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An alternative method to construct a consistent second-order theory on the equilibrium figures of rotating celestial bodies

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ABSTRACT

The main objective of this work is to construct a new method to develop a consistent second-order amplitudes theory to evaluate the potential of a rotating deformable celestial body when the hydrostatic system equilibrium has been achieved. In this case, we have: $\vec{\nabla} P = \rho \vec{\nabla} \Psi$, $\triangle \Psi = -4\pi G\rho + 2\omega^2$, where P is the pressure, ρ is the density, Ψ is the total potential, $\triangle$ is *Laplace* operator, G is the gravitational constant and $\vec{\omega}$ is the angular velocity of the system. To integrate these equations in a general case of mass distribution a state equation relating pressure and density is needed.

To assess the full potential, Ψ, it is necessary to calculate the self-gravitational potential, Ω, and the centrifugal potential, V_c. The equilibrium configuration involves the hydrostatic equilibrium, it is, the rigid rotation of the system corresponding to the minimum potential and, according to Kopal, this state involves the identification of equipotential, isobaric, isothermal and isopycnic surfaces.

To study the structure of the body we define a coordinate system $OXYZ$ where O is the center of mass of the component, OX is an axis fixed in an arbitrary point of the body equator, OZ an axis parallel to angular velocity $\vec{\omega}$ and OY defining a direct trihedron. For an arbitrary point P in the rotating body the Clairaut coordinates are given by (a, θ, λ) where a is the radius of the sphere that contains the same mass that the equipotential surface that contains P and (θ, λ) are the angular spherical coordinates of P.

This problem has been solved in the first order in ω^2 following two techniques: the first one is based on the asymptotic properties of the numerical quadrature formulae. The second is similar to the one used by Laplace to develop the inverse of the distance between two planets. The second-order theory based on the first method has been developed by the authors in a recent paper. In this work we develop a consistent second-order theory about the equilibrium figures of rotating celestial bodies based on the second method.

Finally, to show the performance of the method it is interesting to study a numerical example based on a convective star.

1. Introduction

Let M be a deformable isolated mass with uniform rotation around its mass center, endowed with angular velocity $\vec{\omega}$ and whose mass distribution is given by $\rho(x, y, z)$. Let assume that the mass distribution function is differentiable.

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Let $OXYZ$ be the coordinate system associated with the body, defined as follows:

- O is the mass center of the rotating body.
- OZ is the axis determined by the straight line passing through O and parallel to the angular velocity vector ($\vec{\omega}$) of the rotating body.
- OX is a line passing through O and contained in the OXY plane so that the trihedron $OXYZ$ is direct.

Then, the potential at an inner point whose radius vector has $\vec{r} = (x, y, z)$ as components will be determined by the following expression:

$$\Psi = \Omega + V_c = G \int_M \frac{dm'}{\Delta} + \frac{\omega^2}{2}\left(x^2 + y^2\right). \tag{1}$$

where Ω is the self-gravitational potential, V_c is the centrifugal potential, G is the universal gravitational constant, dm' is the mass element of an arbitrary inner coordinate point (x', y', z') and Δ is the distance between the coordinate points (x, y, z) and (x', y', z').

An inner point of the body in coordinate rotation (x, y, z) is expressed in spherical coordinates as follows:

$$x = r\,\cos\theta\,\cos\lambda, \quad y = r\,\cos\theta\,\sin\lambda, \quad z = r\,\sin\theta.$$

$$r \geq 0, \qquad -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}, \qquad 0 \leq \lambda \leq 2\pi.$$

The self-gravitational potential (Ω) is expressed, according to [1–5], through

$$\Omega = G\int_0^{r_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\rho}{\Delta} r'^2 \cos\theta' d\lambda' d\theta' dr' + G\int_{r_0}^{r_1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\rho}{\Delta} r'^2 \cos\theta' d\lambda' d\theta' dr'. \tag{2}$$

where r_1 is the radius of the smallest O centered sphere containing the rotating body.

The inverse of the distance is developed as follows

$$\frac{1}{\Delta} = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma) & \text{if } r' < r \\ \dfrac{1}{r'}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\gamma) & \text{if } r < r' \end{cases} \tag{3}$$

where γ is the angle between the radii vectors of $\vec{r} = (r, \theta, \lambda)$ and $\vec{r'} = (r', \theta', \lambda')$ of the points (x, y, z) and (x', y', z'), respectively, expressed in spherical coordinates and P_n are the Legendre polynomials.

Then, the self-gravitational potential can be expressed as

$$\Omega = \sum_{n=0}^{\infty}\left\{U_n r^n + V_n r^{-n-1}\right\}, \tag{4}$$

where

$$U_n = G\int_{r_0}^{r_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho\, r'^{1-n} P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' dr', \tag{5}$$

$$V_n = G\int_0^{r_0}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho\, r'^{2+n} P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' dr'. \tag{6}$$

In the Clairaut coordinate system (a, θ, λ) a parameter is constant on each equipotential surface. In this work we have chosen the parameter a so that it is the radius of the sphere centered in O such that it has the same mass as the equipotential surface. The Clairaut coordinate system is related to the spherical coordinate system by the relation $r = r(a, \theta, \lambda)$. In consequence,

$$\begin{aligned} x &= r(a, \theta, \lambda)\cos\theta\,\cos\lambda, \\ y &= r(a, \theta, \lambda)\cos\theta\,\sin\lambda, \\ z &= r(a, \theta, \lambda)\sin\theta. \end{aligned} \tag{7}$$

In the Clairaut coordinate system the radius vector r of an equipotential surface is developed [1–4] as follows:

$$r = a\left\{1 + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{n,m}(a)\, Y_{n,m}(\theta, \lambda)\right\}, \tag{8}$$

where $f_{n,m}(a)$ are the functions of amplitude and $Y_{n,m}(\theta, \lambda)$ the spherical functions in real form [6].

For symmetry reasons, the radius vector r can be developed as follows:

$$r = a\left\{1 + \sum_{m=0}^{\infty} f_{2m}(a) P_{2m}(\sin\theta)\right\}, \tag{9}$$

where P_n are the Legendre polynomials.

When $\vec{\omega}$ is small (in a slow rotation case), the amplitudes $f_{2m}(a)$ are small quantities with respect to the unit.

In order to clarify the notation, we will denote:

$$K_n = \frac{G}{n-2}\int_a^{a_1} \rho \frac{\partial}{\partial a'} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{2-n} P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' da', \quad n \neq 2 \tag{10}$$

$$K_2 = G\int_a^{a_1} \rho \frac{\partial}{\partial a'} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log r' \, P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' da' \tag{11}$$

and

$$W_n = \frac{G}{n+3}\int_0^{a} \rho \frac{\partial}{\partial a'} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r'^{n+3} P_n(\cos\gamma)\cos\theta' d\theta' d\lambda' da' \tag{12}$$

where a_1 is the first root of the equation $\rho(a) = 0$.

To evaluate these integrals up to second order in amplitudes, the following approximations are used:

$$r^k = a^k\left[1 + k\Sigma + \frac{k(k-1)}{2}\Sigma^2\right], \tag{13}$$

$$\log r = \log a + \Sigma - \frac{1}{2}\Sigma^2 \tag{14}$$

where $\Sigma = \sum_{n=0}^{\infty} f_{2n}(a) P_{2n}$.

To develop as a linear combination of Legendre polynomials the powers and cross products of Legendre polynomials that are obtained from the different powers of Σ in (13) and (14) the Adams–Newman formula are used [7].

$$P_n(x)P_m(x) = \sum_{j=0}^{m} \frac{A_{m-j}A_jA_{n-j}}{A_{n+m-j}}\left\{\frac{2n+2m+1-4j}{2n+2m+1-2j}\right\} P_{n+m-2j}(x), \tag{15}$$

$$\text{where} \quad A_j = \frac{(2j-1)!!}{j!}, \; m \leq n.$$

On the other hand, taking into account the spherical harmonics addition theorem in real form

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{n,m}(\theta,\lambda) Y_{n,m}(\theta',\lambda'), \tag{16}$$

from its orthonormality and from the fact that

$$P_s(\sin\theta) = \sqrt{\frac{4\pi}{2s+1}}\, Y_{s,0}(\theta,\lambda), \tag{17}$$

it is obtained

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_n(\cos\gamma) P_s(\sin\theta')\cos\theta' \, d\theta' \, d\lambda' = \frac{4\pi}{2n+1} P_n(\sin\theta)\,\delta_{n,s}, \tag{18}$$

where $\delta_{n,s}$ is Kronecker's delta.

On the one hand, it should be noted that the assumption made by Finlay [1] and Kopal [4] about the fact that (5) and (6) are equivalent to (10), (11) and (12) respectively, is based on the Laplace's desideratum which, as indicated in [5] (volume II, chapter XIX, page 317), is not a proven fact but a conjecture. López [8–10] obtains the correct development in first and second order in amplitudes of U_n(5) and V_n(6). Results are obtained in [9] and [10], respectively, without making use of the Laplace's desideratum.

However, the self-gravitational potential (Ω), developed up to first and second order in amplitudes by Kopal [4] and López, [9] and [10], respectively, coincide. Then, although the developments up to first and second order in amplitudes of the external (U_n) and inner potentials V_n, carried out by Kopal [4] are incorrect, [9] and [10] demonstrate that the development, up to first and second order in amplitudes, of the self-gravitational potentials (Ω) obtained by Kopal [4] and by López, [9] and [10], are identical.

In this work it is proved, by following a completely different way that, without making use of Laplace's desideratum, the classical equations of the potential, up to the second order, and the amplitudes given by Kopal [4], are correct.

This section introduces the problem that is going to be addressed and the research objectives.

In Section 2, by using an analytical method, the calculation of the inner and outer potentials to the equipotential surface, up to the second order, passing by point P, is developed. This method makes unnecessary the use of the Laplace's desideratum.

Section 3 illustrates the power of the Clairaut method by taking a convective star as a sample.

In Section 4, the main conclusions of the work are presented.

For the theoretical foundation of the numerical example, the authors consider it interesting to include an Appendix where the theoretical model to be used is exposed. It will allow us to approximate, almost exactly, a convective star of the main sequence by means of a polytrope with a polytropy index $n = 1.5$. By means of this approximation, calculations are considerably simplified with a negligible loss of accuracy.

2. Algorithm based on the inverse distance analytic development

Let be the self-gravitational potential $\Omega = K + W$, where

$$K = G \int_{Outer} \frac{dm'}{\Delta}\ , \quad W = G \int_{Inner} \frac{dm'}{\Delta}. \tag{19}$$

To calculate the two above integrals, let us define

$$D(a, a') = \frac{1}{\sqrt{a^2 + a'^2 - 2a\,a'\,\cos\gamma}}. \tag{20}$$

Then, developing $\frac{1}{\Delta}$ up to second order with respect to Σ y Σ', around (a, a'); where

$$r - a = a\,\Sigma\ ,\quad r' - a' = a'\,\Sigma'$$

and the subscripts indicating the derivatives with respect to that parameter, it results

$$\frac{1}{\Delta} = D(a,a') + D_a(a,a')\,a\,\Sigma + D_{a'}(a,a')\,a'\,\Sigma' + \frac{1}{2}D_{aa}(a,a')\,a^2\,\Sigma^2 + D_{aa'}(a,a')\,a\,a'\,\Sigma\,\Sigma' + \frac{1}{2}D_{a'a'}(a,a')\,a'^2\,\Sigma'^2. \tag{21}$$

On the other hand, from (13) it follows that

$$\frac{\partial r'}{\partial a'} = 1 + \Sigma' + a'\,\Sigma'_{a'}, \tag{22}$$

$$r'^2 = a'^2\left(1 + 2\Sigma' + \Sigma'^2\right). \tag{23}$$

Hence, by truncating, up to the second order in Σ', the product $\frac{\partial r'}{\partial a'}\,r'^2$ it is obtained

$$\frac{\partial r'}{\partial a'}\,r'^2 = a'^2\left(1 + 3\Sigma' + 3\Sigma'^2 + a'\,\Sigma'_{a'} + 2a'\,\Sigma'\,\Sigma'_{a'}\right). \tag{24}$$

The mass element dm' expressed in Clairaut's coordinates (7) is

$$dm' = \rho(a')\,\frac{\partial r'}{\partial a'}\,r'^2(a',\theta',\lambda')\,\cos\theta\,da'\,d\theta'\,d\lambda'. \tag{25}$$

Then, taking into account (24), the mass element dm' is expressed as

$$dm' = \rho(a')\,a'^2\left(1 + 3\Sigma' + 3\Sigma'^2 + a'\,\Sigma'_{a'} + 2a'\,\Sigma'\,\Sigma'_{a'}\right)\cos\theta'\,d\theta'\,d\lambda'\,da'. \tag{26}$$

Consequently, from (21), (26) and truncating the development up to the second order in Σ and Σ' of the quotient $\frac{dm'}{\Delta}$, it stands

$$\begin{aligned}\frac{dm'}{\Delta} = \Big\{ & D(a,a') + D_a(a,a')\,a\,\Sigma + \left[3D(a,a') + D_{a'}(a,a')\,a'\right]\Sigma' \\ & + \frac{1}{2}D_{aa}(a,a')\,a^2\,\Sigma^2 + \left[D_{aa'}(a,a')\,a\,a' + 3D_a(a,a')\,a\right]\Sigma\,\Sigma' \\ & + \left[\frac{1}{2}D_{a',a'}(a,a')\,a'^2 + 3D_{a'}(a,a')\,a' + 3D(a,a')\right]\Sigma'^2 \\ & + D(a,a')\,a'\,\Sigma'_{a'} + D_a(a,a')\,a\,a'\,\Sigma\,\Sigma'_{a'} \\ & + \left[D_{a'}(a,a')\,a'^2 + 2D(a,a')\,a'\right]\Sigma'\,\Sigma'_{a'}\Big\}\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da'.\end{aligned} \tag{27}$$

The next step is to develop Σ, Σ', $\Sigma'_{a'}$, Σ^2, Σ'^2, $\Sigma\Sigma'$, $\Sigma\Sigma'_{a'}$ and $\Sigma'\Sigma'_{a'}$ until the second order with respect to the amplitudes. Then it is obtained

$$\Sigma = f_0(a) + f_2(a)\,P_2(\sin\theta) + f_4(a)\,P_4(\sin\theta) + \sum_{m=3}^{\infty} f_{2m}(a)\,P_{2m}(\sin\theta), \tag{28}$$

$$\Sigma' = f_0(a') + f_2(a')\,P_2(\sin\theta') + f_4(a')\,P_4(\sin\theta') + \sum_{m=3}^{\infty} f_{2m}(a')\,P_{2m}(\sin\theta'), \tag{29}$$

$$\Sigma'_{a'} = f'_0(a') + f'_2(a')\,P_2(\sin\theta') + f'_4(a')\,P_4(\sin\theta') + \sum_{m=3}^{\infty} f'_{2m}(a')\,P_{2m}(\sin\theta'), \tag{30}$$

$$\Sigma^2 = f_2^2(a)\left(\frac{1}{5} + \frac{2}{7}\,P_2(\sin\theta) + \frac{18}{35}\,P_4(\sin\theta)\right), \tag{31}$$

$$\Sigma'^2 = f_2^2(a')\left(\frac{1}{5} + \frac{2}{7}\,P_2(\sin\theta') + \frac{18}{35}\,P_4(\sin\theta')\right), \tag{32}$$

$$\Sigma\,\Sigma' = f_2(a)f_2(a')\,P_2(\sin\theta)\,P_2(\sin\theta'), \tag{33}$$

$$\Sigma\,\Sigma'_{a'} = f_2(a)f'_2(a')\,P_2(\sin\theta)\,P_2(\sin\theta'), \tag{34}$$

$$\Sigma'\,\Sigma'_{a'} = f_2(a')f'_2(a')\left(\frac{1}{5} + \frac{2}{7}\,P_2(\sin\theta') + \frac{18}{35}\,P_4(\sin\theta')\right) \tag{35}$$

where $f'_n(a') = \frac{d f_n(a')}{da'}$.

Moreover,

$$D(a, a') = \begin{cases} \dfrac{1}{a}\displaystyle\sum_{n=0}^{\infty}\left(\frac{a'}{a}\right)^n P_n(\cos\gamma) & \text{if } a' < a \\[2ex] \dfrac{1}{a'}\displaystyle\sum_{n=0}^{\infty}\left(\frac{a}{a'}\right)^n P_n(\cos\gamma) & \text{if } a < a' \end{cases} \tag{36}$$

where γ is the angle between the radii vectors a and a' and P_n are the Legendre polynomials.

Then, in the inner part of dm' ($a' < a$) equipotential surface, the following developments are obtained:

$$D_a(a, a') = -\sum_{n=0}^{\infty}(n+1)\,\frac{a'^n}{a^{n+2}}\,P_n(\cos\gamma), \tag{37}$$

$$D_{a'}(a, a') = \sum_{n=1}^{\infty} n\,\frac{a'^{n-1}}{a^{n+1}}\,P_n(\cos\gamma), \tag{38}$$

$$D_{aa}(a, a') = \sum_{n=0}^{\infty}(n+1)(n+2)\,\frac{a'^n}{a^{n+3}}\,P_n(\cos\gamma), \tag{39}$$

$$D_{aa'}(a, a') = -\sum_{n=1}^{\infty} n\,(n+1)\,\frac{a'^{n-1}}{a^{n+2}}\,P_n(\cos\gamma), \tag{40}$$

$$D_{a'a'}(a, a') = \sum_{n=2}^{\infty} n\,(n-1)\,\frac{a'^{n-2}}{a^{n+1}}\,P_n(\cos\gamma). \tag{41}$$

Outside the equipotential surface dm' ($a < a'$), the following developments are obtained:

$$D_a(a, a') = \sum_{n=1}^{\infty} n\,\frac{a^{n-1}}{a'^{n+1}}\,P_n(\cos\gamma), \tag{42}$$

$$D_{a'}(a, a') = -\sum_{n=0}^{\infty}(n+1)\,\frac{a^n}{a'^{n+2}}\,P_n(\cos\gamma), \tag{43}$$

$$D_{aa}(a, a') = \sum_{n=2}^{\infty} n\,(n-1)\,\frac{a^{n-2}}{a'^{n+1}}\,P_n(\cos\gamma), \tag{44}$$

$$D_{aa'}(a,a') = -\sum_{n=1}^{\infty} n\,(n+1)\,\frac{a^{n-1}}{a'^{n+2}}\,P_n(\cos\gamma), \tag{45}$$

$$D_{a'a'}(a,a') = \sum_{n=0}^{\infty} (n+1)(n+2)\,\frac{a^{n}}{a'^{n+3}}\,P_n(\cos\gamma). \tag{46}$$

Consequently, taking into account (18), (19) and (27)–(46), in the next step each of the terms of dm' will be truncated in the second order of amplitudes, in the inner and outside parts of the equipotential surface of (27).

The integrals of (27) different terms are shown below:

Inner part of the equipotential surface

$\mathbf{D(a,a')}$

$$\begin{aligned} &G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D(a,a')\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\ &= \frac{4\pi G}{a}\,\delta_{n,0}\int_0^a a'^2\,\rho(a')\,da'. \end{aligned} \tag{47}$$

$\mathbf{D_a(a,a')\,a\,\Sigma}$

$$\begin{aligned} &G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_a(a,a')\,a\,\Sigma\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\ &= -\frac{4\pi G}{a}\left(\sum_{m=0}^{\infty} f_{2m}(a)\,P_{2m}(\sin\theta)\right)\delta_{n,0}\int_0^a a'^2\,\rho(a')\,da'. \end{aligned} \tag{48}$$

$\left[\mathbf{3D(a,a') + D_{a'}(a,a')\,a'}\right]\mathbf{\Sigma'}$

$$\begin{aligned} &G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D(a,a')\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\ &= \frac{12\pi G}{a}\,\delta_{n,0}\int_0^a a'^2\,\rho(a')f_0(a')\,da' \\ &\quad + \frac{12\pi G}{5a^3}\,\delta_{n,2}\,P_2(\sin\theta)\int_0^a a'^4\,\rho(a')f_2(a')\,da' \\ &\quad + \frac{12\pi G}{9a^5}\,\delta_{n,4}\,P_4(\sin\theta)\int_0^a a'^6\,\rho(a')f_4(a')\,da' \\ &\quad + \sum_{m=3}^{\infty}\frac{12\pi G}{(2n+1)a^{n+1}}\,\delta_{n,2m}\,P_{2m}(\sin\theta)\int_0^a a'^{n+2}\,\rho(a')f_{2m}(a')\,da'. \end{aligned} \tag{49}$$

$$\begin{aligned} &G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{a'}(a,a')\,a'\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\ &= \frac{8\pi G}{5a^3}\,\delta_{n,2}\,P_2(\sin\theta)\int_0^a a'^4\,\rho(a')f_2(a')\,da' \\ &\quad + \frac{16\pi G}{9a^5}\,\delta_{n,4}\,P_4(\sin\theta)\int_0^a a'^6\,\rho(a')f_4(a')\,da' \\ &\quad + \sum_{m=3}^{\infty}\frac{4\,n\,\pi G}{(2n+1)a^{n+1}}\,\delta_{n,2m}\,P_{2m}(\sin\theta)\int_0^a a'^{n+2}\,\rho(a')f_{2m}(a')\,da'. \end{aligned} \tag{50}$$

$\mathbf{\frac{1}{2}\,D_{aa}(a,a')a^2\Sigma^2}$

$$\begin{aligned} &G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\,D_{aa}(a,a')\,a^2\,\Sigma^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\ &= \frac{4\pi G}{a}\,f_2^2(a)\left(\frac{1}{5}+\frac{2}{7}\,P_2(\sin\theta)+\frac{18}{35}\,P_4(\sin\theta)\right)\delta_{n,0}\int_0^a a'^2\,\rho(a')\,da'. \end{aligned} \tag{51}$$

$\left[\mathbf{D_{aa'}}(\mathbf{a},\mathbf{a}')\,\mathbf{a}\,\mathbf{a}'+\mathbf{3D_a}(\mathbf{a},\mathbf{a}')\,\mathbf{a}\right]\boldsymbol{\Sigma}\,\boldsymbol{\Sigma}'$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{aa'}(a,a')\,a\,a'\,\Sigma\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' = -\frac{24\pi G}{5a^3}$$
$$\times f_2(a)\left(\frac{1}{5}+\frac{2}{7}P_2(\sin\theta)+\frac{18}{35}P_4(\sin\theta)\right)\delta_{n,2}\int_0^a a'^4\,\rho(a')f_2(a')\,da'. \tag{52}$$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D_a(a,a')\,a\,\Sigma\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' = -\frac{36\pi G}{5a^3}$$
$$\times f_2(a)\left(\frac{1}{5}+\frac{2}{7}P_2(\sin\theta)+\frac{18}{35}P_4(\sin\theta)\right)\delta_{n,2}\int_0^a a'^4\,\rho(a')f_2(a')\,da'. \tag{53}$$

$\left[\frac{1}{2}\,\mathbf{D_{a'a'}}(\mathbf{a},\mathbf{a}')\,\mathbf{a}'^2+\mathbf{3D_{a'}}(\mathbf{a},\mathbf{a}')\,\mathbf{a}'+\mathbf{3D}(\mathbf{a},\mathbf{a}')\right]\boldsymbol{\Sigma}'^2$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}D_{a'a'}(a,a')\,a'^2\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da'$$
$$=\frac{4\pi G}{5a^3}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_0^a a'^4\,\rho(a')f_2^2(a')\,da'$$
$$+\frac{24\pi G}{9a^5}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_0^a a'^6\,\rho(a')f_2^2(a')\,da'. \tag{54}$$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D_{a'}(a,a')\,a'\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da'$$
$$=\frac{24\pi G}{5a^3}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_0^a a'^4\,\rho(a')f_2^2(a')\,da'$$
$$+\frac{48\pi G}{9a^5}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_0^a a'^6\,\rho(a')f_2^2(a')\,da'. \tag{55}$$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D(a,a')\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da'$$
$$=\frac{12\pi G}{5a}\,\delta_{n,0}\int_0^a a'^2\,\rho(a')f_2^2(a')\,da'$$
$$+\frac{12\pi G}{5a^3}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_0^a a'^4\,\rho(a')f_2^2(a')\,da'$$
$$+\frac{12\pi G}{9a^5}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_0^a a'^6\,\rho(a')f_2^2(a')\,da'. \tag{56}$$

$\mathbf{D}(\mathbf{a},\mathbf{a}')\,\mathbf{a}'\,\boldsymbol{\Sigma}'_{\mathbf{a}'}$

$$G\int_0^a\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D(a,a')\,a'\,\Sigma'_{a'}\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da'$$
$$=\frac{4\pi G}{a}\,\delta_{n,0}\int_0^a a'^3\,\rho(a')\,\frac{d f_0(a')}{da'}\,da'$$
$$+\frac{4\pi G}{5a^3}\,P_2(\sin\theta)\,\delta_{n,2}\int_0^a a'^5\,\rho(a')\,\frac{d f_2(a')}{da'}\,da'$$
$$+\frac{4\pi G}{9a^5}\,P_4(\sin\theta)\,\delta_{n,4}\int_0^a a'^7\,\rho(a')\,\frac{d f_4(a')}{da'}\,da'$$
$$+\sum_{m=3}^{\infty}\frac{4\pi G}{(2n+1)\,a^{n+1}}\,P_{2m}(\sin\theta)\,\delta_{n,2m}\int_0^a a'^{n+3}\,\rho(a')\,\frac{d f_{2m}(a')}{da'}\,da'. \tag{57}$$

$\mathbf{D_a(a, a')\, a\, a'\, \Sigma\, \Sigma'_{a'}}$

$$G \int_0^a \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_a(a, a')\, a\, a'\, \Sigma\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' = -\frac{12\pi G}{5a^3}$$
$$\left(\frac{1}{5} + \frac{2}{7} P_2(\sin\theta) + \frac{18}{35} P_4(\sin\theta)\right) f_2(a)\, \delta_{n,2} \int_0^a a'^5\, \rho(a')\, \frac{df_2(a')}{da'}\, da'. \tag{58}$$

$\left[\mathbf{D_{a'}(a, a')\, a'^2 + 2a'\, D(a, a')}\right] \mathbf{\Sigma'\, \Sigma'_{a'}}$

$$G \int_0^a \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{a'}(a, a')\, a'^2\, \Sigma'\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da'$$
$$= \frac{8\pi G}{5a^3}\, \frac{2}{7} P_2(\sin\theta)\, \delta_{n,2} \int_0^a a'^5\, \rho(a') f_2(a')\, \frac{df_2(a')}{da'}\, da'$$
$$+ \frac{16\pi G}{9a^5}\, \frac{18}{35} P_4(\sin\theta)\, \delta_{n,4} \int_0^a a'^7\, \rho(a') f_2(a')\, \frac{df_2(a')}{da'}\, da'. \tag{59}$$

$$G \int_0^a \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a'\, D(a, a')\, \Sigma'\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da'$$
$$= \frac{8\pi G}{5a}\, \delta_{n,0} \int_0^a a'^3\, \rho(a') f_2(a')\, \frac{df_2(a')}{da'}\, da'$$
$$+ \frac{8\pi G}{5a^3}\, \frac{2}{7} P_2(\sin\theta)\, \delta_{n,2} \int_0^a a'^5\, \rho(a') f_2(a')\, \frac{df_2(a')}{da'}\, da'$$
$$+ \frac{8\pi G}{9a^5}\, \frac{18}{35} P_4(\sin\theta)\, \delta_{n,4} \int_0^a a'^7\, \rho(a') f_2(a')\, \frac{df_2(a')}{da'}\, da'. \tag{60}$$

Outer part of the equipotential surface

$\mathbf{D(a, a')}$

$$G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D(a, a')\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da'$$
$$= 4\pi G\, \delta_{n,0} \int_a^{a_1} a'\, \rho(a')\, da'. \tag{61}$$

$\mathbf{D_a(a, a')\, a\, \Sigma}$

$$G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_a(a, a')\, a\, \Sigma\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' = 0. \tag{62}$$

$\left[\mathbf{3D(a, a') + D_{a'}(a, a')\, a'}\right] \mathbf{\Sigma'}$

$$G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D(a, a')\, \Sigma'\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da'$$
$$= 12\pi G\, \delta_{n,0} \int_a^{a_1} a'\, \rho(a') f_0(a')\, da'$$
$$+ \frac{12\pi G\, a^2}{5}\, \delta_{n,2}\, P_2(\sin\theta) \int_a^{a_1} a'^{-1}\, \rho(a') f_2(a')\, da'$$
$$+ \frac{12\pi G\, a^4}{9}\, \delta_{n,4}\, P_4(\sin\theta) \int_a^{a_1} a'^{-3}\, \rho(a') f_4(a')\, da'$$
$$+ \sum_{m=3}^{\infty} \frac{12\pi G\, a^n}{(2n+1)}\, \delta_{n,2m}\, P_{2m}(\sin\theta) \int_a^{a_1} a'^{1-n}\, \rho(a') f_{2m}(a')\, da'. \tag{63}$$

$$G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{a'}(a, a')\, a'\, \Sigma'\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da'$$
$$-4\pi G\, \delta_{n,0} \int_a^{a_1} a'\, \rho(a') f_0(a')\, da'$$

$$
\begin{aligned}
&-\frac{12\pi G a^2}{5}\,\delta_{n,2}\,P_2(\sin\theta)\int_a^{a_1} a'^{-1}\,\rho(a')f_2(a')\,da' \\
&-\frac{20\pi G a^4}{9}\,\delta_{n,4}\,P_4(\sin\theta)\int_a^{a_1} a'^{-3}\,\rho(a')f_4(a')\,da' \\
&-\sum_{m=3}^{\infty}\frac{4(n+1)\pi G a^n}{(2n+1)}\,\delta_{n,2m}\,P_{2m}(\sin\theta)\int_a^{a_1} a'^{1-n}\,\rho(a')f_{2m}(a')\,da'. \qquad (64)
\end{aligned}
$$

$\frac{1}{2}\,\mathbf{D_{aa}(a,a')a^2\Sigma^2}$

$$
G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}D_{aa}(a,a')\,a^2\,\Sigma^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' = 0. \qquad (65)
$$

$\left[\mathbf{D_{aa'}(a,a')\,a\,a' + 3D_a(a,a')\,a}\right]\mathbf{\Sigma\,\Sigma'}$

$$
\begin{aligned}
&G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{aa'}(a,a')\,a\,a'\,\Sigma\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' = -\frac{24\pi G a^2}{5} \\
&\times f_2(a)\left(\frac{1}{5}+\frac{2}{7}P_2(\sin\theta)+\frac{18}{35}P_4(\sin\theta)\right)\delta_{n,2}\int_a^{a_1} a'^{-1}\,\rho(a')f_2(a')\,da'. \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
&G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D_a(a,a')\,a\,\Sigma\,\Sigma'\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' = \frac{24\pi G a^2}{5} \\
&f_2(a)\left(\frac{1}{5}+\frac{2}{7}P_2(\sin\theta)+\frac{18}{35}P_4(\sin\theta)\right)\delta_{n,2}\int_a^{a_1} a'^{-1}\,\rho(a')f_2(a')\,da'. \qquad (67)
\end{aligned}
$$

$\left[\frac{1}{2}\,\mathbf{D_{a'a'}(a,a')\,a'^2 + 3D_{a'}(a,a')\,a' + 3D(a,a')}\right]\mathbf{\Sigma'^2}$

$$
\begin{aligned}
&G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}D_{a'a'}(a,a')\,a'^2\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\
&=\frac{4\pi G}{5}\,\delta_{n,0}\int_a^{a_1} a'\,\rho(a')f_2^2(a')\,da' \\
&\quad+\frac{24\pi G a^2}{5}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_a^{a_1} a'^{-1}\,\rho(a')f_2^2(a')\,da' \\
&\quad+\frac{60\pi G a^4}{9}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_a^{a_1} a'^{-3}\,\rho(a')f_2^2(a')\,da'. \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
&G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D_{a'}(a,a')\,a'\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\
&=-\frac{12\pi G}{5}\,\delta_{n,0}\int_a^{a_1} a'\,\rho(a')f_2^2(a')\,da' \\
&\quad-\frac{36\pi G a^2}{5}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_a^{a_1} a'^{-1}\,\rho(a')f_2^2(a')\,da' \\
&\quad-\frac{60\pi G a^4}{9}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_a^{a_1} a'^{-3}\,\rho(a')f_2^2(a')\,da'. \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
&G\int_a^{a_1}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3D(a,a')\,\Sigma'^2\,\rho(a')\,a'^2\,\cos\theta'\,d\theta'\,d\lambda'\,da' \\
&=\frac{12\pi G}{5}\,\delta_{n,0}\int_a^{a_1} a'\,\rho(a')f_2^2(a')\,da' \\
&\quad+\frac{12\pi G a^2}{5}\,\frac{2}{7}\,P_2(\sin\theta)\,\delta_{n,2}\int_a^{a_1} a'^{-1}\,\rho(a')f_2^2(a')\,da' \\
&\quad+\frac{12\pi G a^4}{9}\,\frac{18}{35}\,P_4(\sin\theta)\,\delta_{n,4}\int_a^{a_1} a'^{-3}\,\rho(a')f_2^2(a')\,da'. \qquad (70)
\end{aligned}
$$

$\mathbf{D(a, a')\, a'\, \Sigma'_{a'}}$

$$
\begin{aligned}
&G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D(a, a')\, a'\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' \\
&= 4\pi G\, \delta_{n,0} \int_a^{a_1} a'^2\, \rho(a')\, \frac{d f_0(a')}{da'}\, da' \\
&\quad + \frac{4\pi G\, a^2}{5}\, P_2(\sin\theta)\, \delta_{n,2} \int_a^{a_1} \rho(a')\, \frac{d f_2(a')}{da'}\, da' \\
&\quad + \frac{4\pi G\, a^4}{9}\, P_4(\sin\theta)\, \delta_{n,4} \int_a^{a_1} a'^{-2}\, \rho(a')\, \frac{d f_4(a')}{da'}\, da' \\
&\quad + \sum_{m=3}^{\infty} \frac{4\pi G\, a^n}{2n+1}\, P_{2m}(\sin\theta)\, \delta_{n,2m} \int_a^{a_1} a'^{2-n}\, \rho(a')\, \frac{d f_{2m}(a')}{da'}\, da'.
\end{aligned}
\tag{71}
$$

$\mathbf{D_a(a, a')\, a\, a'\, \Sigma\, \Sigma'_{a'}}$

$$
\begin{aligned}
&G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_a(a, a')\, a\, a'\, \Sigma\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' = \frac{8\pi G\, a^2}{5} \\
&\times \left(\frac{1}{5} + \frac{2}{7}\, P_2(\sin\theta) + \frac{18}{35}\, P_4(\sin\theta)\right) f_2(a)\, \delta_{n,2} \int_a^{a_1} \rho(a')\, \frac{d f_2(a')}{da'}\, da'.
\end{aligned}
\tag{72}
$$

$\left[\mathbf{D_{a'}(a, a')\, a'^2 + 2a'\, D(a, a')}\right] \mathbf{\Sigma'\, \Sigma'_{a'}}$

$$
\begin{aligned}
&G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D_{a'}(a, a')\, a'^2\, \Sigma'\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' \\
&- \frac{4\pi G}{5}\, \delta_{n,0} \int_a^{a_1} a'^2\, \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da' \\
&- \frac{12\pi G\, a^2}{5}\, \frac{2}{7}\, P_2(\sin\theta)\, \delta_{n,2} \int_a^{a_1} \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da' \\
&- \frac{20\pi G\, a^4}{9}\, \frac{18}{35}\, P_4(\sin\theta)\, \delta_{n,4} \int_a^{a_1} a'^{-2}\, \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da'.
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
&G \int_a^{a_1} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a'\, D(a, a')\, \Sigma'\, \Sigma'_{a'}\, \rho(a')\, a'^2\, \cos\theta'\, d\theta'\, d\lambda'\, da' \\
&= \frac{8\pi G}{5}\, \delta_{n,0} \int_a^{a_1} a'^2\, \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da' \\
&\quad + \frac{8\pi G\, a^2}{5}\, \frac{2}{7}\, P_2(\sin\theta)\, \delta_{n,2} \int_a^{a_1} \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da' \\
&\quad + \frac{8\pi G\, a^4}{9}\, \frac{18}{35}\, P_4(\sin\theta)\, \delta_{n,4} \int_a^{a_1} a'^{-2}\, \rho(a') f_2(a')\, \frac{d f_2(a')}{da'}\, da'.
\end{aligned}
\tag{74}
$$

By adding together the common factor inner potentials $\delta_{n,j}$, with $j = 0, 2, 4$, results can be seen fully coincident with $W_j\, r^{-j-1}$, with $j = 0, 2, 4$, respectively, and by adding the common factor outer potential of $\delta_{n,j}$, with $j = 0, 2, 4$, results can be seen fully coincident with $K_j\, r^j$, with $j = 0, 2, 4$, respectively.

Taking into account all the factors mentioned above, it follows that, up to the second order, the autogravitational potential is expressed by

$$
\Omega = \sum_{n=0}^{2} K_n r^n + \sum_{n=0}^{2} W_n r^{-n-1}. \tag{75}
$$

Consequently, it has been established by using an alternative method to that of Kopal, that the classical equations of the terms of the autogravitational potential, up to second order, obtained by Kopal, are correct.

3. Numerical example

In this section, a numerical example of Clairaut's method is shown to compare the effect of rotation in a main sequence star, whose energy transport model is convective, with an ideal convective star [11–13].

The Clairaut coordinate system is defined in (7) and the equipotential surfaces are determined by the parameter a.

The equations determining the stellar structure, which can be expressed in Clairaut coordinates [14], are given, among others, by Faulkner [15].

The variables appearing in these equations are: M, mass; ρ, density; P, pressure; Ψ, total potential; L, luminosity; ε, energy generation per unit mass; T, temperature in degrees Kelvin, κ, radiative opacity per unit mass; $\vec{omega}$, angular velocity; A, radiation constant; c, speed of light in vacuum; G, constant of universal gravitation; μ, mean molecular mass of plasma; R, gas constant and Γ_2 the second adiabatic exponent.

The management of structure equations is complex. Now, in the case of main sequence stars whose energy transport is purely radiative or convective, these equations can be simplified by using a polytropic model of index 3 for the radiative case and of index 1.5 for the convective case [16].

Given our training as mathematicians and knowing that this work is mainly aimed at mathematicians, we have considered it interesting to include an Appendix where it is shown that an ideal gas composed of isolated particles and whose energy transport is carried out in an adiabatic way, satisfies the equation of a polytrope of index 1.5. Furthermore, it is easy to accept the assumption that the plasma forming a main sequence star behaves like an ideal gas of free particles since nuclei and electrons move freely in plasma and, therefore, the volume occupied by the particles is negligible compared to the star volume. Obviously, the gas temperature is well above the critical temperature.

The use of polytropic models to solve the main sequence star structure equations is justified because it considerably simplifies the calculations and the results obtained differ very little from those obtained by other means.

In main sequence stars the transport of energy is convective when their mass is less than or equal to $0.5M_S$ (being M_S the mass of the Sun). In case of a mass between $0.5M_S$ and $1.5M_S$, the transport is radiative in the nucleus and convective in outer layers. For a mass greater than $1.5M_S$, transport is convective in the nucleus and radiative in the outermost layers.

To verify Clairaut's method goodness of fit by using a numerical example, we have chosen a star in which it is assumed a completely convective energy transport. Consequently, according to what is stated in [10,14] and Appendix, its equation of state is

$$P = K\rho^{1+\frac{1}{n}}, \tag{76}$$

where $n = 1.5$.

Introducing variables Θ and ξ, defined as:

$$\rho = \rho_c \Theta^n, \tag{77}$$

and

$$r = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\xi, \tag{78}$$

it follows that from the Poisson equation the Lane–Emden equation results

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) + \Theta^n = 0, \tag{79}$$

Kopal [14] shows that using the Clairaut coordinates defined above, for the case of equilibrium configurations of polytropic masses in uniform slow rotation and with a small angular velocity ω, and defining ξ by

$$a = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\xi, \tag{80}$$

the perturbed Lane–Emden equation is verified, whose expression is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) + \Theta^n = \nu, \tag{81}$$

where ν is a small parameter given by

$$\nu = \frac{\omega^2}{2\pi G\rho_c}. \tag{82}$$

To solve (81) in first order of disturbance, Kopal [14] develops Θ in first order with respect to ν $\Theta(\xi) = \Theta_0(\xi) + \nu\varphi(\xi)$.

Replacing this last expression in (81) it results that Θ_0 and φ satisfy

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta_0}{d\xi}\right) + \Theta_0 = 0, \tag{83}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\varphi}{d\xi}\right) + n\Theta_0^{n-1}\varphi = 1, \tag{84}$$

with the initial conditions $\Theta_0(0) = 1$, $\Theta_0'(0) = 0$, $\varphi(0) = 0$, $\varphi'(0) = 0$.

The value ξ_1^* where Θ vanishes is obtained from the relation

$$\xi_1^* = \xi_1 - \frac{\varphi(\xi_1)}{-\Theta_0'}. \tag{85}$$

On the other hand, the relationship between the mean and the central density is given by

$$\frac{\overline{\rho}}{\rho_c} = -\frac{3}{\xi_1}\left(\frac{d\varphi}{d\xi}\right)_{\xi_1}, \tag{86}$$

where the values of ξ_1 and $\left(\frac{d\varphi}{d\xi}\right)_{\xi_1}$ for the surface ξ_1 of the polytrope are given by solving the Lane–Emden equation [11] and depend exclusively on the polytropy index.

For $n = 1.5$, Chandrasekhar gives the value $\rho_c/\overline{\rho} = 5.99071$. Finally, the constant K of the polytrope is determined from the relation given by Clayton [12] for the total mass

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c^{(3-n)/2}\left(\xi_1^2\frac{d\varphi}{d\xi}\right)_{\xi_1}, \tag{87}$$

where for $n = 1.5$ it is verified [11] $\left(\xi_1^2\frac{d\varphi}{d\xi}\right)_{\xi_1} = -2.71406$.

Next we consider a star with mass $M = 0.5M_S$ so that the energy transport can be considered fully convective.

To develop the numerical model, the value of the mean density must be determined first. According to Clayton, for a star of the main sequence, this value is $\overline{\rho} = 1.4/M$ expressed in g/cm^3 where M is the mass expressed in M_S.

From this value and taking into account (86), $K = 7.75872 \cdot 10^{13}$.

According to Kopal [14], the amplitude functions f_2 and f_4 satisfy, up to second order in ν, the differential equations:

$$\begin{aligned}&\xi^2 f_2'' + 6D(\xi f_2') - 6f_2 = \frac{2}{7}\left[2(\xi f_2')^2 + 9\xi f_2' f_2 - 9D((\xi^2 f_2')^2 + 18\xi f_2' f_2)\right]\\&+6\nu\frac{\rho_c}{\overline{\rho}}(1-D)(\xi f_2' + f_2),\end{aligned} \tag{88}$$

where $D = \frac{\rho}{\overline{\rho}}$.

$$\xi^2 f_4'' + 6D(\xi f_4') - 20f_4 = \frac{18}{35}\left[1(\xi f_4')^2 + 4f_4' f_4 - 3D3(\xi f_4')^2 + 6\xi f_4' f_4 + 7f_4^2\right]. \tag{89}$$

The functions f_2 and f_4 satisfy at the origin $f_2'(0) = f_4'(0) = 0$.

To complete the system it is required that the boundary conditions are satisfied on the outer surface given by ξ_1^*

$$2f_2 + \xi_1 f_2' + \frac{5}{2}\frac{\rho_c}{\rho_m}\nu = (\xi_1 f_2' + 5f_2)\frac{\rho_c}{\rho_m}\nu + \frac{2}{7}((\xi_1 f_2')^2 + 3\xi f_2' f_2 + 6f_2^2), \tag{90}$$

$$4f_4 + \xi_1 f_4' = \frac{18}{35}((\xi_1 f_2')^2 + 5\xi f_2' f_2 + 6f_2^2), \tag{91}$$

where ρ_m is the mean density of the star.

For a polytrope of order n it is verified

$$\frac{\rho_c}{\overline{\rho}} = -\frac{\xi}{3\Theta'(\xi)}, \quad D = -\frac{\xi\Theta^n(\xi)}{3\Theta'(\xi)}. \tag{92}$$

The integration of these equations can be carried out in stages using a classical fourth-order Runge–Kutta method. Now, since Eqs. (83) and (84) must be continuous in ξ and present a regular singular point at $\xi = 0$, then the value of the first iteration of the method will be taken by making a series approximation, around the origin, of a higher order than the integration method which, according to index Eqs. (83) and (84), results in the following developments:

$$\begin{aligned}\Theta_0(\xi) &= \frac{\xi^2}{6} + \frac{\xi^4}{80} - \frac{\xi^6}{1440},\\ \varphi(\xi) &= \frac{\xi^2}{6} - \frac{\xi^4}{80} + \frac{19\xi^6}{20160}.\end{aligned} \tag{93}$$

Once these equations are integrated and in order to avoid uncertainties around the origin, it is convenient to take as $D(\xi) = \rho(\xi)/\overline{\rho(\xi)}$ the approximation

$$D(\xi) = 1 - \frac{\xi^2}{10} + \frac{\xi^4}{600} - \frac{7\xi^6}{162000}. \tag{94}$$

For the resolution of (88) we find ourselves in a case analogous to that of the polytrope. So, from its index equation and the continuity at the origin, we have up to sixth order in ξ:

$$f_2(\xi) = k_2(1 + \frac{3\xi^2}{70} + \frac{47\xi^4}{25260} + \frac{157\xi^6}{2268000}). \tag{95}$$

Table 1
Lane–Emden and amplitude functions.

ξ	$\Theta(\xi)$	$f_0(\xi)$	$f_2(\xi)$	$f_4(\xi)$
0.00000000	0.00000000	−0.00000104	−0.00228408	0.00000402
0.50000000	0.95930834	−0.00000112	−0.00237129	0.00000444
1.00000000	0.84594504	−0.00000142	−0.00266645	0.00000601
1.50000000	0.68273070	−0.00000214	−0.00327215	0.00001000
2.00000000	0.49851733	−0.00000385	−0.00438795	0.00002028
2.50000000	0.31951480	−0.00000798	−0.00631640	0.00004746
3.00000000	0.16358877	−0.00001772	−0.00941361	0.00011669
3.50000000	0.03862348	−0.00003917	−0.01399503	0.00027639
3.65375369	0.00000000	−0.00004954	−0.01573887	0.00035479

The value of k_2 is determined by imposing on ξ_1^* the boundary condition (90) resulting in $K_2 = -2.28408 \cdot 10^{-3}$. Similarly, from (89), we have for $f_4(\xi)$ that around the origin and up to sixth order in ξ it is verified:

$$f_4(\xi) = k_2^2 \left(-\frac{27}{35} - \frac{503\xi^4}{539000} + \frac{12787\xi^6}{1471470000} \right) + k_4\xi^2 \left(1 + \frac{9\xi^2}{110} + \frac{237\xi^4}{57220} \right). \tag{96}$$

The value of k_4 is obtained by imposing on ξ_1^* the boundary condition (91) resulting in $k_4 = -9.086 \cdot 10^{-8}$.

Table 1 shows the values of $\Theta(\xi), f_0(\xi), f_2(\xi), f_4(\xi)$ for various values of ξ:

4. Concluding remarks

Main conclusions of this work can be summarized as follows:

- The proposed algorithm consists of replacing the development of the inverse of the distance given by a similar method to the one used by Laplace for the case of construction of planetary theories. Laplace, in his algorithm, obtains an approximation to the inverse of the distance up to second order in ω^2 and proceeds to the integration of the terms resulting from K and W.
- By means of this algorithm it is also proved, without using the Laplace's desideratum, that the classical development of the self-gravitational potential is correct up to the second order in $\nu = \omega^2/2\pi G\rho_c$.
- As we can see in this work, moving from a first order theory to a second order theory supposes a considerable increase in algorithmic complexity. Consequently, the number of calculations required to evaluate potentials beyond the second order is so large that it is extremely difficult to achieve. All this reasons lead us to conjecture that the method exposed in this work is extendable to orders above the second.
- The method proposed in the present work does not require the derivability of the density in the case of a barotrope, which is true for a polytropic star.
- The numerical results obtained from applying this theory to a convective star with a mass M equal to 0.5 solar masses is consistent with what is expected, since it presents less deformations than in the case of a convective star whose mass is equal to that of the Sun [10], which validates the theory.

Appendix

As the star matter is in state of plasma due to its temperature we assume that the star behaves like an ideal gas where its particles are free [17]:

- Gases are made up of a large number of particles, which behave like hard spherical objects, in a state of constant and random motion.
- Gas particles move in a straight line until they collide with other particles or with the walls of the container.
- Particles are much smaller than the distances between them. Most of the volume of a gas is therefore empty space.
- There is no force of attraction between the gas particles, nor between these particles and the walls of the container.
- The collisions between the gas particles or between them and the walls of the container are perfectly elastic. Energy can be transferred from one particle to another during a collision, but the total kinetic energy of the particles after the collision is the same as before the collision.
- The average kinetic energy of a set of gas particles depends only on the temperature of the gas.

For a perfect gas, its internal energy U is given by

$$U = nc_vT, \tag{A.1}$$

where n is the molar concentration of the gas, c_v the constant-volume heat of the gas and T the absolute temperature.

The constant-volume heat for a gas whose particles have three degrees of freedom, $c_v = 3\,\text{cal}\,\text{K}^{-1}\,\text{mol}^{-1}$.

On the other hand, we know Mayer's law

$$c_p - c_v = R, \tag{A.2}$$

where c_p is the molar heat at constant pressure and R is the gas constant, being its value $\left(R = 2\,\text{cal}\,\text{K}^{-1}\ \text{mol}^{-1}\right)$.

Defining the adiabatic exponent as $\gamma = v_p/c_v$, we have that $\gamma = 5/3$.

On the other hand, according to the first principle of thermodynamics, we have

$$dU = dQ - dW, \tag{A.3}$$

where dQ is the heat received by the system and dW is the work done by the system. Note that U in (A.3) is a state function. Consequently, its variation is independent of the path followed. However, this is not the case for Q and W.

In an adiabatic transformation we have that $dQ = 0$ and, therefore, $dU = -dW$.

To express the variation of internal energy with respect to the pressure and volume variables, it is enough to take into account the Clapeyron equation

$$PV = nRT, \tag{A.4}$$

in consequence

$$U = c_v \frac{PV}{R}. \tag{A.5}$$

Differentiating (A.5) we have $dU = \frac{c_v PdV + c_v VdP}{R}$ and since $dW = PdV$ is obtained from (A.3)

$$\frac{c_v PdV + c_v VdP}{R} + PdV = 0, \tag{A.6}$$

multiplying by R and grouping gives

$$P(c_v + R)dV + c_v VdP = c_p PdV + c_v VdP = 0, \tag{A.7}$$

from where

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0. \tag{A.8}$$

Integrating this last equation that of the adiabatic equation is obtained

$$PV^{\gamma} = K. \tag{A.9}$$

Taking into account that $\rho = n\mu/V$, (A.9)

$$P = K'\rho^{\gamma}, \tag{A.10}$$

where K' is constant.

On the other hand, a polytrope of order n is defined as that gas in which pressure and density satisfy the relation

$$P = Cte \cdot \rho^{1+\frac{1}{n}}. \tag{A.11}$$

Now, as for the convective case $\gamma = 5/3$, we have that $1 + \frac{1}{n} = \frac{5}{3}$ and therefore $n = 1.5$. Consequently, a convective star can be modeled almost exactly by a polytrope of index $n = 1.5$.

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