LATTICE PATHS WITH GIVEN NUMBER OF TURNS AND SEMIMODULES OVER NUMERICAL SEMIGROUPS

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ABSTRACT. Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. In this article we consider several relations between the so-called Γ-semimodules and lattice paths from $(0, \alpha)$ to $(\beta, 0)$: we investigate isomorphism classes of Γ-semimodules as well as certain subsets of the set of gaps of Γ, and finally syzygies of Γ-semimodules. In particular we compute the number of Γ-semimodules which are isomorphic with their k-th syzygy for some k.

1. INTRODUCTION

In our paper [4] we considered Hilbert series of graded modules over the polynomial ring $R = \mathbb{F}[X, Y]$ with $\deg(X)$ and $\deg(Y)$ being coprime. The central result was an arithmetic criterion for such a series to be the Hilbert series of some finitely generated R–module of positive depth. This criterion is formulated in terms of the numerical semigroup generated by $\deg(X)$ and $\deg(Y)$. For reader's convenience we recall some basic vocabulary of this theory here.

Let Γ be a sub–semigroup of $\mathbb{N}$ such that the greatest common divisor of all its elements is equal to 1. Then the set $\mathbb{N} \setminus \Gamma$ has only finitely many elements, which are called the *gaps* of Γ. Such a semigroup is said to be *numerical*. The crucial notion in [4] was that of a *fundamental couple*: Let $\alpha, \beta > 0$ be coprime integers and let G denote the set of gaps of $\langle \alpha, \beta \rangle$. An (α, β)–*fundamental couple* $[I, J]$ consists of two integer sequences $I = (i_k)_{k=0}^m$ and $J = (j_k)_{k=0}^m$, such that

(0) $i_0 = 0$.

(1) $i_1, \ldots, i_m, j_1, \ldots, j_{m-1} \in G$ and $j_0, j_m \leq \alpha\beta$.

(2) $$\begin{array}{llll} i_k \equiv j_k & \bmod \alpha & \text{and} \quad i_k < j_k & \text{for } k = 0, \ldots, m; \\ j_k \equiv i_{k+1} & \bmod \beta & \text{and} \quad j_k > i_{k+1} & \text{for } k = 0, \ldots, m-1; \\ j_m \equiv i_0 & \bmod \beta & \text{and} \quad j_m \geq i_0. & \end{array}$$

(3) $|i_k - i_\ell| \in G$ for $1 \leq k < \ell \leq m$.

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One of the problems considered in this article will be the counting of sets of integers like those appearing in the first position of a fundamental couple. We coin a name for these sets:

Definition 1.1. Let Γ be a numerical semigroup. A set $\{x_0 = 0, x_1, \ldots, x_n\} \subseteq \mathbb{N}$ is called Γ-*lean* if $|x_i - x_j| \notin \Gamma$ for $0 \leq i < j \leq n$.

The next two sections deal with objects related to Γ-lean sets: we begin with isomorphism classes of Γ-semimodules. In section 3 we consider certain lattice paths; from this section on, Γ is restricted to be generated by two elements. In the last sections we turn our attention to the second position of a fundamental couple: we identify the sequences J appearing there with so-called syzygies of Γ-semimodules and investigate their relation with lattice paths. The process of taking syzygies can be iterated — our final result allows to compute the number of semimodules Δ whose k-th syzygy is isomorphic with Δ for some k.

2. Generators of Γ-semimodules

Let Γ be a numerical semigroup. A Γ-semimodule Δ is a non-empty subset of $\mathbb{N}$ such that $\Delta + \Gamma \subseteq \Delta$. A system of generators of Δ is a subset $\mathcal{E}$ of Δ with

$$\bigcup_{x \in \mathcal{E}} (x + \Gamma) = \Delta.$$

It is called minimal if no proper subset of $\mathcal{E}$ generates Δ. Note that, since $\Delta \setminus \Gamma$ is finite, every Γ-semimodule is finitely generated.

Lemma 2.1. *Every Γ-semimodule Δ has a unique minimal system of generators.*

Proof. Inductively we construct a sequence (x_i) of elements of Δ starting with $x_1 = \min \Delta$ such that any system of generators has to contain this sequence. If $x_1, \ldots, x_n$ are already constructed but do not generate Δ we set $x_{n+1} = \min \Delta \setminus \cup_{i=1}^{n} (x_i + \Gamma)$. After finitely many steps we arrive at a system of generators $x_1, \ldots, x_r$, and by construction it is clear that this system is minimal and that any system of generators must contain $x_1, \ldots, x_r$. □

Lemma 2.2. *Let $x_1, \ldots, x_r$ be the minimal system of generators of a Γ-semimodule. Then $|x_i - x_j|$ is a gap of Γ for all $i \neq j$. Conversely, any subset $\{x_1, \ldots, x_r\}$ of $\mathbb{N}$ with this property minimally generates a Γ-semimodule.*

Proof. We may assume $x_j > x_i$ for $j > i$. Then, by minimality, $x_j - x_i \notin \Gamma$ for all $j > i$. The second assertion is clear since $x_i \notin \cup_{j \neq i} (x_j + \Gamma)$. □

Two Γ-semimodules Δ, Δ' are called isomorphic if there is an integer n such that $x \mapsto x + n$ is a bijection from Δ to Δ'. For every Γ-semimodule Δ there is a unique semimodule $\Delta' \cong \Delta$ containing 0; such a Γ-semimodule is called normalized.

The semimodule $\Delta^\circ := \{x - \min \Delta \mid x \in \Delta\}$ is called the normalization of Δ. It is the unique semimodule isomorphic to Δ and containing 0.

Corollary 2.3. *The minimal system of generators of a normalized Γ-semimodule is Γ-lean, and conversely, every Γ-lean set of $\mathbb{N}$ minimally generates a normalized Γ-semimodule. Hence there is a bijection between the set of isomorphism classes of Γ-semimodules and the set of Γ-lean sets of $\mathbb{N}$.*

3. Lattice paths and $\langle\alpha, \beta\rangle$-lean sets

From now on we only consider numerical semigroups with two generators $\alpha < \beta$. In this case there is a connection between Γ-lean sets and certain lattice paths which allows to deduce a formula for the number of $\langle\alpha, \beta\rangle$-lean sets.

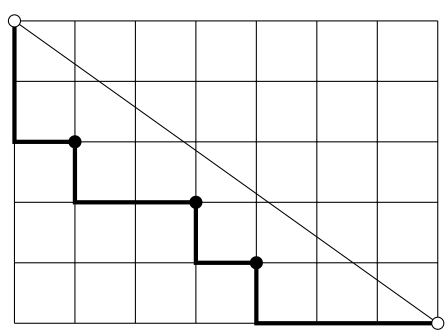
Lemma 3.1 ([7], Lemma 1, resp. [4], Corollary 3.5)**.** *(1) Let $e \in \mathbb{Z}$. Then $e \notin \langle\alpha, \beta\rangle$ if and only if there exist $k, \ell \in \mathbb{N}_{>0}$ such that $e = \alpha\beta - k\alpha - \ell\beta$.*
(2) Any integer $n > 0$ has a unique presentation $n = p\alpha\beta - a\alpha - b\beta$ with integers $p > 0, 0 \le a < \beta$ and $0 \le b < \alpha$.

This result yields a map $G \to \mathbb{N}^2$, $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$ which identifies a gap with a lattice point. Since $\alpha\beta - a\alpha - b\beta > 0$ the point lies inside the triangle with corners $(0, 0), (\beta, 0), (0, \alpha)$.

Lemma 3.2 ([4], Lemma 3.19)**.** *Let $i_1 = \alpha\beta - a_1\alpha - b_1\beta, i_2 = \alpha\beta - a_2\alpha - b_2\beta$ be gaps of $\langle\alpha, \beta\rangle$. Then the difference $|i_1 - i_2|$ is a gap if and only if $(a_2 - a_1)(b_2 - b_1) < 0$.*

Corollary 3.3. *Let $\mathcal{E} := \{0, x_1, \ldots, x_m\} \subseteq \mathbb{N}$ with gaps $x_i = \alpha\beta - a_i\alpha - b_i\beta$ of $\langle\alpha, \beta\rangle$, $i = 1, \ldots, m$ such that $a_1 < a_2 < \cdots < a_m$, then $\mathcal{E}$ is $\langle\alpha, \beta\rangle$-lean if and only if $b_1 > b_2 > \cdots > b_m$.*

Therefore an $\langle\alpha, \beta\rangle$-lean set yields a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the diagonal, where the points identified with the gaps mark the turns from x-direction to y-direction. In the sequel those turns will be called ES-turns for short.



Lattice path for the $\langle 5, 7\rangle$-lean set $\{0, 9, 6, 8\}$.

(Note that in the first component $I = (i_k)_{k=0}^m$ of a fundamental couple the numbering of elements is that of Corollary 3.3 reversed, see [4, Corollary 3.21 a)]. Hence in the path associated to I the ES-turns are numbered from right to left. One could avoid this inversion by considering paths from $(\beta, 0)$ to $(\alpha, 0)$ or by using a different orientation of the diagram, but we prefer our version for typographical reasons.)

Conversely, for every lattice path from $(0, \alpha)$ to $(\beta, 0)$ not crossing the diagonal the points of the ES-turns can be identified with the gaps in an $\langle\alpha, \beta\rangle$-lean set:

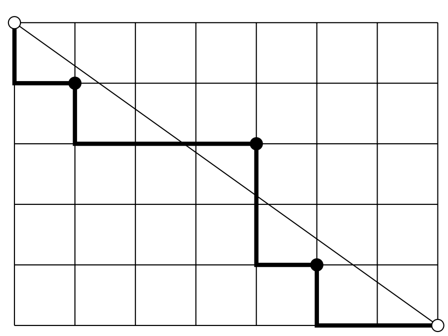
Lemma 3.4. *Let α, β be coprime positive integers. Then there is a bijection between the set of $\langle\alpha, \beta\rangle$-lean sets and the set of lattice paths from $(0, \alpha)$ to $(\beta, 0)$ not crossing the diagonal.*

Therefore counting of $\langle\alpha, \beta\rangle$-lean sets is equivalent to counting of such lattice paths. The latter was considered by Bizley in [2]. The main idea used there even allows to count the paths with a certain number of ES-turns.

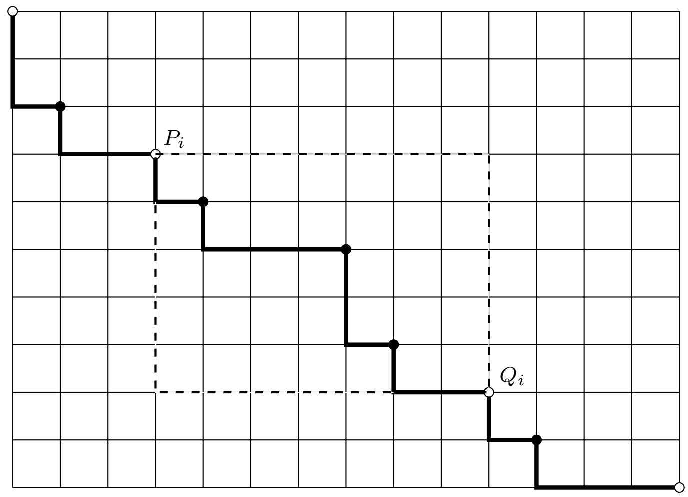
The number of all lattice paths with r ES-turns from $(0, \alpha)$ to $(\beta, 0)$ is easily computed: The r turning points have x-coordinates in the range $\{1, \ldots, \beta - 1\}$ and also y-coordinates in the range $\{1, \ldots, \alpha - 1\}$. Since the sequence of coordinates has to be increasing resp. decreasing there are $\binom{\beta-1}{r}\binom{\alpha-1}{r}$ lattice paths. We have to determine how many of these paths stay below the diagonal. To this end we use the concept of a cyclic permutation of a path.

A lattice path with r ES-turns can also be described by a $2 \times (r + 1)$-matrix where the i-th column contains the numbers of steps downwards and to the right the path takes between the $(i - 1)$-th and the i-th turning points (where the 0-th and $(r + 1)$-th points are to be understood as $(0, \alpha)$ resp. $(\beta, 0)$). For the path in the example above we get the matrix $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$.

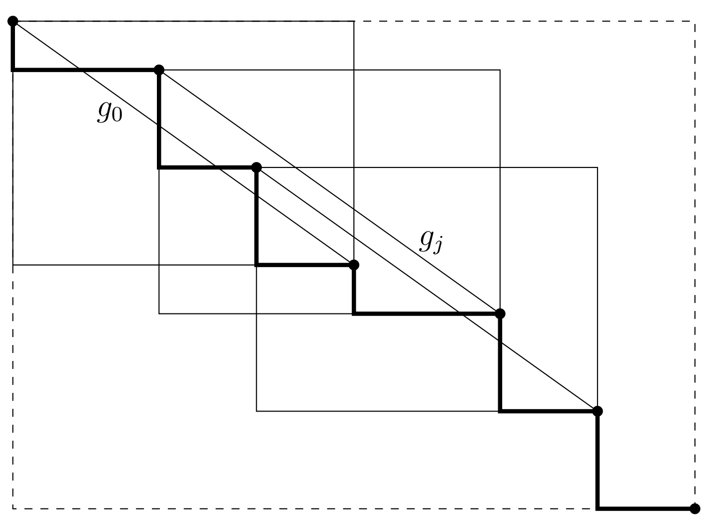
A *cyclic permutation* of the path is a path belonging to the matrix with cyclically permuted columns. The permuted matrix $\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$ yields the path



One can also imagine a cyclic permutation in a different way. We extend the path with turning points $P_i = (x_i, y_i)$ beyond $(\beta, 0)$ with points $Q_i = (x_i + \beta, y_i - \alpha)$, thus amending a second copy of the original path. The cyclic permutations are the paths from P_i to Q_i with turning points $P_{i+1}, \ldots, P_r, (\beta, 0), Q_1, \ldots, Q_{i-1}$:



A lattice path with r turning points admits $r+1$ cyclic permutations. As we will show now there is always exactly one permutation staying below the diagonal: Again we consider the doubled path as described above. Let g_0 denote the line through $P_0 := (0, \alpha)$ and $Q_0 := (\beta, 0)$ and g_i the line through P_i and Q_i for $i = 1, \ldots, r$. Note that all these lines are parallel and that, since α and β are coprime, there is no turning point between P_i and Q_i lying on g_i. Hence there is one line g_j, $j \in \{0, \ldots, r\}$ namely the one with the greatest distance from the origin, such that the path stays below this line.



Since the line g_i yields the diagonal of the i-th cyclic permutation of the original path, indeed exactly one of the permuted paths stays below the diagonal; hence we have proven:

Proposition 3.5. *Let α and β be two coprime positive integers.*

1. *For every lattice path from $(0,\alpha)$ to $(\beta,0)$ there is exactly one cyclic permutation staying below the diagonal.*
2. *The number of $\langle\alpha,\beta\rangle$-lean sets with r gaps equals the number of lattice paths with r ES-turns from $(0,\alpha)$ to $(\beta,0)$ staying below the diagonal, and this number is given by*

$$\frac{1}{r+1}\binom{\alpha-1}{r}\binom{\beta-1}{r}.$$

In combination with Corollary 2.3 and Lemma 3.4 this result yields the following theorem:

Theorem 3.6. *Let $\alpha,\beta,r\in\mathbb{N}$ with $\gcd(\alpha,\beta)=1$. Then the following numbers*

(1) *The number of isomorphism classes of $\langle\alpha,\beta\rangle$-semimodules minimally generated by $r+1$ elements.*
(2) *The number of $\langle\alpha,\beta\rangle$-lean sets with r gaps.*
(3) *The number of lattice paths with r ES-turns from $(0,\alpha)$ to $(\beta,0)$ staying below the diagonal.*

equal

$$L_{\alpha,\beta}(r) := \frac{1}{r+1}\binom{\alpha-1}{r}\binom{\beta-1}{r}.$$

Using standard techniques one can also deduce a formula for $\sum_{r\geq 0} L_{\alpha,\beta}(r)$, recovering results of Bizley, resp. Beauville, Fantechi–Göttsche–van Straten, and Piontkowski:

$$\begin{aligned}
\sum_{r\geq 0} L_{\alpha,\beta}(r) &= \sum_{r\geq 0}\frac{1}{r+1}\binom{\alpha-1}{r}\binom{\beta-1}{r} \\
&= \sum_{r\geq 0}\frac{1}{r+1}\binom{\alpha-1}{r}\frac{r+1}{\beta}\binom{\beta}{r+1} \\
&= \dots
\end{aligned}$$

$$
\begin{aligned}
\dots &= \frac{1}{\beta}\sum_{r\geq 0}\binom{\alpha-1}{r}\binom{\beta}{r+1}\\
&= \frac{1}{\beta}\sum_{r\geq 1}\binom{\alpha-1}{r-1}\binom{\beta}{r} = \frac{1}{\beta}\sum_{r\geq 0}\binom{\alpha-1}{r-1}\binom{\beta}{r}\\
&= \frac{1}{\beta}\sum_{r\geq 0}\binom{\alpha-1}{\alpha-r}\binom{\beta}{r}.
\end{aligned}
$$

The Vandermonde convolution yields

$$
\begin{aligned}
\frac{1}{\beta}\sum_{r\geq 0}\binom{\alpha-1}{\alpha-r}\binom{\beta}{r} &= \frac{1}{\beta}\binom{\alpha+\beta-1}{\alpha} = \frac{1}{\beta}\frac{(\alpha+\beta-1)!}{\alpha!\cdot(\beta-1)!} = \frac{1}{\alpha+\beta}\frac{(\alpha+\beta)!}{\alpha!\cdot\beta!}\\
&= \frac{1}{\alpha+\beta}\binom{\alpha+\beta}{\alpha},
\end{aligned}
$$

which implies the following result.

Theorem 3.7. *Let $\alpha, \beta \in \mathbb{N}$ be coprime. Then the following numbers*

(1) *The number of isomorphism classes of $\langle\alpha,\beta\rangle$-semimodules (cf.* [1], [3], [5]*).*
(2) *The number of $\langle\alpha,\beta\rangle$-lean sets.*
(3) *The number of lattice paths from $(0,\alpha)$ to $(\beta,0)$ staying below the diagonal (cf.* [2]*).*

equal

$$
L_{\alpha,\beta} := \sum_{r\geq 0} L_{\alpha,\beta}(r) = \frac{1}{\alpha+\beta}\binom{\alpha+\beta}{\alpha}.
$$

Remark 3.8. In particular $L_{\alpha,\beta}$ gives the number of (α,β)-fundamental couples and hence the number of inequalities appearing in condition $(\star)$ of Theorem 3.13 in [4], also see [4, Remark 3.12].

Remark 3.9. In the special case $\beta = \alpha + 1$ the numbers $L_{\alpha,\beta}(r)$ and $L_{\alpha,\beta}$ coincide with certain combinatorial numbers. We get

$$
L_{\alpha,\alpha+1}(r) = \frac{1}{\alpha}\binom{\alpha}{r}\binom{\alpha}{r+1} = N(\alpha, r+1),
$$

the so-called *Narayana number* (cf. [6]). Moreover, $L_{\alpha,\alpha+1}$ agrees with the *Catalan number* C_α since

$$
L_{\alpha,\alpha+1} = \sum_{r\geq 0} L_{\alpha,\alpha+1}(r) = \sum_{r\geq 0} N(\alpha, r+1) = C_\alpha,
$$

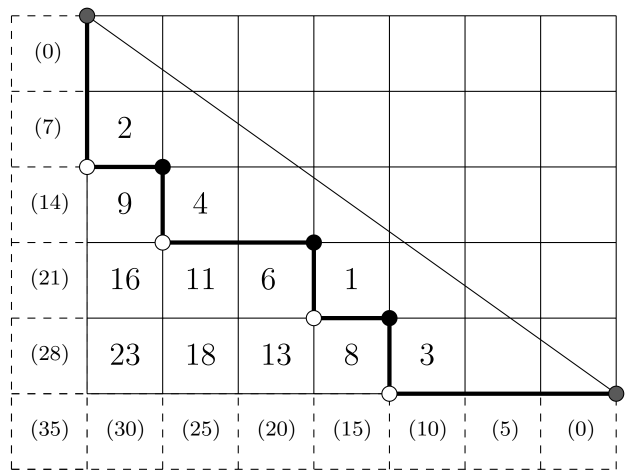
see [8, Exercise 6.36].

4. Syzygies of $\langle\alpha,\beta\rangle$-semimodules and lattice paths

We consider now the sequences appearing in the second position of a fundamental couple. Let $[I, J]$ be a fundamental couple with sequences $I = [i_0 = 0, \ldots, i_n]$ and $J = [j_0, \ldots, j_n]$. By definition, the elements $j_1, \ldots, j_{n-1}$ are gaps of $\langle\alpha,\beta\rangle$ such that

$$j_k \equiv i_k \mod \alpha \quad \text{and } j_k \equiv i_{k+1} \mod \beta.$$

An inspection of the lattice path belonging to I shows that these gaps $j_1, \ldots, j_{n-1}$ correspond to the inner SE-turning points of the path. By extension of the labeling beyond the axis we can even identify j_0 and j_n with the remaining SE-turns. For illustration see again the example of the previous section:



$I = [0, 8, 6, 9]$ and $J = [15, 13, 16, 14]$.

Next we explain the meaning of J in terms of $\langle\alpha,\beta\rangle$-semimodules: Every $\langle\alpha,\beta\rangle$-semimodule Δ yields another $\langle\alpha,\beta\rangle$-semimodule $\mathrm{Syz}(\Delta)$.

Definition 4.1. Let I be an $\langle\alpha,\beta\rangle$-lean set, and let Δ be the $\langle\alpha,\beta\rangle$-semimodule generated by I. The *syzygy* of Δ is the $\langle\alpha,\beta\rangle$-semimodule

$$\mathrm{Syz}(\Delta) := \bigcup_{\substack{i,i' \in I \\ i \neq i'}} \left(\left(i + \langle\alpha,\beta\rangle\right) \cap \left(i' + \langle\alpha,\beta\rangle\right) \right).$$

The semimodule $\mathrm{Syz}(\Delta)$ consists of those elements in Δ which admit more than one presentation of the form $i + x$ with $i \in I, x \in \langle\alpha,\beta\rangle$. The name *syzygy* may be justified by considering an analogue of Δ in the setting of commutative algebra:

Let $R = \mathbb{F}[t^\alpha, t^\beta]$, then Δ can be identified with the R-submodule M of $\mathbb{F}[t]$ generated by $\{t^i \mid i \in I\}$. Let

$$\begin{array}{ccc} \bigoplus_{i\in I} R(-i) & \xrightarrow{\varphi} & M \\ (f_i) & \mapsto & \sum_i f_i t^i \end{array}$$

be the first step in a graded minimal free resolution of M. By [5, Lemma 2.3] the kernel of φ is generated by (homogeneous) elements $v_{i,i'}$ of the form

$$(0, \dots, t^{\gamma_i}, 0, \dots, 0, -t^{\gamma_{i'}}, 0, \dots, 0),$$

with the non-zero entries in positions $i, i' \in I$. Since

$$\deg(v_{i,i'}) = \gamma_i + i = \gamma_{i'} + i' \in (i + \langle\alpha, \beta\rangle) \cap (i' + \langle\alpha, \beta\rangle)$$

the module $\ker\varphi$ ist non-zero exactly in the degrees contained in $\mathrm{Syz}(\Delta)$.

The connection between fundamental couples and syzygies is described in the following theorem:

Theorem 4.2. *Let $[I, J]$ be an $\langle\alpha, \beta\rangle$-fundamental couple and let Δ be the $\langle\alpha, \beta\rangle$-semimodule generated by the elements of I. Then*

$$\mathrm{Syz}(\Delta) = \bigcup_{0\le k<m\le n} \Big((i_k + \langle\alpha, \beta\rangle) \cap (i_m + \langle\alpha, \beta\rangle) \Big) = \bigcup_{k=0}^{n} (j_k + \langle\alpha, \beta\rangle).$$

Proof. By definition of a fundamental couple we have $j_k = i_k + r\alpha = i_{k+1} + s\beta$ with some $r, s \in \mathbb{N}$, hence the inclusion $\supseteq$ is clear. In order to show the other inclusion, we consider $i_k + \langle\alpha, \beta\rangle \cap i_m + \langle\alpha, \beta\rangle$. For every $\gamma_m \in \langle\alpha, \beta\rangle$ we have $i_m + \gamma_m \in i_k + \langle\alpha, \beta\rangle$ if and only if $i_m - i_k + \gamma_m \in \langle\alpha, \beta\rangle$. As mentioned in the previous section, i_k and i_m can be written in the form

$$i_k = \alpha\beta - a_k\alpha - b_k\beta,\ i_m = \alpha\beta - a_m\alpha - b_m\beta;$$

by Lemma 3.2 we may assume $a_k > a_m$ and $b_k < b_m$. Since

$$i_m - i_k = (a_k - a_m)\alpha + (b_k - b_m)\alpha = \alpha\beta - (\beta - a_k + a_m)\alpha - (b_m - b_k)\beta$$

the characterization of $\mathbb{Z} \setminus \langle\alpha, \beta\rangle$ in Lemma 3.1 implies

$$\{\gamma \in \langle\alpha, \beta\rangle \mid \gamma + i_m - i_k \in \langle\alpha, \beta\rangle\} = ((\beta - a_k + a_m)\alpha + \langle\alpha, \beta\rangle) \cup ((b_m - b_k)\beta + \langle\alpha, \beta\rangle).$$

This means

$$\begin{aligned} & (i_m + \langle\alpha, \beta\rangle) \cap (i_k + \langle\alpha, \beta\rangle) \\ = \ & (i_m + (\beta - a_k + a_m)\alpha + \langle\alpha, \beta\rangle) \cup (i_m + (b_m - b_k)\beta + \langle\alpha, \beta\rangle). \end{aligned}$$

Moreover,

$$\begin{aligned} i_m + (\beta - a_k + a_m)\alpha &= \alpha\beta - a_k\alpha - b_m\beta + \alpha\beta \\ &= (\beta - a_1)\alpha + (a_1 - a_k)\alpha + (\alpha - b_m)\beta \\ &= j_0 + (a_1 - a_k)\alpha + (\alpha - b_m)\beta \in j_0 + \langle \alpha, \beta \rangle, \end{aligned}$$

and on the other hand

$$\begin{aligned} i_m + (b_m - b_k)\beta &= \alpha\beta - a_m\alpha - b_m\beta + (b_m - b_k)\beta \\ &= \alpha\beta - a_m\alpha - b_k\beta \\ &= \alpha\beta - a_{k+1}\alpha - b_k\beta + (a_{k+1} - a_m)\alpha \\ &= j_k + (a_{k+1} - a_m)\alpha \in j_k + \langle \alpha, \beta \rangle, \end{aligned}$$

hence $i_m + \langle \alpha, \beta \rangle \cap i_k + \langle \alpha, \beta \rangle \subseteq j_0 + \langle \alpha, \beta \rangle \cup j_k + \langle \alpha, \beta \rangle$. □

Corollary 4.3. *Let $[I, J]$ be an $\langle \alpha, \beta \rangle$-fundamental couple and let Δ be the $\langle \alpha, \beta \rangle$-semimodule generated by the elements of I. We have*

$$\operatorname{Syz}(\Delta) = \bigcup_{k=0}^{n-1} \Big((i_k + \langle \alpha, \beta \rangle) \cap (i_{k+1} + \langle \alpha, \beta \rangle) \Big) \cup \big((i_0 + \langle \alpha, \beta \rangle) \cap (i_n + \langle \alpha, \beta \rangle) \big).$$

Proof. This follows immediately from $j_k \in i_k + \langle \alpha, \beta \rangle \cap i_{k+1} + \langle \alpha, \beta \rangle$ for $k = 1, \ldots, n-1$ resp. $j_n \in i_0 + \langle \alpha, \beta \rangle \cap i_n + \langle \alpha, \beta \rangle$ and the previous theorem. □

5. Orbits

Let $[I, J]$ a fundamental couple and let

$$\begin{pmatrix} y_0 & y_1 & \ldots & y_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix}$$

be the matrix describing the path for the semimodule Δ generated by I. We consider a second lattice path from $(0, b_n)$ (the point associated to j_n) to $(\beta, b_n - \alpha)$ with ES-turns in the SE-turning points of the first path (those points representing $j_{n-1}, \ldots, j_0$). The matrix for this path is given by

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_n & y_0 \\ x_0 & x_1 & \ldots & x_{n-1} & x_n \end{pmatrix}.$$

It is easily seen that—up to a cyclic permutation of columns—this matrix also describes the path belonging to the normalization Δ° of Δ: In terms of lattice paths normalizing Δ means translation of the path such that the ES-turn belonging to $\min J$ is moved to $(0, \alpha)$ and the part of the path left of this point will be appended behind the former end point.

The procedure of building a syzygy can be iterated; we set

$$\operatorname{Syz}^{(k)}(\Delta) := \operatorname{Syz}(\operatorname{Syz}^{(k-1)}(\Delta)), \; k \geq 2.$$

From the matrix description of the path for $\mathrm{Syz}(\Delta)$ it is clear that $\mathrm{Syz}^{(n+1)}(\Delta) \cong \Delta$. Now we consider under which conditions even lower syzygies of Δ are isomorphic with Δ.

Definition 5.1. A sequence $\Delta_1, \ldots, \Delta_\ell$ of distinct $\langle\alpha, \beta\rangle$-semimodules such that $\Delta_k = \mathrm{Syz}(\Delta_{k-1})^\circ$ for $k = 2, \ldots, \ell$, and $\Delta_1 = \mathrm{Syz}(\Delta_\ell)^\circ$ is called an orbit of length ℓ or, for short, an ℓ-orbit (of $\langle\alpha, \beta\rangle$-semimodules). The 1-orbits, i. e. $\langle\alpha, \beta\rangle$-semimodules Δ with $\Delta \cong \mathrm{Syz}(\Delta)$, are called $\langle\alpha, \beta\rangle$-fixed points.

We want to count the number of ℓ-orbits of $\langle\alpha, \beta\rangle$-semimodules with n generators. To this end we investigate the structure of a semimodule Δ with $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$; we may restrict our attention to the case of ℓ dividing n, since the length of an orbit can be viewed as the order of an element in the cyclic group of order n. Let

$$\begin{pmatrix} y_0 & y_1 & \ldots & y_{n-1} \\ x_0 & x_1 & \ldots & x_{n-1} \end{pmatrix}$$

be the matrix for the path belonging to Δ. Then, as mentioned above, this matrix and the matrix

$$\begin{pmatrix} y_\ell & y_{\ell+1} & \ldots & y_{\ell-2} & y_{\ell-1} \\ x_0 & x_1 & \ldots & x_{n-1} & x_n \end{pmatrix},$$

have to be equal up to cyclic permutation of columns. This means that there exists a $k \in \{1, \ldots, n-1\}$ such that

$$\begin{pmatrix} y_{k+\ell} & y_{k+\ell+1} & \ldots \\ x_k & x_{k+1} & \ldots \end{pmatrix} = \begin{pmatrix} y_0 & y_1 & \ldots \\ x_0 & x_1 & \ldots \end{pmatrix},$$

we may assume that k is minimal with this property. Since

$$\begin{pmatrix} y_{k+2\ell} & y_{k+2\ell+1} & \ldots \\ x_k & x_{k+1} & \ldots \end{pmatrix} = \begin{pmatrix} y_\ell & y_{\ell+1} & \ldots \\ x_0 & x_1 & \ldots \end{pmatrix}$$

we have

$$\begin{pmatrix} y_{2k+2\ell} & y_{2k+2\ell+1} & \ldots \\ x_{2k} & x_{2k+1} & \ldots \end{pmatrix} = \begin{pmatrix} y_{k+\ell} & y_{k+\ell+1} & \ldots \\ x_k & x_{k+1} & \ldots \end{pmatrix} = \begin{pmatrix} y_0 & y_1 & \ldots \\ x_0 & x_1 & \ldots \end{pmatrix}.$$

By induction we get $x_j = x_{rk+j}$ for $j \in \{0, \ldots k-1\}$, $r \geq 0$. Hence the bottom row of the matrix is of the form

$$[x_0 \ \ldots \ x_{k-1}] \ [x_0 \ \ldots \ x_{k-1}] \ \ldots \ [x_0 \ \ldots \ x_{k-1}].$$

On the other hand we can find a minimal $m \in \{2, \ldots, n\}$ such that

$$\begin{pmatrix} y_{m-\ell} & y_{m-\ell+1} & \ldots \\ x_m & x_{m+1} & \ldots \end{pmatrix} = \begin{pmatrix} y_0 & y_1 & \ldots \\ x_0 & x_1 & \ldots \end{pmatrix},$$

and with the same reasoning as above, the top row of the matrix is of the form

$$[y_0 \ \ldots \ y_{m-1}] \ [y_0 \ \ldots \ y_{m-1}] \ \ldots \ [y_0 \ \ldots \ y_{m-1}].$$

Therefore the matrix for Δ looks like

$$\begin{pmatrix} [y_0 \ \cdots\cdots \ y_{m-1}] & \cdots & [y_0 \ \cdots\cdots \ y_{m-1}] \\ [x_0 \ \ldots \ x_{k-1}] & \cdots & [x_0 \ \ldots \ x_{k-1}] \end{pmatrix}$$

with m' blocks $[y_0 \ \ldots \ y_{m-1}]$ and k' blocks $[x_0 \ \ldots \ x_{k-1}]$. Since

$$m' \cdot \sum_{j=0}^{m-1} y_j = \alpha \quad \text{and} \quad k' \cdot \sum_{j=0}^{k-1} x_j = \beta, \tag{5.1}$$

the numbers m' and k' divide α resp. β, so in particular they are coprime. By assumption, ℓ is the least positive integer p such that the matrices

$$\begin{pmatrix} y_0 & y_1 & \cdots \\ x_0 & x_1 & \cdots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_p & y_{p+1} & \cdots \\ x_0 & x_1 & \cdots \end{pmatrix}$$

contain the same columns. This is the case if and only if there are $r, s \in \mathbb{N}$ with $p + sm = rk$. This implies that p has to be contained in the ideal generated by k and m. Hence, by minimality of ℓ, we get $\ell = \gcd(k, m)$, and so we may write $k = \tilde{k}\ell$ and $m = \tilde{m}\ell$. From $kk' = n = mm'$ we get $\tilde{k}k' = \tilde{m}m' = \frac{n}{\ell}$; by $\gcd(\tilde{k}, \tilde{m}) = 1$ this implies $\tilde{k} = m'$ and $\tilde{m} = k'$, and by $k' \mid \alpha$ and $m' \mid \beta$ moreover $k = \gcd(\beta, \frac{n}{\ell})$ and $m = \gcd(\alpha, \frac{n}{\ell})$, in particular $\frac{n}{\ell} \mid \alpha\beta$. By now we have shown the following proposition:

Proposition 5.2. *Let Δ be an $\langle\alpha, \beta\rangle$-semimodule with n generators. If Δ is an element of an ℓ-orbit, then $\ell | n$ and $\frac{n}{\ell} | \alpha\beta$. The corresponding matrix is of the form*

$$\begin{pmatrix} [y_0 \ \ldots \ y_{m-1}] & \cdots & [y_0 \ \ldots \ y_{m-1}] \\ [x_0 \ \cdots\cdots \ x_{k-1}] & \cdots & [x_0 \ \cdots\cdots \ x_{k-1}] \end{pmatrix}, \tag{5.2}$$

where $k = \ell \cdot \gcd(\alpha, \frac{n}{\ell})$ and $m = \ell \cdot \gcd(\beta, \frac{n}{\ell})$.

In fact the structure described in the previous proposition is shared by all $\langle\alpha, \beta\rangle$-semimodules Δ with n generators and $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$—not only by elements of ℓ-orbits:

Lemma 5.3. *Let Δ be an $\langle\alpha, \beta\rangle$-semimodule with n generators, let $\ell \in \mathbb{N}$ be a divisor of n. If $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$ then the matrix for Δ can be written in the form mentioned in Proposition 5.2.*

Proof. Let $d = \min\{t \in \mathbb{N} \mid \mathrm{Syz}^{(t)}(\Delta) \cong \Delta\}$. Then $d \mid \ell$, and we only have to consider the case $d < \ell$. By Proposition 5.2 the matrix for Δ can be written in the form

$$\begin{pmatrix} [y_0 \ \ldots \ y_{m_d-1}] & \cdots & [y_0 \ \ldots \ y_{m_d-1}] \\ [x_0 \ \cdots\cdots \ x_{k_d-1}] & \cdots & [x_0 \ \cdots\cdots \ x_{k_d-1}] \end{pmatrix}$$

with $k'_d = \gcd(\beta, \frac{n}{d})$ blocks $[x_0 \ \ldots \ x_{k_d-1}]$ and $m'_d = \gcd(\alpha, \frac{n}{d})$ blocks $[y_0 \ \ldots \ y_{m_d-1}]$.

We want to obtain a matrix with $k'_\ell = \gcd(\beta, \frac{n}{\ell})$ blocks $[x_0 \ \ldots \ x_{k_\ell - 1}]$ and $m'_\ell = \gcd(\alpha, \frac{n}{\ell})$ blocks $[y_0 \ \ldots \ y_{m_\ell - 1}]$. Since $k'_\ell \mid k'_d$ and $m'_\ell \mid m'_d$, this is easily done by concatenating $\frac{k'_d}{k'_\ell}$ x-blocks resp. $\frac{m'_d}{m'_\ell}$ y-blocks. □

Corollary 5.4. *Let $\ell \in \mathbb{N}$ be a divisor of $n < \alpha$. Then there exists an ℓ-orbit of $\langle \alpha, \beta \rangle$-semimodules with n generators.*

Proof. Let k, k', m, m' be as in the proof of Prop. 5.2. Then $\sum y_i = \frac{\alpha}{m'} = m\frac{\alpha}{n} > m$ resp. $\sum x_i > k$. Hence $r := \frac{\alpha}{m'} - m + 1 > 1$ and $s := \frac{\beta}{k'} - k + 1 > 1$, so we may choose the matrix (5.2) to be built of blocks $[y_0 \ldots] = [1 \ 1 \ldots 1 \ r]$, $[x_0 \ldots] = [1 \ 1 \ldots 1 \ s]$. Since these blocks cannot be split into smaller ones, the corresponding semimodule cannot be part of a d-orbit with $d < \ell$. □

Lemma 5.3 allows to count $\langle \alpha, \beta \rangle$-semimodules Δ with $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$ and n generators. Using the same notation as in the deduction of Proposition 5.2, we find the following: By (5.1) the steps in the y-block sum up to $\frac{\alpha}{m'}$, therefore the partial sums $\sum_{j=0}^{r} y_j$ for $r = 0, \ldots, m-2$ (these are the y-coordinates of the first $m-1$ ES-turns in the corresponding path) have to be chosen in the range $1, \ldots, \frac{\alpha}{m'} - 1$. Hence there are $\binom{\frac{\alpha}{m'} - 1}{m-1}$ different y-blocks. Similarly there are $\binom{\frac{\beta}{k'} - 1}{k-1}$ different x-blocks. Any combination of an x- and a y-block yields a matrix of the form (5.2), and so there are $\binom{\frac{\alpha}{m'} - 1}{m-1}\binom{\frac{\beta}{k'} - 1}{k-1}$ of them. But, as in the counting of lattice paths in section 3, only one of the n cyclic permutations of the matrix is admissible, hence we have to divide by n:

Theorem 5.5. *There are*

$$\frac{1}{n}\binom{\frac{\alpha}{\gcd(\frac{n}{\ell},\alpha)} - 1}{\ell \cdot \gcd(\frac{n}{\ell}, \beta) - 1}\binom{\frac{\beta}{\gcd(\frac{n}{\ell},\beta)} - 1}{\ell \cdot \gcd(\frac{n}{\ell}, \alpha) - 1} \tag{5.3}$$

$\langle \alpha, \beta \rangle$-semimodules Δ with $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$ generated by n elements.

In particular we get a formula for the number of $\langle \alpha, \beta \rangle$-fixed points:

Corollary 5.6. *For any integer $n \leq \alpha$ with $n \mid \alpha\beta$ there are*

$$\frac{1}{n}\binom{\frac{\alpha}{\gcd(n,\alpha)} - 1}{\gcd(n, \beta) - 1}\binom{\frac{\beta}{\gcd(n,\beta)} - 1}{\gcd(n, \alpha) - 1} \tag{5.4}$$

$\langle \alpha, \beta \rangle$-fixed points with n generators.

Remark 5.7.

(1) In the case of $n = \ell$ the number provided by Theorem 5.5 agrees with that of all $\langle \alpha, \beta \rangle$-semimodules with n generators (Thm. 3.6), in accordance to the fact that $\mathrm{Syz}^{(n)}(\Delta) \cong \Delta$ for all these semimodules.

(2) For $n = \alpha$ the formula (5.4) yields $\frac{1}{\alpha}\binom{\beta-1}{\alpha-1}$, which equals $L_{\alpha,\beta}(\alpha-1)$. Hence *all* $\langle\alpha,\beta\rangle$-semimodules with maximal number of generators are fixed points—in this case there are only entries "1" in the top row of the matrix. Note that in the case of α and β being prime numbers there are no other fixed points.

One has to keep in mind that formula (5.3) does not only count the elements with n generators in the ℓ-orbits, but *all* semimodules with n generators such that $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$, including those with $\mathrm{Syz}^{(d)}(\Delta) \cong \Delta$ and $d < \ell$, in particular the fixed points. However, it is possible to compute the number of ℓ-orbits using the inclusion-exclusion principle, see the next and closing example.

Example 5.8. Let $\langle\alpha,\beta\rangle = \langle 15, 16\rangle$. We want to compute how many orbits consisting of semimodules with 12 generators exist. Since 12 divides $15 \cdot 16$ there are ℓ-orbits for each divisor ℓ of 12. Denote the set of elements of those orbits by Orb_ℓ and the set of those semimodules Δ with n generators and $\mathrm{Syz}^{(\ell)}(\Delta) \cong \Delta$ by A_ℓ. By Theorem 5.5 we get

$$|A_1| = 1,\ |A_2| = 7,\ |A_3| = 91,\ |A_4| = 455,\ |A_6| = 637,\ |A_{12}| = 41405.$$

We have $\mathrm{Orb}_1 = A_1$, and one easily checks that

$$\begin{aligned}
\mathrm{Orb}_2 &= A_2 \setminus A_1\\
\mathrm{Orb}_3 &= A_3 \setminus A_1\\
\mathrm{Orb}_4 &= A_4 \setminus A_2\\
\mathrm{Orb}_6 &= A_6 \setminus (A_2 \cup A_3)\\
\mathrm{Orb}_{12} &= A_{12} \setminus \cup_{i=1,2,3,4,6}\mathrm{Orb}_i.
\end{aligned}$$

Since $A_2 \cap A_3 = A_1$ we get

$$\begin{aligned}
|\mathrm{Orb}_1| &= 1\\
|\mathrm{Orb}_2| &= |A_2| - |A_1| = 6\\
|\mathrm{Orb}_3| &= |A_3| - |A_1| = 90\\
|\mathrm{Orb}_4| &= |A_4| - |A_2| = 448\\
|\mathrm{Orb}_6| &= |A_6 \setminus (A_2 \cup A_3)| = |A_6| - |A_2| - |A_3| + |A_1| = 540\\
|\mathrm{Orb}_{12}| &= 41405 - 540 - 448 - 90 - 6 - 1 = 40320,
\end{aligned}$$

and thus the following numbers of ℓ-orbits:

ℓ	1	2	3	4	6	12
$\frac{1}{\ell}\lvert\mathrm{Orb}_\ell\rvert$	1	3	30	112	90	3360

Concluding the discussion of this example we determine the single fixed point. By Proposition 5.2 the rows of the corresponding matrix have to consist of three blocks $[y_0, y_1, y_2, y_3]$ resp. four blocks $[x_0, x_1, x_2]$. Since the entries in those blocks sum up to $\frac{15}{3} = 5$ resp. $\frac{16}{4} = 4$ both blocks have to contain entries "1" and a single entry "2" i.e. up to cyclic permutation the matrix of the fixed point looks like

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}.$$

By considering the corresponding lattice path one can deduce that the admissible permutation of the path is described by

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix},$$

and the minimal set of generators of the $\langle 15, 16\rangle$-semimodule Δ belonging to this path is given by $I = \{0, 14, 13, 12, 10, 9, 8, 22, 5, 4, 18, 17\}$; the first syzygy of Δ is generated by $J = \{30, 29, 28, 42, 25, 24, 38, 37, 20, 34, 33, 32\}$.

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