

DUAL TOPOLOGIES ON NON-ABELIAN GROUPS

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Dedicated to Dikran Dikranjan

ABSTRACT. The notion of locally quasi-convex abelian group, introduced by Vilenkin, is extended to maximally almost-periodic non-necessarily abelian groups. For that purpose, we look at certain bornologies that can be defined on the set $\text{rep}(G)$ of all finite dimensional continuous representations on a topological group G in order to associate well behaved group topologies (dual topologies) to them. As a consequence, the poset of all Hausdorff totally bounded group topologies on a group G is shown to be isomorphic to the lattice of certain special subsets of $\text{rep}(G_d)$. Moreover, generalizing some ideas of Namioka, we relate the structural properties of the dual topological groups to topological properties of the bounded subsets belonging to the associated bornology. In like manner, certain type of bornologies that can be defined on a group G allow one to define canonically associated uniformities on the dual object \widehat{G} . As an application, we prove that if for every dense subgroup H of a compact group G we have that \widehat{H} and \widehat{G} are uniformly isomorphic, then G is metrizable. Thereby, we extend to non-abelian groups some results previously considered for abelian topological groups.

1. INTRODUCTION AND MOTIVATION

In this paper we investigate the application of Pontryagin duality methods to non-abelian groups and our main goal is to study general groups using these techniques.

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Although there are many ways of approaching non-abelian duality, here, we shall explore a method that is based on the duality theory of compact groups established by Tannaka and Kreĭn (see [18, 10] and the references therein).

Duality methods are well known and widely used in the study of topological abelian groups and the literature on this subject is vast. This program has not been accomplished for non-abelian groups and there are several reasons that explain this asymmetric development between both theories. No doubt, the non-abelian context is more involved since the *dual object* is no longer a topological group and it seems difficult to equip it with a structure which will be sufficiently general and tractable. Notwithstanding this, it is known, as a consequence of the celebrated Gel'fand and Raĭkov Theorem, that the set of all unitary representations of a locally compact group G contains the information necessary to recover the topological and algebraic structure of the group (see [10]). In this paper we look at the duality theory of Maximally Almost Periodic groups (MAP groups for short) and their finite dimensional representations. Our main goal is to develop this theory in order to apply duality methods for non-abelian groups. A main tool here is the unitary duality introduced by Tannaka, Kreĭn, Hochschild and Mostow for compact groups, and extended subsequently by different authors [5, 19, 26, 28, 27, 30, 13, 14, 16, 29].

In this line, the notion of locally quasi-convex abelian group, introduced by Vilenkin, is extended to maximally almost-periodic non-necessarily abelian groups. For that purpose, we look at certain bornologies that can be defined on the set $\text{rep}(G)$ of all finite dimensional continuous representations on a topological group G in order to associate well behaved group topologies (dual topologies) to them. As a consequence, the lattice

of all Hausdorff totally bounded group topologies on a group G is shown to be isomorphic to the lattice of certain special subsets of $\text{rep}(G_d)$. Moreover, generalizing some ideas of Namioka, we relate the structural properties of the dual topological groups to topological properties of the bounded subsets belonging to the associate bornology. In like manner, certain type of bornologies that can be defined on a group G allow one to define canonically associate uniformities on the dual object \widehat{G} . As an application, we prove that if for every dense subgroup H of a compact group G we have that \widehat{H} and \widehat{G} are uniformly isomorphic, then G is metrizable. Thereby, we extend to non-abelian groups some results previously considered for abelian topological groups.

2. BASIC DEFINITIONS AND TERMINOLOGY

All (topological) groups are assumed to be (Hausdorff and) maximally almost periodic (MAP, for short); that is, groups that can be continuously injected homomorphically into compact groups. For any group G , let e_G denote the neutral element of the group.

Let $\mathbb{U}(n)$ denote the *unitary group of order n* , namely, the group of all complex-valued $n \times n$ matrices A for which $A^{-1} = A^*$. Then $\mathbb{U}(n)$ is a compact Lie group and can be realized as the group of isometries of \mathbb{C}^n . We endow $\mathbb{U}(n)$ with the metric coming from the operator norm. More specifically

$$d(\varphi, \psi) = \|\varphi - \psi\| = \|\varphi\psi^{-1} - I_n\|$$

Set $\mathbb{U} = \bigsqcup_{n < \omega} \mathbb{U}(n)$ (topological/uniform sum).

A *unitary representation* φ of the (topological) group G is a (continuous) homomorphism into the group of all linear isometries of a complex Hilbert space \mathcal{H} . When

$\dim \mathcal{H} < \infty$, we say that φ is a *finite dimensional representation*; in this case, φ is a homomorphism into one of the groups $\mathbb{U}(n)$.

A linear subspace $E \subseteq \mathbb{C}^n$ is an *invariant* subspace for $\mathcal{S} \subseteq \mathbb{U}(n)$ if $ME \subseteq E$ for all $M \in \mathcal{S}$. If there is a subspace E with $\{0\} \subsetneq E \subsetneq \mathbb{C}^n$ which is invariant for \mathcal{S} , then \mathcal{S} is called *reducible*; otherwise \mathcal{S} is *irreducible*. An *irreducible representation* of G is a unitary representation φ such that $\text{ran}(\varphi)$ is irreducible.

If G is a compact group, the Peter-Weyl Theorem (see [20]) implies that the finite dimensional irreducible representations of G yield a complete analysis of G . The set of all n -dimensional continuous unitary representations on G will be denoted by $\text{rep}_n(G)$ and the symbol $\text{rep}(G) = \bigcup_{n < \omega} \text{rep}_n(G)$ denotes the set of all finite dimensional continuous unitary representations. In like manner, the set of all irreducible n -dimensional continuous unitary representations on G will be denoted by $\text{irrep}_n(G)$ and $\text{irrep}(G) = \bigcup_{n < \omega} \text{irrep}_n(G)$.

Two n -dimensional representations φ and ψ are *equivalent* ($\varphi \sim \psi$) when there is M in $\mathbb{U}(n)$ such that $\varphi(x) = M^{-1}\psi(x)M$ for all $x \in G$. This defines an equivalence relation on the set of all n -dimensional representations of the group. The quotient set $\widehat{G}_n = \frac{\text{irrep}_n(G)}{\sim}$ consists of the equivalence classes of irreducible representations of dimension n . Denote by $\widehat{G} = \bigcup_{n < \omega} \widehat{G}_n$ the collection of all equivalence classes $[\varphi]$ of irreducible representations φ of G . One calls \widehat{G} the *dual object* of G . Following von Neumann [25], in order to deal with \widehat{G} , one must make an arbitrary choice of an irreducible representation in each equivalence class.

When convenient, the symbol G_τ will denote the group G equipped with a group topology τ and, more generally, the notions related to G_τ must be understood in this context. For example, the set of all *continuous* n -dimensional unitary representations

on G_τ will be denoted by $\text{rep}_n(G_\tau)$ and so on. The symbol G_d will denote the algebraic group G equipped with the discrete topology.

We now collect some well-known facts about uniform spaces. The basic definitions and terminology are taken from [31].

Let X be a set and $\Delta = \Delta(X) \subseteq X \times X$ the diagonal on X . For $B, C \subseteq X \times X$, $S \subseteq X$ we define $B \circ C = \{(x, z) \in X \times X : (x, y) \in B \text{ and } (y, z) \in C \text{ for some } y \in X\}$, $B^{-1} = \{(x, y) : (y, x) \in B\}$, $B[S] = \{y \in X : (x, y) \in B \text{ for some } x \in S\}$. A *uniformity* on X is a set μ of subsets of $X \times X$ satisfying the following conditions: (i) $B \supseteq \Delta$ for all $B \in \mu$; (ii) if $B \in \mu$, then $B^{-1} \in \mu$; (iii) if $B \in \mu$, there exists $C \in \mu$ such that $C \circ C \subseteq B$; (iv) the intersection of two members of μ also belongs to μ ; (v) any subset of $X \times X$, which contains a member of μ , itself belongs to μ ; (vi) $\Delta = \bigcap \{B : B \in \mu\}$. If the latter condition is not satisfied we say that μ is a *pseudo-uniformity* on X .

The members of μ are called *bands* (of the uniformity). By a *base* for a uniformity μ is meant a subset \mathfrak{B} of μ such that a subset of $X \times X$ belongs to μ if and only if it contains a set belonging to \mathfrak{B} . A *uniform space* μX is a pair comprising a set X and a uniformity μ on X . If μX is a uniform space, one may define a topology τ on X by assigning to each point x of X the neighborhood base comprised of the sets $B[x]$, B ranging over the uniformity. The topological space associated to μX will be denoted by $X_{\tau\mu}$. Given two covers \mathcal{U} and \mathcal{V} of X we say that \mathcal{U} *refines* \mathcal{V} if for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $U \subseteq V$. A cover \mathcal{U} of X is named *μ -uniform* if there exists $B \in \mu$ such that the cover $\{B[x] : x \in X\}$ refines \mathcal{U} .

A topological group G is said to be *totally bounded* when for every neighborhood U of the neutral element in G , there is $F \subseteq G$ with $|F| < \aleph_0$ such that $FU = G$. This notion has also meaning for arbitrary cardinal numbers in the following manner: let κ

be a cardinal number, following the terminology of Arhangel'skii and Tkachenko (see [1]), we say that G is κ -*narrow* when for every neighborhood U of the neutral element in G , there is $F \subseteq G$ with $|F| \leq \kappa$ such that $FU = G$. These definitions extend to uniform spaces in a natural way.

Definition 2.1. Let μX be a uniform space, we say that μX is *totally bounded* (resp. κ -*narrow*) when every μ -uniform cover of X has a subcover of finite cardinality (resp. of cardinality less or equal than κ). The *Lindelöf number* of a uniform space μX is defined as the less cardinal κ such that μX is κ -narrow.

It is known that a uniform space μX is κ -narrow if and only if it contains no uniformly discrete subspace of power κ (see [21, p. 24]). From this fact, it follows:

Proposition 2.2. *Every (uniform) subspace of a κ -narrow uniform space is κ -narrow.*

In this paper, a *bornology* \mathcal{B} on a space X designates an ideal of subsets of X that contains all finite subsets of X (denoted $Fin(X)$). That is to say, \mathcal{B} satisfies the following properties:

- (i) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$;
- (ii) $A \in \mathcal{B}$ and $B \subseteq A$ implies $B \in \mathcal{B}$;
- (iii) $Fin(X) \subseteq \mathcal{B}$.

Let X be a set and let M be a metrizable space. We denote by M^X (resp. $l^\infty(X, M)$) the set of all (resp. *bounded*) functions from X to M . Suppose that \mathcal{B} is a bornology on X and $F \subseteq M^X$ is pointwise bounded. Then we may equip F with the uniformity $\mu_{\mathcal{B}}$, which has as a uniformity base the bands

$$B(K, \epsilon) = \{(f, g) \in F^2 : d(f(x), g(x)) < \epsilon, \forall x \in K\}$$

where $\epsilon > 0$, $K \in \mathcal{B}$ and d is the metric defined on M .

If $F \subseteq l^\infty(X, M)$, there also exists a metric uniformity μ_∞ that can be defined canonically on F , which is generated by the bands $\{B(X, \epsilon) : \epsilon > 0\}$. The metric associated to μ_∞ is defined for all $(f, g) \in F^2$ by

$$d_\infty(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

We assume that $l^\infty(X, M)$ is equipped with this metric in principle.

From here on, we shall make use of the *evaluation mapping* \mathbf{e}_F that is associated to every $F \subseteq M^X$ as follows:

$$\begin{aligned} \mathbf{e}_F : X &\hookrightarrow M^F \\ x &\mapsto \mathbf{e}_F(x)(f) = f(x) \end{aligned}$$

Furthermore, if F is pointwise bounded, it follows that $\mathbf{e}_F(X) \subseteq l^\infty(F, M)$. Therefore, we may equip X with the pseudo-metric d_F inherited from $l^\infty(F, M)$, which is defined by

$$d_F(x, y) = \sup_{f \in F} d(f(x), f(y)) \quad \forall (x, y) \in X^2.$$

The pseudo-uniformity associated to d_F is denoted by μ_F . The symbol X_F denotes the space X equipped with the pseudo-metric d_F . We also identify X_F to $X_{\tau\mu_F}$ for the sake of simplicity.

Given a topological space X and a metrizable space M , let $C(X, M)$ be the set of all continuous functions from X to M . We denote by $\mu_{\mathcal{K}}$ the uniformity on $C(X, M)$ generated by the bornology $\mathcal{B} = \mathcal{K}(X)$ generated by all compact subsets of X .

3. VARIATIONS ON A THEOREM BY NAMIOKA

The starting point of this section is the following generalization of a theorem by Namioka in [24] about the relationship between the metrizability and separability of certain spaces. In order to do this, we need some preliminary notational observations.

Let X , F and M be sets and let $\Phi : X \times F \longrightarrow M$ be an arbitrary map. Identifying the elements f, g in F such that $\Phi(x, f) = \Phi(x, g)$ for all $x \in X$, we assume without loss of generality that every element $f \in F$ is uniquely determined by the map $\Phi_f \in M^X$, defined as $\Phi_f(x) = \Phi(x, f)$. Therefore, with some notational abuse, the set F may be looked at as a subset of M^X . With this assumption the symbol $cl_{M^X} F$ denotes the set $cl_{M^X} \{\Phi_f : f \in F\}$.

The *density* $d(X)$ of a topological space X is the minimal cardinality of a dense subset of X . The *weight* $w(X)$ is the minimal cardinality of an open basis for the topology of X . We have $w(X) = d(X)$ for infinite metrizable spaces X .

Theorem 3.1. *Let X, F be sets and let M be a metrizable space. If $\Phi : X \times F \longrightarrow M$ is a map such that $K = cl_{M^X} F$ is compact. Then the following assertions are true:*

- (a) *If $d(X_F) \leq \kappa$, then $w(K) \leq \kappa$.*
- (b) *If $w(M) \cdot w(K) \leq \kappa$, then $d(X_F) \leq \kappa$.*

(*Sketch*). (a) Because X_F is equipped with the topology of uniform convergence, it follows that F is equicontinuous. Therefore also K is equicontinuous on X . Thus any dense subset A of X_F with $|A| \leq \kappa$ separates the points of K . The proof of (b) is clear. □

Next we present several consequences that follow from the result above. In the rest of this section, F denotes a pointwise bounded subset of M^X .

Corollary 3.2. *Let μX be a κ -narrow uniform space and let M be a metrizable space. If F is a uniformly equicontinuous and relatively pointwise compact subset of M^X , then $w(F) \leq \kappa$.*

Proof. Observe that, if F is uniformly equicontinuous, then $\mathbf{e}_F : \mu X \longrightarrow l^\infty(F, M)$ is uniformly continuous (indeed, for every $\epsilon > 0$ there exists $B_\epsilon \in \mu$ such that if $(x, y) \in B_\epsilon$ then $d(f(x), f(y)) < \epsilon \forall f \in F$, that is, $(\mathbf{e}_F(x), \mathbf{e}_F(y)) \in B(X, \epsilon)$). Therefore the uniformity μ is finer than μ_F , which implies that $\mu_F X$ is κ -narrow. Since $\mu_F X$ is pseudo-metrizable, it follows that $d(X_F) \leq \kappa$. It will suffice now to apply Theorem 3.1 to the map $\Phi(x, f) = f(x)$ in order to finish the proof. \square

In the sequel, we are concerned with a subset Y of M^X which has associated a certain bornology \mathcal{B} and we assume further that X is equipped with the uniformity of uniform convergence on the members of \mathcal{B} . Firstly, we need a technical lemma that we think is probably known. Nevertheless, we include a proof of it here because it is essential in the rest of the paper.

Proposition 3.3. *The product of an arbitrary family of κ -narrow uniform spaces is a κ -narrow uniform space.*

Proof. Since a product has the weak uniformity generated by projections, it will suffice to prove that the product of two κ -narrow uniform spaces is κ -narrow. Now, let μX and νY be κ -narrow. If \mathcal{U} is a $(\mu \times \nu)$ -uniform cover of $X \times Y$, then \mathcal{U} must have a refinement of the form $\mathcal{U}_1 \times \mathcal{U}_2 = \{U_1 \times U_2 : U_i \in \mathcal{U}_i, 1 \leq i \leq 2\}$, where \mathcal{U}_1 is μ -uniform and \mathcal{U}_2 is ν -uniform. By hypothesis there is a subcover \mathcal{V}_i of \mathcal{U}_i such that $|\mathcal{V}_i| \leq \kappa$, $1 \leq i \leq 2$. Then $\mathcal{V}_1 \times \mathcal{V}_2$ is a $(\mu \times \nu)$ -uniform refinement of $\mathcal{U}_1 \times \mathcal{U}_2$ and has cardinality less than or equal to κ . This completes the proof. \square

Theorem 3.4. *Let X be a set, let M be a metrizable space, and let Y be a subset of M^X that is equipped with some bornology \mathcal{B} consisting of pointwise relatively compact*

sets. If $\mu_{\mathcal{B}}$ is the uniformity on X defined by $\mu_{\mathcal{B}} = \sup\{\mu_F : F \in \mathcal{B}\}$ and furthermore $w(M) \leq \kappa$, then $\mu_{\mathcal{B}}X$ is κ -narrow if and only if $w(F) \leq \kappa \forall F \in \mathcal{B}$.

Proof. (\Rightarrow) If $\mu_{\mathcal{B}}X$ is κ -narrow then every $\mu_F X$ is also κ -narrow and, since $\mu_F X$ is pseudo-metrizable, it follows that $d(X_F) \leq \kappa$. By Theorem 3.1, we obtain $w(F) \leq \kappa$ for all $F \in \mathcal{B}$.

(\Leftarrow) Since $w(F) \leq \kappa$ for all $F \in \mathcal{B}$, we deduce by Theorem 3.1 again that $d(X_F) \leq \kappa$ for all $F \in \mathcal{B}$. Thus $\mu_F X$ is κ -narrow for all $F \in \mathcal{B}$. Since $\mu_{\mathcal{B}}X$ is the diagonal (uniform) subspace of $\prod_{F \in \mathcal{B}} \mu_F X$, it follows by the preceding proposition that $\mu_{\mathcal{B}}X$ is κ -narrow too. \square

As a consequence, we obtain the following corollary for general topological spaces.

Corollary 3.5. *If X is a κ -Lindelöf space and F is an equicontinuous subset of $C(X, M)$ then $w(F) \leq \kappa$.*

4. DUALY REPRESENTABLE MAP GROUPS

Assume that our overall goal now is to study the group topologies that can be defined using finite dimensional representations. Because we are interested in Hausdorff topologies, it will suffice to deal only with groups whose finite dimensional representations separate points (MAP groups). Firstly, it is necessary to remember which collections of representations identify a set of *continuous* representations for some group topology. Our approach here is based on the celebrated Tannaka-Kreĭn duality, where we have followed the well known text by Hewitt and Ross [18] although this theory is currently treated using categorical methods. We recommend the article by Joyal and Street [22] that is illuminating in this regard.

Let G be a topological group and $\Gamma \subseteq \text{rep}(G)$ be a separating subset, that is, for all $e_G \neq x \in G$ there exists $\varphi \in \Gamma$ such that $\varphi(x)$ is not the identity matrix. If Γ_n denotes the set of all n -dimensional elements in Γ , we have $\Gamma = \bigcup_{n < \omega} \Gamma_n$ and we say that Γ is a *representation space* for G if it satisfies:

- (1) $\varphi_1 \in \Gamma_n, \varphi_2 \in \Gamma_m \Rightarrow \varphi_1 \oplus \varphi_2 \in \Gamma_{n+m}, \varphi_1 \otimes \varphi_2 \in \Gamma_{nm}$;
- (2) if $\varphi \in \Gamma_n$ and $M \in \mathbb{U}(n)$, then $\bar{\varphi}$ and $M^{-1}\varphi M$ belong to Γ_n ;
- (3) if $\Gamma \ni \varphi = \bigoplus_{j=1}^m \varphi_j$ then $\varphi_j \in \Gamma$ for all $1 \leq j \leq m$;
- (4) $1 \in \Gamma_1$.

We notice that (4) follows from (1)-(3). It is also possible to define representation spaces using only equivalence classes of irreducible representations. Moreover, the notion of *Kreĭn algebra* makes possible to define the notion of representation space without referring to any specific group G .

A bornology \mathcal{B} of subsets in Γ (and consequently $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}(\Gamma_n)$ is a bornology in Γ_n) is named (*representation*) *bornology* when it satisfies the following properties:

- (1) for every $P \in \mathcal{B}$, there is a natural number N_P such that $P = \bigcup_{n < N_P} P_n$, with $P_n \in \mathcal{B}_n$ and some P_n may be the empty set;
- (2) if $P \in \mathcal{B}_n, Q \in \mathcal{B}_m \Rightarrow P \oplus Q \in \mathcal{B}_{n+m}, P \otimes Q \in \mathcal{B}_{nm}$.

The pair (Γ, \mathcal{B}) allows us to equip G with a topology $t(\mathcal{B})$ that has as a neighborhood base at the identity the family of sets $[P, \epsilon]$ where

- (1) $[P, \epsilon] = G$ if $P = \emptyset$;
- (2) $[P, \epsilon] = \{g \in G : \|\varphi(g) - I_n\| \leq \epsilon \text{ for all } \varphi \in P\}$ if $\emptyset \neq P$ belongs to \mathcal{B}_n ; and
- (3) $[P, \epsilon] = \bigcap_{n < N_P} [P_n, \epsilon]$ if $P = \bigcup_{n < N_P} P_n \in \mathcal{B}$.

Here I_n is the identity matrix in $\mathbb{U}(n)$ and $\epsilon > 0$ for every $n \in \mathbb{N}$.

For instance, we shall make use of the topology $t(\text{Fin}(\text{rep}(G)))$ associated to the bornology $\text{Fin}(\text{rep}(G))$ consisting of all finite subsets of $\text{rep}(G)$. In case that G is a compact group, the celebrated Peter-Weyl Theorem establishes that the groups G and $G_{t(\text{Fin}(\text{rep}(G)))}$ are topologically isomorphic. Observe that the bornology $\text{Fin}(\text{rep}(G))$ defines the initial, or weak, (group) topology generated by $\text{rep}(G)$. In this case, we may say that $\text{Fin}(\text{rep}(G))$ is the smallest bornology on $\text{rep}(G)$. Moreover, for any algebraic group G , there exists a biggest bornology \mathcal{B}^+ that is generated by the collection $\{\text{rep}_n(G_d) : n < \omega\}$. The group $G_{t(\mathcal{B}^+)}$ is not necessarily discrete as it will be seen below (cf. [16]). If G_τ is a topological group and \mathcal{E} denotes the bornology of equicontinuous subsets on $\text{rep}(G)$, then $t(\text{Fin}(\text{rep}(G))) \subseteq t(\mathcal{E}) \subseteq \tau$.

The proof of the following result is clear.

Proposition 4.1. *$(G, t(\mathcal{B}))$ is a Hausdorff topological group.*

A topology $t(\mathcal{B})$ on a group G which is associated to a bornology \mathcal{B} on a representation space Γ is named *dually induced topology* and we say that $G_{t(\mathcal{B})}$ is a *dually induced topological group*. This notion has been extensively studied for abelian groups and it turns out that most important classes of topological abelian groups possesses dually induced topologies. The main goal of this paper is the understanding of the dually induced group topologies for non-abelian MAP groups.

It is not difficult to verify that for every group G and representation space Γ , the bornology $\text{Fin}(\Gamma)$ defines a totally bounded topology on G . Conversely, we show next that every totally bounded group topology is a dually induced topology. This is our version of a well known result given by Comfort and Ross for totally bounded abelian groups (cf. [8]).

Theorem 4.2. *For any group G , the poset of (Hausdorff) totally bounded group topologies on G is bijectively isomorphic to the poset of representation spaces of G .*

Proof. Let $\mathcal{HTB}(G)$ denote the poset of Hausdorff totally bounded group topologies on G and let $\mathcal{RS}(G)$ be the poset of representation spaces on G . We define

$$\Phi : \mathcal{HTB}(G) \longrightarrow \mathcal{RS}(G) \text{ by } \Phi(\tau) = \text{rep}(G_\tau)$$

and

$$\Psi : \mathcal{RS}(G) \longrightarrow \mathcal{HTB}(G) \text{ by } \Psi(\Gamma) = t(\text{Fin}(\Gamma)).$$

Clearly, these maps are well defined because the groups $\mathbb{U}(n)$ are compact. It will suffice to verify that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ coincide with the identity mapping in order to finish the proof. Suppose that $\tau \in \mathcal{HTB}(G)$ and let K be the Weil completion of G_τ , that is a compact group. The groups K and G_τ possess the same set of continuous representations $\text{rep}(G_\tau)$ and, by Peter-Weyl Theorem, it follows that K is equipped with the topology $t(\text{Fin}(\text{rep}(G_\tau)))$. Therefore, since G_τ is dense in K , we have $\tau = t(\text{Fin}(\text{rep}(G_\tau)))$, which shows that $\Psi \circ \Phi$ is the identity map. Now, let $\Gamma \in \mathcal{RS}(G)$ and set $\tau = t(\text{Fin}(\Gamma)) = \Psi(\Gamma)$. Again, let K be the compact group obtained as the Weil completion of G_τ . By continuity, it is clear that every element $\varphi \in \Gamma$ can be extended to a continuous representation φ^K on K . Thus, we have that the representation space $\Gamma^K = \{\varphi^K : \varphi \in \Gamma\}$ is canonically included in $\text{rep}(K)$. Moreover, if $e \neq x \in K$, there is a neighbourhood of the neutral element U such that $x \notin U^2$. Take $F \in \text{Fin}(\Gamma)$ and $\epsilon > 0$ such that $[F, \epsilon] \subseteq U \cap G$. Using a standard continuity argument, it is readily seen that $\|\varphi^K(x) - I\| \geq \epsilon/2$. Thus the set Γ^K satisfies (1)-(4) and separates the points in K . By Tannaka duality this means that $\Gamma^K = \text{rep}(K)$ (see [22]). Since K and G_τ possess the same continuous representations, it follows that $\Phi \circ \Psi$ is the identity and this completes the proof. \square

Corollary 4.3. *The bornology $Fin(rep(G_d))$ gives the largest totally bounded group topology on G .*

Unfortunately, the situation is not so nice for more general classes of groups. In the following example (due to Moran) it is shown that there are discrete MAP groups that are not dually induced topological groups.

Let (p_n) be an infinite sequence of distinct primes numbers ($p_n > 2$) and let G_n be the projective special linear group of dimension two over the Galois field $GF(p_n)$ of order p_n . If $G = \prod_{n=1}^{\infty} G_n$ equipped with discrete topology, then $G_{t(\mathcal{B}^+)} = \prod_{n=1}^{\infty} G_n$ (with product topology).

In general, if G is a compact group such that $bG_d = G$, then G_d is not a dually induced topological group. Indeed, it is known (see [14, 13]) that, for these groups, $rep(G_d) = rep(G)$ and moreover each $(\widehat{G_d})_n$ is finite for all $n < \omega$. As a consequence, the same is true for any maximal subset of pairwise non-equivalent collection of n -dimensional irreducible representations, say $\Gamma(G_d)_n$. Since every finite-dimensional representation is a direct sum of irreducible representations, this means that the group $G_{t(\mathcal{B}^+)}$ associated to the biggest bornology \mathcal{B}^+ on (the algebraic group) G is topologically isomorphic to $G_{t(Fin(rep(G)))}$ which, in turn, is topologically isomorphic to the compact group G by Peter-Weyl Theorem. This implies that there is just one dually induced topology on the (algebraic) group G ; namely, its compact group topology. Therefore, the group G_d is not a dually induced topological group.

The situation for finitely generated discrete groups has been clarified completely by A. Thom (see [32] and [15]).

Theorem 4.4 (Thom, 2010). *A finitely generated discrete group G is a dual topological group if and only if G is abelian by finite.*

We do not know in general which discrete groups are dual topological groups. But we notice that there are infinite discrete groups which are dual topological groups (see [16]). We denote by $\exp(G)$ the exponent of a group G .

Theorem 4.5. *Let $G = \sum_{i \in I} F_i$, where F_i is a finite simple non-abelian group for each $i \in I$. Then the discrete group G is dual if and only if the set $\{\exp(F_i) : i \in I\}$ is bounded.*

The examples above show that it is necessary to consider infinite dimensional representations if one wants to apply duality methods for *all* locally compact groups. Our methods here only deal with groups whose topology may be understood using finite dimensional representations.

We will now look at several methods of defining dually induced topologies. In the sequel, G is a MAP group, Γ a representation space for G and \mathcal{B} is a bornology on Γ . For any cardinal number κ , we define \mathcal{B}_κ to be the ideal of subsets generated by the collection $\{F \in \mathcal{B} : w(F_{t(Fin(G))}) \leq \kappa\}$. It is readily seen that \mathcal{B}_κ is a bornology on Γ . As a consequence of Theorem 3.4, we obtain:

Proposition 4.6. *$G_{t(\mathcal{B}_\kappa)}$ is κ -narrow.*

Now, we are going to extend this approach in order to establish the existence of some dually induced topologies associated to every topological group.

Theorem 4.7. *Let G be a MAP group. Then the following assertions are satisfied.*

- (i) *For every cardinal κ , there is a largest, dually induced, κ -narrow topology $t(\kappa)$ on G . In particular, there is a largest dually induced topology on G .*

(ii) If τ is a group topology on G , then there is a largest, dually induced, κ -narrow topology τ_κ that is included in τ . In particular, there is a largest dually induced topology that is included in τ .

Proof. (i) It suffices to apply Proposition 4.6 to the representation space $\Gamma = \text{rep}(G_d)$ and the largest bornology \mathcal{B}^+ generated by the collection $\{\text{rep}_n(G_d) : n < \omega\}$. By Theorem 3.4, the topology $t(\mathcal{B}_\kappa^+)$ is κ -narrow and furthermore it is the finest κ -narrow, dually induced, topology on G . On the other hand, the biggest bornology \mathcal{B}^+ , induces the topology t^+ that is the largest dually induced topology on G .

(ii) Let κ be a cardinal number and let $\Gamma_\tau = \text{rep}(G_\tau)$. On Γ_τ consider the bornology $\mathcal{E}(\Gamma_\tau)_\kappa$ generated by the collection of all equicontinuous subsets A of $(\Gamma_\tau)_n$ for some $n < \omega$ and $w(A_{t(\text{Fin}(G))}) \leq \kappa$. These defines a dually induced topology $t(\mathcal{E}(\Gamma_\tau)_\kappa)$ on G that is weaker than τ .

On the other hand, the collection of all equicontinuous subsets A of Γ_τ such that $A \subseteq (\Gamma_\tau)_n$ for some $n < \omega$ defines a bornology $\mathcal{E}(\Gamma_\tau)$ on Γ_τ such that the topology τ^+ , induced by $\mathcal{E}(\Gamma_\tau)$ on G , is the largest dually induced topology on G that is weaker than τ . □

We recall that if G is an abelian topological group, the dual object \widehat{G} is also a topological group which coincides with $\text{irrep}(G) = \text{irrep}_1(G) = \text{CHom}(G, \mathbb{T})$.

Let G be an abelian group with no topology initially assumed on it. Given a subgroup Γ of $\text{Hom}(G, \mathbb{T})$, it is said that a topology τ on G is *compatible* with the pair $\langle G, \Gamma \rangle$ when $\widehat{G}_\tau = \Gamma$ algebraically. In recent times, there is an interest in finding compatible topologies for a given pair $\langle G, \Gamma \rangle$. We notice that if $\Gamma = \widehat{G}_\tau$ for some topology τ on G , then $t(\text{Fin}(\Gamma)) \subseteq t(\mathcal{E}(\Gamma)_\kappa) \subseteq \tau$ for each cardinal number κ . Since the

largest and the coarsest topologies in this chain have the same algebraic dual group, it follows that the topologies $t(\mathcal{E}(\Gamma)_\kappa)$ are all compatible with $\langle G, \Gamma \rangle$.

Now, suppose that G is a Čech-complete group (for instance, locally compact or complete metrizable) and let $\Gamma = \widehat{G}$. It is known that every pointwise compact subset of Γ must be equicontinuous on G (see [12]). Therefore, we obtain the following result.

Proposition 4.8. *Let G be an abelian Čech-complete group and let $\Gamma = \widehat{G}$. If \mathcal{B} is the bornology of all pointwise compact subsets of Γ , then the dually induced topologies $t(\mathcal{B}_\kappa)$ are compatible with $\langle G, \Gamma \rangle$ for all cardinal κ .*

The latter results extend in a straightforward manner to non-abelian groups. In this case, it is said that a topology τ on G is *compatible* with the representation space Γ when $\text{rep}(G_\tau) = \Gamma$.

5. UNIFORMITIES ON THE REPRESENTATION SPACES

In this section, we look at the uniformities on the dual object of a topological group that are associated to certain bornologies defined on it. Let G be a (topological) group and let \mathcal{B} be a bornology on (the set) G . We say that \mathcal{B} is a *group bornology* on G when satisfies the following properties:

- (1) if $A \in \mathcal{B}$ then $A^{-1} \in \mathcal{B}$;
- (2) if A, B belong to \mathcal{B} then AB belongs to \mathcal{B} .

In most cases \mathcal{B} means the collection of all relatively compact (resp. finite, precompact) subsets of G .

Assume now that Γ is a representation space for G and \mathcal{B} is a group bornology on the group. Since the groups $\mathbb{U}(n)$ are compact, it follows that $\Gamma_n \subseteq C(G, \mathbb{U}(n)) \cap l^\infty(G, \mathbb{U}(n))$. Therefore, we may equip Γ_n with the uniformity $\mu_{\mathcal{B}}$ associated to \mathcal{B} as

described in Section 2. The representation space Γ is now equipped with the free uniformity sum of the family $\{\mu_{\mathcal{B}}\Gamma_n : n < \omega\}$. We denote by $\mu_{\mathcal{B}}\Gamma$ the set Γ equipped with this uniformity.

The procedure introduced above allows us to define also a uniformity on the dual space \widehat{G} associated to a bornology \mathcal{B} on G . Indeed, if \mathcal{B} is a bornology on G , the symbol $\mu_{\mathcal{B}}\widehat{G}_n$ designates the set \widehat{G}_n equipped with the final uniformity induced by the canonical quotient $Q_n : \mu_{\mathcal{B}}\text{irrep}_n(G) \rightarrow \widehat{G}_n$ and $\mu_{\mathcal{B}}\widehat{G}$ is the dual space equipped with the free uniformity sum of the family $\{\mu_{\mathcal{B}}\widehat{G}_n : n < \omega\}$. In particular, if G is equipped with some group bornology \mathcal{B} and κ is a cardinal number, the family $\mathcal{B}_{\kappa} := \{K \subseteq \mathcal{B} : w(K_{t(\text{Fin}(\text{rep}(G))))} \leq \kappa\}$ defines a group bornology on G and $\mu_{\mathcal{B}_{\kappa}}\widehat{G}$ denotes the dual space \widehat{G} equipped with the uniformity $\mu_{\mathcal{B}_{\kappa}}$. The proof of the following result is a direct consequence of Theorem 3.4.

Proposition 5.1. $\mu_{\mathcal{B}_{\kappa}}\widehat{G}$ is κ -narrow.

Proof. It is enough to observe that $\mu_{\mathcal{B}_{\kappa}}\text{rep}_n(G)$ is κ -narrow for all $n < \omega$. \square

6. DETERMINED GROUPS

This section is dedicated to applying the results obtained previously to a question that has attracted the attention of workers interested in duality theory. Let us assume for the moment that G is an abelian group so that $\widehat{G} = \text{irrep}(G) = \text{irrep}_1(G)$ is also a topological group. When D is a dense subgroup of a topological abelian group G , then the restriction homomorphism $R|_D : \widehat{G} \rightarrow \widehat{D}$ of the dual groups is a continuous isomorphism, but need not be a topological isomorphism. According to Comfort, Raczkowski and Trigos-Arrieta [7], a (dense) subgroup D of a topological abelian group G *determines* G if the homomorphism $R|_D : \widehat{G} \rightarrow \widehat{D}$ is a topological isomorphism. If every

dense subgroup of G determines it, then G is called *determined*. The cornerstone in this topic is the following theorem due to Aussenhofer [2] and, independently, Chasco [3]:

Theorem 6.1 (Außenhofer, Chasco). *A metrizable abelian group G is determined.*

Comfort, Raczkowski and Trigos-Arrieta [7] noticed that this theorem fails for non-metrizable groups G even when G is compact. More precisely, they proved that every non-metrizable compact group G of weight $\geq \mathfrak{c}$ contains a dense subgroup that does not determine G . Hence, under the assumption of the continuum hypothesis, every determined compact group G is metrizable. Subsequently, it was shown in [17] that the result also holds without assuming the continuum hypothesis. Furthermore, Dikranjan and Shakmatov [9] proved that, for a compact abelian group G , no subgroup with cardinality smaller than $w(G)$ may determine it. However, this result does not extend to non-compact groups (see [4] for further information about this topic).

Example 6.2. *Let $G = \mathbb{R}^+$ be the group of reals equipped with the Bohr topology; that is, the topology of pointwise convergence on the elements of the dual group that, incidently, coincides with the group \mathbb{R} equipped with the standard topology. We have the following canonical isomorphisms*

$$\widehat{\mathbb{R}^+} \cong \widehat{\mathbb{R}} \cong \mathbb{R} \cong \widehat{\mathbb{Q}} \cong \widehat{\mathbb{Q}^+}.$$

This means that the dense subgroup of rational numbers determine G in spite of the fact that

$$w(\mathbb{R}^+) = |\mathbb{R}| = \mathfrak{c} \not\geq |\mathbb{Q}|.$$

Separately, in [23], Lukács considered the first approach to non-abelian determined groups. He extended the Außenhofer-Chasco theorem mentioned above by

proving the following: if G is a metrizable group, H is a dense subgroup, and K is a compact Lie group, then the spaces $\text{CHom}(G, K)_{\mathcal{K}(G)}$ and $\text{CHom}(H, K)_{\mathcal{K}(H)}$ are homeomorphic. Here $\text{Hom}(G, K)$ (resp. $\text{CHom}(G, K)$) denotes the set of all group homomorphisms (resp. continuous group homomorphisms) of G into K . In this paper, we look at this question with a more geometric view point.

Suppose that G is a compact metrizable abelian group and H is dense in G . From the Außenhofer-Chasco theorem, we deduce the existence of a compact subset K of H such that

$$K^\triangleright = \{\chi \in \widehat{G} : \|\chi(x) - 1\| \leq \sqrt{2} \forall x \in K\} = \{e\}.$$

Therefore, there is a compact subset K in H that equips \widehat{G} with a metric d_K as follows

$$d_K(\chi, \rho) = \sup\{|\chi(x) - \rho(x)| : \forall x \in K\}$$

for all χ, ρ in \widehat{G} . The metric d_K defines the discrete topology on \widehat{G} and, furthermore, it is easily verified that the restriction mapping

$$R|_H : (\widehat{G}, d_G) \longrightarrow (\widehat{H}, d_K)$$

is a *bi-uniform isomorphism*. Thus, using the results proved by different authors, the following theorem summarizes what is known for compact abelian groups concerning the notion of determined group, so far.

Theorem 6.3. *Let G be a compact abelian group. The following assertions are equivalent:*

- (a) *G is metrizable*
- (b) *For every dense subgroup H of G , the restriction mapping $R|_H : \widehat{G} \longrightarrow \widehat{H}$ is a homeomorphism.*

- (c) For every dense subgroup H of G , the restriction mapping $R|_H : (\widehat{G}, d_G) \longrightarrow (\widehat{H}, d_K)$ is a bi-uniform isomorphism.

In this section, our lofty goal is to extend this result to the class of non-abelian groups. However, non-abelian duality is far more complicated and several notions, which fit naturally for abelian groups, need of an appropriate modification in order to translate them to the non-abelian context. In this section, we deal with the applications of the techniques studied in the previous sections in order to obtain that (c) implies (a). The implication (a) \Rightarrow (c) is left for a subsequent paper [11]. At present, we do not know how (b) is placed between (a) and (c) (see the final remarks at the end of this manuscript). On the other hand, Proposition 6.6, and Corollaries 6.7, 6.8 and 6.9 below are also new for abelian groups, as far as we know.

Let G be a topological group and let $\mu_{\mathcal{K}}\text{irrep}(G)$ denote the set $\text{irrep}(G)$ equipped with the uniformity generated by the bornology $\mathcal{K}(G)$ on G . If $\pi : \text{irrep}(G) \longrightarrow \widehat{G}$ is the canonical quotient mapping, then $\mu_{\mathcal{K}}\widehat{G}$ designates the dual object \widehat{G} equipped with the final uniformity $\pi[\mu_{\mathcal{K}}]$.

Definition 6.4. We say that a subgroup H of a group G (*uniformly*) *determines* G when the restriction mapping $R|_H : \mu_{\mathcal{K}}\widehat{G} \longrightarrow \mu_{\mathcal{K}}\widehat{H}$ is an isomorphism of uniform spaces. G is (*uniformly*) *determined* if every dense subgroup of G determines G .

The main result we are concerned with here is the following:

A compact group is determined if and only if is metrizable

The proof of this result (as we have approached it) relies on the structure theory of compact groups and their continuous representations (sufficiency), and the methods

developed in the previous sections (necessity). Next we are going to introduce the basic definitions for non-abelian groups and we will prove that every determined compact group is metrizable. We leave the proof that every metrizable compact group is determined for a subsequent paper [11].

Firstly, we present some necessary conditions for a group to be determined. The following definition is needed.

Definition 6.5. *The compact weight $w_\kappa(X)$ of a topological space X is the cardinal number $w_\kappa(X) = \sup\{w(K) : K \text{ is compact in } G\}$*

Proposition 6.6. *Let G be a topological group. If H is a dense subgroup of G such that $|H| \leq \kappa$, then $\mu_\kappa \widehat{H}$ is κ -narrow.*

Proof. If $|H| \leq \kappa$, then $w(C) \leq \kappa$ for all $C \in \mathcal{K}(H)$. This means that $\mu_\kappa \text{irrep}_n(H)$ is κ -narrow for all $n < \omega$. Therefore the quotient uniform space $\mu_\kappa \widehat{H}_n$ must also be κ -narrow for all $n < \omega$. This completes the proof. □

As a consequence we obtain

Corollary 6.7. *Let G be a topological abelian group. If H is a dense subgroup of G such that $\kappa = |H| < w_\kappa(G)$, then H does not determine G .*

Proof. For abelian groups G , we have $\text{irrep}(G) = \text{irrep}_1(G) = \widehat{G}$. Thus $\mu_\kappa \widehat{H}$ is κ -narrow. On the other hand, since $\kappa < w_\kappa(G)$, it follows that $\mu_\kappa \text{irrep}(G) = \mu_\kappa \widehat{G}$ may not be κ -narrow by Theorem 3.4. This completes the proof. □

Corollary 6.8. *Let G be a compact group. If H is a dense subgroup of G such that $\kappa = |H| < w(G)$, then H does not determine G .*

Proof. Since G is compact, it follows that $\mu_{\mathcal{K}}\widehat{G}_n$ is uniformly discrete for all $n < \omega$ (see [19]). On the other hand, by Peter-Weyl Theorem, we know that $w(G) = \sup\{|\widehat{G}_n| : n \in \mathbb{N}\}$. Thus there is some natural number N such that $\kappa < |\widehat{G}_N| \leq w(G)$. Combining this fact with Theorem 3.4, we obtain that $\mu_{\mathcal{K}}\widehat{G}_N$ may not be κ -narrow. Now, $\kappa = |H|$ implies that $\mu_{\mathcal{K}}\widehat{H}_n$ is κ -narrow for all $n < \omega$. This completes the proof. \square

As a consequence, we obtain.

Corollary 6.9. *Let G be a compact or a topological abelian group. If $2^{|H|} < |G|$ then H does not determine G .*

In order to prove that every compact determined group is metrizable we need the following useful fact about quotients of determined compact groups.

Lemma 6.10. *Every quotient of a determined compact group is a determined compact group.*

Proof. Let G be a determined compact group and let $\pi : G \longrightarrow G/N$ be a canonical quotient map, where N is a closed normal subgroup of G . Given an arbitrary dense subgroup H of G/N , we set $L = \pi^{-1}(H)$ that is a dense subgroup of G . Firstly, observe that the map $i_1 : \mu_{\mathcal{K}}\text{irrep}_n(H) \longrightarrow \mu_{\mathcal{K}}\text{irrep}_n(L)$, defined by $i_1(\varphi) = \varphi \circ \pi$ for all $\varphi \in \text{irrep}_n(H)$, is a uniform embedding (i.e., a uniform isomorphism onto its image). Indeed, it is clear that i_1 is one-to-one. On the other hand, if $C \in \mathcal{K}(L)$ then $\pi(C) \in \mathcal{K}(H)$ and $(i_1 \times i_1)(B_H(\pi(C), \epsilon) \cap i_1(\text{irrep}_n(H)))^2 \subseteq B_L(C, \epsilon)$. Therefore i_1 is uniformly continuous. Conversely, if $K \in \mathcal{K}(H)$, since $\pi^{-1}(K) \in \mathcal{K}(L)$, it follows that $(i_1 \times i_1)^{-1}(B_L(\pi^{-1}(K), \epsilon) \cap i_1(\text{irrep}_n(H)))^2 = B_H(K, \epsilon)$. Therefore i_1 is a uniform embedding as claimed.

Now, let us verify that the map $\widehat{i}_1 : \mu_{\mathcal{K}}\widehat{H}_n \longrightarrow \mu_{\mathcal{K}}\widehat{L}_n$, defined by $\widehat{i}_1([\varphi]) = [\varphi \circ \pi]$ for all $\varphi \in \text{irrep}_n(H)$, is also a uniform embedding. Indeed, consider the diagram

$$\begin{array}{ccc} \mu_{\mathcal{K}}\text{irrep}_n(H) & \xrightarrow{Q_n^{(1)}} & \mu_{\mathcal{K}}\widehat{H}_n \\ \downarrow i_1 & & \downarrow \widehat{i}_1 \\ \mu_{\mathcal{K}}\text{irrep}_n(L) & \xrightarrow{Q_n^{(2)}} & \mu_{\mathcal{K}}\widehat{L}_n \end{array}$$

where $Q_n^{(i)}$, $1 \leq i \leq 2$, are quotient maps. Then $\widehat{i}_1 \circ Q_n^{(1)} = Q_n^{(2)} \circ i_1$ is uniformly continuous by the commutativity of the diagram. As a consequence, \widehat{i}_1 is uniformly continuous because it is defined on a quotient (uniform) space. On the other hand, since G is determined, the restriction map $R|_L : \mu_{\mathcal{K}}\widehat{G}_n \longrightarrow \mu_{\mathcal{K}}\widehat{L}_n$ is a uniform isomorphism and we have that $\mu_{\mathcal{K}}\widehat{L}_n$ is uniformly discrete because $\mu_{\mathcal{K}}\widehat{G}_n$ is (see [19]). Therefore $\widehat{i}_1^{-1}|_{\widehat{i}_1(\widehat{H}_n)}$ is also necessarily uniformly continuous. In like manner, the map $\widehat{i}_2 : \mu_{\mathcal{K}}(\widehat{G/N})_n \longrightarrow \mu_{\mathcal{K}}\widehat{G}_n$ is a uniform embedding as well. Now, we have the following commutative diagram

$$\begin{array}{ccc} \mu_{\mathcal{K}}(\widehat{G/N})_n & \xrightarrow{R|_H} & \mu_{\mathcal{K}}\widehat{H}_n \\ \downarrow \widehat{i}_2 & & \downarrow \widehat{i}_1 \\ \mu_{\mathcal{K}}\widehat{G}_n & \xlongequal{R|_L} & \mu_{\mathcal{K}}\widehat{L}_n \end{array}$$

where $R|_H = \widehat{i}_1^{-1}|_{\widehat{i}_1(\widehat{H}_n)} \circ R|_L \circ \widehat{i}_2$ is a uniform isomorphism. This completes the proof. \square

The next follows the main result in this section.

Theorem 6.11. *Every determined compact group is metrizable.*

Proof. Let G be a compact group of uncountable weight. By the Peter-Weyl Theorem, \widehat{G}_n must be uncountable for some $n \in \omega$. Fix a set Γ of n -dimensional irreducible continuous representations with $|\Gamma| = \aleph_1$. If $N = \bigcap \{\ker(\varphi) : \varphi \in \Gamma\}$, then the quotient

group G/N has weight \aleph_1 because it is embedded in $\mathbb{U}(n)^{\aleph_1}$. It is well known that G/N contains a countable dense subgroup H (see [6]). By Lemma 6.8, the subgroup H does not determine G/N and by Lemma 6.10 this means that G may not be a determined group. \square

6.1. Final remarks. The notion of compact determined group admits some variants that deserve to be further investigated. According to the approach in this paper, a compact group G is (uniformly) determined when, for every dense subgroup H of G , the spaces $\mu_{\mathcal{K}}\widehat{G}$ and $\mu_{\mathcal{K}}\widehat{H}$ are isomorphic as uniform spaces. However, from the results proved by Lukács [23], it follows that if G is a compact metrizable group and H is dense in G , then the spaces \widehat{G} and \widehat{H} are homeomorphic when they are equipped with appropriate dual topologies (i.e. $\widehat{G} = \bigsqcup_{n \in \mathbb{N}} \widehat{G}_n$ and each \widehat{G}_n possesses the quotient topology of $\frac{\text{irrep}_n(G)}{\sim}$ when $\text{irrep}_n(G)$ is equipped with the compact open topology; see [11]). Therefore, we may say that a compact group G is (*topologically*) *determined* when for every dense subgroup H , the spaces \widehat{G} and \widehat{H} are homeomorphic. In order to obtain a better understanding of the dual structure of compact groups, it would be very useful to know the answer to the following question.

Question 6.12. *Is every topologically determined compact group uniformly determined?*

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