Application of polynomial algebras to non-linear equation solvers

J. Canela[♭] and D. Pérez-Palau[♮1]

(♭) Insitut Universitari de Matemàtiques i Aplicacions de Castelló and Departament de Matemàtiques,
Universitat Jaume I,
Av. Vicent Sos Baynat, s/n 12071 Castelló de la Plana, Spain.
(♮) Escuela Superior de Ingeniería y Tecnología,
Universidad Internacional de la Rioja (UNIR),
Av. La Paz 137, 26006 Logroño, Spain.

1 Introduction

Polynomial algebras emerged during the decade of the 1990 [3] as a tool to propagate efficiently sets of points around an initial condition x_0. They were first used in the study of particle accelerators.

The main idea of a polynomial algebra approach is to approximate a function $g(x)$ by a polynomial $P(\xi)$ in such a way that $P(\xi) \approx g(x_0 + \xi)$. This technique is usually applied to functions like Poincaré maps, flow maps, etc. The polynomial approximation of functions is equivalent to the floating-point approximation of real values. As it is the case of the floating-point, where the real numbers are truncated up to a given number of digits, the polynomials are truncated up to a given order N. Therefore, Jets (or truncated polynomials) are used to approximate functions. The larger is the order N the better is the approximation of the function.

The use of polynomial arithmetics has been widely extended for the last decades. Some examples of its implementation can be found in [8, 9]. Using such arithmetics, given two truncated series S_a and S_b one can compute the jet of all the classic operations, like products $S_a \cdot S_b$, additions $S_a + S_b$, or divisions S_a/S_b (assuming that the independent term of S_b does not banish), as well as exponentials, logarithms or trigonometric operations, among other. For instance, if $S_a = \sum_{i=0}^{n} a_i\xi^i$ and $S_b = \sum_{i=0} b_i\xi^i$. Then, the coefficients of $S_c = S_a \cdot S_b = \sum_{i=0} c_i\xi^i$ and $S_d = S_a/S_b = \sum_{i=0} d_i\xi^i$ are given by:

$$c_i = \sum_{j=0}^{i} a_{i-j} b_j, \qquad c_n = \frac{a_n - \sum_{j=1}^{n} c_{n-j} b_j}{b_0}. \tag{1}$$

These techniques have been used in several applications. The main one is in the propagation of ordinary differential equations. In [3] the bases of the technique were introduced. Later on, [1,2,10] used it to study and propagate trajectories and its uncertainties in the evolution of spacecrafts and asteroids. In [4, 5] the same ideas were applied in the modification of a Kalman filter and in the orbit determination with uncertainties. From a theoretical point of view, [6] used the jet transport technique to compute periodic orbits of delayed differential equations and [7] computed high order expansions of invariant manifolds around tori of high dimension.

[1] daniel.perez@unir.net

As can be observed in the bibliography, the Jet Transport is widely used in the context of the propagation of differential equations. However, there are other fields where it can be used. One of those applications is in the solution of non-linear equations with parameters.

Let us consider a non-linear equation $f(x) = 0$. Usually, such equation can be solved numerically by means of iterative methods such as Newton's method:

$$x_{k+1} = N(x_k) = x_k - \frac{f(x_k)}{f_x(x_k)}.$$

If the initial seed x_0 is close enough to the solution $x^\star$ of $f(x) = 0$, the iteration converges to $x^\star$.

In the following, let us consider a function f which depends on a parameter c. Hence, we want to solve $f(x, c) = 0$. Therefore, we look for a solution $x(c)$ such that $f(x(c), c) = 0$. It follows from the Implicit Function Theorem that $x(c)$ exists under regularity conditions. We want to find $x(c) = \sum a_i(c_0 - c)^i$ such that $f(x(c), c) = 0$. However, such solution $x(c)$ may require an infinite sum. Hence, we will restrict to a finite order polynomial approximation $x_N(c) = \sum_{i=0}^{N} a_i(c_0 - c)^i$.

In Section 2 we show how to use the jet arithmetics to solve an example of such problem. Let $c = c_0 + \xi$, and assume that x_0 is a solution of $f(x_0, c_0)$. The first step is to apply Newton's method to the function $f(x, c_0 + \xi)$ with the initial seed x_0. We have that

$$x_{k+1}(\xi) = x_k(\xi) - \frac{f(x_k(\xi), c + \xi)}{f_x(x_k(\xi), c + \xi)}.$$

We show that at each step there is an increasing number of coefficients that correctly approximate $x(c)$. In Section 3 we give the main result of the paper. There, we proof that at each iterate of the procedure the number of vanishing coefficients of the Taylor expansion of $f(x_k(\xi), c_0 + \xi)$ doubles. Finally, in Section 5 we give some conclusions.

2 Preliminary example

Given the function $f(x, c) = x^2 + x + c$, where $c \in \mathbb{R}$, We want to find a parametric solution of $f(x, c) = 0$ around $c = c_0 = 0$. Notice that $x = 0$ is a solution of $f(x, c_0) = 0$ ($f(0, 0) = 0$). The exact solution is given by $x(c) = \dfrac{-1 + \sqrt{1 - 4c}}{2}$. The Taylor series of the previous function can be computed by derivation, obtaining

$$x(c_0 + \delta c) = -\delta c - \delta c^2 - 2\delta c^3 - 5\delta c^4 - 14\delta c^5 - 42\delta c^6 - 132\delta c^7 - 429\delta c^8 - 1430\delta c^9 + o(\delta c^{10}).$$

Instead of using derivation, we can obtain the Taylor series of x_c (up to a given order) applying the Newton Polynomial Technique

$$x_{k+1} = x_k - \frac{f(x_k, 0 + \xi)}{f_x(x, 0 + \xi)}.$$

We look for a functional solution of $f(x, c(\xi)) = 0$, where $c(\xi) = c_0 + \xi = 0 + \xi$. We take as initial approximation of the constant solution $x_0(\xi) = 0$, since $f(x_0(\xi), c(0)) = 0$. After iterating the procedure once, we obtain

$$x_1(\xi) = x_0(\xi) - \frac{f(x_0(\xi), 0 + \xi)}{f_x(x_0(\xi), 0 + \xi)} = 0 + \frac{0^2 + 0 + 0 + \xi}{2 \cdot 0 + 1} = -\xi.$$

Repeating the procedure and applying the polynomial algebra formulas 1 to the fraction we get

$$x_2(\xi) = x_1(\xi) - \frac{f(x_1(\xi), 0 + \xi)}{f_x(x_1(\xi), 0 + \xi)} = 0 + \frac{(-\xi)^2 - \xi + 0 + \xi}{2(-\xi) + 1} = -\xi - \xi^2 - 2\xi^2 + o(\xi^3)$$

Counting the constant term, the power series of $x_1(\xi)$ has 2 exact terms. After applying the procedure, $x_2(\xi)$ has 4 exact terms. As we will see, at each step of new Newton Polynomial Technique we double the number of exact terms. Indeed, for $x_3(\xi)$ we get

$$x_3(\xi) = -\xi - \xi^2 - 2\xi^3 - 5\xi^4 - 14\xi^5 - 42\xi^6 - 132\xi^7 + o(\xi^8).$$

3 Main Theorem

In this section we formalize the results discussed up to now. This is the content of Theorem A. Before stating the theorem, we define the concept of polynomial zero of a (sufficiently smooth) function $f(x)$.

Definition 1. *A polynomial $P(\xi)$ is a* **polynomial zero of order** *n of a function f if*

$$f(P(\xi)) = 0 + \mathcal{O}(\xi^n).$$

Theorem A. *Let $f(x,c)$ be a sufficiently smooth function depending on a parameter $c \in \mathbb{R}$. Let $N(x)$ denote the Newton method using polynomial iterates,*

$$P_{k+1}(\xi) = N(P_k(\xi)) = x - \frac{f(P_k(\xi), c_0 + \xi)}{f_x(P_k(\xi), c_0 + \xi)},$$

and let $P_0(\xi) = x_0$ be the initial seed with $f(x_0, c_0) = 0$. Then, the iterate $P_k(\xi)$ is a polynomial zero of order 2^k of $f(x, c+\xi)$.

Proof. We will prove the result by induction. Notice that the (constant) polynomial $P_0(\xi) = x_0$ is a polynomial zero of order $1 = 2^0$. Indeed, by hypothesis we have that $f(x_0, c_0 + \xi) = 0 + \mathcal{O}(\xi)$.

For the sake of clarity, next prove that P_1 is a polynomial zero of order 2^1. We have:

$$\begin{aligned} P_1(\xi) &= P_0(\xi) - \frac{f(P_0(\xi), c_0 + \xi)}{f_x(P_0(\xi), c_0 + \xi)} \\ &= x_0 - \frac{f(x_0, c_0 + \xi)}{f_x(x_0, c_0 + \xi)}. \end{aligned}$$

Evaluating f using its Taylor expansion with respect to x around x_0 we get:

$$\begin{aligned} f(P_1(\xi), c_0 + \xi) =& f\left(x_0 - \frac{f(x_0, c_0 + \xi)}{f_x(x_0, c_0 + \xi)}, c_0 + \xi\right) \\ =& f(x_0, c_0 + \xi) - f_x(x_0, c_0 + \xi)\left(\frac{f(x_0, c_0 + \xi)}{f_x(x_0, c_0 + \xi)}\right) \\ &+ \frac{1}{2} f_{xx}(x_0, c_0 + \xi)\left(-\frac{f(x_0, c_0 + \xi)}{f_x(x_0, c_0 + \xi)}\right)^2 + \mathcal{O}\left(\left(-\frac{f(x_0, c_0 + \xi)}{f_x(x_0, c_0 + \xi)}\right)^3\right). \end{aligned}$$

Using that $f(x_0, c_0 + \xi) = 0 + \alpha_1 \xi + o(\xi)$ we get that

$$f(P_1(\xi), c_0 + \xi) = 0 + \mathcal{O}(\xi^2).$$

We conclude that $P_1(\xi)$ is a polynomial zero of order $2 = 2^1$.

We will use induction to prove the general case. The **induction hypothesis** is that the polynomial P_k is a polynomial zero of order 2^k, that is:

$$f(P_k(\xi), c_0 + \xi) = \mathcal{O}\left(\xi^{2^k}\right).$$

The polynomial P_{k+1} is given by

$$P_{k+1}(\xi) = P_k(\xi) - \underbrace{\frac{f(P_k, c_0+\xi)}{f_x(P_k, c_0+\xi)}}_{\tilde{P}_{k+1}(\xi)}.$$

Evaluating $f(x, c_0+\xi)$ at P_{k+1} and using the Taylor expansion of $f(x, c_0+\xi)$ with respect to x around $P_k(\xi)$ we get

$$\begin{aligned} f(P_{k+1}(\xi), c_0+\xi) =& f\left(P_k(\xi) - \tilde{P}_{k+1}(\xi), c_0+\xi\right) \\ =& f(P_k(\xi), c_0+\xi) - f_x(P_k(\xi), c_0+\xi)\frac{f(P_k(\xi), c_0+\xi)}{f_x(P_k, c_0+\xi)} \\ &+ \frac{1}{2} f_{xx}(P_k(\xi), c_0+\xi)(-\tilde{P}_{k+1}(\xi))^2 + \mathcal{O}\left(\left(\tilde{P}_{k+1}(\xi)\right)^3\right). \end{aligned}$$

By induction hypothesis we have $f(P_k, c_0+\xi) = \mathcal{O}\left(\xi^{2^k}\right)$ and, therefore

$$\tilde{P}_{k+1}(\xi) = \frac{f(P_k, c_0+\xi)}{f_x(P_k, c_0+\xi)} = \mathcal{O}\left(\xi^{2^k}\right).$$

Finally, we get

$$\begin{aligned} f(P_{k+1}(\xi), c_0+\xi) =& \frac{1}{2} f_{xx}(P_k, c_0+\xi)\left(\frac{f(P_k(\xi), c_0+\xi)}{f_x(P_k(\xi), c_0+\xi)}\right)^2 + \mathcal{O}\left(\left(\tilde{P}_{k+1}(\xi)\right)^3\right) \\ =& \frac{1}{2} f_{xx}(P_k(\xi), c_0+\xi)\left(\mathcal{O}\left(\xi^{2^k}\right)\right)^2 + \mathcal{O}\left(\left(\mathcal{O}\left(\xi^{2^k}\right)\right)^3\right) \\ =& \mathcal{O}\left(\xi^{2^{k+1}}\right). \end{aligned}$$

■

4 Conclusions

This paper studies methods for solving non-linear equations using a polynomial algebra. The method computes the solution, $x(c_0+\xi)$, of the non-linear equation $f(x, c_0+\xi) = 0$. This is done by approximating $x(c_0+\xi)$ as a jet, $P(\xi)$, and working with polynomial arithmetics in a Newton's method scheme. The main theorem of the paper shows that each iterate of the procedure, $P_k(\xi)$, multiplies by two the coefficients that vanish in $f(P_k(\xi), c_0+\xi)$.

References

[1] Elisa Maria Alessi, Ariadna Farres, Arturo Vieiro, Angel Jorba, and Carles Simó. Jet transport and applications to neos. In *Proceedings of the 1st IAA Planetary Defense Conference, Granada, Spain*, pages 10–11, 2009.

[2] Roberto Armellin and Pierluigi Di Lizia. Probabilistic optical and radar initial orbit determination. *Journal of Guidance, Control, and Dynamics*, 41(1):101–118, 2018.

[3] Martin Berz. *Modern map methods in particle beam physics.* Academic Press, 1999.

[4] Jianlin Chen, Josep Masdemont, Gerard Gomez, and Jianping Yuan. Analysis of jet transport-based geostationary trajectory uncertainty propagation. *Journal of Guidance, Control, and Dynamics*, 43, 04 2020.

[5] Jianlin Chen, Josep Masdemont, Gerard Gomez, and Jianping Yuan. An efficient statistical adaptive order-switching methodology for kalman filters. *Communications in Nonlinear Science and Numerical Simulation*, 93, 09 2020.

[6] Joan Gimeno and Angel Jorba. Using automatic differentiation to compute periodic orbits of delay differential equations. *Discrete & Continuous Dynamical Systems - B*, 22, 01 2017.

[7] Joan Gimeno, Angel Jorba, Begoña Nicolás, and Estrella Olmedo. Numerical computation of high-order expansions of invariant manifolds of high-dimensional tori. *SIAM Journal on Applied Dynamical Systems*, 21:1832–1861, 09 2022.

[8] Àlex Haro, Marta Canadell, Jordi-Lluis Figueras, Alejandro Luque, and Josep María Mondelo. *The Parameterization Method for Invariant Manifolds: From rigorous results to effective computations*. Springer, 2016.

[9] Angel Jorba and Maorong Zou. A software package for the numerical integration of odes by means of high-order taylor methods. *Experimental Mathematics*, 14(1):99–117, 2005.

[10] Monica Valli, Roberto Armellin, Pierluigi Di Lizia, and Michèle R Lavagna. Nonlinear filtering methods for spacecraft navigation based on differential algebra. *Acta Astronautica*, 94(1):363–374, 2014.