KOROVKIN-TYPE RESULTS ON CONVERGENCE OF SEQUENCES OF POSITIVE LINEAR MAPS ON FUNCTION SPACES

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Abstract. In this paper we deal with the convergence of sequences of positive linear maps to a (not assumed to be linear) isometry on spaces of continuous functions. We obtain generalizations of known Korovkin-type results and provide several illustrative examples.

1. Introduction

One of the most impressive results in approximation theory is, without doubt, Korovkin's theorem on convergence of positive linear operators on a space of continuous functions. More explicitly, Korovkin's theorem (often called Korovkin's first theorem) states that if a sequence $\{T_n\}$ of positive linear maps on $C_{\mathbb{R}}[0,1]$ converges to the identity operator on the quadratic polynomials, then T_nf converges to f for all $f \in C_{\mathbb{R}}[0,1]$ ([8]). This result arose from a generalization of the well-known proof of Weierstrass's approximation theorem given by S. Bernstein. Its strength and simplicity have produced, as it is clearly imaginable, a wide range of applications and generalizations. One of them deals with substituting the identity operator by other operators and the closed interval $[0,1]$ by other spaces. Others center on finding subsets of function spaces, known as Korovkin sets or test functions, which guarantee that the convergence of a sequence of positive linear maps holds on the whole space provided it holds on them. For more details and other aspects of this topic, we refer to the monographs [2, 6], the recent survey paper by Altomare [1], and the references therein.

Let X and Y be compact Hausdorff spaces, M be a unital subspace of $C(X)$, and S be a function space included in M. In [7], the authors studied the convergence of a sequence of unital linear contractions towards a fixed linear isometry. Indeed, they proved that, under certain assumptions, if each $T_n : M \longrightarrow C(Y)$ ($n \in \mathbb{N}$) is a unital linear contraction and $T_\infty : M \longrightarrow C(Y)$ is a linear isometry such that $\{T_nf\}$ converges to $T_\infty f$ for all $f \in S$, then $\{T_nf\}$ converges to $T_\infty f$ for all $f \in M$, not only pointwise but also uniformly. In this paper we deal with the convergence of sequences of (not necessarily contractions) positive linear maps to a (not assumed to be linear) isometry on spaces of continuous functions by combining ideas given in [7] and in the original proof

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of Korovkin's theorem. In particular, we obtain proper generalizations of [7, Theorems 3.1 and 4.1] and of several classical Korovkin-type results, and provide several illustrative examples.

2. Preliminaries

For any compact Hausdorff space X, let $C(X)$ denote the space of continuous real or complex-valued functions on X, equipped with the uniform norm $\|\cdot\|$. Note that we write $C_{\mathbb{R}}(X)$ instead of $C(X)$ when we want to consider only real-valued case. A unital subspace S of $C(X)$ is called a *function space* on X if S separates the points of X in the sense that for each $x, x' \in X$ with $x \neq x'$ there exists a function $f \in S$ such that $f(x) \neq f(x')$.

Let S be a subspace of $C(X)$, which we always assume to be linear. We denote by $\mathcal{B}_{S^*}$ the closed unit ball of the dual space of $(S, \|\cdot\|)$. A nonempty subset E of X is called a *boundary* for S if each function in S attains its maximum modulus within E. The *Choquet boundary* $Ch(S)$ of S is the non-empty set of all points $x \in X$ for which δ_x, the evaluation functional at x, is an extreme point of the closed unit ball $\mathcal{B}_{S^*}$. Namely, we have $ext(\mathcal{B}_{S^*}) = \mathbb{T}\, Ch(S) = \{\alpha x : \alpha \in \mathbb{T} \,\text{and}\, x \in Ch(S)\}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is known that $Ch(S)$ is a boundary for S. In particular, one can obtain the following remark immediately:

Remark 2.1. If for each $x \in X$ there is a function $h \in S$ such that $h(x) = 1$ and $|h(y)| < 1$ for any $y \neq x$, then $Ch(S) = X$. For example, as in Korovkin's original theorem, if we assume $X = [0, 1]$ and $S = Span\{1, x, x^2\}$, then $h(x) := 1 - (x - a)^2$, $a \in [0, 1]$, yields $Ch(S) = [0, 1]$.

In the sequel, unless otherwise stated, it is assumed that X and Y are compact Hausdorff spaces, M is a *self-conjugate* subspace of $C(X)$ in the sense that $\bar{f} \in M$ whenever $f \in M$, and S is a function space included in M.

A linear map $T : M \longrightarrow C(Y)$ is called *positive* if $Tf \geq 0$ holds for all $f \geq 0$.

Let $f, f_1, f_2, ... \in C(X)$ and $X_0 \subseteq X$. If $\{f_n\}$ converges pointwise to f on X_0, we write $f_n \longrightarrow f$ on X_0. Also, we omit X_0 when $X_0 = X$.

Given $f, g \in C(X)$, we shall write $f \otimes 1 + 1 \otimes g$ to denote the function in $C(X \times X)$ such that $(f \otimes 1 + 1 \otimes g)(x, x') := f(x) + g(x')$. Furthermore, if $T, T' : S \subseteq C(X) \longrightarrow C(Y)$, then we set $(T \otimes T1\, T')(f \otimes 1 + 1 \otimes g)(y) := Tf(y) + T1(y)T'g(y)$ for all $f, g \in S$ and $y \in Y$.

Finally let us state the following lemma which is used in the proofs of our results.

Lemma 2.2. [5, Theorem 2.2.6] *Let S be a function space on X and $x_0 \in X$. Then $x_0 \in Ch(S)$ if and only if for any $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$ and any open neighborhood U of x_0, there is a function $f \in S$ such that* $\mathrm{Re} f \leq 0$ *on X,* $\mathrm{Re} f < -\beta$ *on U^c and* $\mathrm{Re} f(x_0) > -\alpha$.

3. Results

Theorem 3.1. *Suppose that $\{T_n\}$ is a sequence of positive linear maps from M into $C(Y)$, and T_∞ is an isometry from M onto a subspace $T_\infty(M)$ of $C(Y)$.*

(a) If $T_n f \longrightarrow T_\infty f$ for all $f \in S$, then $T_n f \longrightarrow T_\infty f$ on $Ch(T_\infty(S))$ for all $f \in M$.

(b) Let $N := Span \bigcup_{1\le n\le\infty} T_n(M)$. If, in part (a), $Ch(N) \subseteq Ch(T_\infty(S))$ and the set $\{T_n 1 : n \in \mathbb{N}\}$ is bounded, then $T_n f \longrightarrow T_\infty f$ for all $f \in M$.

Proof. We will base the proof of (a) through the following steps.

Step 1. For each triple of distinct points $x, x', z \in Ch(M)$, there exists a function $h \in M$ such that $|h(x)| \neq |h(x')|$ and $h(z) = 0$.

Since M is a self-conjugate function space we can find a real-valued function $f \in M$ such that $f(x) = 1$ and $f(x') = 0$. Now we consider the following cases based on the value of f at z:

- $f(z) = 1$. Clearly, $h = 1 - f$ is the desired function.
- $f(z) \neq 1, \frac{1}{2}$. Take $h = f - f(z)$.
- $f(z) = \frac{1}{2}$. In this case we choose a non-negative function g in M with $g(x), g(x') > 3$ and $g(z) < \frac{1}{2}$, by Lemma 2.2. If $g(x') - g(x) = 2$, then $h = g - g(z)$ is the desired function. Otherwise, we can see that $h = 2f + g - g(z) - 1$ satisfy the requested properties.

Step 2. T_∞ is a linear isometry.

Note that $T_\infty 0 = \lim T_n 0 = 0$. Then according to the Mazur-Ulam theorem [10], T_∞ is a real-linear isometry. Hence now we only need to consider the complex case. Let us point out that $T_\infty 1 = \lim T_n 1 \geq 0$. Taking into account Step 1, from [9, Theorem 2.3] it follows that $T_\infty 1 = 1$ and there exist a (possibly empty) clopen subset K of $Ch(T_\infty(M))$, and a continuous surjective map $\phi : Ch(T_\infty(M)) \longrightarrow Ch(M)$ such that for all $f \in M$,

$$T_\infty f = \begin{cases} f \circ \phi & \text{on } K, \\ \overline{f \circ \phi} & \text{on } Ch(T_\infty(M)) \setminus K. \end{cases}$$

But $T_\infty i = \lim T_n i = i \lim T_n 1 = i T_\infty 1 = i$, which implies that $K = Ch(T_\infty(M))$. Hence taking into account that $Ch(T_\infty(M))$ is a boundary for $T_\infty(M)$, we deduce that T_∞ is a linear isometry.

Step 3. For each $f \in M$, $T_n f \longrightarrow T_\infty f$ on $Ch(T_\infty(S))$.

By [7, Lemma 2.5] (or [3, Corollary 3.2]), there is a continuous surjection $\varphi : Ch(T_\infty(S)) \longrightarrow Ch(S)$ such that

$$T_\infty f(y) = f(\varphi(y)) \ \ (f \in S, y \in Ch(T_\infty(S))).$$

Let $f \in M$ and $\epsilon > 0$. Then we can define a function in $C(X \times X)$ as $F := f \otimes 1 - 1 \otimes f$. Clearly, $F = 0$ on the subset $\triangle_X = \{(x, x) : x \in X\}$ of $X \times X$. Then there is an open neighborhood U of $\triangle_X$ with $|F| < \epsilon$ on U.

Let $y' \in Ch(T_\infty(S))$ and $x' = \varphi(y')$. Choose an open neighborhood $V_{x'}$ of x' such that $V_{x'} \times V_{x'} \subseteq U$. By Lemma 2.2, we find a function $f_{y'} \in S$ such that

$$\mathrm{Re} f_{y'} \geq 0 \text{ on } X, \ \mathrm{Re} f_{y'} \geq 1 \ \text{ on } V_{x'}^c, \ \mathrm{Re} f_{y'}(x') < \epsilon.$$

Put $F_{y'} = f_{y'} \otimes 1 + 1 \otimes f_{y'}$. It is clear that $\mathrm{Re} F_{y'} \geq 0$ on $X \times X$ and $\mathrm{Re} F_{y'} \geq 1$ on U^c. Hence we have

$$\mathrm{Re} F \leq \|F\| \leq \|F\| \mathrm{Re} F_{y'} \ \text{ on } U^c,$$

which yields $|\mathrm{Re} F| \leq 1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'}$ on $X \times X$. In other words,

$$-(1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'}) \leq \mathrm{Re} F \leq 1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'} \ \text{ on } X \times X.$$

Hence for each $y \in X$ we get

$$-\epsilon - 2\|F\| \mathrm{Re} f_{y'} - \|F\| \mathrm{Re} f_{y'}(y) + \mathrm{Re} f(y) \leq \mathrm{Re} f - \|F\| \mathrm{Re} f_{y'} \leq \epsilon + \|F\| \mathrm{Re} f_{y'}(y) + \mathrm{Re} f(y).$$

Since $\{T_n\}$ is a sequence of linear positive maps, it follows that

$$-2\|F\| T_n(\mathrm{Re} f_{y'}) + (-\epsilon - \|F\| \mathrm{Re} f_{y'}(y) + \mathrm{Re} f(y)) T_n 1 \leq T_n(\mathrm{Re} f) - \|F\| T_n(\mathrm{Re} f_{y'}) \leq$$

$$T_n 1(\epsilon + \|F\| \mathrm{Re} f_{y'}(y) + \mathrm{Re} f(y))$$

for each $y \in X$. Now, from the representation of T_∞ on M (Step 2), we deduce that

$$-2\|F\| T_n(\mathrm{Re} f_{y'})(z) + T_\infty(-\epsilon - \|F\| \mathrm{Re} f_{y'} + \mathrm{Re} f)(z') T_n 1(z) \leq T_n(\mathrm{Re} f)(z) - \|F\| T_n(\mathrm{Re} f_{y'})(z) \leq$$

$$T_n 1(z) T_\infty(\epsilon + \|F\| \mathrm{Re} f_{y'} + \mathrm{Re} f)(z')$$

for any $z \in Y$ and $z' \in Ch(T_\infty(M))$. Thus, again since $T_\infty 1 = 1$, T_∞ is a positive linear map and also $Ch(T_\infty(M))$ is a boundary for $T_\infty(M)$, it is observed that the above relation holds for all $z, z' \in Y$. Therefore, especially we get

$$-||F|| T_n(\mathrm{Re} f_{y'}) - T_n 1 T_\infty(\epsilon + \|F\| \mathrm{Re} f_{y'}) \leq T_n(\mathrm{Re} f) - T_n 1 T_\infty(\mathrm{Re} f) \leq T_n 1 T_\infty(\epsilon + \|F\| \mathrm{Re} f_{y'}) + ||F|| T_n(\mathrm{Re} f_{y'})$$

on Y. Rewriting the above inequality adopted to our notation in Section 2 we have

$$-(T_n \otimes T_n 1 T_\infty)(1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'}) \leq (T_n \otimes T_n 1 T_\infty)(\mathrm{Re} F) \leq (T_n \otimes T_n 1 T_\infty)(1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'}),$$

equivalently,

$$|(T_n \otimes T_n 1 T_\infty)(\mathrm{Re} F)| \leq (T_n \otimes T_n 1 T_\infty)(1 \otimes \epsilon + \|F\| \mathrm{Re} F_{y'}).$$

Consequently, from the fact that each T_n is a positive linear map and the representation of T_∞, it follows that

$$\begin{aligned}
|\mathrm{Re}(T_n \otimes T_n 1 T_\infty)(F)| &= |\mathrm{Re}T_n f - \mathrm{Re}(T_n 1 T_\infty f)| \\
&= |T_n(\mathrm{Re}f) - T_n 1 T_\infty(\mathrm{Re}f)| \\
&= |(T_n \otimes T_n 1 T_\infty)(\mathrm{Re}F)| \\
&\leq (T_n \otimes T_n 1 T_\infty)(1 \otimes \epsilon + \|F\| \mathrm{Re}F_{y'}) \\
&= (T_n \otimes T_n 1 T_\infty)(1 \otimes \epsilon) + (T_n \otimes T_n 1 T_\infty)(\|F\| \mathrm{Re}F_{y'}) \\
&= \epsilon T_n 1 + \|F\|(T_n(\mathrm{Re}f_{y'}) + T_n 1 T_\infty(\mathrm{Re}f_{y'})) \\
&= \epsilon T_n 1 + \|F\|(\mathrm{Re}T_n f_{y'} + T_n 1 \mathrm{Re}T_\infty f_{y'}) \\
&\leq \epsilon T_n 1 + \|F\|(|T_n f_{y'} - T_\infty f_{y'}| + T_n 1 \mathrm{Re}T_\infty f_{y'} + \mathrm{Re}T_\infty f_{y'}),
\end{aligned}$$

which is to say,

$$|\mathrm{Re}(T_n \otimes T_n 1 T_\infty)(F)| \leq \epsilon T_n 1 + \|F\|(|T_n f_{y'} - T_\infty f_{y'}| + T_n 1 \mathrm{Re}T_\infty f_{y'} + \mathrm{Re}T_\infty f_{y'}).$$

Thus, from the latter inequality, the representation of T_∞ and for any sufficiently large integer n, we get

$$\begin{aligned}
|\mathrm{Re}T_n f(y') - \mathrm{Re}T_\infty f(y')| &\leq |\mathrm{Re}T_n f(y') - T_n 1(y')\mathrm{Re}T_\infty f(y')| + |T_n 1(y')\mathrm{Re}T_\infty f(y') - \mathrm{Re}T_\infty f(y')| \\
&\leq \epsilon T_n 1(y') + \|F\|(|T_n f_{y'}(y') - T_\infty f_{y'}(y')| + T_n 1(y')\mathrm{Re}f_{y'}(x') + \\
&\quad \mathrm{Re}f_{y'}(x')) + |\mathrm{Re}T_\infty f(y')||T_n 1(y') - 1| \\
&\leq 2\epsilon + \|F\|(\epsilon + 2\epsilon + \epsilon) + \|f\|\epsilon \\
&= (2 + 4\|F\| + \|f\|)\epsilon.
\end{aligned}$$

Hence $\mathrm{Re}T_n f \longrightarrow \mathrm{Re}T_\infty f$ on $Ch(T_\infty(S))$. By replacing f by $-if$, we see that $\mathrm{Im}T_n f \longrightarrow \mathrm{Im}T_\infty f$ on $Ch(T_\infty(S))$. Therefore, $T_n f \longrightarrow T_\infty f$ on $Ch(T_\infty(S))$, which completes the proof of part (a).

(b) We first claim that $\|T_n\| \leq \sqrt{2}\|T_n 1\|$, where $\|T_n\|$ is the operator norm of T_n (for each $n \in \mathbb{N}$). To see this, assume that $g \in M$ is real-valued and has supremum norm at most 1. Then $-1 \leq g \leq 1$ and thus, $-T_n 1 \leq T_n g \leq T_n 1$, which implies that $\|T_n g\| \leq \|T_n 1\|$. In the real case, this shows that T_n is continuous and the claim holds. In the complex case, from this argument and the fact that M is self-conjugate, it easily follows that $\|T_n\| \leq \sqrt{2}\|T_n 1\|$.

Let $f \in M$. Taking into account the above claim and the boundedness of $\{T_n 1 : n \in \mathbb{N}\}$, we deduce that the set $\{T_n f : n \in \mathbb{N}\}$ is bounded. Now one can follow the last part of the proof of [7, Theorem 3.3] to conclude that $T_n f \longrightarrow T_\infty f$ on Y and we include it for completeness. Assume that $\sim$ is the equivalence relation on Y defined by

$$y \sim y' \Leftrightarrow g(y) = g(y') \qquad \forall g \in N.$$

The quotient space of Y by $\sim$ is denoted by $Y/\sim$, and $\hat{y}$ will stand for the image of $y \in Y$ under the canonical map $\hat{}$ from Y onto $Y/\sim$. Moreover, we define $\hat{g}(\hat{y}) = g(y)$ for all g in N and y in $\hat{Y} = \{\hat{y} : y \in Y\}$. It is apparent that $\hat{N} = \{\hat{g} : g \in N\}$ is a function space on the compact space $\hat{Y}$.

By [4, Section V] and [12, Section 4], for any $y \in Y$, there exits a positive measure μ on the σ-ring of subsets of $\mathcal{B}_{\hat{N}^*}$ generated by $ext(\mathcal{B}_{\hat{N}^*})$ and the Baire subsets of $\mathcal{B}_{\hat{N}^*}$ which represents $\hat{y}$ and $\mu(\mathcal{B}_{\hat{N}^*}) = 1$. From part (a), it is clear that $\widehat{T_n f} \longrightarrow \widehat{T_\infty f}$ on $Ch(\widehat{T_\infty(S)})$. Hence, since $ext(\mathcal{B}_{\hat{N}^*}) = \mathbb{T}\, Ch(\hat{N}) \subseteq \mathbb{T}\, Ch(\widehat{T_\infty(S)})$ and the set $\{T_n f : n \in \mathbb{N}\}$ is bounded, from the Lebesgue's dominated convergence theorem we get

$$T_n f(y) = \widehat{T_n f}(\hat{y}) = \int_{\mathcal{B}_{\hat{N}^*}} \widehat{T_n f} \longrightarrow \int_{\mathcal{B}_{\hat{N}^*}} \widehat{T_\infty f}\, d\mu = \widehat{T_\infty f}(\hat{y}) = T_\infty f(y).$$

Therefore, $T_n f \longrightarrow T_\infty f$, as desired. □

Let us recall here the famous Arzela-Ascoli theorem, which will be used in the proof of the next result.

Theorem (Arzela-Ascoli). Given a subset A of $C(X)$, the following statements are equivalent:

(1) A is a compact subset of $(C(X), \|\cdot\|)$.

(2) A is closed, bounded, and equicontinuous in the sense that for each $x \in X$ and $\epsilon > 0$, there exists a neighborhood V of x such that $|f(y) - f(x)| < \epsilon$ for all $f \in A$ and $y \in V$.

Theorem 3.2. *Let $\{T_n\}$ be a sequence of positive linear maps from M into $C(Y)$, and T_∞ be an isometry from M onto a subspace $T_\infty(M)$ of $C(Y)$.*

(a) If $\{T_n f\}$ converges uniformly to $T_\infty f$ for all $f \in S$, then $\{T_n f\}$ converges uniformly to $T_\infty f$ on each compact subset of $Ch(T_\infty(S))$ for all $f \in M$.

(b) If, furthermore, either $Ch(T_\infty(S))$ or $Ch(N)$ is compact and $Ch(N) \subseteq Ch(T_\infty(S))$, then $\{T_n f\}$ converges uniformly to $T_\infty f$ for any $f \in M$, where N is as in Theorem 3.1.

Proof. (a) As in the proof of Theorem 3.1, there is a continuous surjection $\varphi : Ch(T_\infty(S)) \longrightarrow Ch(S)$ such that for all $f \in S$,

$$T_\infty f(y) = f(\varphi(y)) \;\; (f \in S, y \in Ch(T_\infty(S))).$$

Suppose that K is a compact subset of $Ch(T_\infty(S))$. Let $f \in M$, $y' \in K$ and $\epsilon > 0$. Put $F = f \otimes 1 - 1 \otimes f$ and $x' = \varphi(y')$. As before, we choose an open neighborhood $V_{x'}$ of x' and a function

$f_{y'} \in S$ such that $\mathrm{Re} f_{y'} \geq 0$ on X, $\mathrm{Re} f_{y'} \geq 1$ on $V_{x'}^c$ and $\mathrm{Re} f_{y'}(x') < \epsilon$, and we also have

$$|\mathrm{Re} T_n f - \mathrm{Re} T_\infty f| \leq \epsilon T_n 1 + \|F\|(|T_n f_{y'} - T_\infty f_{y'}| + \mathrm{Re} T_\infty f_{y'} + T_n 1 \mathrm{Re} T_\infty f_{y'}) + |\mathrm{Re} T_\infty f||T_n 1 - 1|,$$

on Y. Now, we prove the following claim.

Claim: The set $\{T_n f : n \in \mathbb{N}\}$ is equicontinuous at y'.

Since $\{T_n f_{y'}\}$ and $\{T_n 1\}$ converge uniformly to $T_\infty f_{y'}$ and 1, respectively, there is an integer n_0 such that for each $n \geq n_0$, $\|T_n f_{y'} - T_\infty f_{y'}\| < \epsilon$ and $\|T_n 1 - 1\| < \epsilon$. On the other hand, $\mathrm{Re} T_\infty f_{y'}(y') < \epsilon$ and so, from the continuity of $\mathrm{Re} T_\infty f_{y'}$ and $T_\infty f$, we can choose a neighborhood $W_{y'}$ of y' so that the inequalities $\mathrm{Re} T_\infty f_{y'} < \epsilon$ and $|T_\infty f - T_\infty f(y')| < \epsilon$ hold on $W_{y'}$. Hence, letting $\eta = \sup_{i\in\mathbb{N}} \|T_i 1\|$, for each $y \in W_{y'}$ and $n \geq n_0$ we get

$$\begin{aligned}
|\mathrm{Re} T_n f(y) - \mathrm{Re} T_n f(y')| &\leq |\mathrm{Re} T_n f(y) - \mathrm{Re} T_\infty f(y)| + |\mathrm{Re} T_n f(y') - \mathrm{Re} T_\infty f(y')| + \\
&|\mathrm{Re} T_\infty f(y) - \mathrm{Re} T_\infty f(y')| \leq \eta\epsilon + \|F\|(|T_n f_{y'}(y) - T_\infty f_{y'}(y)| + \mathrm{Re} T_\infty f_{y'}(y) + \\
&\eta T_\infty f_{y'}(y)) + \|f\||T_n 1(y) - 1| + \eta\epsilon + \|F\|(|T_n f_{y'}(y') - T_\infty f_{y'}(y')| + \\
&\mathrm{Re} T_\infty f_{y'}(y') + \eta T_\infty f_{y'}(y')) + \|f\||T_n 1(y') - 1| + |\mathrm{Re} T_\infty f(y) - \mathrm{Re} T_\infty f(y')| \\
&\leq \eta\epsilon + \|F\|(\epsilon + \epsilon + \eta\epsilon) + \|f\|\epsilon + \eta\epsilon + \|F\|(\epsilon + \epsilon + \eta\epsilon) + \|f\|\epsilon + \epsilon \\
&= \epsilon(2\eta + 2\|f\| + 4\|F\| + 2\eta\|F\|) + \epsilon.
\end{aligned}$$

Now, from the continuity of $T_1 f, ..., T_{n_0} f$, it follows that the set $\{\mathrm{Re} T_n f : n \in \mathbb{N}\}$ is equicontinuous at y'. Similarly, the set $\{\mathrm{Im} T_n f : n \in \mathbb{N}\}$ is equicontinuous at y', and, as a consequence, $\{T_n f : n \in \mathbb{N}\}$ is equicontinuous at y', as claimed.

Moreover, as observed in the proof of Theorem 3.1(b), $\{T_n f : n \in \mathbb{N}\}$ is bounded. Therefore, from the Arzela-Ascoli theorem and Theorem 3.1(a), it follows that each subsequence $\{T_n f\}$ has a uniformly convergent sequence to $T_\infty f$ on K. This argument shows that $\{T_n f\}$ converges uniformly to $T_\infty f$ on the compact set K.

(b) When either $Ch(T_\infty(S))$ or $Ch(N)$ is compact, then, from the above discussion, we deduce that $\{T_n f\}$ converges uniformly to $T_\infty f$ on $Ch(N)$. Next, since $Ch(N)$ is a boundary for N, it is immediately seen that $\{T_n f\}$ converges uniformly to $T_\infty f$ (on Y). □

Remark 3.3. We would like to remark that the sequential version of Korovkin's theorem does not yield its net version (see [14]). However, it can be easily checked that our techniques hold true when we replace the sequence $\{T_n\}$ by a net of positive linear maps.

In the following corollary, we obtain the main results of [7], namely, [7, Theorem 3.3] and [7, Theorem 4.1] as consequences of Theorems 3.1 and 3.2.

Corollary 3.4. *Let M be a subspace of $C(X)$, $S \subseteq M$ be a function space, $\{T_n\}$ be a sequence of unital linear contractions from M into $C(Y)$, T_∞ be a linear isometry from M into $C(Y)$, and $Ch(N) \subseteq Ch(T_\infty(S))$, where $N := Span \bigcup_{1\leq n\leq\infty} T_n(M)$.*

(a) If $T_n f \longrightarrow T_\infty f$ for all $f \in S$, then $T_n f \longrightarrow T_\infty f$ for all $f \in M$.

(b) If $\{T_n f\}$ converges uniformly to $T_\infty f$ for all $f \in S$, then $\{T_n f\}$ converges uniformly to $T_\infty f$ on each compact subset of $Ch(T_\infty(S))$ for any $f \in M$. If, furthermore, $Ch(T_\infty(S))$ or $Ch(N)$ is compact, then $\{T_n f\}$ converges uniformly to $T_\infty f$ for all $f \in M$.

Proof. In the context of real-valued function spaces, since every linear map $\mathcal{T}$ with $\|\mathcal{T}\| = \mathcal{T}(1) = 1$ is positive ([13]), the result follows immediately from Theorems 3.1 and 3.2. Now let us consider the complex case. We note that

$$M + \overline{M} = \{f + \overline{g} : f, g \in M\}$$

is a self-conjugate subspace of $C(X)$. According to [7, Lemma 2.5] (or [3, Corollary 3.2]), there is a continuous surjection $\varphi : Ch(T_\infty(M)) \longrightarrow Ch(M)$ such that

$$T_\infty f(y) = f(\varphi(y)) \ \ (f \in M, y \in Ch(T_\infty(M))).$$

Since $Ch(T_\infty(M) + \overline{T_\infty(M)}) = Ch(T_\infty(M))$ and $Ch(M + \overline{M}) = Ch(M)$ ([7, Lemma 2.3]) are boundaries, T_∞ can be extended to a linear isometry $\widetilde{T}_\infty : M + \overline{M} \longrightarrow C(Y)$ such that

$$\widetilde{T}_\infty(f + \overline{g})(y) = f(\varphi(y)) + \overline{g(\varphi(y))} \ \ (f, g \in M, y \in Ch(T_\infty(M))).$$

Moreover, by [7, Lemma 3.2], each T_n can be extended to a positive linear map $\widetilde{T}_n$ from $\overline{M} + M$ into $C(Y)$. Now, we get the result from Theorems 3.1 and 3.2. □

4. Examples

In this section we provide several examples which show how our results can be applied.

Example 4.1. Let $k \in \mathbb{N}\cup\{0, \infty\}$ and $C^{(k)}(I)$ denote the space of k-times continuously differentiable functions on the interval $I = [0, 1]$ which is a self-conjugate space. Suppose that $\{T_n\}$ is a sequence of positive linear maps from $C^{(k)}(I)$ into $C(I)$ satisfying

$$T_n 1 \longrightarrow 1, \ \ T_n x \longrightarrow x, T_n x^2 \longrightarrow x^2.$$

For each $a \in I$, the function $h(x) = 1 - (x - a)^2$ belongs to the function space $S = Span\{1, x, x^2\}$. Since $h(a) = 1$ and $|h(y)| < 1$ for any $y \neq a$, we infer $Ch(S) = I$, by Remark 2.1. Now from Theorem 3.1, we conclude that $T_n f \longrightarrow f$ for all $f \in C^{(k)}(I)$. Meantime, by Theorem 3.2, the same result holds true for "uniformly convergence" instead of "pointwise convergence", which can be also obtained from Korovkin's first theorem.

Example 4.2. Let Ω be a non-empty open subset of $\mathbb{R}^p$ and K be a compact subset of Ω. The term *multi-index* denotes an ordered p-tuple $\alpha = (\alpha_1, ..., \alpha_p)$ of nonnegative integers α_i. For each multi-index α, consider the differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_p}\right)^{\alpha_p},$$

if $\alpha \neq 0$, and $D^\alpha f = f$ if $\alpha = 0$. A function f on Ω is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for all multi-index α. By $\mathcal{D}_K$ we denote the space $\{f|_K : f \in C^\infty(\Omega)\}$. Since $\mathcal{D}_K$ may be considered as a function space on K, from our results we deduce the following.

If $\{T_n : \mathcal{D}_K \longrightarrow C(K) : n \in \mathbb{N}\}$ is a sequence of positive linear maps such that $T_n 1 \longrightarrow 1$, $T_n(P_k) \longrightarrow P_k$, $T_n(\sum_{k=1}^p P_k^2) \longrightarrow \sum_{k=1}^p P_k^2$, where P_k is the projection

$$P_k(x) = x_k \text{ for } x = (x_1, ..., x_p),$$

then $T_n f \longrightarrow f$ for all $f \in \mathcal{D}_K$. A similar result holds true for "uniformly convergence" instead of "pointwise convergence".

Let us remark that for any $a = (a_1, ..., a_p) \in K$, the function

$$h(x) = b_1 - (P_1(x) - a_1)^2 + ... + b_k - (P_p(x) - a_p)^2 \quad (x = (x_1, ..., x_p) \in \Omega),$$

where $b_i > \max\{|P_i(x) - a_i| : x \in K\}$, $i = 1, ..., p$, implies that a belongs to the Choquet boundary of $S = Span\{1, P_1, ..., P_p, P_1^2, ..., P_p^2\}$ by Remark 2.1.

The following example includes the complex Korovkin theorem.

Example 4.3. If $\{T_n : C(\mathbb{T}) \longrightarrow C(\mathbb{T}) : n \in \mathbb{N}\}$ is a sequence of positive linear maps such that $T_n 1 \longrightarrow 1$ and $T_n z \longrightarrow z$, then $T_n f \longrightarrow f$ for all $f \in C(\mathbb{T})$. Notice that here if $z_0 \in \mathbb{T}$, then the function $h(z) = \frac{z+z_0}{2}$ works for Remark 2.1 ($S = Span\{1, z\}$).

Let D be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ and $\{T_n : C(D) \longrightarrow C(D) : n \in \mathbb{N}\}$ be a sequence of positive linear maps such that $T_n 1 \longrightarrow 1$, $T_n z \longrightarrow z$, $T_n |z|^2 \longrightarrow |z|^2$, then $T_n f \longrightarrow f$ for all $f \in C(D)$.

It should be noted that since T_n is positive, it is easily seen that $T_n \bar{z} = \overline{T_n z}$, which yields $T_n \bar{z} \longrightarrow \bar{z}$. Hence for each $z_0 \in D$, the function $h(z) = 1 - \frac{|z-z_0|^2}{4} = 1 - \frac{|z|^2 - \bar{z}z_0 - \bar{z}_0 z + |z_0|^2}{4}$, which belongs to $S = Span\{1, z, \bar{z}, |z|^2\}$, is the appropriate function for Remark 2.1.

The two above results holds true for "uniformly convergence" instead of "pointwise convergence".

Remark 4.4. From our theorems, one can obtain the Korovkin-type results of [11] and [15] (with respect to both "uniformly convergence" and "pointwise convergence"), which are generalizations of Korovkin's second theorem on convergence of a sequence of positive linear maps for the space of real-valued continuous 2π-periodic functions on $\mathbb{R}$.

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