ℓ_p-MAXIMAL REGULARITY FOR A CLASS OF FRACTIONAL DIFFERENCE EQUATIONS ON UMD SPACES: THE CASE $1 < \alpha \leq 2$.

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ABSTRACT. By using Blunck's operator-valued Fourier multiplier theorem, we completely characterize the existence and uniqueness of solutions in Lebesgue spaces of sequences for a discrete version of the Cauchy problem with fractional order $1 < \alpha \leq 2$. This characterization is given solely in spectral terms on the data of the problem, whenever the underlying Banach space belongs to the UMD-class.

1. INTRODUCTION

Our concern in this paper is the ℓ_p- maximal regularity of solutions for the abstract non homogeneous Cauchy problem of fractional order

$$\begin{cases} \Delta^{\alpha}u(n) & = Tu(n) + f(n), \quad n \in \mathbb{Z}_+, \ n \geq 2, \quad 1 < \alpha \leq 2; \\ u(0) & = 0 \\ u(1) & = 0, \end{cases} \tag{1.1}$$

where $1 < p < \infty$, T is a bounded linear operator defined on a Banach space X and $f : \mathbb{Z}_+ \to X$ is given. Here, the discrete fractional operator Δ^{α} corresponds to sampling, by means of the Poisson distribution, of the Riemann-Liouville fractional derivative D_t^{α} of order α on $\mathbb{R}_+$

$$\Delta^{\alpha}u(n) = \int_0^{\infty} \frac{t^{n-2}}{(n-2)!} e^{-t} D_t^{\alpha}u(t)dt, \quad n \in \mathbb{Z}_+, \ n \geq 2.$$

See the reference [19], where recently this remarkable connection between the discrete and continuous fractional operators is proved. Note that (1.1) is an abstract way to write the modelling of classes of fractional integro-differential equations in discrete time. There are many situations where this type of mixed equations appear. In the unidimensional case, some of them are called lattices models in the literature, for instance the discrete Nagumo equation [14]. A different example is given by the nonconvolution equation

$$\Delta^{\alpha}u(n,x) = \int k(x,s)u(n,s)ds + f(n,x), \quad n \in \mathbb{N}_0, \ x \in \Omega \subset \mathbb{R}^N,$$

where the kernel k is a complex-valued measurable function and f is a suitable forcing term. It admits the form (1.1) where

$$Tf(x) = \int k(x,s)f(s)ds$$

is a bounded operator. In the non fractional case, such equations arise in a variety of contexts. From a numerical point of view, our analysis refers to schemes that are discretized only in time. For instance, Strikwerda and Lee [23] discussed the accuracy of the fractional step projection method for the incompressible Navier-Stokes equations restricting the analysis to schemes that are discretized only in time. From another point of view, one step time-discrete equations naturally appear in some fields of Physics [9] and in fracture mechanics and biology [20, Section 5]. Cardiac cells provide another suitable context [17]. A recent rich source of examples is provided by the Master Equation for an animal in behavioural ecology. See [22] formulas (24) and (25). Note that in such examples the operator T may be also unbounded.

On the other hand, in the last years the existence and qualitative properties of discrete solutions for fractional difference equations began to be studied. See for instance [2, 6] and [7]. However, most of these studies refers only to scalar situations, that is when $X = \mathbb{R}$ or $\mathbb{C}$. The study of abstract models in general Banach spaces, that includes the analysis of mixed partial differential equations and integral equations, is a very recent and promising area of research. See [19, 18] and [24]. We note that because of the nature of this area of investigation, the notion of fractional difference may vary.

Maximal regularity is an important tool in the investigation of the existence and uniqueness of solutions to evolution equations. For a recent review of this topic in the context of discrete models on Banach spaces, see the monograph [3] and references therein. Although research in this area has been done, there are many interesting questions related to the study of fractional difference equations that remain unanswered. Recently, in the paper [18], the maximal regularity property on Lebesgue spaces of sequences was studied for problem (1.1) when $0 < \alpha \le 1$. However, this study was left open for any other values of α.

The main objective of this paper is to provide a complete answer to this open problem. We have success into solve it in the full range $1 < \alpha \le 2$ by means of a *characterization* of maximal regularity for the solutions of the equation (1.1) in Lebesgue vector-valued spaces defined on the set $\mathbb{Z}_+$. In order to solve this problem, we will introduce a special sequence of bounded operators, called α-resolvent families, which will play a central role in the representation of the solution of the problem (1.1). Then, we use Blunck's operator-valued multiplier Theorem [3, Section 2.4], [10] in order to obtain the desired characterization. One remarkable fact is that such characterization is obtained solely in terms of the data of the problem. More precisely, for $1 < \alpha \le 2$, $p > 1$ and X a UMD space, suppose that $\{z^{2-\alpha}(z-1)^\alpha\}_{\{|z|=1,\, z\neq 1\}} \subset \rho(T)$ holds, where $T \in B(X)$ and $\rho(T)$ denotes the resolvent set of T. Then the following assertions are equivalent

(i) Equation (1.1) has ℓ_p- maximal regularity;

(ii) The set

$$\{z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1} : |z| = 1, z \neq 1\} \text{ is } R - \text{bounded}.$$

Compared with [18], this result is different. Here, the set $\Omega_2 := \{(z^{2-\alpha}(z-1)^\alpha)\}_{|z|=1,z\neq1}$ must lie in the resolvent set of T instead of the set $\Omega_1 := \{(z^{1-\alpha}(z-1)^\alpha)\}_{|z|=1,z\neq1}$ corresponding to the case $0 < \alpha \leq 1$. This last set has, in a certain sense, dual geometry compared with Ω_2. In other words, whereas the set Ω_1 lies mainly in the left hand side of the complex plane for values of α near to 1, we have that the set Ω_2 lies mainly in the right hand side of the complex plane for values of α near to 2. See [18, Figs. 1 and 2] and Figs. 1 and 2 below. The transition between both geometries has a jump in the border case $\alpha = 1$, although have a symmetry with respect to the imaginary axis. Concerning the method of proof of our main result (Theorem 4.2), we want to point out that we consider in the present paper a different class of sequences of bounded operators than those considered in [18, Definition 3.1]. See Definition 3.1 below. This is due to the consideration of two initial values in equation (1.1) instead of only one. In case $\alpha = 2$, Definition 3.1 can be compared with the notion of discrete time cosine function. In [18], the case $\alpha = 1$ corresponds to the concept of discrete time semigroup (powers of a bounded operator). In this way, the representation of the solution in the cases $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ varies (compare [18, Theorem 3.7] with Theorem 3.8 below). This representation, in case $1 < \alpha \leq 2$, was difficult to obtain and hence the results of the present paper were not considered in our earlier paper. Therefore, the present manuscript can be considered as a companion work to the published paper [18].

We point out that characterizations of maximal regularity for evolution equations using methods of operator valued Fourier multiplier theorems has been already studied (see for example [3]). For instance, S. Bu in [11] and [12] used Fourier multipliers to characterize the Lebesgue maximal regularity of fractional evolution equations in compact intervals. The corresponding study in Hölder spaces was done by Ponce in [21].

Our paper is organized as follows: in Section 2 we introduce some basic concepts related to the study of fractional differences that will be later needed.

In Section 3, we introduce a special sequence of bounded operators that we call α-resolvent sequence, denoted by $S_\alpha(n)$, which will play a very important role in the study of ℓ_p-maximal regularity. Then, we provide an explicit representation of the solution for the fractional difference equation (1.1) with initial values $u(0) = x$ and $u(1) = y$ namely

$$u(n) = S_\alpha(n)u(0) + (S_\alpha * h_\alpha)(n-1)[u(1) - u(0)] + (S_\alpha * h_\alpha * f)(n-2), \quad n \geq 2,$$

see Theorem 3.8 below. Here h_α is defined by the sequence $h_\alpha(n) = (\alpha-1)^n$. It is interesting to observe that in case of $\alpha = 2$ the resolvent sequence $S_2(n)$ coincides with the notion of discrete time cosine function introduced by Chojnacki [13] who studied it also in the context of UMD-spaces. Finally, in Section 4 we show our main result: Theorem 4.2. There, we prove the above mentioned characterization of ℓ_p-maximal regularity. A simple criteria in the special case of Hilbert space is also provided. This is given only in terms of an spectral property of a normal

operator T. Namely, we show that if $T \in \mathcal{B}(H)$ is a normal operator defined on a Hilbert space H and

$$\sigma(T) \subset \{ z \in \mathbb{C} \ : \ |z| > 2^\alpha \ \},$$

then the equation (1.1) has ℓ^p-maximal regularity. We finish this paper with a concrete example on a nonconvolution integral equation arising in the study of numerical methods on polygonal domains, highlighting the role of the fractional parameter in the treatment of additive perturbations for the given equation.

2. Preliminaries

In this section, we recall some necessary concepts related to UMD spaces, R-boundedness, fractional differences and operator-valued Fourier multipliers. See also the recent monograph [3].

From now on, given a a real number, we denote by $\mathbb{N}_a := \{a, a+1, a+2, ...\}$, and $s(\mathbb{N}_a; X)$ the vector space consisting of all vector-valued sequences $f : \mathbb{N}_a \to X$. We recall that the forward Euler operator $\Delta_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For each $m \in \mathbb{N}_2$, we define recursively the m-th order forward difference operator $\Delta_a^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a.$$

In particular, we have $(\Delta_0^1 f)(n) = f(n+1) - f(n)$, $n \in \mathbb{N}_0$. The following definition of fractional sum was formally introduced in [19], after previous work of Atici, Eloe and Abdeljawad (see [2, 6] and [7]).

Definition 2.1. Let $\alpha > 0$ be given and $f : \mathbb{N}_0 \to X$. We define the fractional sum of order α as follows

$$\Delta^{-\alpha} f(n) = \sum_{k=0}^{n} k^\alpha(n-k) f(k), \quad n \in \mathbb{N}_0, \tag{2.1}$$

where

$$k^\alpha(j) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0.$$

Concerning the development of discrete fractional calculus, we observe that Holm in [16] is among the first authors who employed the technique of Laplace transform for discrete fractional calculus in the arena of fractional difference equations. In [15] Goodrich studied the existence of positive solutions and geometrical properties. Applications of discrete fractional calculus for several biological and physical problems have been studied in [8].

Now, we recall from [19] the discrete analogous concept to the definition of a fractional derivative in the sense of Riemann-Liouville, see also [6]. In that paper, it is shown their strong connection, by means of the Poisson distribution, with the Riemann-Liouville fractional derivative on $\mathbb{R}_+$. We refer the reader also to the recent papers [1] and [18] where it is shown their usefulness in different contexts of research.

Definition 2.2. The fractional difference operator of order $\alpha > 0$ (in the sense of Riemann-Liouville) is defined by

$$\Delta^\alpha f(n) := \Delta_0^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied. We also recall the concept of finite convolution $*$ of two sequences $f(n)$ and $g(n)$:

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j), \quad n \in \mathbb{N}_0.$$

The discrete time Fourier transform (DTFT) of a vector valued sequence $f \in s(\mathbb{Z}; X)$ is given by

$$\widehat{f}(z) := \sum_{j=-\infty}^{\infty} z^{-j} f(j), \text{ where } z = e^{it}, \quad t \in (-\pi, \pi),$$

whenever it exists. We now recall the definition of the UMD class of Banach spaces. For more details [4, Section III.4.3-III.4.5].

Definition 2.3. A Banach space X is said to have the Unconditional Martingale Difference property (UMD) if for each $p \in (1, \infty)$ there exists a constant $C_p > 0$ such that for any martingale $(f_n)_{n \geq 0} \subset L^p(\Omega, \Sigma, \mu; X)$ and any choice of signs $(\xi_n)_{n \geq 0} \subset \{-1, 1\}$ and any $N \in \mathbb{Z}_+$ the following estimate holds

$$\left\| f_0 + \sum_{n=1}^{N} \xi_n (f_n - f_{n-1}) \right\|_{L^p(\Omega,\Sigma,\mu;X)} \leq C_p \|f_N\|_{L^p(\Omega,\Sigma,\mu;X)}.$$

To end this section we recall the Fourier multiplier theorem for operator valued symbols that provides necessary and sufficients condition for the R-boundedness property due to Blunck in [10]. We will first need the notion of an R-bounded set.

Definition 2.4. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called R-bounded if there is a constant $c \geq 0$ such that

$$\|(T_1 x_1, ..., T_n x_n)\|_R \leq c \|(x_1, ..., x_n)\|_R, \tag{2.2}$$

for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$ where

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|, \quad x_1, ..., x_n \in X.$$

Let now $\mathbb{T} := (-\pi, \pi) \setminus \{0\}$.

Theorem 2.5. [10, Theorem 1.3] *Let $p \in (1, \infty)$ and let X be a UMD space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be differentiable and such that the set*

$$\left\{ M(t), (z-1)(z+1)M'(t) : z = e^{it}, \ t \in \mathbb{T} \right\}$$

is R-bounded. Then there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that

$$\widehat{(T_M f)}(z) = M(t)\widehat{f}(z), \text{ for all } z = e^{it}, \ t \in \mathbb{T}. \tag{2.3}$$

The converse of Blunck's theorem also holds without any restriction on the Banach space X, as follows:

Theorem 2.6. *([10]) Let $p \in (1, \infty)$ and let X be a Banach space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be an operator valued function. Suppose that there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that the identity* (2.3) *holds. Then the set*

$$\{M(t) : t \in \mathbb{T}\}$$

is R-bounded.

3. RESOLVENT SEQUENCES: $1 < \alpha \leq 2$

Let $T \in \mathcal{B}(X)$ be given. In this section, we introduce an operator theoretical method to study the linear fractional difference equation

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}, \tag{3.1}$$

with initial conditions $u(0) = x, u(1) = y \in X$ and $1 < \alpha \leq 2$. We observe that the case $0 < \alpha \leq 1$ was previously studied in [18]. Therefore, our analysis in this paper complement the results given in [18] and provide new insights in the case $1 < \alpha \leq 2$.

Definition 3.1. Let T be a bounded operator defined on a Banach space X and $\alpha > 1$. We call T the generator of an α-resolvent sequence if there exists a sequence of bounded and linear operators $\{S_\alpha(n)\}_{n \in \mathbb{N}_0} \subset B(X)$ such that satisfies the following properties:

(i) $S_\alpha(0) = I$,
(ii) $S_\alpha(1) = I$,
(iii) $S_\alpha(n+2) - S_\alpha(n+1) = T(S_\alpha * k^{\alpha-1})(n) + k^{\alpha-1}(n+2)I - (\alpha-1)k^{\alpha-1}(n+1)I$, $n \in \mathbb{N}_0$.

In this case, $S_\alpha(n)$ is called the α-resolvent sequence generated by T.

Remark 3.2. Observe that (iii) can be rewritten as follows

$(iii)'$ $\quad \Delta S_\alpha(n+1) = T(S_\alpha * k^{\alpha-1})(n) + \Delta k^{\alpha-1}(n+1) + (2-\alpha)k^{\alpha-1}(n+1)$, $n \in \mathbb{N}_0$.

and therefore this property is comparable with [18, Definition 3.1] except for the extra term $(2-\alpha)k^{\alpha-1}(n+1)$.

The following Lemma follows easily from the definition.

Lemma 3.3. *If T generates an α-resolvent sequence, then it is unique.*

Proof. Let $S_\alpha(n)$ and $Q_\alpha(n)$ two α- resolvent sequences generated by T. Let us define $P_\alpha(n) = S_\alpha(n) - Q_\alpha(n)$. Then $P_\alpha(0) = 0$, $P_\alpha(1) = 0$ and $P_\alpha(n+2) - P_\alpha(n+1) = T\sum_{j=0}^{n} k^{\alpha-1}(n-j)P_\alpha(j)$, for all $n \in \mathbb{N}_0$, which implies that $P_\alpha(n) = P_\alpha(1) = 0$ for all $n \in \mathbb{N}_0$.

$\square$

Example 3.4. In the case $\alpha = 2$ we have

$$k^1(j) = \frac{\Gamma(1+j)}{\Gamma(1)\Gamma(j+1)} = 1, \quad j \in \mathbb{N}_0.$$

$$S_2(n+2) - S_2(n+1) = T(S_2 * k^1)(n) + k^1(n+2)I - k^1(n+1)I = T\sum_{j=0}^{n} S_2(j), \quad n \in \mathbb{N}_0.$$

Since $S_2(0) = I$ and $S_2(1) = I$ we get:

$$S_2(n) = \sum_{k=0}^{[n/2]} \binom{n}{2k} T^k.$$

Remark 3.5. Let $1 < \alpha \leq 2$ be given. Suppose that for all $z \in \mathbb{C}$ with $|z| = 1$ we have $z^{2-\alpha}(z-1)^\alpha \in \rho(T)$, the resolvent set of T. Then, the following holds:

$$\hat{S}_\alpha(z) = z(z - (\alpha - 1))\Big(z^{2-\alpha}(z-1)^\alpha - T\Big)^{-1}.$$

In particular, in case $\alpha = 2$ we have $\hat{S}_2(z) = z(z-1)((z-1)^2 - T)^{-1}$, and therefore we obtain from [3, Proposition 1.4.2] that

$$S_2(n) = \mathcal{C}(n), \quad n \in \mathbb{N}_0,$$

where $\mathcal{C}$ is the discrete time cosine operator sequence generated by T. From [3, Corollary 1.4.6] it follows that $\mathcal{C}$ satisfies

$$\mathcal{C}(n+m) + \mathcal{C}(n-m) = 2\mathcal{C}(n)\mathcal{C}(m), \quad n, m \in \mathbb{Z}.$$

We observe that the notion of cosine sequence of operators was first introduced by Chojnacki [13].

Now, we present the following useful Lemma:

Lemma 3.6. *Let* $1 < \alpha \leq 2$, $a : \mathbb{N}_0 \to \mathbb{C}$ *and* $S : \mathbb{N}_0 \to X$ *be given then:*

$$\Delta^\alpha(a*S)(n) = \sum_{j=0}^{n} \Delta^\alpha S(n-j)a(j) + S(0)a(n+2) - \alpha S(0)a(n+1) + S(1)a(n+1).$$

$$(3.2)$$

Proof. By definition we have

$$\Delta^\alpha(a * S)(n) = \Delta^2(\Delta^{-(2-\alpha)}a * S)(n)$$

$$= \Delta^{-(2-\alpha)}(a * S)(n+2) - 2\Delta^{-(2-\alpha)}(a * S)(n+1)$$

$$+ \Delta^{-(2-\alpha)}(a * S)(n)$$

$$= (k^{2-\alpha} * a * S)(n+2) - 2(k^{2-\alpha} * a * S)(n+1)$$

$$+ (k^{2-\alpha} * a * S)(n)$$

$$= \sum_{j=0}^{n+2}(k^{2-\alpha} * S)(n+2-j)a(j) - 2\sum_{j=0}^{n+1}(k^{2-\alpha} * S)(n+1-j)a(j)$$

$$+ \sum_{j=0}^{n}(k^{2-\alpha} * S)(n-j)a(j).$$

Therefore

$$\Delta^\alpha(a * S)(n) = \sum_{j=0}^{n}[(k^{2-\alpha} * S)(n+2-j) - 2(k^{2-\alpha} * S)(n+1-j)$$

$$+ (k^{2-\alpha} * S)(n-j)]a(j) + (k^{2-\alpha} * S)(1)a(n+1)$$

$$+ (k^{2-\alpha} * S)(0)a(n+2) - 2(k^{2-\alpha} * S)(0)a(n+1)$$

$$= \sum_{j=0}^{n}\Delta^2(k^{2-\alpha} * S)(n-j)a(j) + (k^{2-\alpha} * S)(1)a(n+1)$$

$$+ k^{2-\alpha}(0)S(0)a(n+2) - 2k^{2-\alpha}(0)S(0)a(n+1).$$

Hence,

$$\Delta^\alpha(a * S)(n) = \sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j) + k^{2-\alpha}(1)S(0)a(n+1)$$

$$+ k^{2-\alpha}(0)S(1)a(n+1) + S(0)a(n+2) - 2S(0)a(n+1)$$

$$= \sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j) + (2-\alpha)S(0)a(n+1) + S(1)a(n+1)$$

$$+ S(0)a(n+2) - 2S(0)a(n+1)$$

$$= \sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j) + S(0)a(n+2)$$

$$- \alpha S(0)a(n+1) + S(1)a(n+1),$$

proving the Lemma.   □

Remark 3.7. In the case $S = S_\alpha$, Lemma 3.6 states:

$$\Delta^\alpha(a * S_\alpha)(n) = \sum_{j=0}^{n}\Delta^\alpha S_\alpha(n-j)a(j) + a(n+2) - (\alpha - 1)a(n+1), \quad (3.3)$$

because $S_\alpha(0) = I$ and $S_\alpha(1) = I$ by definition.

For each $1 < \alpha \le 2$ we define

$$h_\alpha(n) = \begin{cases} (\alpha - 1)^n & n \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

The following theorem is the main result of this section.

Theorem 3.8. *Let $1 < \alpha \le 2$ and $f : \mathbb{N} \to X$ be given. The unique solution of* (3.1) *with initial conditions $u(0) = x$, $u(1) = y$ is given by:*

$$u(n) = S_\alpha(n)u(0) + (S_\alpha * h_\alpha)(n-1)[u(1) - u(0)] + (S_\alpha * h_\alpha * f)(n-2), \quad n \ge 2. \tag{3.5}$$

Proof. Applying the operator Δ^α to (3.5) we obtain

$$\Delta^\alpha u(n) = \Delta^\alpha S_\alpha(n)u(0) + \Delta^\alpha(S_\alpha * h_\alpha)(n-1)[u(1) - u(0)] + \Delta^\alpha(S_\alpha * h_\alpha * f)(n-2). \tag{3.6}$$

By Definition 3.1 we get

$$\Delta^\alpha S_\alpha(n) = \Delta^\alpha S_\alpha(n - 1) + T\Delta^\alpha(S_\alpha * k^{\alpha-1})(n - 2) + \Delta^\alpha k^{\alpha-1}(n)I$$
$$- (\alpha - 1)\Delta^\alpha k^{\alpha-1}(n - 1)I. \tag{3.7}$$

Note that $\Delta^\alpha k^{\alpha-1}(n) = \Delta^2 \Delta^{2-\alpha} k^{\alpha-1}(n) = \Delta^2 k^1(n) = 0$ for all $n \in \mathbb{N}$. Then

$$\Delta^\alpha S_\alpha(n) = \Delta^\alpha S_\alpha(n - 1) + T\Delta^\alpha(S_\alpha * k^{\alpha-1})(n - 2).$$

Using Lemma 3.6 we get:

$$\Delta^\alpha(S_\alpha * k^{\alpha-1})(n) = (\Delta^\alpha k^{\alpha-1} * S_\alpha)(n) + S_\alpha(n + 2) - \alpha S_\alpha(n + 1)$$
$$+ (\alpha - 1)S_\alpha(n + 1)$$
$$= S_\alpha(n + 2) - S_\alpha(n + 1)$$
$$= \Delta S_\alpha(n + 1).$$

Replacing the above identity in equation (3.7) we have

$$\Delta^\alpha S_\alpha(n) = \Delta^\alpha S_\alpha(n - 1) + \Delta T S_\alpha(n - 1), \tag{3.8}$$

equivalently

$$\Delta \Delta^\alpha S_\alpha(n - 1) = \Delta T S_\alpha(n - 1).$$

We claim that $\Delta^\alpha S_\alpha(n - 1) = T S_\alpha(n - 1)$. Indeed, we observe that this happens if and only if $\Delta^\alpha S_\alpha(0) = T S_\alpha(0) = T$. Now, we will prove this last assertion. By Definitions 2.1 and 2.2, we have

$$\Delta S_\alpha(n) = \Delta^2(k^{2-\alpha} * S_\alpha)(n) = (k^{2-\alpha} * S_\alpha)(n+2) - 2(k^{2-\alpha} * S_\alpha)(n+1) + (k^{2-\alpha} * S_\alpha)(n).$$

For $n = 0$:

$$\Delta S_\alpha(0) = \Delta^2(k^{2-\alpha} * S_\alpha)(0) = (k^{2-\alpha} * S_\alpha)(2) - 2(k^{2-\alpha} * S_\alpha)(1) + (k^{2-\alpha} * S_\alpha)(0). \tag{3.9}$$

Note that
$$(k^{2-\alpha} * S_\alpha)(2) = k^{2-\alpha}(0)S_\alpha(2) + k^{2-\alpha}(1)S_\alpha(1) + k^{2-\alpha}(2)S_\alpha(0)$$
$$= S_\alpha(2) + (2-\alpha)I + \frac{(3-\alpha)(2-\alpha)}{2}I$$
$$= I + T + \frac{\alpha(\alpha-1)}{2}I - (\alpha-1)^2 I + (2-\alpha)I + \frac{(3-\alpha)(2-\alpha)}{2}I$$
$$= T + (5-2\alpha)I,$$

$$(3.10)$$

as well as
$$(k^{2-\alpha} * S_\alpha)(1) = (2-\alpha)I + I = (3-\alpha)I \text{ and } (k^{2-\alpha} * S_\alpha)(0) = I. \quad (3.11)$$
Replacing (3.10) and (3.11) in (3.9) we get:
$$\Delta^\alpha S_\alpha(0) = T + (5-2\alpha)I - 2(3-\alpha)I + I = T.$$
So, the claim is proved and we have
$$\Delta^\alpha S_\alpha(n) = TS_\alpha(n), \quad (3.12)$$
for all $n \in \mathbb{N}_0$. By Lemma 3.6,
$$\Delta^\alpha(S_\alpha * h_\alpha)(n) = (\Delta^\alpha S_\alpha * h_\alpha)(n) + h_\alpha(n+2) - (\alpha-1)h_\alpha(n+1)$$
$$= (\Delta^\alpha S_\alpha * h_\alpha)(n) \quad (3.13)$$
$$= T(S_\alpha * h_\alpha)(n).$$
Moreover, again using Lemma 3.6,
$$\Delta^\alpha(S_\alpha * h_\alpha * f)(n) = (\Delta^\alpha(S_\alpha * h_\alpha) * f)(n) + (S_\alpha * h_\alpha)(0)f(n+2)$$
$$- \alpha(S_\alpha * h_\alpha)(0)f(n+1) + (S_\alpha * h_\alpha)(1)f(n+1)$$
$$= (\Delta^\alpha S_\alpha * h_\alpha * f)(n) + f(n+2) - \alpha f(n+1) + \alpha f(n+1)$$
$$= (\Delta^\alpha S_\alpha * h_\alpha * f)(n) + f(n+2)$$
$$= T(S_\alpha * h_\alpha * f)(n) + f(n+2).$$

$$(3.14)$$

Replacing equations (3.12), (3.13) and (3.14) in (3.6) we finally obtain:
$$\Delta^\alpha u(n) = T[S_\alpha(n)u(0) + (\Delta^\alpha S_\alpha * h_\alpha)(n-1)[u(1) - u(0)]$$
$$+ (S_\alpha * h_\alpha * f)(n-2)] + f(n)$$
$$= Tu(n) + f(n),$$
for all $n \in \mathbb{N}_0$ proving the theorem.

$\square$

In the border case $\alpha = 2$ we have $h_2(j) = 1$ for all $j \in \mathbb{N}_0$ and hence we recover the following result proved in [3, Proposition 1.3.1] by a different method.

Corollary 3.9. *Let $T \in \mathcal{B}(X)$ be given, then the unique solution of the equation*
$$\begin{cases} \Delta^2 u(n) &= Tu(n) + f(n), \quad n \in \mathbb{Z}_+; \\ u(0) &= x, \\ u(1) &= y, \end{cases} \quad (3.15)$$

is given by

$$u(n) = S_2(n)x + (S_2 * h_2)(n-1)(y-x) + (S_2 * h_2 * f)(n-2), \quad n \geq 2,$$

where $S_2(n)$ coincides with the discrete time cosine operator function and

$$(S_2 * h_2)(n) = \sum_{j=0}^{n} S_2(j)$$

coincides with the discrete time sine operator function. See [3].

4. A characterization of maximal ℓ_p- regularity

Let $T \in \mathcal{B}(X)$ be given and $f : \mathbb{Z}_+ \to X$ be a vector valued sequence. In this section we consider the discrete time evolution equation of fractional order

$$\begin{cases} \Delta^\alpha u(n) &= Tu(n) + f(n), \quad n \in \mathbb{N}, \\ u(0) &= 0, \\ u(1) &= 0. \end{cases} \tag{4.1}$$

where $1 < \alpha \leq 2$. By Theorem 3.8 the solution of equation (4.1) can be represented by

$$u(n) = (S_\alpha * h_\alpha * f)(n-2), \quad n \in \mathbb{N}, n \geq 2.$$

Note that:

$$\Delta^\alpha u(n) = T(S_\alpha * h_\alpha * f)(n-2) + f(n). \tag{4.2}$$

The following definition is motivated by the case $\alpha = 2$ which, in turn, comes from [10] following the continuous case.

Definition 4.1. We say that equation (4.1) has maximal ℓ_p-regularity if

$$(\mathcal{K}_\alpha f)(n) = T \sum_{j=0}^{n} (S_\alpha * h_\alpha)(n-j)f(j)$$

defines a bounded operator $K_\alpha \in \mathcal{B}(\ell_p(\mathbb{Z}_+; X))$ for some $p \in (1, \infty)$.

In other words, and in view of the relation (4.2), the question is if $f \in \ell_p(\mathbb{N}_0, X)$ implies $u, \ \Delta^\alpha u \in \ell_p(\mathbb{N}_0, X)$.
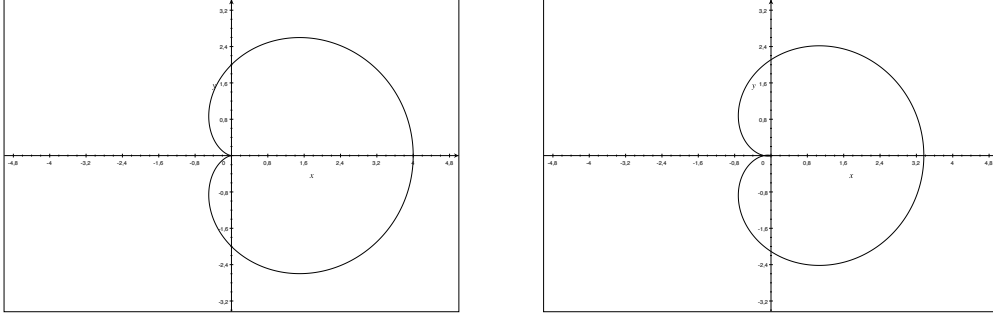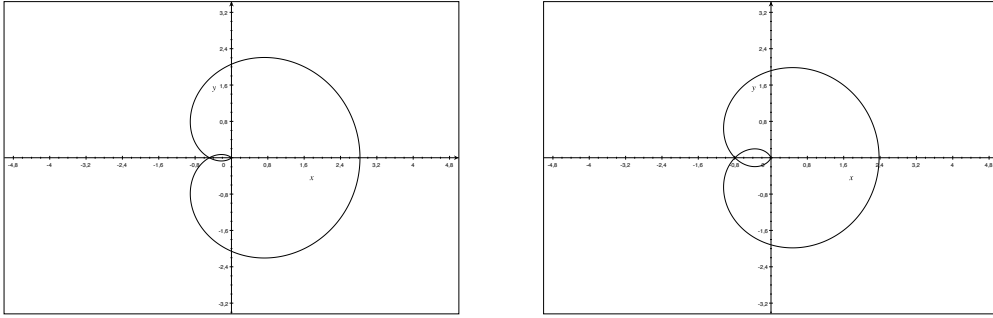
We will consider the following hypothesis that will be denoted by $(H)_\alpha$:

$(H)_\alpha$ The operator $(z^{2-\alpha}(z-1)^\alpha - T)$ is invertible for all $|z| = 1, z \neq 1$.

Denote $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ and define the set

$$\Omega_\alpha := \{z \in \mathbb{C} : z = (w-1)^\alpha w^{2-\alpha}, \ w \in \partial \mathbb{D}, \ w \neq 1\}.$$

Some cases are illustrated in the figures below.

FIGURE 1. left: $\alpha = 2$;   right: $\alpha = 1.75$

FIGURE 2. left: $\alpha = 1.5$;   right: $\alpha = 1.25$

We now present our main theorem:

Theorem 4.2. *Let $1 < \alpha \leq 2$, $p > 1$ and X be a UMD space. Let $T \in B(X)$ be given and let us suppose that $(H)_\alpha$ holds then the following assertions are equivalent:*

(i) *Equation* (4.1) *has maximal ℓ_p-regularity;*

(ii) *The following set*

$$\{z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1} : |z| = 1, z \neq 1\},$$

is R-bounded.

Proof. By hypothesis $(H)_\alpha$ we can define $M(t) := z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$ for all $z = e^{it}$, $t \in (-\pi, \pi)$. Also, we define $f_\alpha(t) := e^{2it}(1 - e^{-it})^\alpha$. Then $M(t) = f_\alpha(t)(f_\alpha(t) - T)^{-1}$.

Suppose (ii). Observe that $M'(t) = \frac{f_\alpha'(t)}{f_\alpha(t)}M(t) - \frac{f_\alpha'(t)}{f_\alpha(t)}M(t)^2$. Moreover, we have

$$f_\alpha'(t) = 2if_\alpha(t) + \alpha i(1 - e^{it})^{\alpha-1}e^{2it}e^{-it} = 2if_\alpha(t) + \frac{\alpha i f_\alpha(t)}{e^{it} - 1} = 2if_\alpha(t) + \frac{i\alpha}{z - 1}f_\alpha(t).$$

Therefore,

$$(z-1)(z+1)M'(t) = [2i(z-1)(z+1) + i\alpha(z+1)]M(t)$$
$$- [2i(z-1)(z+1) + i\alpha(z+1)]M(t)^2,$$

where $a_\alpha(t) := 2i(z-1)(z+1) + i\alpha(z+1)$ is bounded for $z = e^{it}, t \in (-\pi, \pi)$. We conclude from [3, Proposition 2.2.5] that the set $\{(z-1)(z+1)M'(t) : z = e^{it}, t \in (-\pi, \pi)\}$ is R- bounded. Then, by Theorem 2.5, there exists an operator $T_\alpha \in \mathcal{B}(\ell_p(\mathbb{Z}, X))$ such that:

$$\widehat{(T_\alpha f)}(z) = M(t)\hat{f}(z), \text{ for all } z = e^{it}, t \in (-\pi, \pi) \text{ and } f \in \ell_p(\mathbb{Z}, X). \quad (4.3)$$

From the identity

$$T(z^{2-\alpha}(z-1)^\alpha - T)^{-1} = z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1} - I. \quad (4.4)$$

and from (4.3) we have that the left hand side of the following identity

$$T(z^{2-\alpha}(z-1)^\alpha - T)^{-1}\hat{f}(z) = z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1}\hat{f}(z) - \hat{f}(z)$$
$$= M(t)\hat{f}(e^{it}) - \hat{f}(e^{it}). \quad (4.5)$$

defines a bounded operator on $\ell_p(\mathbb{Z}, X)$ given by $R_\alpha f(n) := T_\alpha f(n) - f(n)$, $n \in \mathbb{Z}$. For $f \in \ell_p(\mathbb{Z}, X)$ we define the operator:

$$K_\alpha f(n) = \begin{cases} T(S_\alpha * h_\alpha * f)(n), & n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

By $(H)_\alpha$, Remark 3.5 and definition of h_α we have that the Z-transform of $S_\alpha * h_\alpha(z)$ is $z(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$. Then, the identity (4.5) shows that the discrete time Fourier transform of $K_\alpha f(n-1)$ coincides with the time discrete time Fourier transform of $R_\alpha f(n)$ for all $n \in \mathbb{N}$. Therefore $K_\alpha f(n-1) = R_\alpha f(n)$ for all $n \in \mathbb{N}$ by uniqueness. It proves (i).

Now, we suppose that (i) holds. We define the operator

$$K_\alpha f(n) = \begin{cases} \mathcal{K}_\alpha f(n), & n \in \mathbb{N}; \\ 0 & \text{otherwise,} \end{cases}$$

where by hypothesis $\mathcal{K}_\alpha f(n) = T(S_\alpha * h_\alpha * f)(n), n \in \mathbb{Z}_+$ is given in Definition 4.1. Define $T_\alpha f(n) := K_\alpha f(n-1) + f(n), n \in \mathbb{Z}$. Given $z = e^{it}, t \in (-\pi, \pi)$, we have

$$\widehat{T_\alpha f}(z) = \sum_{j\in\mathbb{Z}} z^{-j}T_\alpha f(j) = \sum_{j=1}^\infty z^{-j}\mathcal{K}_\alpha f(j-1) + \sum_{j\in\mathbb{Z}} z^{-j}f(j)$$
$$= z^{-1}\sum_{j=0}^\infty z^{-j}\mathcal{K}_\alpha f(j) + \hat{f}(z).$$

By hypothesis $(H)_\alpha$ the Z- transform of $(S_\alpha * h_\alpha)(z)$ which is equal to $z(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$ exists for $|z| = 1$ and therefore:

$$
\begin{aligned}
\widehat{T_\alpha f}(z) &= z^{-1}T(\widehat{S_\alpha * h_\alpha})(z)\hat{f}(z) + \hat{f}(z) \\
&= T(z^{2-\alpha}(z-1)^\alpha - T)^{-1}\hat{f}(z) + \hat{f}(z) \\
&= z^{2-\alpha}(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1}\hat{f}(z) - \hat{f}(z) + \hat{f}(z) = M(t)\hat{f}(z),
\end{aligned}
\tag{4.6}
$$

where we have used the identity (4.4) and the definition of $M(t)$ is given at the beginning of the proof. An application of Theorem 2.6 shows that (ii) holds and the proof is complete.

$\square$

Remark 4.3. Under the hypothesis that equation (4.1) has l_p-maximal regularity, we deduce that the operator $(z^{2-\alpha}(z-1)^\alpha - T)$ in $(H)_\alpha$ is always surjective. Indeed, given $x \in X$ we define

$$
f(n) = \begin{cases} x & n = 0, \\ \\ 0 & otherwise. \end{cases}
$$

Hence, by hypothesis we obtain that there exists $u_x \in l_p(\mathbb{Z}, X)$ such that $(z^{2-\alpha}(z-1)^\alpha - T)\widehat{u_x}(z) = \hat{f}(z) = x$ where $z = e^{it}, t \in (-\pi, \pi)$.

Remark 4.4. In case that X is a Hilbert space, condition (ii) can be replaced by

$$
(ii)' \qquad \sup_{|z|=1, z \neq 1} \|(z-1)^\alpha(z^{2-\alpha}(z-1)^\alpha - T)^{-1}\| < \infty.
$$

In the case of Hilbert spaces, the hypothesis of R-boundedness can be replaced by boundedness. In such case, we obtain an interesting alternative condition on the operator T in order to have l_p-maximal regularity.

Theorem 4.5. *Let $T \in \mathcal{B}(H)$ be a normal operator defined on a Hilbert space H and assume that*

$$
\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha \}.
$$

Then for each $f \in l_p(\mathbb{Z}_+, X), p > 1$, there is a unique $u \in l_p(\mathbb{Z}_+, X)$ such that

$$
\begin{cases} \Delta^\alpha u(n) &= Tu(n) + f(n), \quad n \in \mathbb{N}, \\ u(0) &= 0, \\ u(1) &= 0, \end{cases}
\tag{4.7}
$$

for any $1 < \alpha \leq 2$.

Proof. We define $f_\alpha(z) = z^{2-\alpha}(z-1)^\alpha$ for $z = e^{it}, t \in (-\pi, \pi)$. Observe that

$$
f_\alpha(z) = z^{2-\alpha}(z-1)^\alpha = (1 - \frac{1}{z})^\alpha z^2 = (1 - \cos t + i\sin t)^\alpha e^{2it+2ki\pi}
$$

$$
= (2 - 2\cos t)^{\frac{\alpha}{2}} e^{i[\alpha \arctan\left(\frac{\sin t}{1-\cos t}\right) + 2t + 2k\pi]}, \; k \in \mathbb{Z}.
$$

We now consider the function $m(\alpha, t) = (2 - 2\cos t)^{\frac{\alpha}{2}}$ that represents the modulus of $f_\alpha(e^{it})$ as t varies on $(-\pi, \pi)$. Then

$$\sup_{t \in (-\pi, \pi)} |m(\alpha, t)| = \sup_{t \in (-\pi, \pi)} |(2 - 2\cos t)^{\frac{\alpha}{2}}| = 4^{\alpha/2}.$$

Since $\sigma(T) \subset \{z \in \mathbb{C} : |z| > 2^\alpha\}$, we have $f_\alpha(z) \in \Omega_\alpha$ for all $z \in \partial\mathbb{D}$, $z \neq 1$ and therefore condition (H_α) is satisfied. Moreover there exists $\epsilon > 0$ such that $d(f_\alpha(z), \sigma(T)) > \epsilon > 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Since T is normal, it follows that

$$\|(z - 1)^\alpha (f_\alpha(z) - T)^{-1}\| \leq \frac{2}{d(f_\alpha(z), \sigma(T))} < \frac{2}{\epsilon}$$

for all $|z| = 1, z \neq 1$. It follows from Remark 4.4 that condition (ii) in Theorem 4.2 is satisfied and therefore the assertion of the theorem is proved. $\qquad\square$

Remark 4.6. From the proof of the above theorem, we observe that the maximum value of the function $m(\alpha, t) = (2 - 2\cos t)^{\frac{\alpha}{2}}$ is attained at the points $t = \pm\pi$.

We encourage the reader to compare Theorem 4.5 with the characterization given for $0 < \alpha \leq 1$ in [18, Corollary 4.5].

We finish this work with the following simple example that highlight the role of the fractional difference in a given equation when we are dealing with additive perturbations.

Example 4.7. Let $1 < \alpha \leq 2$ and $\epsilon > 0$ be given. We consider the equation

$$\begin{cases} \Delta^\alpha u(n, x) &= \int_0^1 \frac{k(x/t)}{t} u(n, t)dt + (2^\alpha + \epsilon)u(n, x) + F(n, x), \\ u(0, x) &= 0, \\ u(1, x) &= 0, \end{cases} \tag{4.8}$$

where $n \in \mathbb{N}_0$, $x \in [0, 1]$, and

$$k(u) = \frac{1}{\pi}\left(\frac{\sin(\pi - \sigma)}{u + u^{-1} - 2\cos(\pi - \sigma)}\right), \quad u > 0,$$

and $0 < \sigma < \pi$. The kernel k appeared in [5] associated to boundary integral equations on polygonal domains. For each $f \in C([0, 1])$ we define

$$T_\alpha f(x) = \int_0^1 \frac{k(x/t)}{t} f(t)dt + (2^\alpha + \epsilon)f(x), \quad x \in [0, 1].$$

The operator $T_\alpha \in \mathcal{B}(C([0, 1]))$ corresponds to an additive perturbation of the integral operator

$$Kf(x) := \int_0^1 \frac{k(x/t)}{t} f(t)dt.$$

It can be shown (see [5, Section 7, Formula (7.4)]) that the spectrum of K is $[0, 1 - \sigma/\pi]$. Therefore,

$$\sigma(T_\alpha) = [2^\alpha + \epsilon, 2^\alpha + \epsilon + 1 - \frac{\sigma}{\pi}] \subset \{z \in \mathbb{C} : |z| > 2^\alpha\}.$$

From Theorem 4.5 we can conclude that if

$$\sum_{j=0}^{\infty} \Big(\sup_{x \in [0,1]} |F(j,x)| \Big)^2 < \infty,$$

then there exists a unique solution $u(n,x)$ of (4.8) satisfying

$$\sum_{j=0}^{\infty} \Big(\sup_{x \in [0,1]} |u(j,x)| \Big)^2 < \infty.$$

In particular, such solution satisfy $|u(n,x)| \to 0$ as $n \to \infty$, uniformly for $x \in [0,1]$.

Let us consider the limit case $\epsilon = 0$. Observe that $T_\alpha \to K + 4I$ as $\alpha \to 2$ and $T_\alpha \to K + 2I$ as $\alpha \to 1$. Therefore, beginning with $\alpha = 2$ and as α approaches 1 the additive perturbation of (4.8) is better in the sense that $\|T_\alpha - K\| = 2^\alpha < 4$ for $1 < \alpha < 2$.

Remark 4.8. Compared with the case $0 < \alpha \le 1$ the obtained characterization in Theorem 4.2 is not continuous at $\alpha = 1$. This is due to the discrete character of the equation (4.1), and also to the structure used in the right hand side of (4.1). In other words, the spectral structure obtained in Figures 1 and 2, changes according to the consideration of $\Delta^\alpha u(n) = Tu(n)$ or $\Delta^\alpha u(n) = Tu(n+1)$, for instance. It should be noted that in the last case, the use of closed linear operators instead of only bounded operators is important (see [19]) and therefore deserves further investigation. This will be done in a forthcoming work.

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