MULTI-LINEAR ISOMETRIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper we study multilinear isometries defined on certain subspaces of vector-valued continuous functions. We provide conditions under which such maps can be properly represented. Our results contain all known results concerning linear and bilinear isometries defined between spaces of continuous functions. The key result is a vector-valued version of the additive Bishop's Lemma, which we think that has interest in itself.

1. INTRODUCTION

Throughout this paper, for a compact Hausdorff space X and a normed space E, we denote the space of all E-valued continuous functions on X by $C(X, E)$. We set $C(X)$ if E is the scalar field.

Let $A_1, ..., A_k$ be subspaces of continuous functions on compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Z be a compact Hausdorff space. A k-linear map $T : A_1 \times ... \times A_k \longrightarrow C(Z)$ is called a *multilinear (or k-linear) isometry* if

$$\|T(f_1, ..., f_k)\|_\infty = \prod_{i=1}^{k} \|f_i\|_\infty \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k),$$

where $\| \cdot \|_\infty$ denotes the supremum norm.

An important generalization of the famous Banach-Stone theorem, which characterizes surjective linear isometries of the space $C(X)$, was given by Holsztyński in [6] (see also [1]), where he studied non-surjective linear isometries from $C(X)$ to $C(Y)$, that is, 1-linear isometries. More recently, in [9], Moreno and Rodriguez proved the following bilinear (or 2-linear) version of Holsztyński's theorem:

Let $T : C(X) \times C(Y) \longrightarrow C(Z)$ be a bilinear isometry. Then there exist a closed subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a norm-one continuous function $a \in C(Z)$ such that $T(f, g)(z) = a(z)f(\pi_X(h(z)))g(\pi_Y(h(z)))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$,

where π_X and π_Y are projection maps. The proof of this result relies heavily on the powerful Stone-Weierstrass theorem.

In [7] (see also [4]), based on a new version of the additive Bishop's Lemma, the authors extended the above results to multilinear isometries of function algebras on locally compact Hausdorff spaces, a context where the Stone-Weierstrass theorem is not applicable.

On the other hand, the concept of multilinear isometry can be naturally extended to the context of spaces of vector-valued continuous functions. In this context, Jerison [8] investigated a formulation of the Banach-Stone theorem for $C(X, E)$-spaces, where E is a strictly convex Banach space. In 1978, a vector analogue of Holsztyński's theorem was obtained by Cambern ([3]). Namely, he considered 1-linear isometries from $C(X, E)$ into $C(Y, F)$ assuming that E and F are normed spaces and F is strictly convex. In this vector-valued context, examples of bilinear isometries can be found, for instance, in [10, Proposition 5.2], where the author provided certain compact spaces X and Banach spaces E for which there exists a bilinear isometry $T : C(X, E) \times C(X, E) \longrightarrow C(Y, E)$. In [5], the authors obtained conditions under which a representation of such bilinear isometries on this setting can be obtained, which is to say, a vector-valued version of the results in [9]. They proved that, if F is a strictly convex Banach space and if $T : C(X, E_1) \times C(Y, E_2) \longrightarrow C(Z, F)$ is a bilinear isometry which is *stable on constants* (see Remark 4.3 for its definition), then there exists a nonempty subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a continuous function $\omega : Z_0 \longrightarrow Bil(E_1 \times E_2, F)$ such that

$$T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z))))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$, where $Bil(E_1 \times E_2, F)$ denotes the space of jointly continuous bilinear maps from $E_1 \times E_2$ into F equipped with the strong operator topology.

In this paper we focus on multilinear isometries defined on certain subspaces of vector-valued continuous functions. We provide a new weaker condition than the *stability on constants* considered in [5], which allows us to obtain a complete representation of such isometries. In particular, we describe multilinear isometries of $C(X, E)$-spaces for normed spaces E which are not necessarily Banach spaces (compare with [5]). Our results include, basically, all known results concerning linear and bilinear isometries defined between spaces of (both scalar-valued and vector-valued) continuous functions. The key result is a vector-valued version of the additive Bishop's Lemma, which we think that has interest in itself.

2. Preliminaries

Let X be a compact Hausdorff space and E be a normed space. For each $e \in E$, let $\widehat{e}$ denote the function in $C(X, E)$ which is constantly e on X. Moreover, for any $f \in C(X, E)$, set $M_f := \{x \in X : \|f(x)\| = \|f\|_\infty\}$ which is a nonempty compact subset of X.

For any normed space E, S_E denotes the unit sphere of E. A normed space E is called *strictly convex* if each $e \in S_E$ is an extreme point of the closed unit ball of E. Especially, for each $e_1, e_2 \in E \setminus \{0\}$, we have $\|e_1\|, \|e_2\| < \max\{\|e_1 + e_2\|, \|e_1 - e_2\|\}$.

Let X be a compact Hausdorff space and E be a Banach space. We say a subspace $A(X, E)$ of $C(X, E)$ is $\star$-*regular* if for each $e \in S_E$, $x \in X$ and neighborhood V of x, there exists a function $f \in A(X, E)$ such that $f(x) = e$, $\|f\|_\infty = 1$ and $f = 0$ on $X \setminus V$. Note that, among others, the space of vector-valued continuous functions, (little) Lipschitz functions, n-times continuously differentiable functions ($n \in \mathbb{N} \cup \{\infty\}$), continuous functions of bounded variation and absolutely continuous functions on appropriate compact spaces X, are $\star$-regular.

A normed space E is called *strictly convex* if each $e \in S_E$ is an extreme point of the closed unit ball of E. Especially, for each $e_1, e_2 \in E \setminus \{0\}$, we have $\|e_1\|, \|e_2\| < \max\{\|e_1 + e_2\|, \|e_1 - e_2\|\}$.

In the sequel, unless otherwise stated, we will assume that $X_1, ..., X_k, Y$ are compact Hausdorff spaces, $E_1, ..., E_k$ are Banach spaces, and F is a strictly convex Banach space. Furthermore, by $Mul(E_1 \times ... \times E_k, F)$ we mean the space of jointly continuous multilinear maps from $E_1 \times ... \times E_k$ to F, endowed with the strong operator topology (SOT). For each $i \in \{1, ..., k\}$, let $A(X_i, E_i)$ be a $\star$-regular subspace of $C(X_i, E_i)$ containing the constant functions.

Our aim in this paper is to study multilinear isometries $T : A(X_1, E_1) \times ... \times A(X_k, E_k) \longrightarrow C(Y, F)$. We note that T can be extended naturally to $T : \overline{A(X_1, E_1)} \times ... \times \overline{A(X_k, E_k)} \longrightarrow C(Y, F)$, where for each $i \in \{1, ..., k\}$, $\overline{A(X_i, E_i)}$ is the uniform closure of $A(X_i, E_i)$ in $C(X_i, E_i)$. So we assume, without loss of generality, that each $A(X_i, E_i)$ $(i = 1, ..., k)$ is uniformly closed.

For any $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ and $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$, we set

$$C_{x_i}^{e_i} := \{f \in A(X_i, E_i) : \|f\|_\infty = 1, f(x_i) = e_i\} \quad (i \in \{1, ..., k\}).$$

Moreover, we define

$$\mathcal{I}_{x_1, ..., x_k}^{e_1, ..., e_k} := \{y \in Y : y \in M_{T(f_1, ..., f_k)} \text{ for all } (f_1, ..., f_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}\}.$$

3. Required Lemmas

We deduce our main result (Theorem 4.1) through several lemmas. The first key lemma is an additive version of Bishop's Lemma adapted to the context of spaces of vector-valued continuous functions.

Lemma 3.1. *Assume that $A(X, E)$ is a $\star$-regular closed subspace of $C(X, E)$, $e \in S_E$ and $x_0 \in X$. If $f \in A(X, E)$ with $f(x_0) = 0$, then there exists $h \in C_{x_0}^e$ such that $\|\,\|f\| + 2\|f\|_\infty \|h\|\,\|_\infty = 2\|f\|_\infty$. In particular, $\|f \pm 2\|f\|_\infty h\|_\infty = 2\|f\|_\infty$.*

Proof. We apply an argument similar to the proof of [11, Lemma 1]. For each $n \in \mathbb{N}$, put

$$V_n := \left\{ x \in X : \|f(x)\| < \frac{\|f\|_\infty}{2^{n+1}} \right\}.$$

Clearly, V_n is a neighborhood of x_0 and $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Since $A(X, E)$ is a $\star$-regular, we can choose $h_n \in A(X, E)$ such that $h_n(x_0) = e$, $\|h_n\|_\infty = 1$ and $h_n = 0$ on $X \setminus V_n$. Define $h = \sum_{n=1}^{\infty} \frac{h_n}{2^n}$. Obviously, $h \in A(X, E)$, $h(x_0) = e$ and $\|h\|_\infty = 1$. We now show that $\|\,\|f\| + 2\|f\|_\infty \|h\|\,\|_\infty = 2\|f\|_\infty$. If $x \in \bigcap_{n=1}^{\infty} V_n$, then it is apparent that $f(x) = 0$ and so $\|f(x)\| + 2\|f\|_\infty \|h(x)\| \leq 2\|f\|_\infty$. If $x \notin \bigcup_{n=1}^{\infty} V_n$, then $h(x) = 0$ and so $\|f(x)\| + 2\|f\|_\infty \|h(x)\| \leq 2\|f\|_\infty$. Finally, if x belongs to $V_1, ..., V_{n-1}$ but not to V_n, then we have

$$\|f(x)\| + 2\|f\|_\infty \|h(x)\| < \frac{\|f\|_\infty}{2^n} + 2\|f\|_\infty \left(\sum_{i=1}^{n-1} \frac{1}{2^i} \right) < 2\|f\|_\infty.$$

Then, from the above arguments and since $(\|f\| + 2\|f\|_\infty \|h\|)(x_0) = 2\|f\|_\infty$, we conclude that $\|\,\|f\| + 2\|f\|_\infty \|h\|\,\|_\infty = 2\|f\|_\infty$. In particular, it is evident that $\|f \pm 2\|f\|_\infty h\|_\infty = 2\|f\|_\infty$.
$\qquad\square$

Lemma 3.2. *For any $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ and $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$, the set $\mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k}$ is nonempty.*

Proof. Since for each $(f_1, ..., f_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$, the set $M_{T(f_1,...,f_k)}$ is a compact subset of Y, then it is enough to show that the family $\{ M_{T(f_1,...,f_k)} : (f_1, ..., f_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k} \}$ has the finite intersection property. To see, assume that $(f_1^1, ..., f_k^1), ..., (f_1^n, ..., f_k^n)$ belong to $C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$. Define

$$f_i := \frac{1}{n} \sum_{j=1}^{n} f_i^j, \quad i \in \{1, ..., k\}.$$

It is clear that $(f_1, ..., f_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$, whence $\|T(f_1, ..., f_k)\|_\infty = \|f_1\|_\infty ... \|f_k\|_\infty = 1$. Then there is a point $y_0 \in Y$ such that

$$1 = \|T(f_1, ..., f_k)(y_0)\| = \frac{1}{n^k} \left\| \sum_{1 \leq i_1,...,i_k \leq n} T(f_1^{i_1}, ..., f_k^{i_k})(y_0) \right\|.$$

Since for each $1 \leq i_1, ..., i_k \leq n$, $(f_1^{i_1}, ..., f_k^{i_k}) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$, $\|T(f_1^{i_1}, ..., f_k^{i_k})\|_\infty = 1$, and so we conclude that $\|T(f_1^{i_1}, ..., f_k^{i_k})(y_0)\| = 1$. In particular, $y_0 \in \bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)}$. Therefore $\bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)} \neq \emptyset$, as desired.
$\qquad\square$

Lemma 3.3. *Let $(x_1, ..., x_k) \in X_1 \times ... \times X_k$, $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$, let also I, J be two disjoint sets with $I \neq \emptyset$ and $I \cup J = \{1, ..., k\}$. If for each $j \in J$, $h_j \in C_{x_j}^{e_j}$ and for each $i \in I$, $f_i \in A(X_i, E_i)$ with $f_i(x_i) = 0$, then $T(F_1, ..., F_k)(y) = 0$ for all $y \in \mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k}$, where $F_t = f_t$ if $t \in I$ and $F_t = h_t$ if $t \in J$.*

Proof. Contrary to what we claim, suppose that there exists $y_0 \in \mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k}$ such that $T(F_1, ..., F_k)(y_0) = e \neq 0$. For each $i \in I$, by Lemma 3.1, there exists a function $h_i \in C_{x_i}^{e_i}$ such that $\|\, \|f_i\| + r_i\|h_i\|\, \|_\infty = \|f_i \pm r_i h_i\|_\infty = r_i$, where $r_i = 2\|f_i\|_\infty$. Moreover, $T(h_1, ..., h_k)(y_0) = e_0 \in S_F$ since $(h_1, ..., h_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$.

Let us suppose that $I = \{1\}$. Then taking into account that F is strictly convex, we have

$$r_1 = \|f_1 \pm r_1 h_1\|_\infty \|h_2\|_\infty ... \|h_k\|_\infty = \|T(f_1 \pm r_1 h_1, h_2, ..., h_k)\|_\infty$$

$$\geq \|T(f_1, h_2, ..., h_k)(y_0) \pm r_1 T(h_1, ..., h_k)(y_0)\|$$

$$= \|e \pm r_1 e_0\| > \|r_1 e_0\| = r_1,$$

and it is a contradiction showing that $T(F_1, ..., F_k)(y) = 0$ for all $y \in \mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k}$. A similar method implies that the result holds for all cases where $card(I) = 1$.

Next suppose that the result is true when $card(I) = l - 1$ and $2 \leq l \leq k$ and we show that the result is held if $card(I) = l$. Let us first assume that $l < k$. Without loss of generality, suppose that $I = \{x_1, ..., x_l\}$. Similar to the above argument, from strict convexity of F it follows that

$$r_1 r_2 ... r_l = \|f_1 \pm r_1 h_1\|_\infty \|f_2 + r_2 h_2\|_\infty ... \|f_l + r_l h_l\|_\infty \|h_{l+1}\|_\infty ... \|h_k\|_\infty$$

$$= \|T(f_1 \pm r_1 h_1, f_2 + r_2 h_2, ..., f_l + r_l h_l, h_{l+1}, ..., h_k)\|_\infty$$

$$\geq \|T(f_1 \pm r_1 h_1, f_2 + r_2 h_2, ..., f_l + r_l h_l, h_{l+1}, ..., h_k)(y_0)\|$$

$$= \|T(f_1, ..., f_l, h_{l+1}, ..., h_k)(y_0) \pm r_1 r_2 ... r_l T(h_1, ..., h_k)(y_0)\|$$

$$= \|e \pm r_1 ... r_l e_0\| > r_1 ... r_l,$$

which is impossible. This argument shows the validity of the result for the case where $card(I) < k$.

Now assume that $I = \{x_1, ..., x_k\}$. We have

$$r_1 ... r_k = \|f_1 \pm r_1 h_1\|_\infty \|f_2 + r_2 h_2\|_\infty ... \|f_k + r_k h_k\|_\infty$$

$$= \|T(f_1 \pm r_1 h_1, f_2 + r_2 h_2, ..., f_k + r_k h_k)\|_\infty$$

$$\geq \|T(f_1, ..., f_k)(y_0) \pm r_1 ... r_k T(h_1, ..., h_k)(y_0)\|$$

$$= \|e \pm r_1 ... r_k e_0\| > r_1 ... r_k,$$

which is again a contradiction showing that $T(f_1, ..., f_k)(y) = 0$ for all $y \in \mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k}$. $\quad\square$

Lemma 3.4. *Let $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ be distinct elements in $X_1 \times ... \times X_k$, and $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$, then $\mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k} \bigcap \mathcal{I}^{e_1,...,e_k}_{x'_1,...,x'_k} = \emptyset$.*

Proof. Contrary to what we claim, suppose that there exists a point $y_0 \in \mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k} \bigcap \mathcal{I}^{e_1,...,e_k}_{x'_1,...,x'_k}$. Since $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ are distinct, the set $L = \{i : 1 \leq i \leq k, x_i \neq x'_i\}$ is nonempty. For each $i \in L$, we can choose a function $f_i \in C^{e_i}_{x_i}$ such that $f_i(x'_i) = 0$ because $A(X_i, E_i)$ is $\star$-regular. Moreover, for each $j \in \{1, ..., k\} \setminus L$, we take a function f_j in $C^{e_j}_{x_j}$. Now according to Lemma 3.3, $T(f_1, ..., f_k)(y_0) = 0$ since $y_0 \in \mathcal{I}^{e_1,...,e_k}_{x'_1,...,x'_k}$. On the other hand, $\|T(f_1, ..., f_k)(y_0)\| = 1$ because $y_0 \in \mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k}$. But it is impossible and so $\mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k} \bigcap \mathcal{I}^{e_1,...,e_k}_{x'_1,...,x'_k} = \emptyset$. $\qquad\square$

Definition 3.5. *For any $(x_1, ..., x_k) \in X_1 \times ... \times X_k$, let*

$$\mathcal{I}_{x_1,...,x_k} := \bigcap_{(e_1,...,e_k) \in S_{E_1} \times ... \times S_{E_k}} \mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k}.$$

It should be noted that according to the above lemma, for any distinct elements $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ in $X_1 \times ... \times X_k$, $\mathcal{I}_{x_1,...,x_k} \bigcap \mathcal{I}_{x'_1,...,x'_k} = \emptyset$.

Although for each $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ and $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$, $\mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k}$ is non-empty (Lemma 3.2), we do not know if the set $\mathcal{I}_{x_1,...,x_k}$ is also non-empty. So we need to introduce an additional property as follows:

Definition 3.6. *We say that a k-linear isometry T satisfies N-property if for each $(x_1, ..., x_k) \in X_1 \times ... \times X_k$, $\mathcal{I}_{x_1,...,x_k} \neq \emptyset$ or, equivalently, there exists $y \in Y$ such that $\|T(f_1, ..., f_k)(y)\| = 1$ for all functions $(f_1, ..., f_k)$ with $f_i(x_i) \in S_{E_i}$ and $\|f_i\|_\infty = 1$.*

Proposition 3.7. *Assume that for each $(f_1, ..., f_k) \in A(X_1, E_1) \times ... \times A(X_k, E_k)$ and $e_i, e'_i \in S_{E_i}, i = 1, ..., k$, we have*

$$M_{T(f_1,...,f_{i-1},e_i,f_{i+1},...,f_k)} = M_{T(f_1,...,f_{i-1},e'_i,f_{i+1},...,f_k)}.$$

Then

$$\mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k} = \mathcal{I}^{e'_1,...,e'_k}_{x_1,...,x_k}$$

for all $(e_1, ..., e_k), (e'_1, ..., e'_k) \in S_{E_1} \times ... \times S_{E_k}$ and each $(x_1, ..., x_k) \in X_1 \times ... \times X_k$.

Proof. Fix $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ and let $y \in \mathcal{I}^{e_1,...,e_k}_{x_1,...,x_k}$, which exists by Lemma 3.2, $(g_1, ..., g_k) \in C^{e_1}_{x_1} \times ... \times C^{e_k}_{x_k}$ and $(f_1, ..., f_k) \in C^{e'_1}_{x_1} \times ... \times C^{e'_k}_{x_k}$. From Lemma 3.3, we have $T(f_1 - e'_1, g_2, ..., g_k)(y) = 0$ and so $T(f_1, g_2, ..., g_k)(y) = T(e'_1, g_2, ..., g_k)(y)$. Since $M_{T(e'_1, g_2, ..., g_k)} = M_{T(e_1, g_2, ..., g_k)}$ and $y \in M_{T(e_1, g_2, ..., g_k)}$, then we infer that $y \in M_{T(f_1, g_2, ..., g_k)}$. This argument shows that $y \in \mathcal{I}^{e'_1, e_2, ..., e_k}_{x_1, x_2, ..., x_k}$. Thus again by Lemma 3.3, we have $T(f_1, f_2 - e'_2, g_3, ..., g_k)(y) = 0$, and so $T(f_1, f_2, g_3, ..., g_k)(y) = T(f_1, e'_2, g_3, ..., g_k)(y)$. Now taking into account that $M_{T(f_1, e'_2, g_3, ..., g_k)} = M_{T(f_1, e_2, g_3, ..., g_k)}$ and $y \in$

$\mathcal{I}_{x_1,x_2,...,x_k}^{e'_1,e_2,...,e_k}$, it follows that $y \in M_{T(f_1,f_2,g_3,...,g_k)}$. According to this discussion, we deduce that $y \in \mathcal{I}_{x_1,x_2,x_3,...,x_k}^{e'_1,e'_2,e_3,...,e_k}$. By continuing this process, finally it is concluded that $y \in \mathcal{I}_{x_1,...,x_k}^{e'_1,...,e'_k}$. Therefore, $\mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k} \subseteq \mathcal{I}_{x_1,...,x_k}^{e'_1,...,e'_k}$. Similarly, $\mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k} \supseteq \mathcal{I}_{x_1,...,x_k}^{e'_1,...,e'_k}$ and so $\mathcal{I}_{x_1,...,x_k}^{e_1,...,e_k} = \mathcal{I}_{x_1,...,x_k}^{e'_1,...,e'_k}$. $\square$

It is clear that if T satisfies the assumption in the above proposition, then T satisfies N-property. It should be also noted that N-property is weaker than the condition (stability on constants) proposed in [5] for bilinear isometries (see Remark 4.3).

Lemma 3.8. *Let $(x_1,...,x_k) \in X_1 \times ... \times X_k$, I and J be two disjoint sets with $I \neq \emptyset$ and $I \cup J = \{1,...,k\}$. If for each $i \in I$, $f_i \in A(X_i, E_i)$ with $f_i(x_i) = 0$, and for each $j \in J$, $f_j \in A(X_j, E_j)$, then $T(f_1,...,f_k)(y) = 0$ for all $y \in \mathcal{I}_{x_1,...,x_k}$.*

Proof. Let $y \in \mathcal{I}_{x_1,...,x_k}$. We can assume, without loss of generality, that $f_j(x_j) \in S_{E_j}$ for all $j \in J$. Moreover, for each $i \in I$, let e_i be a point in S_{E_i}. Then $(e_1,...,e_k) \in S_{E_1} \times ... \times S_{E_k}$, where $e_j = f_j(x_j)$ for each $j \in J$, and we choose $(h_1,...,h_k) \in C_{x_1}^{e_1} \times ... \times C_{x_k}^{e_k}$.

First consider the case where $card(J) = 1$. We suppose, without loss of generality, that $J = \{1\}$. By Lemma 3.3, $T(h_1, f_2,...,f_k)(y) = 0$ and $T(h_1 - f_1, f_2,...,f_k)(y) = 0$. Then from k-linearity of T it follows that $T(f_1,...,f_k)(y) = 0$. Now assume that $card(J) = 2$ and $J = \{1,2\}$. Again from Lemma 3.3 we have $T(f_1 - h_1, f_2 - h_2, f_3,...,f_k)(y) = 0$, $T(f_1 - h_1, h_2, f_3,...,f_k)(y) = 0$, $T(h_1, f_2 - h_2, f_3,...,f_k)(y) = 0$, and $T(h_1, h_2, f_3,...,f_k)(y) = 0$, that specially the latter three equations guarantee that $T(f_1, h_2, f_3,...,f_k)(y) = 0$ and $T(h_1, f_2,...,f_k)(y) = 0$. Now, k-linearity of T easily yields that $T(f_1,...,f_k)(y) = 0$. A similar discussion holds for all cases where $card(J) = 2$.

By continuing this progress we conclude that the result is also true for all cases where $card(J) > 2$. $\square$

Lemma 3.9. *If $(x_1,...,x_k) \in X_1 \times ... \times X_k$, $(f_1,...,f_k) \in A(X_1, E_1) \times ... \times A(X_k, E_k)$ and $y \in \mathcal{I}_{x_1,...,x_k}$, then $T(f_1,...,f_k)(y) = T(\widehat{f_1(x_1)},...,\widehat{f_k(x_k)})(y)$.*

Proof. From the previous lemma, we have $T(f_1 - \widehat{f_1(x_1)}, f_2,...,f_k)(y) = 0$, which implies that

$$T(f_1,...,f_k)(y) = T(\widehat{f_1(x_1)}, f_2,...,f_k)(y).$$

Again by the previous lemma, $T(\widehat{f_1(x_1)}, f_2 - \widehat{f_2(x_2)}, f_3,...,f_k)(y) = 0$, and so $T(\widehat{f_1(x_1)}, f_2,...,f_k)(y) = T(\widehat{f_1(x_1)}, \widehat{f_2(x_2)}, f_3,...,f_k)(y)$. Consequently we have proved that

$$T(f_1,...,f_k)(y) = T(\widehat{f_1(x_1)}, \widehat{f_2(x_2)}, f_3,...,f_k)(y).$$

By continuing this progress we derive that $T(f_1,...,f_k)(y) = T(\widehat{f_1(x_1)},...,\widehat{f_k(x_k)})(y)$. $\square$

4. Main Result

Now we can state our main result.

Theorem 4.1. *Let $T : A(X_1, E_1) \times ... \times A(X_k, E_k) \longrightarrow C(Y, F)$ be a k-linear isometry satisfying N-property. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow X_1 \times ... \times X_k$, a continuous function $\omega : Y_0 \longrightarrow Mul(E_1 \times ... \times E_k, F)$ such that*

$$T(f_1, ..., f_k)(y) = \omega(y)(f_1(\pi_1(\varphi(y))), ..., f_k(\pi_k(\varphi(y)))),$$

for all $(f_1, ..., f_k) \in A(X_1, E_1) \times ... \times A(X_k, E_k)$ and $y \in Y_0$, where π_i is the ith projection map.

Proof. Set $Y_0 := \{y \in Y : y \in \mathcal{I}_{x_1,...,x_k} \text{ for some } (x_1, ..., x_k) \in X_1 \times ... \times X_k\}$, which is a nonempty subset of Y by N-property.

We define the map $\varphi : Y_0 \longrightarrow X_1 \times ... \times X_k$ by $\varphi(y) := (x_1, ..., x_k)$ if $y \in \mathcal{I}_{x_1,...,x_k}$. From Lemma 3.4, it is apparent that φ is well-defined. Meantime, since T satisfies N-property, we infer that φ is surjective.

Next, define the map $\omega : Y_0 \longrightarrow Mul(E_1 \times ... \times E_k, F)$ as $\omega(y)(e_1, ..., e_k) = T(\widehat{e_1}, ..., \widehat{e_k})(y)$ for all $(e_1, ..., e_k) \in E_1 \times ... \times E_k$ and $y \in Y_0$. We now prove that ω is continuous. Let (y_α) be a net in Y_0 converging to $y \in Y_0$. Fix $(e_1, ..., e_k) \in E_1 \times ... \times E_k$. From the definition of ω we deduce that

$$\|\omega(y_\alpha)(e_1, ..., e_k) - \omega(y)(e_1, ..., e_k)\| = \|T(\widehat{e_1}, ..., \widehat{e_k})(y_\alpha) - T(\widehat{e_1}, ..., \widehat{e_k})(y)\| \to 0,$$

because of the continuity of $T(\widehat{e_1}, ..., \widehat{e_k})$. Hence ω is continuous.

Now if $(f_1, ..., f_k) \in A(X_1, E_1) \times ... \times A(X_k, E_k)$ and $y \in \mathcal{I}_{x_1,...,x_k}$ for some $(x_1, ..., x_k) \in X_1 \times ... \times X_k$, then by Lemma 3.9 we have

$$T(f_1, ..., f_k)(y) = T(\widehat{f_1(x_1)}, ..., \widehat{f_k(x_k)})(y)$$

$$= \omega(y)(f_1(x_1), ..., f_k(x_k)).$$

Indeed, we get

$$T(f_1, ..., f_k)(y) = \omega(y)(f_1(\pi_1(\varphi(y))), ..., f_k(\pi_k(\varphi(y)))),$$

for all $(f_1, ..., f_k) \in A(X_1, E_1) \times ... \times A(X_k, E_k)$ and $y \in Y_0$.

Finally, we show the continuity of φ. Assume that $y \in Y_0$ and $\varphi(y) = (x_1, ..., x_k)$. Let $V_1 \times ... \times V_k$ be a neighborhood of $(x_1, ..., x_k)$ in $X_1 \times ... \times X_k$. For each i, $i = 1, ..., k$, choose $f_i \in C_{x_i}^{e_i}$ such that $f_i = 0$ on $X_i \setminus V_i$ for some $(e_1, ..., e_k) \in S_{E_1} \times ... \times S_{E_k}$. Define

$$W := \{z \in Y_0 : \|T(f_1, ..., f_k)(z)\| > \frac{1}{2}\}.$$

Clearly, W is a neighborhood of y since $\varphi(y) = (x_1, ..., x_k)$ and so $\|T(f_1, ..., f_k)(y)\| = 1$. Moreover we claim that $\varphi(W) \subseteq V_1 \times ... \times V_k$. To this end, let $z \in W$ and $\varphi(z) = (x_1', ..., x_k')$. From the representation of T, we conclude that

$$\frac{1}{2} < \|T(f_1, ..., f_k)(z)\| = \|\omega(z)(f_1(\pi_1(\varphi(z))), ..., f_k(\pi_k(\varphi(z))))\|$$
$$= \|\omega(z)(f_1(x_1'), ..., f_k(x_k'))\|$$
$$= \|T(\widehat{f_1(x_1')}, ..., \widehat{f_k(x_k')})(z)\|$$
$$\leq \|T(\widehat{f_1(x_1')}, ..., \widehat{f_k(x_k')})\|_\infty$$
$$= \|f_1(x_1')\|...\|f_k(x_k')\|,$$

which especially shows that for any $i \in \{1, ..., k\}$, $f_i(x_i') \neq 0$, and consequently, $x_i' \in V_i$. Hence $(x_1', ..., x_k') \in V_1 \times ... \times V_k$, as asserted. Therefore, φ is continuous. $\qquad \square$

The next result describes multilinear isometries of $C(X, E)$-spaces for normed spaces E which are not necessarily Banach spaces.

Corollary 4.2. *If $E_1, ..., E_k, F$ are normed spaces and $T : C(X_1, E_1) \times ... \times C(X_k, E_k) \longrightarrow C(Y, F)$ is a k-linear isometry satisfying N-property, then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow X_1 \times ... \times X_k$, a continuous function $\omega : Y_0 \longrightarrow Mul(E_1 \times ... \times E_k, F)$ such that for each $(f_1, ..., f_k) \in C(X_1, E_1) \times ... \times C(X_k, E_k)$ and $y \in Y_0$,*

$$T(f_1, ..., f_k)(y) = \omega(y)(f_1(\pi_1(\varphi(y))), ..., f_k(\pi_k(\varphi(y)))).$$

Proof. According to the notations from the proof of Lemma 3.1, we can choose $k_n \in C(X)$ such that $k_n(x_0) = 1 = \|k_n\|_\infty$ and $k_n = 0$ on $X \setminus V_n$. Now putting $h_n = k_n e$ and defining $h = \sum_{n=1}^\infty \frac{h_n}{2^n}$, we can obtain Lemma 3.1. Then following the rest of steps, we can obtain the result. $\qquad \square$

Remark 4.3. (1) The next example shows that our assumption (see Definition 3.6) is weaker than the assumption *stability on constants* given in [5]. Let us first recall that a bilinear isometry $T : C(X_1, E_1) \times C(X_2, E_2) \longrightarrow C(Y, F)$ is said to be *stable on constants* if for each $(f, g) \in C(X_1, E_1) \times C(X_2, E_2)$ and $y \in Y$ we have

$$\|T(f, \widehat{e_2})(y)\| = \|T(f, \widehat{e_2'})(y)\| \quad (e_2, e_2' \in S_{E_2}),$$

$$\|T(\widehat{e_1}, g)(y)\| = \|T(\widehat{e_1'}, g)(y)\| \quad (e_1, e_1' \in S_{E_1}).$$

In particular, we have $M_{T(f, \widehat{e_2})} = M_{T(f, \widehat{e_2'})}$ and $M_{T(\widehat{e_1}, g)} = M_{T(\widehat{e_1'}, g)}$ for all $(f, g) \in C(X_1, E_1) \times C(X_2, E_2)$ and $(e_1, e_2), (e_1', e_2') \in S_{E_1} \times S_{E_2}$. By Proposition 3.7, we infer that $\mathcal{I}_{x_1, x_2} = \mathcal{I}_{x_1, x_2}^{e_1, e_2}$ for all $(e_1, e_2) \in S_{E_1} \times S_{E_2}$ and $(x_1, x_2) \in X_1 \times X_2$.

Let $X_1 = \{x_1\}, X_2 = \{x_2\}$, and $Y =$ the one point compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$. Define $T : C(X_1, c_0) \times C(X_2) \longrightarrow C(Y, c_0)$ by $T(f_1, f_2)(1) = T(f_1, f_2)(2) = f_1(x_1)f_2(x_2), T(f_1, f_2)(\infty) = 0$, and for each $n \geq 3$, $T(f_1, f_2)(n) = < f_1(x_1), e_n > f_2(x_2)e_n$, where $e_n = (0, ..., 0, 1, 0, ..)$, that is, all coordinates are zero except for a 1 in the nth coordinate, and $< f_1(x_1), e_n >$ is the nth term of $f_1(x_1)$. It is apparent that T is a bilinear isometry and we have

$\mathcal{I}_{x_1, x_2}^{e_1, 1} = \{1, 2\}, \mathcal{I}_{x_1, x_2}^{e_3, 1} = \{1, 2, 3\}, \mathcal{I}_{x_1, x_2}^{e_4, 1} = \{1, 2, 4\}, \mathcal{I}_{x_1, x_2}^{e_1, i} = \{1, 2\}, \cdots$.

Clearly, $\mathcal{I}_{x_1, x_2} = \{1, 2\}$. Consequently, T satisfies N-property while T is not stable on constants by the previous paragraph.

(2) By considering $k = 2$ and $A(X_i, E_i) = C(X_i, E_i)$ in Corollary 4.2, we obtain the main result in [5] under a weaker condition (as shown in (1)) and a different method for the case where E_1 and E_2 are not necessarily Banach spaces.

We note that for the case where $E_1 = ... = E_k = \mathbb{C}$ or $\mathbb{R}$, from the k-linearity of T it follows easily that for each $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ and $(e_1, ..., e_k) \in \mathbb{T}^k$ (or $\{1, -1\}^k$), $\mathcal{I}_{x_1, ..., x_k} = \mathcal{I}_{x_1, ..., x_k}^{e_1, ..., e_k}$, which is a non-empty set. The following result, which is an easy consequence of Theorem 4.1, is a generalization of the main theorems in [7] and [9] for certain function spaces.

Corollary 4.4. *Suppose that A_i is a $\star$-regular subspace of $C(X_i)$ $(i = 1, ..., k)$ and $T : A_1 \times ... \times A_k \longrightarrow C(Y)$ is a k-linear isometry. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow X_1 \times ... \times X_k$, and a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ such that*

$$T(f_1, ..., f_k)(y) = \omega(y) \prod_{i=1}^{k} f_i(\pi_i(\varphi(y)))$$

for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $y \in Y_0$, where π_i is the ith projection map.

Let us consider the case where $k = 1$. It is worth pointing out that in this case our assumption can be dropped. Actually, from Lemma 3.3, we obtain this key result: if $x \in X$ and $f(x) = 0$, then $Tf(y) = 0$ for all $y \in \bigcup_{e \in S_E} \mathcal{I}_x^e$, which allows us to define ω as in the proof of Theorem 4.1.

Corollary 4.5. *Let $A(X, E)$ be a $\star$-regular subspace of $C(X, E)$ and $T : A(X, E) \longrightarrow C(Y, F)$ be a linear isometry. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow X$, a continuous function ω from Y_0 to the space of continuous linear operators of E into F such that $T(f)(y) = \omega(y)(f(\varphi(y)))$ for all $f \in A(X, E)$ and $y \in Y_0$.*

We finally remark that when $A(X, E) = C(X, E)$, Corollaries 4.2 and 4.5 reduce to Cambern's result [3] which is the vector-valued analogue of the celebrated Holsztyński's theorem.

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