DISJOINTNESS PRESERVING MAPS BETWEEN VECTOR-VALUED GROUP ALGEBRAS

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ABSTRACT. Let G be a locally compact abelian group and B be a commutative Banach algebra. Let $L^1(G, B)$ be the Banach algebra of B-valued Bochner integrable functions on G. In this paper we provide a complete description of continuous disjointness preserving maps on $L^1(G, B)$-algebras based on a scarcely used tool: the vector-valued Fourier transform. We also present necessary and sufficient conditions for these operators to be compact.

1. INTRODUCTION

Linear maps between Banach algebras, Banach lattices, or Banach spaces preserving certain properties have been of a considerable interest for many years. The most classical question concerns isometries, although more recently, maps that preserve spectrum, spectral radius, commutativity, normal elements, self-adjoint elements, nilpotents, idempotents, linear rank, disjointness of cozeroes, or other properties have been intensely investigated.

Among them, maps that preserve the disjointness of cozeroes defined between spaces of scalar-valued continuous functions on locally compact and compact spaces, as a generalization of the concept of homomorphism, have a long history in functional analysis in the context of rings, algebras, or vector lattices under several names such as Lamperti operators, separating maps, disjointness preserving operators, etc. (see, for example, [1, 2, 3, 4, 5, 7, 8, 13, 16]). In recent years, certain attention has been given to such maps when defined on spaces of vector-valued continuous functions (see, e.g., [10, 14]). However, we do not know much about disjointness preserving maps on vector-valued settings in comparison with scalar-valued contexts and something similar can be said with regard to (algebra) homomorphisms between vector-valued group algebras.

In this paper we focus on the study of disjointness preserving maps defined between vector-valued group algebras. Banach algebras of vector-valued functions date back to the early moments of the theory of Banach algebras and play a natural role in functional analysis. Among them, spaces of

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vector-valued continuous functions and vector-valued group algebras are perhaps the most studied ones. Homomorphisms of algebras ([11]) and multipliers ([22]) on group algebras of vector-valued functions are examples of disjointness preserving maps. Here we provide a weighted composition representation of continuous disjointness preserving maps on vector-valued group algebras and a characterization of compact disjointness preserving maps on the same context. Let us recall that the study of continuous disjointness preserving maps on vector-valued function spaces and the compactness dates back to [15], where Jamison and Rajagopalan described continuous disjointness preserving maps on the Banach space $C(X,E)$ of all continuous functions from a compact space X into a Banach space E, and gave criteria of compactness for these maps. Shortly after, in [6], the results of [15] were extended to $C_0(X,E)$ for locally compact X. It is worth mentioning that our results are based on the vector-valued Fourier transform, a scarcely used tool in the literature, which acts as a "vector-valued Gelfand transform". This technique contrasts with the one used in most previous papers dealing with algebras of vector-valued functions which are based on the scalar-valued Gelfand transform (see e.g., [14] and the references therein).

2. Preliminaries

Let G be a locally compact abelian group with the Haar measure m and B be a commutative Banach algebra.

Let $L^1(G,B)$ be the *Bochner algebra* of G, i.e., the commutative Banach algebra of integrable functions from G to B endowed with the convolution product

$$(f*g)(t)=\int_G f(t-s)\cdot g(s)dm(s)$$

for all $f,g\in L^1(G,B)$, $t\in G$, and the norm

$$||f||_1=\int_G ||f(t)||dm(t)$$

for all $f\in L^1(G,B)$. We shall write $L^1(G)$ if B is chosen as the complex numbers. Next we provide the main properties of $L^1(G,B)$ and $L^1(G)$, which can be found, basically, in [17, Section 4.13] and [20, Chapter 2].

Let $\hat{G}$ be the dual group of G. Given $f\in L^1(G,B)$, its vector-valued Fourier transform is defined as

$$\hat{f}(\gamma):=\int_G f(t)\cdot\gamma(-t)dm(t)$$

for a given $\gamma\in\hat{G}$.

Let $A(\hat{G},B)$ be the vector-valued Fourier algebra associated to $L^1(G,B)$, that is,

$$A(\hat{G},B):=\{\hat{f}\,:\,f\in L^1(G,B)\}.$$

Let us recall that $C_0(\hat{G}, B)$ is the Banach space of all continuous B-valued functions on $\hat{G}$ vanishing at infinity. It is known that $A(\hat{G}, B) \subset C_0(\hat{G}, B)$ separates the points of $\hat{G}$, and that $||\hat{f}||_\infty \leq ||f||_1$, that is, the Fourier transform, considered as a map from $L^1(G, B)$ into $C_0(\hat{G}, B)$, is a continuous linear injection. Besides,

(1) $f = 0$ if and only if $\hat{f} \equiv 0$ (Uniqueness Principle).

(2) If $f \in L^1(G, B)$ (or $L^1(G)$) and $g \in L^1(G, B)$, then $f * g \in L^1(G, B)$ and $\widehat{f * g} = \hat{f}\hat{g}$.

Let $f, g \in L^1(G)$ and $b_1, b_2 \in B$. Let us recall that the tensor product of f and b_1 is defined as $(f \otimes b_1)(\gamma) := f(\gamma)b_1$ for all $\gamma \in G$. Here we present some properties of the tensor product:

(1) $f \otimes b_1 \in L^1(G, B)$ and $||f \otimes b_1||_1 = ||f||_1||b_1||$.

(2) $\widehat{f \otimes b_1}(\gamma) = (\hat{f}b_1)(\gamma) = \hat{f}(\gamma)b_1$.

(3) $(f \otimes b_1) * (g \otimes b_2) = (f * g) \otimes b_1b_2$.

A classical result of Grothendieck asserts that $L^1(G, B)$ is isometrically isomorphic to the projective tensor product $L^1(G)\hat{\otimes}B$ of $L^1(G)$ and B. Namely, we can identify

$$L^1(G, B) = \left\{ \sum_{i=1}^{\infty} f_i \otimes b_i : f_i \in L^1(G), b_i \in B, \sum_{i=1}^{\infty} \|f_i\|_1 \|b_i\| < \infty \right\}.$$

3. Disjointness preserving maps on vector-valued group algebras

For $i = 1, 2$, let G_i be locally compact abelian groups and B_i be commutative Banach algebras.

Definition 3.1. *Let $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ be a linear mapping. It is said that T is zero product preserving if $Tf * Tg = 0$ whenever $f * g = 0$ for every $f, g \in L^1(G_1, B_1)$.*

Associated to a zero product preserving mapping T we can define a mapping $\hat{T} : A(\hat{G}_1, B_1) \longrightarrow A(\hat{G}_2, B_2)$ defined as $\hat{T}\hat{f} := \widehat{Tf}$ for all $f \in L^1(G_1, B_1)$. It is apparent, due to the Uniqueness Principle, that $\hat{T}$ is zero product preserving if and only if so is T.

Definition 3.2. *Let $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ be a linear mapping. It is said that T is separating or disjointness preserving if $coz(\hat{f}) \cap coz(\hat{g}) = \emptyset$ yields $coz(\hat{T}\hat{f}) \cap coz(\hat{T}\hat{g}) = \emptyset$ for every $f, g \in L^1(G_1, B_1)$. If T is a separating bijection whose inverse is also separating, then it is said to be biseparating.*

Unless otherwise specified, in the sequel $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ will stand for a disjointness preserving map.

It is apparent that, unlike the scalar case, in this vector valued setting, preserving zero products and preserving disjointness of cozeros are different concepts. However if we, for instance, assume that B_1 and B_2 are integral domains (i.e., they have no divisors of zero), then both concepts agree.

Definition 3.3. *Let* $\hat{G}_{20} := \{\xi \in \hat{G}_2 : \hat{T}\hat{f}(\xi) \neq 0 \; for\ some\ f \in L^1(G_1, B_1)\}$. *A point* $\gamma \in \hat{G}_1 \cup \{\infty\}$ *is said to be a support point for* $\xi \in \hat{G}_{20}$ *if for any neighborhood* U *of* γ, *there is* $f \in L^1(G_1, B_1)$ *with* $coz(\hat{f}) \subset U$ *and* $\hat{T}\hat{f}(\xi) \neq 0$.

Let us remark that if T is onto, then $\hat{G}_{20} = \hat{G}_2$.

Lemma 3.4. *Given* $\xi \in \hat{G}_{20}$, *there exists a unique support point for* ξ *in* $\hat{G}_1 \cup \{\infty\}$.

Proof. Suppose, contrary to what we claim, that there is no support point for a certain $\xi \in \hat{G}_{20}$, that is, for every $\gamma \in \hat{G}_1 \cup \{\infty\}$, there exists a neighborhood U such that if $f \in L^1(G_1, B_1)$ and $coz(\hat{f}) \subset U$, then $\hat{T}\hat{f}(\xi) = 0$. Since such neighborhoods form a cover of the compact $\hat{G}_1 \cup \{\infty\}$, there exists a finite subcover, say $\{U_1, ..., U_n\}$. By [9, Lemma 1], we can find $\{f_1, ..., f_n\} \subset L^1(G_1)$ such that $coz(\hat{f}_i) \subset U_i$ for $i = 1, ..., n$ and $\sum_{i=1}^n \hat{f}_i = 1$. Hence, given any $f \in L^1(G_1, B_1)$, we have $\hat{f} = \sum_{i=1}^n \hat{f}\hat{f}_i$ and $\hat{T}(\hat{f}_i\hat{f})(\xi) = 0$ for $i = 1, ..., n$. Therefore, $\hat{T}\hat{f}(\xi) = 0$ for all $f \in L^1(G_1, B_1)$, which contradicts the fact that $\xi \in \hat{G}_{20}$.

Let us suppose that γ_1 and γ_2 are two distinct support points for $\xi \in \hat{G}_{20}$. Let V_1 and V_2 be disjoint neighborhoods of γ_1 and γ_2, respectively. Then there exists $f_1, g_1 \in L^1(G_1, B_1)$ such that $coz(\hat{f}_1) \subset V_1$ and $coz(\hat{f}_2) \subset V_2$, with $\hat{T}\hat{f}_1(\xi) \neq 0$ and $\hat{T}\hat{f}_2(\xi) \neq 0$, which contradicts the disjointness preserving property of $\hat{T}$. □

Lemma 3.4 enables us to define a map $h : \hat{G}_{20} \longrightarrow \hat{G}_1 \cup \{\infty\}$ which sends any $\xi \in \hat{G}_{20}$ to its support point.

Proposition 3.5. *The map* $h : \hat{G}_{20} \longrightarrow \hat{G}_1 \cup \{\infty\}$ *is continuous.*

Proof. Let (ξ_d) be a net in $\hat{G}_{20}$ converging to some $\xi_0 \in \hat{G}_{20}$. Let $(h(\xi_{d'}))$ be a subnet of $(h(\xi_d))$ which converges to some γ_0 in the compact space $\hat{G}_1 \cup \{\infty\}$. Suppose, contrary to what we claim, that $h(\xi_0) \neq \gamma_0$. Let U and V be disjoint neighborhoods of $h(\xi_0)$ and γ_0, respectively. Then there exists $f \in L^1(G_1, B_1)$ such that $\hat{T}\hat{f}(\xi_0) \neq 0$ and $coz(\hat{f}) \subset U$. On the other hand, as $\hat{T}\hat{f}$ is a continuous function, there must exist an index d_0 such $\hat{T}\hat{f}(\xi_{d_0}) \neq 0$ and $h(\xi_{d_0}) \in V$. Let $g \in L^1(G_1, B_1)$ such that $\hat{T}\hat{g}(\xi_{d_0}) \neq 0$ and $coz(\hat{g}) \subset V$. Consequently, $coz(\hat{f}) \cap coz(\hat{g}) = \emptyset$, but $\hat{T}\hat{f}(\xi_{d_0}) \neq 0$ and $\hat{T}\hat{g}(\xi_{d_0}) \neq 0$, which contradicts the disjointness preserving property of $\hat{T}$. □

Proposition 3.6. *Let* U *be an open subset of* $\hat{G}_1 \cup \{\infty\}$ *and let* $f \in L^1(G_1, B_1)$. *If* $\hat{f}|_U \equiv 0$, *then* $\hat{T}\hat{f}|_{h^{-1}(U)} \equiv 0$.

Proof. Assume that $\hat{f}$ vanishes on an open subset U of $\hat{G}_1 \cup \{\infty\}$. If we take $\xi \in h^{-1}(U)$, then there exists $f' \in L^1(G_1, B_1)$ with $coz(\hat{f}') \subset U$ and $\hat{T}\hat{f}'(\xi) \neq 0$. Since $coz(\hat{f}) \cap coz(\hat{f}') = \emptyset$, we infer that $\hat{T}\hat{f}(\xi) = 0$. □

The following example, adapted from [10], shows that, in the vector-valued setting, unlike in the complex-valued case ([9]), the automatic continuity of T cannot be obtained from its disjointness preserving property, even if T is a biseparating bijection.

Example 3.7. Let G be a trivial group consisting only of an identity element and let c_0 be the Banach algebra of all sequences which converge to zero. Then it is apparent that $A(\hat{G}, c_0) = C(\hat{G}, c_0)$. Let $e_n := (0, ..., 0, 1, 0, ...) \in c_0$ and define a linear functional ϕ on c_0 such that $\phi(e_n) = n$, and a linear bijection $\Phi : c_0 \longrightarrow c_0$ defined as $\Phi(\alpha_1, \alpha_2, ...) := (\alpha_1 + \phi(\alpha_1, \alpha_2, ...), \alpha_2, ...)$. It is apparent that both are unbounded. Hence, if ξ stands for the only element in $\hat{G}$, then we can define an unbounded biseparating bijection $\hat{T} : C(\hat{G}, c_0) \longrightarrow C(\hat{G}, c_0)$ as $\hat{T}\hat{f}(\xi) = \Phi[\hat{f}(\xi)]$ for all $\hat{f} \in C(\hat{G}, c_0)$.

Proposition 3.8. *Assume that the disjointness preserving map T is continuous and $\xi_0 \in \hat{G}_{20}$. Then $h(\xi_0) \in \hat{G}_1$ and $\hat{T}\hat{f}(\xi_0) = 0$ for all $f \in L^1(G_1, B_1)$ with $\hat{f}(h(\xi_0)) = 0$.*

Proof. Assume, contrary to what we claim, that $h(\xi_0) = \infty$. We first show that $\hat{T}(\hat{f}_0 b)(\xi_0) = 0$ for any $f_0 \otimes b \in L^1(G_1) \otimes B_1$. By [12, 33.13], we know that $L^1(G_1)$ is Tauberian, that is, the set $\{f \in L^1(G_1) : \hat{f}$ has compact support$\}$ is a dense ideal in $L^1(G_1)$. Hence, given any $f_0 \otimes b \in L^1(G_1, B_1)$, there exists a sequence of functions $(f_n) \subset L^1(G_1)$ whose Fourier transforms have compact support such that $(f_n \otimes b)$ converges to $f_0 \otimes b$. For every $n \in \mathbb{N}$, we know that $\hat{f}_n b$ vanishes on a certain neighborhood of ∞. Hence, by Proposition 3.6, we have $\hat{T}(\hat{f}_n b)(\xi_0) = 0$ for every $n \in \mathbb{N}$. Meantime, from the continuity of T we get

$$\|\hat{T}(\hat{f}_n b) - \hat{T}(\hat{f}_0 b)\|_\infty \leq \|T(f_n \otimes b) - T(f_0 \otimes b)\|_1 \longrightarrow 0,$$

which shows that $\hat{T}(\hat{f}_0 b)(\xi_0) = \lim_{n\to\infty} \hat{T}(\hat{f}_n b)(\xi_0) = 0$. Now for arbitrary $f \in L^1(G_1, B_1)$, as mentioned in Section 2, we have $f = \sum_{i=1}^{\infty} g_i \otimes b_i$, where $g_i \in L^1(G_1)$, $b_i \in B_1$ and $\sum_{i=1}^{\infty} \|g_i\|_1 \|b_i\| < \infty$. Thus, similarly, from the continuity of T we conclude that $\hat{T}\hat{f}(\xi_0) = \sum_{i=1}^{\infty} \hat{T}(\hat{g}_i b_i)(\xi_0) = 0$. Therefore we have $\hat{T}\hat{f}(\xi_0) = 0$ for all $f \in L^1(G_1, B_1)$, which is a contradiction showing that $h(\xi_0) \neq \infty$.

Now we prove that $\hat{T}\hat{f}(\xi_0) = 0$ for any $f \in L^1(G_1, B_1)$ with $\hat{f}(h(\xi_0)) = 0$. As above, it is enough to consider the functions of the form $f_1 \otimes b_1$ in the algebraic tensor product $L^1(G_1) \otimes B_1$. Assume, contrary to what we claim, that there exists $f_0 \otimes b \in L^1(G_1, B_1)$ such that $(\hat{f}_0 b)(h(\xi_0)) = 0$ and $\hat{T}(\hat{f}_0 b)(\xi_0) \neq 0$. From [20, Theorem 2.6.3], we can find, for each $n \in \mathbb{N}$, a function k_n in $L^1(G_1)$ such that

(1) $\hat{k_n} \equiv 1$ on a neighborhood V_n of $h(\xi_0)$,

(2) $||\hat{k_n} \hat{f}_0||_\infty \leq ||k_n * f_0||_1 < 1/n^2$.

Next we can define $g_0 := \sum_{n=1}^{\infty} ((k_n * f_0) \otimes b)$. Since $||(k_n * f_0) \otimes b||_1 < ||b||/n^2$ for every $n \in \mathbb{N}$, we infer that g_0 belongs to the Banach algebra $L^1(G_1, B_1)$. Furthermore, since $\hat{k_n} \hat{f}_0 b \equiv \hat{f}_0 b$ on

V_n, which is a neighborhood of $h(\xi_0)$, by Proposition 3.6 and the additivity of $\hat{T}$, we deduce that $\hat{T}(\hat{k_n}\hat{f}_0 b)(\xi_0) = \hat{T}(\hat{f}_0 b)(\xi_0)$. As a consequence, from the continuity of T it follows that

$$||\hat{T}\hat{g_0}(\xi_0)|| = \left\| \hat{T}\left(\sum_{n=1}^{\infty} \hat{k_n}\hat{f}_0 b\right)(\xi_0) \right\| = \left\| \sum_{n=1}^{\infty} \hat{T}(\hat{f}_0 b)(\xi_0) \right\| = \infty,$$

which is a contradiction.

□

Definition 3.9. *Given $\xi \in \hat{G}_{20}$, let $w_\xi : B_1 \longrightarrow B_2$ be defined as $w_\xi(b) := \hat{T}(\hat{e}b)(\xi)$ where $e \in L^1(G_1)$ and $\hat{e} \equiv 1$ on a certain neighborhood of $h(\xi)$. Furthermore, for any $\xi \in \hat{G}_2 \setminus \hat{G}_{20}$, we define $w_\xi \equiv 0$.*

We remark that by [20, Theorem 2.6.2], we can always find such function e and by Proposition 3.6, we infer that the definition of w_ξ does not depend on the choice of such e. So we can define the function w by $w(\xi) = w_\xi$, and we will see in the next result that w_ξ belongs to the space $\mathcal{L}(B_1, B_2)$ of continuous linear operators of B_1 into B_2 with the strong operator topology.

Meantime, we extend h from $\hat{G}_{20}$ to $\hat{G}_2$, which we keep denoting by h, by assigning to ξ, for each $\xi \in \hat{G}_2 \setminus \hat{G}_{20}$, an arbitrary point in $\hat{G}_1$.

Theorem 3.10. *Let $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ be a continuous disjointness preserving map. Then, there exist maps $h : \hat{G}_2 \longrightarrow \hat{G}_1$ and $w : \hat{G}_2 \longrightarrow \mathcal{L}(B_1, B_2)$ such that for any $\xi \in \hat{G}_2$ and any $f \in L^1(G_1, B_1)$, we have*

$$\hat{T}\hat{f}(\xi) = w_\xi[\hat{f}(h(\xi))].$$

Moreover, w and h are continuous on $\hat{G}_{20}$.

Proof. Let h and w be defined as above. We claim that, for each $\xi \in \hat{G}_{20}$, the linear map w_ξ is continuous, indeed, $w_\xi \in \mathcal{L}(B_1, B_2)$. To this end, let $e \in L^1(G)$ such that $\hat{e}(h(\xi)) = 1$ and $||e||_1 < 2$ (see [20, Theorem 2.6.3]). Then, if $b \in B_1$, we have $||w_\xi(b)|| = ||w_\xi[(\hat{e}b)(h(\xi))]|| = ||\hat{T}(\hat{e}b)(\xi)|| \leq ||\hat{T}(\hat{e}b)||_\infty \leq ||T(e \otimes b)||_1 \leq ||T||||e \otimes b||_1 \leq 2||T||||b||$ thanks to the boundedness of T. Hence, $||w_\xi|| \leq 2||T||$ for every $\xi \in \hat{G}_{20}$.

We now obtain the representation of $\hat{T}$. Assume first that $\xi \in \hat{G}_{20}$. Choose $e \in L^1(G)$ such that $\hat{e} \equiv 1$ on a certain neighborhood of $h(\xi)$ and let $f \in L^1(G_1)$. Since $((\hat{f} - \hat{f}(h(\xi))\hat{e})b)(h(\xi)) = (\hat{f}b - \hat{f}(h(\xi))\hat{e}b)(h(\xi)) = 0$, we deduce, by Proposition 3.8, that $\hat{T}((\hat{f} - \hat{f}(h(\xi))\hat{e})b)(\xi) = 0$. Hence $\hat{T}(\hat{f}b)(\xi) = \hat{T}(\hat{f}(h(\xi))\hat{e}b)(\xi) = w_\xi[(\hat{f}b)(h(\xi))]$.

Given $f \in L^1(G_1, B_1)$, we know, as mentioned in Section 2, that $f = \sum_{i=1}^{\infty} f_i \otimes b_i$, where each $f_i \in L^1(G_1)$, each $b_i \in B_1$ and $\sum_{i=1}^{\infty} \|f_i\|_1 \|b_i\| < \infty$. Hence, for any $\xi \in \hat{G}_{20}$, from the continuity of

T, w_ξ and the Fourier transform and also the above argument, we have

$$\begin{aligned}\hat{T}\hat{f}(\xi) &= \hat{T}\left(\sum_{i=1}^{\infty}\hat{f}_i b_i\right)(\xi) = \sum_{i=1}^{\infty}\hat{T}(\hat{f}_i b_i)(\xi) \\ &= \sum_{i=1}^{\infty} w_\xi[(\hat{f}_i b_i)(h(\xi))] = w_\xi\left[\sum_{i=1}^{\infty}(\hat{f}_i b_i)(h(\xi))\right] \\ &= w_\xi[\hat{f}(h(\xi))].\end{aligned}$$

If we assume that $\xi \in \hat{G}_2 \setminus \hat{G}_{20}$, then it is clear that $\hat{T}\hat{f}(\xi) = 0 = w_\xi[\hat{f}(h(\xi))]$.

Finally, to show the continuity of w, let (ξ_α) be a net in $\hat{G}_{20}$ converging to $\xi_0 \in \hat{G}_{20}$. Choose $e \in L^1(G_1)$ such that $\hat{e} \equiv 1$ on a neighborhood V of $h(\xi_0)$, by [20, Theorem 2.6.2]. Since h is continuous on $\hat{G}_{20}$ by Proposition 3.5, we can assume, without loss of generality, that for each α, $h(\xi_\alpha) \in V$. Hence, for each $b \in B_1$, from the continuity of $\hat{T}(\hat{e}b)$ it follows that

$$w_{\xi_\alpha}(b) = \hat{T}(\hat{e}b)(\xi_\alpha) \longrightarrow \hat{T}(\hat{e}b)(\xi_0) = w_{\xi_0}(b),$$

which shows that $w_{\xi_\alpha} \longrightarrow w_{\xi_0}$ is continuous strongly. Hence w is continuous on $\hat{G}_{20}$. □

It is worth mentioning that if G_2 is discrete, or equivalently, $\hat{G}_2$ is compact, then w is continuous on $\hat{G}_2$. Indeed, in this case we have $w_\xi(b) = \hat{T}(b)(\xi)$ for all $\xi \in \hat{G}_2$ and $b \in B_1$, which easily yields the continuity of w on $\hat{G}_2$.

As a corollary of the above theorem, we show that continuous disjointness preserving maps $T : L^1(G_1, B_1) \longrightarrow B_2$ can be written as the composition of a linear map and an algebra homomorphism.

Corollary 3.11. *Let $T : L^1(G_1, B_1) \longrightarrow B_2$ be a continuous disjointness preserving map. Then $T = W \circ H$, where $W : B_1 \longrightarrow B_2$ is a linear map and $H : L^1(G_1, B_1) \longrightarrow B_1$ is an algebra homomorphism.*

Proof. We assume, without loss of generality, that $T \neq 0$. We can identify B_2 with $L^1(\{g\}, B_2)$ for a certain singleton $\{g\}$. By Theorem 3.10, we know that $\hat{T}\hat{f} = w_g(\hat{f}(h(g)))$ for any $f \in L^1(G_1, B_1)$. Let us define a homomorphism $H : L^1(G_1, B_1) \longrightarrow B_1$ as follows: $H(f) = \hat{f}(h(g)))$. It is clear that $ker(H) \subseteq ker(T)$.

Take an element b_1 in the range of H, that is, there exists $f_1 \in L^1(G_1, B_1)$ such that $H(f_1) = b_1$. Define $w_1(b_1) := T(f_1)$, which is well defined and linear since $ker(H) \subseteq ker(T)$. Hence w_1 is a linear map defined from the range of H into B_2 such that $T = w_1 \circ H$. Such w_1 can be extended to a linear map W which coincides with w_1 on the range of H, vanishes on its complement and $T = W \circ H$.

□

Theorem 3.12. *Let $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ be a continuous biseparating map. Then, there exist a homeomorphism $h : \hat{G}_2 \longrightarrow \hat{G}_1$ and a continuous function $w : \hat{G}_2 \longrightarrow \mathcal{L}(B_1, B_2)$ such that for any $\xi \in \hat{G}_2$ and any $f \in L^1(G_1, B_1)$, we have*

$$\hat{T}\hat{f}(\xi) = w_\xi[\hat{f}(h(\xi))].$$

Moreover, for each $\xi \in \hat{G}_2$, w_ξ is a bijective homeomorphism and especially, B_1 and B_2 are isomorphic as vector spaces.

Proof. Since T is onto, then it is clear that $\hat{G}_{20} = \hat{G}_2$. Furthermore, by Theorem 3.10, there exist continuous maps $h : \hat{G}_2 \longrightarrow \hat{G}_1$ and $w : \hat{G}_2 \longrightarrow \mathcal{L}(B_1, B_2)$ such that $\hat{T}\hat{f}(\xi) = w_\xi[\hat{f}(h(\xi))]$ for all $\xi \in \hat{G}_2$ and $f \in L^1(G_1, B_1)$.

From the Open Mapping theorem we deduce that T^{-1} is continuous. Then for the continuous disjointness preserving map T^{-1}, there exist two continuous maps $h' : \hat{G}_1 \longrightarrow \hat{G}_2$ and $w' : \hat{G}_1 \longrightarrow \mathcal{L}(B_2, B_1)$ defined similarly as h and w for T such that $\hat{T}^{-1}\hat{g}(\zeta) = w'_\zeta[\hat{g}(h'(\zeta))]$ for all $\zeta \in \hat{G}_1$ and $g \in L^1(G_2, B_2)$, by Theorem 3.10.

We claim that $h^{-1} = h'$. To see this, we first show that for each $\xi \in \hat{G}_2$, $h'(h(\xi)) = \xi$. Suppose, on the contrary, that $\xi \in \hat{G}_2$ and $h'(h(\xi)) \neq \xi$. Since $A(\hat{G}_2, B_2)$ separates the points of $\hat{G}_2$, there is $g \in L^1(G_2, B_2)$ such that $\hat{g}(h'(h(\xi))) = 0$ and $\hat{g}(\xi) \neq 0$. From the representation of $\hat{T}^{-1}$ it follows that $\hat{T}^{-1}\hat{g}(h(\xi)) = 0$. Then $\hat{g}(\xi) = 0$ by the representation of $\hat{T}$, a contradiction which yields $h'(h(\xi)) = \xi$. Similarly, $h(h'(\zeta)) = \zeta$ for all $\zeta \in \hat{G}_1$. Therefore, $h^{-1} = h'$ which shows that h is a homeomorphism.

We next prove that, for each $\xi \in \hat{G}_2$, w_ξ is bijective. For this purpose, let $b \in B_2 \setminus \{0\}$, and choose $g \in L^1(G_2, B_2)$ such that $\hat{g}(\xi) = b$. Then we have

$$\begin{aligned} b &= \hat{T}(\hat{T}^{-1}\hat{g})(\xi) = w_\xi[(\hat{T}^{-1}\hat{g})(h(\xi))] \\ &= w_\xi[w'_{h(\xi)}(\hat{g}(\xi))] = w_\xi(w'_{h(\xi)}(b)), \end{aligned}$$

and consequently, $b = w_\xi(w'_{h(\xi)}(b))$. Thus $w_\xi \circ w'_{h(\xi)}$ is the identity operator on B_2. Similarly, one can see that $w'_{h(\xi)} \circ w_\xi$ is the identity operator on B_1. Therefore, w_ξ is a bijective map and so B_1 and B_2 are isomorphic as vector spaces. Moreover, by the Open Mapping theorem, w_ξ^{-1} is continuous, i.e., w_ξ is a homeomorphism. □

4. Compact disjointness preserving maps on vector-valued group algebras

First let us state, adapted to our context, a vector-valued version of the Arzela-Ascoli theorem ([21, Theorem 2.1]):

Theorem 4.1. *A subset H of $C_0(\hat{G}, B)$ is relatively compact (or precompact) if and only if*

(1) *H is equicontinuous, that is, for every $\xi_1 \in \hat{G}$ and every net $\{\xi_\alpha\}$ in $\hat{G}$ converging to ξ_1,*

$$\lim_{\alpha} \sup_{f \in H} \{||f(\xi_\alpha) - f(\xi_1)||\} = 0.$$

(2) *$H(\xi) := \{f(\xi) : f \in H\}$ is precompact in B for every $\xi \in \hat{G}$.*

(3) *For every $\epsilon > 0$, there exists a compact subset K of $\hat{G}$ such that $||f(\xi)|| < \epsilon$ for all $f \in H$ and all $\xi \in \hat{G} \setminus K$.*

We will say that T is *compact* if $\hat{T}$ transforms bounded sets into relatively compact ones. Let w_ξ, $\hat{G}_{20}$, h and w be defined as in the previous results. Then we can obtain the following characterization of compact continuous disjointness preserving maps:

Theorem 4.2. *Let $T : L^1(G_1, B_1) \longrightarrow L^1(G_2, B_2)$ be a continuous disjointness preserving map. Then T is compact if and only if*

(1) *w_ξ is compact for every $\xi \in \hat{G}_{20}$.*

(2) *h is locally constant on $\hat{G}_{20}$.*

(3) *For every $\epsilon > 0$, there exists a compact subset K of $\hat{G}_2$ such that $||w_\xi(\hat{f}(h(\xi)))|| < \epsilon$ for all $f \in L^1(G_1, B_1)$ with $||f||_1 \leq 1$ and all $\xi \in \hat{G}_2 \setminus K$. Equivalently, the map $\xi \longrightarrow \|w_\xi\|$ vanishes at infinity.*

(4) *The map $w : \hat{G}_2 \longrightarrow \mathcal{L}(B_1, B_2)$ is continuous when $\mathcal{L}(B_1, B_2)$ is equipped with the operator norm topology.*

Proof. For the necessity, assume that T is compact. Fix $\xi \in \hat{G}_{20}$ and let $\{b_n\}$ be a sequence in $\{b \in B_1 : ||b|| \leq 1\}$. Let $f \in L^1(G_1)$ such that $\hat{f} \equiv 1$ on a certain neighborhood of $h(\xi)$ with $||f||_1 < 2$ ([20, Theorem 2.6.3]), and define $f_n := f \otimes b_n$. It is apparent that $\{\hat{f}_n\}$ is a bounded sequence. Since T is compact, $\{\hat{T}(\hat{f}_n)\}$ is relatively compact. Hence by Theorems 3.10 and 4.1, we deduce that $\{\hat{T}(\hat{f}_n)(\xi)\} = \{w_\xi(\hat{f}_n(h(\xi)))\} = \{w_\xi(b_n)\}$ is relatively compact.

Assume, contrary to what we claim, that there exists $\xi_0 \in \hat{G}_{20}$ such that h is not constant on any open neighborhood U of ξ_0. If we direct a neighborhood base at ξ_0 by inclusion, there exists a net $\{h(\xi_U)\}$ in $\hat{G}_{20}$ converging to ξ_0 and such that $h(\xi_U) \neq h(\xi_0)$ for all U. By [20, Theorem 2.6.3] we can find, for each U, $f_U \in L^1(G_1)$ with $||f_U||_1 < 2$, $\hat{f_U}(h(\xi_0)) = 1$ and $\hat{f_U}(h(\xi_U)) = 0$. Choose $b \in B_1$ such that $w_{\xi_0}(b) \neq 0$ and let $\mathcal{A} = \{f_U \otimes b\}_U$. It is clear that $\hat{\mathcal{A}}$ is bounded and, hence, $\hat{T}(\hat{\mathcal{A}})$ is relatively compact. Consequently, by Theorem 4.1, $\hat{T}(\hat{\mathcal{A}})$ is equicontinuous, but, for each U, we have

$$||\hat{T}(\hat{f_U}b)(\xi_U) - \hat{T}(\hat{f_U}b)(\xi_0)|| = ||w_{\xi_0}(b)|| > 0,$$

a contradiction.

Let $\mathcal{B} = \{f \in L^1(G_1, B_1) : ||f||_1 \leq 1\}$. Since $\mathcal{B}$ is bounded and the Fourier transform is continuous, $\hat{\mathcal{B}}$ is also bounded. Now from the compactness of $\hat{T}$, we infer that $\hat{T}(\hat{\mathcal{B}})$ is relatively compact in $C_0(\hat{G}_2, B_2)$. By Theorem 4.1, we know that for every $\epsilon > 0$, there exists a compact subset K of $\hat{G}_2$ such that $||\hat{T}\hat{f}(\xi)|| = ||w_\xi(\hat{f}(h(\xi)))|| < \epsilon$ for all $f \in \mathcal{B}$ and $\xi \in \hat{G}_2 \setminus K$.

Now if $\xi \in \hat{G}_2 \setminus K$ and $b \in B_1$ with $\|b\| = 1$, then by considering $f = g \otimes b$, where g is a function in $L^1(G_1)$ with $\|g\|_1 \leq 2$ and $\hat{g}(h(\xi)) = 1$ ([20, Theorem 2.6.3]), from the previous paragraph it follows that

$$\|w_\xi(b)\| = \|w_\xi(\hat{g}(h(\xi))b)\| = \|\hat{T}\hat{g}(\xi)\| < 2\epsilon,$$

and consequently, $\|w_\xi\| \leq 2\epsilon$. Hence the map $\xi \longrightarrow \|w_\xi\|$ vanishes at infinity.

Next, we apply arguments similar to those in [15, Theorem 2] to show that condition (4) is valid. Contrary to what we claim, we assume that w is not continuous at $\xi_0 \in \hat{G}_2$. Hence there exists $\epsilon > 0$ such that for each compact neighborhood V of ξ_0 in $\hat{G}_2$, we can find $\xi_V \in V$ with $\|w_{\xi_V} - w_{\xi_0}\| \geq \epsilon$. Consequently, there is a net $\{e_V\}_V$ in B_1 with $\|e_V\| = 1$ and $\|w_{\xi_V}(e_V) - w_{\xi_0}(e_V)\| \geq \epsilon$. Furthermore for each compact neighborhood V of ξ_0 in $\hat{G}_2$, we can choose $g_V \in L^1(G_1, B_1)$ such that $\hat{g}_V(h(\xi_V)) = \hat{g}_V(h(\xi_0)) = e_V$ and $\|g_V\|_1 \leq 2$ ([20, Theorem 2.6.3]). Then from the representation of $\hat{T}$, it follows that

$$\hat{T}(\hat{g}_V)(\xi_V) = w_{\xi_V}(e_V) \quad \text{and} \quad \hat{T}(\hat{g}_V)(\xi_0) = w_{\xi_0}(e_V).$$

Since T is compact, there is a subnet of $\{\hat{g}_V\}_V$, which we keep denoting by $\{\hat{g}_V\}_V$, and a function f in $L^1(G_2, B_2)$ such that $\|\hat{T}(\hat{g}_V) - \hat{f}\|_\infty \longrightarrow 0$. Hence we have

$$\begin{aligned}\|\hat{T}(\hat{g}_V)(\xi_V) - \hat{T}(\hat{g}_V)(\xi_0)\| &\leq \|\hat{T}(\hat{g}_V)(\xi_V) - \hat{f}(\xi_V)\| + \|\hat{f}(\xi_V) - \hat{f}(\xi_0)\| + \|\hat{f}(\xi_0) - \hat{T}(\hat{g}_V)(\xi_0)\| \\ &\leq \|\hat{T}(\hat{g}_V) - \hat{f}\|_\infty + \|\hat{f}(\xi_V) - \hat{f}(\xi_0)\| + \|\hat{f}(\xi_0) - \hat{T}(\hat{g}_V)(\xi_0)\| \longrightarrow 0,\end{aligned}$$

while

$$\|\hat{T}(\hat{g}_V)(\xi_V) - \hat{T}(\hat{g}_V)(\xi_0)\| = \|w_{\xi_V}(e_V) - w_{\xi_0}(e_V)\| \geq \epsilon,$$

which is a contradiction. Therefore w is continuous.

In order to prove the sufficiency, we must show that $\hat{T}(\hat{\mathcal{B}})$ is relatively compact in $C_0(\hat{G}_2, B_2)$ by checking the conditions (1)-(3) in Theorem 4.1. Fix $\xi_0 \in \hat{G}_{20}$. Since the set $\{\hat{f}(h(\xi_0)) : f \in \mathcal{B}\}$ is bounded in B_1 and w_{ξ_0} is compact, we infer that $\{w_{\xi_0}(\hat{f}(h(\xi_0))) : f \in \mathcal{B}\}$ is relatively compact in B_2, which is to say that $\{\hat{T}\hat{f}(\xi_0) : f \in \mathcal{B}\}$ is relatively compact in B_2.

Fix $\xi_1 \in \hat{G}_{20}$. Let $\{\xi_\alpha\}$ be a net in $\hat{G}_2$ converging to ξ_1. Since $\hat{G}_{20}$ is an open subset of $\hat{G}_2$, then we can assume, without loss of generality, that for each α, $\xi_\alpha \in \hat{G}_{20}$. Let U be a neighborhood of ξ_1 where h is constant. Hence, from a certain α_0, $h(\xi_\alpha) = h(\xi_1)$ and, for all $f \in \mathcal{B}$, we have

$$||\hat{T}\hat{f}(\xi_\alpha) - \hat{T}\hat{f}(\xi_1)|| = ||w_{\xi_\alpha}(\hat{f}(h(\xi_\alpha))) - w_{\xi_1}(\hat{f}(h(\xi_1)))|| \leq ||w_{\xi_\alpha} - w_{\xi_1}||$$

for all $\alpha > \alpha_0$. It is, therefore, apparent, due to the continuity of w, that

$$\lim_{\alpha} \sup_{f\in\mathcal{B}} \{||\hat{T}\hat{f}(\xi_\alpha) - \hat{T}\hat{f}(\xi_1)||\} = 0,$$

which yields the equicontinuity of $\hat{T}(\hat{\mathcal{B}})$ in ξ_1, which is arbitrary in $\hat{G}_{20}$.

Now assume that $\xi_1 \in \hat{G}_2 \setminus \hat{G}_{20}$. If $\{\xi_\alpha\}$ is a net in $\hat{G}_2$ converging to ξ_1, then for each $f \in \mathcal{B}$, we have

$$||\hat{T}\hat{f}(\xi_\alpha) - \hat{T}\hat{f}(\xi_1)|| = ||w_{\xi_\alpha}(\hat{f}(h(\xi_\alpha)))|| \leq ||w_{\xi_\alpha}||$$

for all α. Again from condition (4) it follows that $\lim_\alpha \sup_{f\in\mathcal{B}}\{||\hat{T}\hat{f}(\xi_\alpha)||\} = 0$.

Finally, it is clear that condition (3) yields condition (3) in Theorem 4.1. As a consequence, $\hat{T}(\hat{\mathcal{B}})$ is relatively compact in $C_0(\hat{G}_2, B_2)$ and we are done. □

Remark 4.3. It is known that when $\hat{G}$ is assumed to be compact, condition (3) in Theorem 4.1 is redundant (see e.g., [19, Theorem 47.1]). However, this is not the case in our (locally compact) context as the following example shows. Let us consider the following family of Fejer kernels in $L^1(\mathbb{R})$:

$$H = \left\{ f_n(t) = n \left(\frac{sin(n\pi t)}{n\pi t} \right)^2 : n = 1, 2, ... \right\}.$$

It is known (see e.g., [18, p.139]) that the family, $\hat{H}$, of Fourier transforms of the functions in H turn out to be the following functions in $C_0(\mathbb{R})$:

$$\hat{f}_n(\xi) = 1 - \frac{|\xi|}{n}$$

for $|\xi| < n$ and 0 otherwise. It can be easily checked both that $\hat{H}$ satisfies only conditions (1) and (2) in Theorem 4.1 and contains no convergent subsequence, which is to say that it cannot be relatively compact.

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