt{9\lambda^2 - 20}}{10}} \quad \text{and} \quad x_1 := \sqrt{\frac{-3\lambda + \sqrt{9\lambda^2 - 20}}{10}} \quad \text{if } -2 \leq \lambda \leq -\frac{2\sqrt{5}}{3},$$

and  $x_0$  if  $\lambda \leq -2$ . Notice that

$$q_\lambda(x_0) = \frac{-3\lambda^2 + 20 - \lambda\sqrt{9\lambda^2 - 20}}{20}x_0$$

$$q_\lambda(x_1) = \frac{-3\lambda^2 + 20 + \lambda\sqrt{9\lambda^2 - 20}}{20}x_1.$$

It is easy to check that  $|q_\lambda(x_0)| \geq |q_\lambda(x_1)|$  for  $-2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}$ , which implies that

$$\|Q_{1,\lambda}\| = \begin{cases} \max\{|2 + \lambda|, |q_\lambda(x_0)|\} & \text{if } -2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}, \\ |2 + \lambda| & \text{otherwise.} \end{cases}$$

The equation  $|2 + \lambda| = |q_\lambda(x_0)|$  turns out to have only two roots, namely  $\lambda_0 \approx -2.2654$  and  $\lambda_1 \approx -1.6779$ . By continuity, it is easy to prove that  $|2 + \lambda| \leq |q_\lambda(x_0)|$  only if  $-2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}$ , which concludes the proof.  $\square$

As mentioned above, we have numerical evidence showing that

$$D_{\mathbb{R},6}(2) = D_{\mathbb{R},6}(F) = \frac{\left(2 + |\lambda_0|^{\frac{12}{7}}\right)^{\frac{12}{7}}}{|2 + \lambda_0|} \approx 10.7809.$$

In any case we do have that

$$D_{\mathbb{R},6}(2) \geq 10.7809.$$

As we did in the previous cases, it would be interesting to know if we can improve numerically our best lower bound on  $H_{\mathbb{R},\infty}$  by considering powers of

$$P_6(x, y) = Q_{1,\lambda_0}(x, y) = x^5y + \lambda_0x^3y^3 + xy^5,$$

with  $\lambda_0$  as in Theorem 3.6 ( $\lambda_0 \approx -2.2654$ ). If  $\mathbf{a}_n$  is the vector of the coefficients of  $P_6^n$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.4) \quad D_{\mathbb{R},6n}(2) \geq \frac{|\mathbf{a}_n|^{\frac{12n}{6n+1}}}{\|P_6\|^n}.$$

Using (3.4) with  $n = 100$  and estimating  $\|P_6\|$  according to Remark 3.1 we obtain

$$D_{\mathbb{R},600}(2) \geq (1.58432)^{600},$$

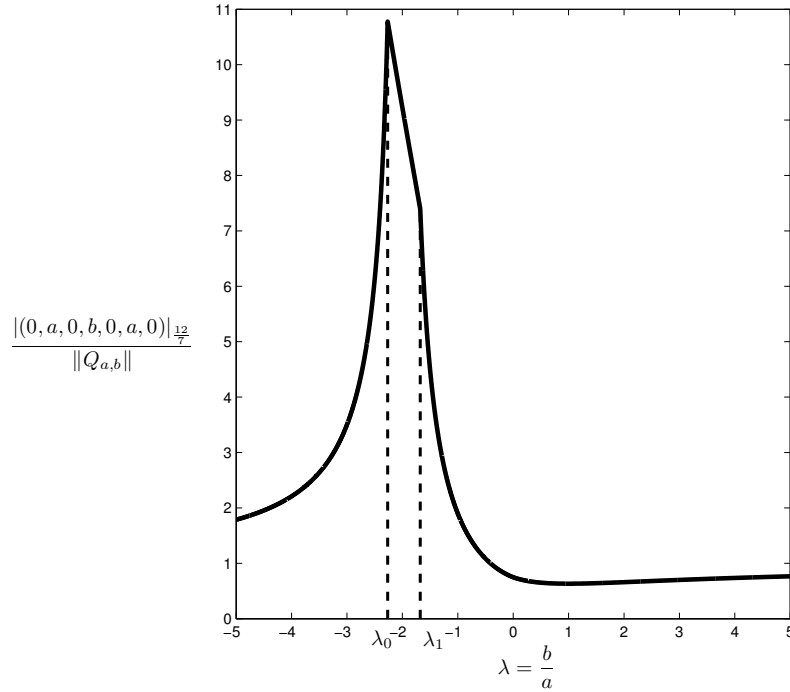


FIGURE 3. Graph of the quotient  $\frac{|(0,a,0,b,0,a,0)|_{\frac{12}{7}}}{\|Q_{a,b}\|}$  as a function of  $\lambda = \frac{b}{a}$ .

which suggests that (see Figure 4)

$$H_{\infty, \mathbb{R}}(2) \geq 1.58432.$$

**3.5. Numerical calculation of  $D_{\mathbb{R},7}(2)$ .** Let us define the polynomial

$$P_7(x, y) = -ax^7 + bx^6y + cx^5y^2 - dx^4y^3 - dx^3y^4 + cx^2y^5 + bxy^6 - ay^7,$$

with

$$a = 0.05126,$$

$$b = 0.22070,$$

$$c = 0.50537,$$

$$d = 0.71044.$$

It can be proved numerically (using Remark 3.1) that

$$\|P_7\| \approx 0.07138,$$

up to 5 decimal places. The authors have numerical evidence showing that

$$D_{\mathbb{R},7}(2) \approx \frac{|(-a, b, c, -d, -d, c, b, -a)|_{\frac{7}{4}}}{\|P_7\|} \approx 19.96308.$$

If  $\mathbf{a}_n$  is the vector of the coefficients of  $P_7^n$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.5) \quad D_{\mathbb{R},7n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{14n}{7n+1}}}{\|P_7\|^n}.$$

Moreover, if we put  $n = 86$  in (3.5) we obtain

$$D_{\mathbb{R},602}(2) \geq (1.61725)^{602},$$

suggesting that

$$H_{\mathbb{R},\infty}(2) \geq 1.61725.$$

**3.6. Numerical calculation of  $D_{\mathbb{R},8}(2)$ .** Let us define the polynomial

$$P_8(x, y) = -ax^7y + bx^5y^3 - bx^3y^5 + axy^7,$$

with

$$\begin{aligned} a &= 0.15258, \\ b &= 0.64697. \end{aligned}$$

It can be established numerically (see Remark 3.1) that

$$\|P_8\| \approx 0.02985,$$

up to 5 decimal places. The authors have numerical evidence showing that

$$D_{\mathbb{R},8}(2) \approx \frac{|(0, -a, 0, b, 0, -b, 0, a, 0)|_{\frac{16}{9}}}{\|P_8\|} \approx 33.36323.$$

If  $\mathbf{a}_n$  is the vector of the coefficients of  $P_8^n$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.6) \quad D_{\mathbb{R},8n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{16n}{8n+1}}}{\|P_8\|^n}.$$

Moreover, using (3.6) with  $n = 75$  we obtain

$$D_{\mathbb{R},600}(2) \geq (1.64042)^{600},$$

which suggests that

$$H_{\mathbb{R},\infty}(2) \geq 1.64042.$$

**3.7. Numerical calculation of  $D_{\mathbb{R},10}(2)$ .** In this case our numerical estimates show that there exists an extreme polynomial in the Bohnenblust-Hille polynomial inequality in  $\mathcal{P}^{(10)}\ell_\infty^2(\mathbb{R})$  of the form

$$P_{10}(x, y) = ax^9y + bx^7y^3 + x^5y^5 + bx^3y^7 + axy^9,$$

with

$$\begin{aligned} a &= 0.0938, \\ b &= -0.5938. \end{aligned}$$

It can be computed numerically (see Remark 3.1) that

$$\|P_{10}\| \approx 0.01530,$$

up to 5 decimal places. The authors have numerical evidence showing that

$$D_{\mathbb{R},10}(2) \approx \frac{|(0, a, 0, b, 0, 1, 0, b, 0, a, 0)|_{\frac{20}{11}}}{\|P_{10}\|} \approx 90.35556.$$

If  $\mathbf{a}_n$  is the vector of the coefficients of  $P_{10}^n$  for each  $n \in \mathbb{N}$ , then we know that

$$(3.7) \quad D_{\mathbb{R},10n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{20n}{10n+1}}}{\|P_{10}\|^n}.$$

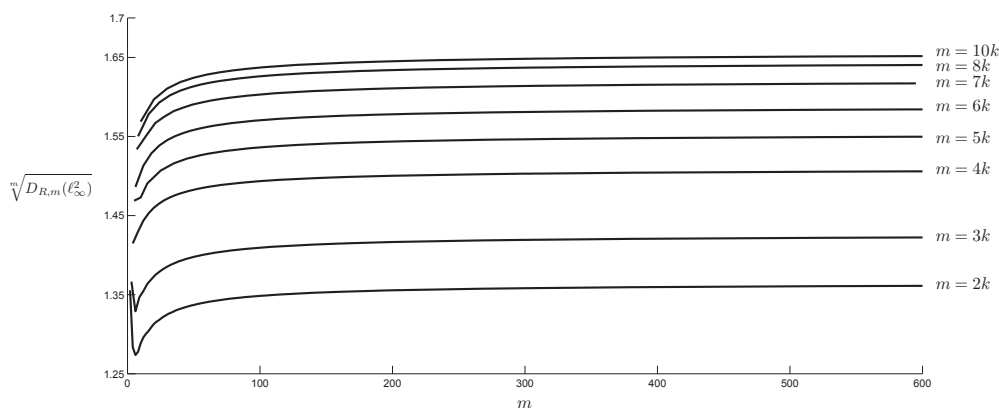


FIGURE 4. Graphs of the estimates on  $\sqrt[m]{D_{\mathbb{R},m}(2)}$  obtained by using (3.1) through (3.7).

If we set  $n = 60$  in (3.7) then we obtain

$$D_{\mathbb{R},600}(2) \geq (1.65171)^{600},$$

which suggests that

$$H_{\mathbb{R},\infty}(2) \geq 1.65171.$$

We have sketched in Figure 4 a summary of the numerical results obtained in this section.

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