FINITE p-SOLVABLE GROUPS WITH THREE p-REGULAR CONJUGACY CLASS SIZES

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Abstract   Let G be a finite p-solvable group. We describe the structure of the p-complements of G when the set of p-regular conjugacy classes has exactly three class sizes. For instance, when the set of p-regular class sizes of G is $\{1, p^a, p^a m\}$ or $\{1, m, p^a m\}$ with $(m, p) = 1$, then we show that $m = q^b$ for some prime q and the structure of the p-complements of G is determined.

Keywords: finite p-solvable groups; conjugacy class sizes; p-regular elements

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1. Introduction

It is known that the structure of a finite group is strongly related to its set of conjugacy class sizes. In particular, in some papers it has been proved that certain properties of the sizes of p-regular conjugacy classes also affect the p-structure of G. In [1], Alemany *et al.* proved that if the set of conjugacy class sizes of p'-elements of a finite group G is $\{1, m\}$, then p-complements of G are nilpotent. In [3], the structure of the p-complements of a p-solvable group G has been described for the case in which the set of p-regular conjugacy class sizes of G is $\{1, m, n\}$ for arbitrary coprime integers $m, n > 1$. In fact, it is shown that G is solvable and the p-complements of G are quasi-Frobenius groups in which the inverse image of the kernel and complement are abelian. Also, in [7], it is proved that, if the set of conjugacy class sizes of all p'-elements of a finite p-solvable group G is $\{1, m, p^a, mp^a\}$, where m is a positive number not divisible by p, then m is a prime power and, furthermore, the p-complements of G are nilpotent.

We note that studying the p-structure of a finite group G from its set of p-regular conjugacy class sizes may be more difficult, even if one considers the p-solvability of G, since if H is a p-complement of G and $x \in H$, then $|x^H|$ does not divide $|x^G|$ in general. Furthermore, we are handicapped by the fact that there is no information on the elements whose order is divisible by p.

In this paper, we study the structure of the p-complements of a p-solvable group G, with $\{1, p^a, mp^a\}$ or $\{1, m, mp^a\}$ as the set of conjugacy class sizes of p'-elements of G, where $(p, m) = 1$. In fact, we prove the following two theorems, which are extensions for p-solvable groups of Itô's Theorem on groups with two class sizes (see, for example, [9, 33.6]).

Theorem A. *Let G be a p-solvable group with $\{1, m, p^a m\}$ as the set of conjugacy class sizes of p'-elements, where $(p, m) = 1$. Then $m = q^b$ for some prime q, and any p-complement H of G satisfies $H = Q \times K$ with Q a Sylow q-subgroup and K abelian.*

Theorem B. *Let G be a p-solvable group. If the set of conjugacy class sizes of p'-elements of G is $\{1, p^a, p^a m\}$, with $(p, m) = 1$, then $m = q^b$ for some prime q and some integer $b \geq 0$, and every p-complement H of G is either*

(i) *$H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or*

(ii) *$H = QK$, with Q a normal abelian Sylow q-subgroup, K abelian and $QO_p(G) \trianglelefteq G$.*

Notice that the solvability of G is an easy consequence of both Theorem A and Theorem B. We also note that the methods we employ for proving Theorems A and B are quite different. In the proof of Theorem A we use the classification of the finite $\mathfrak{M}$-groups due to Schmidt, that is, those non-abelian groups in which all centralizers of non-central elements are abelian. In the proof of Theorem B a more detailed analysis is required.

We remark that the information obtained on the p-structure of a group G from its set of p-regular class sizes has important applications when studying the conjugacy class sizes of G in the ordinary case (see, for example, [5] and [6], in which the information is used to obtain the solvability or nilpotency of certain groups) and, as a consequence, in determining the structure of G.

Throughout this paper all groups are finite. If x is any element of a group G, we denote by x^G the conjugacy class of x in G and $|x^G|$ is called the conjugacy class size of x and also the index of x in G. If p is a prime number and n is an integer, then we use the notation n_p for the p-part of n, i.e. $n_p = p^\alpha$, where p^α divides n and $p^{\alpha+1}$ does not divide n. We will denote the set of p'-elements of G by $G_{p'}$ and the set of conjugacy classes of p'-elements of G by $\mathrm{cs}_{p'}(G)$. All further unexplained notation is standard.

2. Preliminary results

We will need some results on conjugacy class sizes of p-regular elements and of π-elements for a suitable set of primes π.

Lemma 2.1. *Let G be a finite group. All the conjugacy class sizes in $G_{p'}$ are then p-numbers if and only if G has abelian p-complements.*

Proof. See Lemma 2 of [4]. $\qquad \square$

Lemma 2.2. *Suppose that G is a finite group and that p is not a divisor of the sizes of p-regular conjugacy classes. Then $G = P \times H$, where P is a Sylow p-subgroup and H is a p-complement of G.*

Proof. This is exactly Lemma 1 of [8]. $\qquad \square$

Lemma 2.3. *Let x and y be a q-element and a q'-element, respectively, of a group G, such that $C_G(x) \subseteq C_G(y)$. Then $O_q(G) \subseteq C_G(y)$.*

Proof. It is enough to apply Thompson's $P \times Q$-Lemma [11, 8.2.8] to the action of $\langle x \rangle \times \langle y \rangle$ on $O_q(G)$. $\qquad \square$

Lemma 2.4. *Let G be a π-separable group. If $x \in G$ with $|x^G|$ a π-number, then $x \in O_{\pi\pi'}(G)$.*

Proof. See Theorem C of [2]. $\qquad \square$

The following result is an extension for p-regular elements of Itô's Theorem on groups having two class sizes.

Theorem 2.5. *Let G be a finite group. If the set of p-regular conjugacy class sizes of G is exactly $\{1, m\}$, then $m = p^a q^b$, with q a prime distinct from p and $a, b \geq 0$. If $b = 0$, then G has an abelian p-complement. If $b \neq 0$, then $G = PQ \times A$, with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A \subseteq Z(G)$. Furthermore, if $a = 0$, then $G = P \times Q \times A$.*

Proof. This is Theorem A of [2]. $\qquad \square$

Theorem 2.6. *Let G be a finite p-solvable group and let $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of the conjugacy classes of $G_{p'}$ are π-numbers. Then G is solvable, it has abelian π-complements and every p-complement of G has a normal Sylow q-subgroup.*

Proof. This is Theorem 5 of [7]. $\qquad \square$

The following result extends Theorem 6 of [7] with an easier proof.

Theorem 2.7. *Let G be a finite p-solvable group and let $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of p-regular classes in G are π-numbers. Let q^b be the highest power of the prime q that divides the sizes of classes of p-regular elements in G. Suppose that there exists some q-element $x \in G$ such that $|x^G| = p^a q^b$, where $a, b \geq 0$. Then G has nilpotent p-complements and they have abelian Sylow subgroups for all primes distinct from q.*

Proof. We know that G is solvable by Theorem 2.6. Let K be a π-complement of G such that $K \subseteq C_G(x)$. Notice that K is abelian by Theorem 2.6, and furthermore it can be assumed to be non-central in G, otherwise the result is trivial. Let y be any element in K and observe that $K \subseteq C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$. Notice that the index of xy in $C_G(x)$ is a q'-number, and consequently a p-number. If we choose KQ_0 to be a p-complement of $C_G(xy)$, then it is also a p-complement of $C_G(x)$. Note that $x \in KQ_0$. Now, let H be a p-complement of G such that $KQ_0 \subseteq H$. We can then write $H = KQ$ with Q a Sylow q-subgroup of G that is normal in H by Theorem 2.6. Therefore, $Q_0 \subseteq Q$ and

$$C_Q(xy) = Q \cap C_G(xy) = Q_0 \quad \text{and} \quad C_Q(x) = Q \cap C_G(x) = Q_0.$$

Then $C_Q(x) = C_Q(xy) \subseteq C_Q(y)$, so we can apply Thompson's Lemma to get $Q \subseteq C_G(y)$ for all $y \in K$. Consequently, $H = Q \times K$ as desired. $\square$

Theorem 2.8. *Let G be a finite group and let π be a set of primes. Suppose that the conjugacy class size of every π-element of G is a power of p for some fixed prime $p \notin \pi$. Then G has an abelian Hall π-subgroup H and $HO_p(G) \trianglelefteq G$.*

Proof. This is part (a) of Theorem A of [4]. $\square$

We use the above theorem to give a simplified proof of Theorem 2.9, which is moreover an extension of Theorem 7 of [7].

Theorem 2.9. *Let G be a p-solvable group whose conjugacy class sizes of p'-elements are $\{1, p^{a_1}, \ldots, p^{a_r}, p^{c_1}q^b, \ldots, p^{c_s}q^b\}$, where q is a prime distinct from p and $c_i \geqslant 0$, $b, a_i \geqslant 0$ for all i. Then any p-complement H of G is either*

(i) $H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or

(ii) $H = QK$, with Q a Sylow q-subgroup, Q and K both abelian, $Q \trianglelefteq H$ and $QO_p(G) \trianglelefteq G$.

Proof. If there exists a q-element of index $q^b p^{c_i}$ for some i, then case (i) follows by Theorem 2.7. Otherwise, the index of every q-element is a p-number. So, by Theorem 2.8, G has an abelian Sylow q-subgroup Q and $QO_p(G) \trianglelefteq G$ and thus, if H is a p-complement of G, containing Q, then $Q \trianglelefteq H$. Also by Theorem 2.6 we get that G is solvable and it has an abelian $\{p, q\}$-complement. So we have case (ii). $\square$

Corollary 2.10. *Let G be a p-solvable group.*

(a) *If $\mathrm{cs}_{p'}(G) = \{1, q^b, p^a q^b\}$, where q is a prime distinct from p, then any p-complement H of G satisfies $H = Q \times K$ with Q a Sylow q-subgroup and K abelian.*

(b) *If $\mathrm{cs}_{p'}(G) = \{1, p^a, p^a q^b\}$, where q is a prime distinct from p, then every p-complement H of G, is either*

(i) $H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or

(ii) $H = QK$, with Q a normal abelian Sylow q-subgroup, K abelian and $QO_p(G) \trianglelefteq G$.

Proof. Case (a) is an immediate consequence of Theorem 2.7 and case (b) is a consequence of Theorem 2.9. $\square$

3. Groups with three p-regular class sizes

In order to prove Theorems A and B, it is sufficient to show that when the set of p-regular class sizes of a group G is $\{1, p^a, p^a m\}$ or $\{1, m, p^a m\}$, with $(m, p) = 1$, then m is a prime power, q^b. We shall prove this in Theorems 3.1 and 3.2. The main results then follow by Corollary 2.10. Note that if $a = 0$, then G has two p-regular class sizes, so Theorems A and B are immediate consequences of Theorem 2.5. Also note that if $b = 0$, then G has abelian p-complements by Lemma 2.1.

Theorem 3.1. *If G is a p-solvable group such that $\mathrm{cs}_{p'}(G) = \{1, m, p^a m\}$ with $(p, m) = 1$, then $m = q^b$ for some prime q.*

Proof. Take H to be a p-complement of G. We prove the theorem in the following steps.

Step 1. For every non-central p-regular element x of G, we may assume that there exist at least two primes q and r, distinct from p, such that r and q divide $|C_G(x)|/|Z(G)|$.

Suppose that there exists a non-central p-regular element $x \in G$ such that, for some prime $q \neq p$, $|C_G(x)|_{p'}/|Z(G)|_{p'}$ is a q-number. On the other hand, we may assume that there exists a non-central r-element y in G, with r distinct from p and q, since otherwise G is the direct product of a $\{p, q\}$-group and a central factor, and so the result follows. Therefore, r is a divisor of $|C_G(y)|_{p'}/|Z(G)|_{p'}$. Note that $|C_G(x)|_{p'} = |G|_{p'}/m = |C_G(y)|_{p'}$, and so we get a contradiction.

Step 2. If x is a non-central element of H such that $|x^G| = m$, then $|x^H| = m$. Moreover, $C_H(x) = T_x Q_x$, with Q_x a Sylow q-subgroup of $C_H(x)$, where q is a prime divisor of the order of x, and T_x a normal abelian q'-subgroup of $C_H(x)$. Furthermore, $C_H(x)$ is a p-complement of $C_G(x)$.

Let x be a non-central element of H with $|x^G| = m$. So $G = HC_G(x)$, and consequently $|H : C_H(x)| = |G : C_G(x)| = m$, as desired. Also, $|C_G(x) : C_H(x)| = |G : H|$ implies that $C_H(x)$ is a p-complement of $C_G(x)$. By the minimality of the class size of x, we can certainly assume that x is a q-element for some prime q distinct from p. Let y be a non-central $\{p, q\}'$-element in $C_G(x)$ (note that by Step 1 such elements exist). Then $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$, and so by the hypotheses, y has index 1 or p^a in $C_G(x)$. Now, we apply Theorem 2.9, so $C_G(x)$ has abelian $\{p, q\}$-complements, and $T_x O_p(C_G(x)) \trianglelefteq C_G(x)$, for every $\{p, q\}$-complement T_x of $C_G(x)$. Since $C_H(x)$ is a p-complement of $C_G(x)$, we may assume that $T_x \subseteq C_H(x)$, and, as a consequence, $T_x \trianglelefteq C_H(x)$. So we get the result.

Step 3. H is an $\mathfrak{M}$-group.

Let $x \in H$ be a non-central element. We distinguish two possibilities for the index of x in G.

First suppose that $|x^G| = m$. We may assume that x is a q-element for some prime q. By Step 2 we have $C_H(x) = T_x Q_x$, where T_x is the normal abelian q-complement of $C_H(x)$ and Q_x is a Sylow q-subgroup of $C_H(x)$. Let $y \in T_x$ be a non-central r-element for some prime $r \neq q$.

Assume first that $|y^G| = m$. Then $|C_H(x)| = |C_H(y)|$ and $C_H(y) = L_y R_y$, where L_y is the normal abelian r-complement of $C_H(y)$ and R_y is a Sylow r-subgroup of $C_H(y)$, by Step 2. So $Q_y \subseteq L_y$ is the normal abelian Sylow q-subgroup of $C_H(y)$. Therefore, $x \in Q_y$ and hence $Q_y \subseteq C_H(x) \cap C_H(y)$. Also, since T_x is abelian, $T_x \subseteq C_H(x) \cap C_H(y)$, which implies that $C_H(x) = C_H(y) = Q_y \times T_x$, and we deduce that $C_H(x)$ is abelian.

Now we assume that $|y^G| = p^a m$. Then by the minimality of $|C_G(y)|$ we have $C_G(y) = P_y R_y \times K_y$, where P_y and R_y are some Sylow p-subgroup and r-subgroup of $C_G(y)$, respectively, and K_y is abelian. On the other hand, since $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(y)$, the minimality of $|C_G(y)|$ implies that $C_G(xy) = C_G(y) \subseteq C_G(x)$. So we may assume that $R_y \subseteq T_x$ and, consequently, R_y is abelian. Hence, $R_y \times K_y$ is an abelian p-complement of $C_G(y)$. Also $C_G(y) \subseteq C_G(x)$, whence every p-complement of $C_G(y)$ is a p-complement of $C_G(x)$. Now, from the fact that $C_H(x)$ is a p-complement of $C_G(x)$ we get that $C_H(x)$ is abelian.

Suppose that $|x^G| = p^a m$. First we assume that there exists some non-central p-regular element α such that $C_G(x) \subsetneq C_G(\alpha)$. So $|\alpha^G| = m$. Since $C_H(x)$ is a p'-subgroup of $C_G(\alpha)$, there exists $g \in G$ such that $C_H(x) \subseteq C_{H^g}(\alpha)$, where $C_{H^g}(\alpha)$ is a p-complement of $C_G(\alpha)$. By the above argument, $C_{H^g}(\alpha)$ is abelian, whence $C_H(x)$ is abelian too.

Suppose that there exists no non-central p'-element α such that $C_G(x) \subsetneq C_G(\alpha)$. Hence, we may certainly assume that x is an r-element for some prime $r \neq p$. So we can write $C_G(x) = P_x R_x \times K_x$, where P_x and R_x are some Sylow p-subgroup and r-subgroup of $C_G(x)$ and K_x is abelian. By Step 1 there exists a non-central q-element $w \in K_x$ for some prime $q \notin \{p, r\}$. Thus, $C_G(wx) = C_G(x) \cap C_G(w) = C_G(x) \subseteq C_G(w)$, which implies that $C_G(x) = C_G(w)$, by the hypotheses. On the other hand, we have $C_G(w) = P_w Q_w \times L_w$, where P_w and Q_w are some Sylow p-subgroup and q-subgroup of $C_G(w)$ and L_w is abelian. So $R_w \subseteq L_w$ is the normal abelian Sylow r-subgroup of $C_G(x)$. Hence, $C_G(x)$ has abelian p-complements. Consequently, every p'-subgroup of $C_G(x)$ is abelian and, in particular, $C_H(x)$ is abelian too.

Step 4. For every non-central element $x \in H$, m is a divisor of $|x^H|$. In particular, if H is a normal subgroup of G, then the theorem follows.

Let $x \in H$ be a non-central element. Since $C_H(x)$ is a p'-subgroup of $C_G(x)$, there exists $g \in G$ such that $C_H(x) \subseteq C_{H^g}(x)$, where $C_{H^g}(x)$ is a p-complement of $C_G(x)$. On the other hand, $m = |G : C_G(x)|_{p'} = |H^g : C_{H^g}(x)|$. Therefore, m divides $|x^H|$, as desired. If H is a normal subgroup of G, then $|x^H|$ divides $|x^G|$ for every non-central element $x \in H$, and by the fact that m is a divisor of $|x^H|$ we get that $\mathrm{cs}(H) = \{1, m\}$; thus, by applying Ito's Theorem on groups with two class sizes (see [9, 33.6]), we obtain that m is a prime power.

Step 5. Conclusion.

As we proved in Step 3, H is an $\mathfrak{M}$-group. So by applying the classification of finite $\mathfrak{M}$-groups (see [13, Theorem 9.3.12]), we have the following possibilities, each of which leads either to the fact that m is a prime power or to a contradiction.

- Assume that $H = A \times Q$, where A is abelian and Q is a q-group, for some prime q. Let x be an element in $Q \setminus Z(H)$. So $|x^G|$ is a $\{p, q\}$-number, whence m is a q-power, as desired.

- Assume that H is non-abelian and has a normal abelian subgroup N of index q for some prime q distinct from p. Notice that $N \not\subseteq Z(H)$, and hence, if $x \in N \setminus Z(H)$, then $|x^G|$ is a $\{p, q\}$-number, whence m is a q-power.

- Suppose that $H/Z(H)$ is a Frobenius group, with Frobenius kernel $K/Z(H)$ and Frobenius complement $L/Z(H)$, where K and L are abelian. It follows that $\mathrm{cs}(H) = \{1, |K/Z(H)|, |L/Z(H)|\}$, and so, by Step 4, $m = 1$, the theorem is trivially true.

- Suppose that $H/Z(H)$ is a Frobenius group, with Frobenius kernel $K/Z(H)$ and Frobenius complement $L/Z(H)$, where K is abelian and $L/Z(H)$ is a q-group for some prime q. It is easy to see that $\mathrm{cs}(H) = \{1, |L/Z(H)|, |K/Z(H)| |x^L| : x \in L \setminus Z(H)\}$, and so, by Step 4, m is a q-power.

- Assume that $H/Z(H) \cong S_4$, and $V/Z(H)$ is the Klein 4-group, where V is non-abelian. Then for every $x \in H \setminus Z(H)$ we have that $C_H(x)/Z(H)$ is a maximal cyclic subgroup of $H/Z(H)$ (see [13, p. 521]). One can then easily obtain $\mathrm{cs}(H) = \{1, 6, 8, 12\}$, whence $m = 2$ by Step 4.

- Let $H/Z(H) \cong \mathrm{PSL}(2, q^h)$ for some prime q. Note that $Z(H) = Z(G)_{p'}$. By Lemma 2.4, if x is a p-regular element in G such that $|x^G| = m$, then $x \in O_{p'}(G)$. Therefore, $O_{p'}(G)/Z(H)$ is a non-trivial normal subgroup of $\mathrm{PSL}(2, q^h)$, so $H = O_{p'}(G)$ is a normal subgroup of G, and the result follows by Step 4.

- Finally, assume that $H/Z(H) \cong \mathrm{PGL}(2, q^h)$ for some prime q. Note that $Z(H) = Z(G)_{p'}$. Since $O_{p'}(G)$ can be assumed to be a proper non-central subgroup of H, we deduce that $O_{p'}(G)/Z(H) \cong \mathrm{PSL}(2, q^h)$. Therefore, any class size of $\mathrm{PSL}(2, q^h)$ divides a p-regular class size of G and, consequently, their least common multiple, which is $|\mathrm{PSL}(2, q^h)|$, divides m. Let $x \in H \setminus Z(G)$, such that $|x^G| = m$; we then have $|x^H| = m$. Since $|\mathrm{PSL}(2, q^h)| = |O_{p'}(G)|/|Z(H)|$ divides m, there exists an integer t such that $|O_{p'}(G) : Z(H)|t = m = |H : C_H(x)|$. This implies that $|C_H(x) : Z(H)|t = |H : O_{p'}(G)| = 2$, and so $t = 1$. Therefore, $|H/Z(H)| = |H/O_{p'}(G)| |O_{p'}(G)/Z(H)| = 2m$. Let y be a non-central element in H. There exists $g \in G$ such that $C_H(y) \subseteq C_{H^g}(y)$, where $C_{H^g}(y)$ is a p-complement of $C_G(y)$, so $m = |G : C_G(y)|_{p'} = |H^g : C_{H^g}(y)|$. Taking into account that $Z(H) \subset C_H(y) \subseteq C_{H^g}(y) \subseteq H^g$ and $2m = |H^g/Z(H)|$, it follows that $C_H(y) = C_{H^g}(y)$, so $m = |H^g : C_H(y)| = |H : C_H(y)|$ for every $y \in H \setminus Z(H)$. By Ito's Theorem on groups with two class sizes, m is a prime power, which contradicts the fact that $|\mathrm{PSL}(2, q^h)|$ divides m. $\qquad\square$

Theorem 3.2. *Let G be a p-solvable group and suppose that $\mathrm{cs}_{p'}(G) = \{1, p^a, p^a m\}$ with $(p, m) = 1$. Then $m = q^b$ for some prime q.*

Proof. We will proceed by minimal counterexample to prove that m is a prime power. Let G be a group of minimal order satisfying the hypotheses and such that m is not a prime power. Notice that if w is a p'-element of index p^a, then by minimality of its index, w certainly can be assumed to be a q-element for some prime $q \neq p$. For the rest of the proof, we will fix the prime q and a q-element w of index p^a. Let H be a p-complement of G such that $H \subseteq C_G(w)$.

Step 1. If $y \in H$ is a q'-element, then $|y^H| = 1$ or m. As a consequence, $H = QR \times A$, where Q and R are Sylow q- and r-subgroups of H, respectively, and A is abelian, and $m = q^b r^c$ with $b, c > 0$ for some prime $r \neq p, q$.

Let y be any q'-element of H. Then $C_G(wy) = C_G(w) \cap C_G(y) \subseteq C_G(w)$, so by the hypotheses y may have index 1 or m in $C_G(w)$. Now, since $C_G(w) = H C_G(wy)$ and $C_H(wy) = C_H(y)$, it follows that $|H : C_H(y)| = |C_G(w) : C_G(wy)| = 1$ or m. If every q'-element of H has index 1 in H, then H has a central q-complement. Therefore, every element of H is centralized by a $\{p, q\}$-complement of G, so its index is a $\{p, q\}$-number and m would be a power of q, a contradiction. Therefore, both numbers, 1 and m, appear as indexes of q'-elements in H, so we can apply Theorem 2.5. Since we have assumed that m is not a prime power, this completes the step.

Step 2. If x is an s-element for any prime $s \neq q, p$ and y is a q-element such that both x and y have index p^a, then $C_G(x) = C_G(y)^g$ for some $g \in G$.

Let H_1 be a p-complement of G contained in $C_G(y)$. It is clear that there exists some $g \in G$ such that $H_1^g \subseteq C_G(x)$. Then $y^g \in C_G(x)$ and, clearly, $|C_G(x) : C_G(y^g x)|$ must be equal to 1 or m. As m is a p' number, we can take P_x to be a Sylow p-subgroup of $C_G(x)$ such that $P_x \subseteq C_G(y^g x)$. In particular, we have $P_x \subseteq C_G(y^g)$. By considering the orders, P_x is a Sylow p-subgroup of $C_G(y^g)$ and thus $C_G(y^g) = H_1^g P_x = C_G(x)$, as desired.

Step 3. Every s-element of G has index 1 or $p^a m$ in G for any prime $s \neq p, q$. Also, for every s-element x, we have $C_G(x) = P_x S_x \times T_x$, where P_x and S_x are a Sylow p-subgroup and a Sylow s-subgroup of $C_G(x)$, respectively, and T_x is abelian.

Suppose that ρ is a non-central s-element such that $|\rho^G| = p^a$. Then by the last step we have $C_G(w) = C_G(\rho)^g$ for some $g \in G$.

Let $z \in H$ be an element of prime power order. If $(o(z), q) = 1$, then, by Step 1, we conclude that the index of z in H is 1 or m. Let z be a q-element. Since $z \in C_G(w) = C_G(\rho)^g$, we conclude that $C_G(z\rho^g) = C_G(z) \cap C_G(\rho^g) \subseteq C_G(\rho^g)$. Therefore, $|C_G(\rho^g) : C_G(z\rho^g)| = 1$ or m, and as a consequence $C_G(\rho^g) = H C_G(z\rho^g)$. Now it is easy to see that $|z^H| = |H : C_H(z)| = |C_G(\rho^g) : C_G(z\rho^g)| = 1$ or m.

Now we shall prove that m is a prime power, which is a contradiction. By Step 1 we have that H is solvable, which means that there must exist some prime q such that $Z(H)_q < O_q(H)$. If every r-element of H is central in H for every prime r dividing $|H|$ distinct from q, then m is certainly a q-power. So, let x be a non-central r-element of

H for such a prime r. We take Q to be a Sylow q-subgroup of $C_H(x)$. Let us consider the action of $Q \times \langle x \rangle$ on $Q_0 = O_q(H)$. We claim that $C_{Q_0}(Q) \subseteq C_{Q_0}(x)$. In fact, if $z \in C_{Q_0}(Q)$ is non-central in H, then $\langle Q, z \rangle \leqslant C_H(z) < H$. However, by the above paragraph, $|C_H(z)|_q = |C_H(x)|_q = |Q|$, so, in particular, $z \in Q \cap Q_0 \subseteq C_{Q_0}(x)$ as claimed. We apply Thompson's $P \times Q$-Lemma to get $x \in C_H(O_q(H))$, and thus show that every Sylow r-subgroup of H lies in $C_H(O_q(H))$ for every $r \neq q$. This means that $|H : C_H(O_q(H))|$ is a q-number. However, if we take $w \in O_q(H) \setminus Z(H)$, then $C_H(O_q(H)) \subseteq C_H(w)$ and, consequently, m is a q-number too, as desired.

Now let x be a non-central s-element, in which case we have $|x^G| = p^a m$. If y is an $\{s, p\}'$-element in $C_G(x)$, then $C_G(yx) = C_G(y) \cap C_G(x) = C_G(x) \subseteq C_G(y)$, which implies that $C_G(x) = P_x S_x \times T_x$, where P_x and S_x are some Sylow p-subgroup and s-subgroup of $C_G(x)$, respectively, and T_x is abelian.

Step 4. Every non-central $\{r, p\}'$-element has class size p^a. As a consequence, G is a $\{p, q, r\}$-group.

First we claim that every q-element has class size 1 or p^a.

Suppose that α is a q-element of index $p^a m$. Take a p-complement H_1 of G such that $C_{H_1}(\alpha)$ is a p-complement of $C_G(\alpha)$. Note that $\alpha \in H_1$. By using Step 3, there exists a non-central r-element $\beta \in G$ such that $|\beta^G| = p^a m$. Hence, $|C_G(\alpha)| = |C_G(\beta)|$ and $|C_{H_1}(\alpha)/(Z(G) \cap H_1)|_r = |C_G(\beta)/Z(G)|_r > 1$. So we conclude that there exists a non-central r-element $\gamma \in C_{H_1}(\alpha)$, whence $|\gamma^G| = p^a m$. Moreover, $C_G(\alpha\gamma) = C_G(\alpha) \cap C_G(\gamma)$, and by the maximality of the index of α and γ, we conclude that $C_G(\alpha) = C_G(\gamma) = C_G(\alpha\gamma)$.

Now consider the action of $\langle \alpha \rangle \times \langle \gamma \rangle$ on $O_q(H_1)$ and $O_r(H_1)$ and, by Lemma 2.3, we deduce that $O_q(H_1) \times O_r(H_1) \subseteq C_G(\alpha) = C_G(\gamma)$. In particular, $\gamma \in C_{H_1}(F(H_1)) \subseteq F(H_1)$, since, by Step 1, H_1 is a solvable group that can be described as $H_1 = Q_1 R_1 \times A_1$, where Q_1 and R_1 are some Sylow q- and r-subgroups of H_1, respectively, and A_1 is abelian. Therefore, $\gamma \in O_r(H_1)$.

Now we shall show that $R_1 \subseteq C_G(\alpha)$, which provides a contradiction, since α has index $p^a m$, which is divisible by r.

Let $\eta \in R_1$ be a non-central r-element. Then, by Step 3, $C_G(\eta) = P_\eta R_\eta \times T_\eta$, where P_η and R_η are some Sylow p-subgroup and r-subgroup of $C_G(\eta)$, respectively, and T_η is abelian. So $C_{H_1}(\eta) \subseteq (R_\eta \times T_\eta)^x$ for some $x \in C_G(\eta)$. Since $|H_1 : C_{H_1}(\eta)| = m = |G : C_G(\eta)|_{p'}$, by Step 1 we deduce that $|C_{H_1}(\eta)| = |C_G(\eta)|_{p'}$. Therefore, $C_{H_1}(\eta) = (R_\eta \times T_\eta)^x$. By changing the notation we may assume that $C_{H_1}(\eta) = R_\eta \times T_\eta$. Now we consider the action of $R_\eta \times T_\eta$ on $O_r(H_1)$ by conjugation. We claim that $C_{O_r(H_1)}(R_\eta) \subseteq C_{O_r(H_1)}(T_\eta)$.

If z is a non-central element in $C_{O_r(H_1)}(R_\eta)$, then $\langle R_\eta, z \rangle \subseteq C_G(z)$ and, since $|C_G(z)|_r = |C_G(\eta)|_r = |R_\eta|$, we deduce that $z \in R_\eta$ and hence $z \in C_{O_r(H_1)}(T_\eta)$. So it follows that $C_{O_r(H_1)}(R_\eta) \subseteq C_{O_r(H_1)}(T_\eta)$. Now, by using Thompson's $P \times Q$-Lemma, we have $T_\eta \subseteq C_{H_1}(O_r(H_1)) \subseteq C_{H_1}(\gamma)$, which implies that $\alpha \in T_\gamma = T_\eta$, where T_γ is the $\{r, p\}$-complement of $C_G(\gamma)$, and so $\alpha \in C_G(\eta)$ and hence $R_1 \subseteq C_G(\alpha)$, as we claimed.

Now let g be any $\{r, p\}'$-element of G, which can be assumed to belong to H. Then we have $g = g_q z$, where g_q is the q-part of g and z is an element in A. Since $z \in C_G(g_q)$

and g_q has index p^a in G, we deduce that there exists $t \in G$ such that $z \in H^t \subseteq C_G(g_q)$. So $z \in A^t$ and, by the fact that A^t is central in H^t, we have $H^t \subseteq C_G(z)$. Then $H^t \subseteq C_G(z) \cap C_G(g_q) = C_G(g)$, which implies that $|g^G| = 1$ or p^a.

Therefore, by Step 3 and the above argument, we get that every s-element of G is central for every $s \notin \{p, q, r\}$. Hence, the $\{p, q, r\}$-complement of G is central, and so by minimal counterexample we conclude that G is a $\{p, q, r\}$-group.

Step 5. Let P_w be a Sylow p-subgroup of $C_G(w)$. Then any p'-element of G centralizes some conjugate of P_w.

Let h be any p'-element of G, which can be assumed to belong to $H \subseteq C_G(w)$. We factorize $h = h_r h_q$ with $h_r \in R$ and $h_q \in Q$. As we proved in Step 3, h_r has index 1 or $p^a m$. Assume first that h_r has index $p^a m$. Since $h_r \in H \subseteq C_G(w)$, we conclude that $C_G(w h_r) = C_G(h_r) = C_G(h)$, which implies that $C_G(h) \subseteq C_G(w)$. But $|C_G(w) : C_G(h)|$ is m and we obtain that $C_G(h)$ contains some Sylow p-subgroup of $C_G(w)$, and consequently h centralizes some conjugate of P_w. Therefore, we may assume that h_r is central in G, whence h can be assumed to be a q-element. Thus, by applying Step 4, h has index p^a. Since by Step 3 any r-element of G has index 1 or $p^a m$, we can choose an r-element $t \in C_G(h)$ of index $p^a m$. By minimality of the order of the centralizer of t in G, we have $C_G(th) = C_G(t)$, so $C_G(t) \subseteq C_G(h)$. On the other hand, t lies in some p-complement $H^g \subseteq C_G(w^g)$ and similarly $C_G(t) \subseteq C_G(w^g)$. Moreover, $|C_G(w^g) : C_G(t)|$ is necessarily m, so some conjugate of P_w must lie in $C_G(t)$ and, therefore, also in $C_G(h)$ and this case is finished.

Step 6 ($O^p(G) = G$). Suppose that $O^p(G) < G$. Let ρ be a p-regular element of $O^p(G)$ such that $|\rho^G| = p^a$. We have

$$\frac{|G|}{|O^p(G)|} \frac{|O^p(G)|}{|C_{O^p(G)}(\rho)|} = \frac{|G|}{|C_G(\rho)|} \frac{|C_G(\rho)|}{|C_{O^p(G)}(\rho)|}.$$

Let P_ρ be a Sylow p-subgroup of $C_G(\rho)$. The fact that $|C_G(\rho) : C_{O^p(G)}(\rho)|$ is a p-number implies that

$$\frac{|G|}{|O^p(G)|} \frac{|O^p(G)|}{|O^p(G) \cap C_G(\rho)|} = \frac{|G|}{|C_G(\rho)|} \frac{|P_\rho|}{|P_\rho \cap O^p(G)|} = \frac{|G|}{|C_G(\rho)|} \frac{|P_\rho O^p(G)|}{|O^p(G)|},$$

and thus

$$\frac{|O^p(G)|}{|C_{O^p(G)}(\rho)|} = p^a \frac{|P_\rho O^p(G)|}{|G|} = p^k,$$

where $k \geq 0$.

By Step 5 there exists $g \in G$ such that $P_w^g \subseteq C_G(\rho)$. Since $|C_G(\rho)|_p = |C_G(w)|_p$, that is, P_ρ is G-conjugate to P_w^g, we deduce that p^k is constant in the above equation for any element ρ of index p^a.

Now, let ρ be a p-regular element of $O^p(G)$ with $|\rho^G| = p^a m$. So we can write $\rho = \rho_r \rho_q$, where ρ_r and ρ_q are the r-part and q-part of ρ, respectively. By Step 4, ρ_r cannot be central. Thus it is easy to see that $C_G(\rho) = C_G(\rho_r)$. Hence, we may assume that ρ is an

r-element. There exists $g \in G$ such that $\rho \in C_G(w^g)$ and hence $C_G(\rho) \subseteq C_G(w^g)$. Then $|C_G(w^g) : C_G(\rho)| = m$. As $(m, p) = 1$, we have $C_G(w^g) = C_G(\rho) C_{O^p(G)}(w^g)$ and

$$|O^p(G) : C_{O^p(G)}(\rho)| = |O^p(G) : C_{O^p(G)}(w^g)| |C_{O^p(G)}(w^g) : C_{O^p(G)}(\rho)|$$
$$= p^k |C_G(w^g) : C_G(\rho)|$$
$$= p^k m.$$

Therefore, the set of p-regular class sizes of $O^p(G)$ is $\{1, p^k, p^k m\}$. If $k \neq 0$, then by minimal counterexample m is a prime power, which is a contradiction. Thus, $k = 0$ and $\{1, m\}$ are the p-regular conjugacy class sizes of $O^p(G)$. This forces m to be a $\{p, q\}$-number by Theorem 2.5, which is a contradiction by Step 1.

Step 7. There exists N a proper normal subgroup of G such that the index $|G : N|$ is a p'-number and $Z(G) \subseteq O_{pp'}(G) \subseteq N$.

First we show that $O_{pp'}(G) < G$. Otherwise, G has a normal Sylow p-subgroup P. Then $G = PH$, and it is easy to see that $C_G(h) = C_P(h) C_H(h)$ for all $h \in H$. This implies that

$$|G : C_G(h)| = |P : C_P(h)| |H : C_H(h)|,$$

which is 1, p^a or $p^a m$. Therefore, $|H : C_H(h)|$ is 1 or m for every $h \in H$. By Itô's Theorem on groups with two class sizes [9, Theorem 33.6], m is a prime power, which is a contradiction. Hence, $O_{pp'}(G) < G$.

Take N to be the maximal proper subgroup in the upper pp'-series of G and note that the index $|G : N|$ is a p'-number, since $O^{p'}(G) < G$ by Step 6. Moreover, it is obvious that $Z(G) \subseteq O_{pp'}(G) \subseteq N < G$.

Step 8. If Q is a Sylow q-subgroup of H, then $QO_p(G) \trianglelefteq G$, $Q \trianglelefteq H$ and Q is abelian. Moreover, $\bar{R} = R/Z(G)_r$ has exponent r, where R is a Sylow r-subgroup of G.

As we proved in Step 4, every q-element of G has class size 1 or p^a. So, by using Theorem 2.9, G has an abelian Sylow q-subgroup Q, and $QO_p(G) \trianglelefteq G$. Also, by using the fact that $Q \subseteq H$, it easily follows that $Q \trianglelefteq H$, as required.

Now we shall show that $\bar{R} = R/Z(G)_r$ has exponent r. Let $x \in H \setminus QZ(G)_r$. Then we factorize $x = x_r x_q$, where x_r and x_q are the r-part and q-part of x, respectively. Note that $x_r \notin Z(G)_r$. So $C_G(x) \subseteq C_G(x_r)$, and if we also take into account that x_r has index $p^a m$ in G, we conclude that $C_G(x) = C_G(x_r)$. Therefore, $C_H(x) = C_H(x_r)$, whence $|x^H| = m$, by using Step 1. Now we apply Isaacs's Theorem on groups having a normal subgroup such that the class sizes of the elements not in the normal subgroup are equal (see [10]). So we conclude that $H/(QZ(G)_r)$, which is isomorphic to $\bar{R}$, is cyclic, or has exponent r. However, $Z(R) = Z(G)_r$ and R cannot be abelian by Lemma 2.1, so we conclude that $\bar{R}$ has exponent r.

Step 9. If η is a non-central r-element of G, then $C_G(\eta) = P_\eta \times \langle \eta \rangle Z(G)_r \times Q_\eta$, where P_η and Q_η are the Sylow p-subgroup and q-subgroup of $C_G(\eta)$, respectively.

We may assume that H is a p-complement of G, such that $C_H(\eta)$ is a p-complement of $C_G(\eta)$. So by Step 3, $C_H(\eta) = R_\eta \times Q_\eta$ for some Sylow r-subgroup R_η and Sylow q-subgroup Q_η of $C_G(\eta)$. Hence by the fact that $Q \trianglelefteq H$, R_η acts on Q. Since Q is abelian

and this action is coprime, it follows that $Q = [Q, R_\eta] \times C_Q(R_\eta)$ (see, for example, [9, Theorem 14.5]). On the other hand, we consider the action of $\bar{R}_\eta = R_\eta/Z(G)_r$ on $[Q, R_\eta]$, and we claim that this action has no fixed points. Otherwise, there exist $x \in [Q, R_\eta]$ and $y \in R_\eta$ such that $x^{\bar{y}} = x$. Therefore, $x^y = x$, and as a consequence $x \in C_G(y) = P_y R_y \times Q_y$, where P_y and R_y are some Sylow p-subgroup and Sylow r-subgroup of $C_G(y)$, respectively, and Q_y is abelian, by Step 3. Since x is a q-element, it is obvious that $x \in Q_y$. On the other hand, from the fact that $y \in R_\eta$, we conclude that $Q_\eta \subseteq C_G(y)$, and so $Q_\eta = Q_y$, by considering the order equality. Thus $x \in Q_\eta$, and consequently $x \in C_Q(R_\eta)$. Hence, $x \in [Q, R_\eta] \cap C_Q(R_\eta) = 1$, and our claim is proved. So it is well known that $\bar{R}_\eta$ is cyclic or is a generalized quaternion group. By considering Step 8, $\bar{R}_\eta$ is cyclic of order r, and therefore $R_\eta = \langle \eta \rangle Z(G)_r$, so the result follows by the obtained fact in Step 3, that is, $C_G(\eta) = P_\eta R_\eta \times Q_\eta$, where P_η is a Sylow p-subgroup of $C_G(\eta)$.

Step 10. $\bar{R} = R/Z(G)_r$ has order r^2, and consequently it is elementary abelian.

Let N be the normal subgroup introduced in Step 7 and let M be a maximal normal subgroup containing N. Recall that $|G : N|$ is a p'-number. We shall show that $|G/M| = r$. In Step 8 we proved that $QO_p(G) \trianglelefteq G$. So it is easy to conclude that $QO_p(G) \subseteq O_{pp'}(G) \subseteq N$. As a consequence, $|G : N|$ is an r-number. Therefore, $|G : M|$ is an r-number, and since G/M is simple, it follows that $|G/M| = r$.

In the following we shall show that $m_r = r$, and so, by using Step 9, it is obvious that $|\bar{R}| = r^2$, whence $\bar{R}$ is abelian and, as a consequence of Step 8, elementary, as desired.

Let x be a non-central p-regular element of M. Then

$$\frac{|G|}{|M|}\frac{|M|}{|C_M(x)|} = \frac{|G|}{|C_G(x)|}\frac{|C_G(x)|}{|C_M(x)|}.$$

Let us consider a Sylow r-subgroup R_x of $C_G(x)$; the above equality then becomes

$$\frac{|G|}{|M|}\frac{|M|}{|C_M(x)|} = \frac{|G|}{|C_G(x)|}\frac{|R_x|}{|R_x \cap M|} = \frac{|G|}{|C_G(x)|}\frac{|R_x M|}{|M|}$$

and we have the following equality:

$$\frac{|M|}{|C_M(x)|} = \frac{|G|}{|C_G(x)|}\frac{|R_x M|}{|G|}.$$

First suppose that $|x^G| = p^a$. Therefore, R_x is a Sylow r-subgroup of G, whence $G = R_x M$. So the above equation implies that $|x^M| = |x^G| = p^a$.

Now suppose that $|x^G| = p^a m$. We factorize $x = x_r x_q$, where x_r and x_q are the r-part and q-part of x, respectively. By Step 4, x_r is a non-central element. Then $C_G(x) = C_G(x_r) \cap C_G(x_q) \subseteq C_G(x_r)$, and by the fact that x_r has class size $p^a m$ by Step 3, it follows that $C_G(x) = C_G(x_r)$. Also, by Step 9, the Sylow r-subgroup of $C_G(x_r)$ is $R_{x_r} = \langle x_r \rangle Z(G)_r$, which is a subgroup of M, since $x_r \in M$. On the other hand, the equality $C_G(x) = C_G(x_r)$ implies that R_{x_r} is the Sylow r-subgroup of $C_G(x)$, that is, R_x. So $R_x M = M$ and, consequently, $|x^M| = p^a m/r$.

Thus $cs_{p'}(M) = \{1, p^a, p^a m/r\}$, and by minimal counterexample it follows that m/r must be a prime power, whence $m_r = r$, and this completes the step.

Step 11. $N_G(P_x) = C_G(P_x)P_x$, where P_x is the Sylow p-subgroup of $C_G(x)$ for every non-central r-element x of G.

In the following, we will show that $N_G(P_w) = C_G(P_w)P_w$, where P_w is a Sylow p-subgroup of $C_G(w)$. Then by using the fact that there exists some $t \in G$ such that $P_x = P_w^t$, for every r-element x of G, which is a consequence of Step 5, our claim will be proved.

First we show that $G = \bigcup_{h \in G}(C_G(P_w)P_w)^h \cup N$, where N is the subgroup that is mentioned in Step 7. Let g be a non-central element of G and write $g = g_p g_{p'}$. If $g_{p'} \in Z(G) \subseteq N$, then, since $g_p \in N$, it follows that $g \in N$, as required. If $|g_{p'}^G| = p^a$, then by applying Lemma 2.4 we get $g_{p'} \in N$, and similarly we conclude that $g \in N$. So we may assume that $|g_{p'}^G| = p^a m$ and write $g_{p'} = g_q g_r$, where g_q and g_r are the q-part and r-part of g, respectively. Therefore, $g_r \notin Z(G)$, by Step 4, and since $C_G(g_{p'}) = C_G(g_q) \cap C_G(g_r) \subseteq C_G(g_r)$, we conclude that $C_G(g_{p'}) = C_G(g_r)$. By using Step 5, there exists $h \in G$ such that $P_w^h \subseteq C_G(g_r) = C_G(g_{p'})$, whence $g_{p'} \in C_G(P_w)^h$. On the other hand, $g_p \in C_G(g_r)$, and by the fact that P_w^h is the only Sylow p-subgroup of $C_G(g_r)$ by Step 9, we conclude that $g_p \in P_w^h$. Thus we have $g \in (C_G(P_w)P_w)^h$, as required.

The above equality implies that

$$|G| \leqslant |G : N_G(C_G(P_w)P_w)|(|C_G(P_w)P_w| - 1) + |N|,$$

and as a consequence

$$1 \leqslant \frac{|C_G(P_w)P_w| - 1}{|N_G(C_G(P_w)P_w)|} + \frac{|N|}{|G|}.$$

We set $|N_G(C_G(P_w)P_w)| = n$. If $C_G(P_w)P_w < N_G(C_G(P_w)P_w)$, then

$$1 \leqslant \frac{1}{2} - \frac{1}{n} + \frac{1}{2},$$

which is a contradiction. Therefore, $N_G(C_G(P_w)P_w) = C_G(P_w)P_w$, and so it is easy to obtain $N_G(P_w) = C_G(P_w)P_w$, as desired.

Step 12. Let R be a Sylow r-subgroup of H. Then there exists a Sylow p-subgroup P_w of $C_G(w)$ such that $R \subseteq C_G(P_w)$.

Let $x \in R$ be a non-central r-element. Since $R \subseteq C_G(w)$, we obtain $C_G(wx) = C_G(w) \cap C_G(x)$, so we conclude that $C_G(x) \subseteq C_G(w)$. Therefore, there exists a Sylow p-subgroup of $C_G(w)$, say P_w, such that $P_w \in \mathrm{Syl}_p(C_G(x))$.

Now let $\alpha \in R$ be a non-central element. Since $R/Z(G)_r$ is abelian, we have $[x, \alpha] \in Z(G)$. It follows that $x^\alpha = xz$ for some element $z \in Z(G)$. Therefore, $C_G(x)^\alpha = C_G(x)$ and so $\alpha \in N_G(C_G(x))$ and we deduce that $\alpha \in N_G(P_w)$. Therefore, by using the previous step we get $\alpha \in C_G(P_w)P_w$. By the fact that $C_G(P_w)$ is a normal subgroup of $N_G(P_w) = C_G(P_w)P_w$ whose index is a p-number, we conclude that it contains all p'-elements of $N_G(P_w)$. In particular, $\alpha \in C_G(P_w)$, and so $R \subseteq C_G(P_w)$, as required.

Step 13. G is r-nilpotent.

Set $\bar{G} = G/\boldsymbol{Z}(G)_r$. Also, in the following we use $\bar{T} = T/\boldsymbol{Z}(G)_r$. Take R to be a Sylow r-subgroup of G. We shall show that

$$\bar{G} = \bigcup_{\bar{h} \in \bar{G}} \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}}) \cup \bar{N},$$

where N is the normal subgroup mentioned in Step 7.

Let $g = g_p g_{p'}$ be an element of G. If $g_{p'} \in \boldsymbol{Z}(G)$, then $\bar{g} \in \bar{N}$. So assume that $g_{p'} \notin \boldsymbol{Z}(G)$. If $|g_{p'}^G| = p^a$, then by Lemma 2.4 we have $g_{p'} \in \boldsymbol{O}_{pp'}(G) \subseteq N$, so $\bar{g} \in \bar{N}$. Thus we assume that $|g_{p'}^G| = p^a m$ with $g_{p'} = g_q g_r$, where g_q and g_r are the q-part and r-part of g, respectively. So $g_r \notin \boldsymbol{Z}(G)$, and we deduce that $\boldsymbol{C}_G(g_{p'}) = \boldsymbol{C}_G(g_r) \subseteq \boldsymbol{C}_G(g_q)$. There then exists $h \in G$ such that $g_r \in R^h \subseteq \boldsymbol{C}_G(g_q)$, whence $\bar{g}_q \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. Moreover, $\bar{g}_r \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$, since $\bar{R}^{\bar{h}}$ is abelian. We conclude that $\bar{g}_{p'} \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. On the other hand, there exists a Sylow p-subgroup P_w of $\boldsymbol{C}_G(w)$ such that $R \subseteq \boldsymbol{C}_G(P_w)$, by Step 12. So $g_r \in R^h \subseteq \boldsymbol{C}_G(P_w)^h$, which implies that P_w^h is the Sylow p-subgroup of $\boldsymbol{C}_G(g_r)$, and by Step 9 we have $g_p \in \boldsymbol{C}_G(R^h)$, and hence $\bar{g}_p \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. So $\bar{g} \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$, as desired. Thus, we have proved that

$$\bar{G} = \bigcup_{\bar{h} \in \bar{G}} \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}}) \cup \bar{N}.$$

This implies that

$$|\bar{G}| \leqslant |\bar{G} : \boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))|(|\boldsymbol{C}_{\bar{G}}(\bar{R})| - 1) + |\bar{N}|,$$

and hence

$$1 \leqslant \frac{|\boldsymbol{C}_{\bar{G}}(\bar{R})| - 1}{|\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))|} + \frac{|\bar{N}|}{|\bar{G}|}.$$

We set $|\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))| = n$. If we assume that $\boldsymbol{C}_{\bar{G}}(\bar{R}) < \boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))$, then we obtain the following contradiction:

$$1 \leqslant \frac{1}{2} - \frac{1}{n} + \frac{1}{2}.$$

Therefore, $\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R})) = \boldsymbol{C}_{\bar{G}}(\bar{R})$ and consequently $\boldsymbol{N}_{\bar{G}}(\bar{R}) = \boldsymbol{C}_{\bar{G}}(\bar{R})$. Now, by using Burnside's Theorem (see, for example, [**12**, 10.1.8]), we get that $\bar{G}$ is r-nilpotent. So G is r-nilpotent too, as required.

Step 14. Final contradiction.

Let R be a Sylow r-subgroup of H. By Step 12 there exists a Sylow p-subgroup P_w of $\boldsymbol{C}_G(w)$ such that $R \subseteq \boldsymbol{C}_G(P_w)$, whence $R \subseteq \boldsymbol{N}_G(P_w)$. On the other hand, by Step 13, G has a normal r-complement K, and so it is obvious that $K \cap \boldsymbol{N}_G(P_w)$ is normal in $\boldsymbol{N}_G(P_w)$. Hence, R acts coprimely on $K \cap \boldsymbol{N}_G(P_w)$. By coprime action properties, there exists an R-invariant Sylow p-subgroup of $\boldsymbol{N}_G(P_w)$, say P_1. Note that P_w is a normal subgroup of $\boldsymbol{N}_G(P_w)$ and so P_w is contained in P_1. Hence, $P_w \subseteq P_1 \subseteq P$ for some Sylow p-subgroup P of G, and consequently $P_1 = \boldsymbol{N}_P(P_w)$.

Note that $\boldsymbol{N}_P(P_w)/P_w$ is non-trivial. Otherwise $\boldsymbol{N}_P(P_w) = P_w$, and P_w would therefore be a Sylow p-subgroup of G, which is impossible because $|w^G| = p^a$ and $a > 0$. We

claim that $\bar{R} = R/\boldsymbol{Z}(G)_r$ acts fixed-point-freely on

$$\widetilde{\boldsymbol{N}_P(P_w)} = \boldsymbol{N}_P(P_w)/P_w,$$

and so, by a well-known result, $\bar{R}$ is either cyclic (which is impossible) or a generalized quaternion group, which contradicts Step 10.

Suppose that $\tilde{x}^{\bar{t}} = \tilde{x}$ for some $x \in \boldsymbol{N}_P(P_w)$ and some $t \in R$. We can assume that x belongs to $\boldsymbol{C}_P(P_w)$ since, using Step 11, we have $\boldsymbol{N}_P(P_w) = \boldsymbol{C}_P(P_w)P_w$. Then $[x,t] \in P_w$. In particular, $[x,t]$ centralizes x and t. Moreover, as x is a p-element and t is an r-element, we have $1 = [x, t^{o(t)}] = [x^{o(t)}, t] = [x,t]^{o(t)}$. However, $[x,t]$ is a p-element, and this implies that $[x,t] = 1$, that is, $x \in \boldsymbol{C}_G(t)$. By the fact that $t \in R \subseteq \boldsymbol{C}_G(P_w)$, we deduce that P_w is the only Sylow p-subgroup of $\boldsymbol{C}_G(t)$, and so $x \in P_w$, that is, $\tilde{x} = 1$, and the action is fixed-point-free, as desired. $\square$

Examples. In the following we give some examples of the cases of Theorems A and B.

- Let $G = \boldsymbol{Z}_5 \rtimes Q_8$ be the semidirect product of the group $\boldsymbol{Z}_5 = \langle x \rangle$ acted on by the quaternion group $Q_8 = \langle y, z : y^4 = 1, y^2 = z^2, y^z = y^{-1} \rangle$ such that $x^y = x^{-1}$ and $x^z = x$. Then it is easy to see that the set of 5-regular conjugacy class sizes of G is equal to $\{1, 2, 10\}$. This provides an example of a group described in Theorem A.

- Let $G = (\boldsymbol{Z}_7 \times Q_8) \rtimes \boldsymbol{Z}_3$, and further let $\boldsymbol{Z}_7 = \langle x \rangle$, $Q_8 = \langle y, z : y^4 = 1, y^2 = z^2, y^z = y^{-1} \rangle$ and $\boldsymbol{Z}_3 = \langle w \rangle$, where $x^w = x^2$, $y^w = z^5$ and $z^w = z^3 y$. One can easily check that the set of the conjugacy class sizes of 3-regular elements of G is $\{1, 3, 6\}$, which is an example of case (i) of Theorem B.

- The group $\Gamma(8)$, whose set of 7-regular class sizes is exactly $\{1, 7, 28\}$ (see, for example, [**9**, p. 147]), provides an example of case (ii) of Theorem B.

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A HIGHER CATEGORY APPROACH TO TWISTED ACTIONS ON C^*-ALGEBRAS

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Abstract C^*-algebras form a 2-category with $*$-homomorphisms or correspondences as morphisms and unitary intertwiners as 2-morphisms. We use this structure to define weak actions of 2-categories, weakly equivariant maps between weak actions and modifications between weakly equivariant maps. In the group case, we identify the resulting notions with known ones, including Busby–Smith twisted actions and the equivalence of such actions, covariant representations and saturated Fell bundles. For 2-groups, weak actions combine twists in the sense of Green, and Busby and Smith.

The Packer–Raeburn Stabilization Trick implies that all Busby–Smith twisted group actions of locally compact groups are Morita equivalent to classical group actions. We generalize this to actions of strict 2-groupoids.

1. Introduction

An automorphism of a C^*-algebra A is called *inner* if it is of the form $\mathrm{Ad}_u \colon a \mapsto uau^*$ for some unitary multiplier u of A. Inner automorphisms act trivially on K-theory and all other interesting invariants for C^*-algebras; more precisely, they act trivially on a functor F if the corner embedding $F(A) \to F(\mathbb{M}_2(A))$ is invertible for all A. If two automorphisms α_1 and α_2 of a C^*-algebra A differ by an inner automorphism, $\alpha_2 = \mathrm{Ad}_u \circ \alpha_1$, then their crossed product C^*-algebras are isomorphic.

While these statements suggest that we may simply ignore inner automorphisms, this is false for representations of more general groups. For instance, any automorphism of the