HILBERT SERIES OF MODULES OVER POSITIVELY GRADED POLYNOMIAL RINGS

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ABSTRACT. In this note, we give examples of formal power series satisfying certain conditions that cannot be realized as Hilbert series of finitely generated modules. This answers to the negative a question raised in a recent article by the second and the third author. On the other hand, we show that the answer is positive after multiplication with a scalar.

1. INTRODUCTION

Let $\mathbb{K}$ be a field, and let $R = \mathbb{K}[X_1, \ldots, X_n]$ be the positively $\mathbb{Z}$-graded polynomial ring with $\deg X_i = d_i \geq 1$ for every $i = 1, \ldots, n$. Consider a finitely generated graded R–module $M = \bigoplus_k M_k$ over R. The graded components M_k of M are finitely dimensional $\mathbb{K}$-vector spaces, and, since R is positively graded, $M_k = 0$ for $k \ll 0$. The formal Laurent series

$$H_M(t) := \sum_{k\in\mathbb{Z}} (\dim_K M_k) t^k \in \mathbb{Z}[[t]][t^{-1}]$$

is called the Hilbert series of M. Obviously every coefficient of this series is nonnegative. Moreover, it is well-known that $H_M(t)$ can be written as a rational function with denominator $(1-t^{d_1})\cdots(1-t^{d_n})$. In fact, in the standard graded case (i.e. $d_1 = \cdots = d_n = 1$) these two properties characterize the Hilbert series of finitely generated R-modules among the formal Laurent series $\mathbb{Z}[[t]][t^{-1}]$, cf. Uliczka [4, Cor. 2.3].

In the non-standard graded case, the situation is more involved. A characterization of Hilbert series was obtained by the second and third author in [2]:

Theorem 1.1 (Moyano-Uliczka)**.** *Let $P(t) \in \mathbb{Z}[[t]][t^{-1}]$ be a formal Laurent series which is rational with denominator $(1-t^{d_1})\cdots(1-t^{d_n})$. Then P can be realized as Hilbert series of some finitely generated R-module if and only if it can be written in the form*

$$P(t) = \sum_{I\subseteq\{1,\ldots,n\}} \frac{Q_I(t)}{\prod_{i\in I}\left(1-t^{d_i}\right)} \tag{1.1}$$

with nonnegative $Q_I \in \mathbb{Z}[t, t^{-1}]$.

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However, it remained an open question in [2, Remark 2.3] if the condition of the Theorem is satisfied by *every* rational function with the given denominator and nonnegative coefficients. In this paper we answer this question to the negative. In Section 3 we provide examples of rational functions that do not admit a decomposition (1.1) and are thus not realizable as Hilbert series. On the other hand, we show the following in Corollary 2.5 and Theorem 2.6:

Theorem 1.2. *Assume that the degrees $d_1,\dots,d_n$ are pairwise either coprime or equal. Then the following holds:*

(1) If $n=2$, then every rational function $P(t)\in\mathbb{Z}[\![t]\!][t^{-1}]$ with the given denominator and nonnegative coefficients admits a decomposition as in (1.1)

(2) In general, the same still holds up to multiplication with a scalar.

In particular, there is a formal Laurent series $P(t)$ with integral coefficients such that $2P(t)$, but not $P(t)$, is the Hilbert series of a finitely generated graded R–module, cf. Example 3.1. Moreover, we will provide an example (Example 3.3) showing that the conclusion does not hold without the assumption on the degrees being pairwise coprime.

2. Proofs of the main results

As general references for further details about Hilbert series the reader is referred to Bruns and Herzog [1]. Furthermore, we are going to use some well-known facts about quasipolynomials and power series expansions of rational functions. For details about these topics, we refer the reader to Chapter 4 of Stanley [3].

We first show three lemmas before we present the proof of our main results. The following notation will be useful. For $\delta\in\mathbb{N}$ and $0\le j\le\delta-1$ set

$$e_{\delta,j}(h):=\begin{cases}1 & \text{if } h\equiv j \mod \delta,\\ 0 & \text{otherwise.}\end{cases}$$

Obviously, the functions $e_{\delta,0},\dots,e_{\delta,\delta-1}$ form a basis of the space of δ-periodic functions $\mathbb{N}\to\mathbb{Q}$.

Lemma 2.1. *Let $c_1,\dots,c_r:\mathbb{N}\to\mathbb{Q}$ be periodic functions of periods $\delta_1,\dots,\delta_r$, such that their sum takes nonnegative values. Then there exist nonnegative periodic functions $\tilde{c}_1,\dots,\tilde{c}_r:\mathbb{N}\to\mathbb{Q}$ of the same periods such that $\sum_i c_i=\sum_i\tilde{c}_i$. Moreover, if the sum of the c_i takes nonnegative integral values, then the $\tilde{c}_i$ can be chosen to be integral valued.*

Proof. Let us define the coefficients $\mu(i,j)$ by requiring

$$c_i=\sum_{j=0}^{\delta_i-1}\mu(i,j)e_{\delta_i,j}.$$

For each $i>1$, let m_i be the minimum of the $\mu(i,1),\dots,\mu(i,\delta_i)$ and choose a k_i such that $m_i=\mu(i,k_i)$. Set $\tilde{\mu}(i,j):=\mu(i,j)+m_i$ for $1<i\le r$, $\tilde{\mu}(1,j):=\mu(1,j)-\sum_i m_i$ and define $\tilde{c}_i:=\sum_{j=0}^{\delta_i-1}\tilde{\mu}(i,j)e_{\delta_i,j}$. Using the relation

$$\sum_{j=0}^{\delta-1}e_{\delta,j}=\sum_{j=0}^{\delta'-1}e_{\delta',j},$$

which holds for all $\delta, \delta' \in \mathbb{N}$, one easily sees that

$$\sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \sum_{j=0}^{\delta_i - 1} \mu(i,j) e_{\delta_i, j} = \sum_{i=1}^{r} \sum_{j=0}^{\delta_i - 1} \tilde{\mu}(i,j) e_{\delta_i, j} = \sum_{i=1}^{r} \tilde{c}_i.$$

By construction we have $\tilde{\mu}(i,j) \geq 0$ for $i > 1$ and all j, and we claim that also $\tilde{\mu}(1,j) \geq 0$ for all j. To prove this, assume for contrary that there exists an index j_0 such that $\tilde{\mu}(1,j_0) < 0$. Note that by construction $\tilde{\mu}(i,k_i) = 0$ for $1 < i \leq r$. By the Chinese remainder theorem there exists an $0 \leq h < \delta_1 \delta_2 \cdots \delta_r$ such that $h \equiv j_0 \mod \delta_1$ and $h \equiv k_i \mod \delta_i$ for $i > 1$. Then

$$\begin{aligned}\sum_{i=1}^{r} c_i(h) &= \sum_{i=1}^{r} \tilde{c}_i(h) \\ &= \tilde{\mu}(1,j_0) + \tilde{\mu}(2,k_2) + \tilde{\mu}(3,k_3) + \cdots + \tilde{\mu}(r,k_r) \\ &= \tilde{\mu}(1,j_0) < 0,\end{aligned}$$

contradicting the assumption.

Now we turn to the case that $\sum_{i=1}^{r} c_i(h) \in \mathbb{Z}$ for all $h \in \mathbb{N}$. By the same argument as above, for $1 \leq j \leq \delta_1 - 1$ there exists an $h \in \mathbb{N}$ such that $\sum_{i=1}^{r} c_i(h) = \tilde{\mu}(1,j)$, hence $\tilde{\mu}(1,j) \in \mathbb{Z}$ for all j. Further, for each $1 < i \leq r$ and each $1 \leq j \leq \delta_i - 1$, there exists an $h \in \mathbb{N}$ such that $h \equiv j \mod \delta_i$ and $h \equiv k_\ell \mod \delta_\ell$ for each $1 \leq \ell \leq r$, with $\ell \neq i$. Thus $\sum_{i=1}^{r} c_i(h) = \tilde{\mu}(1,j_0) + \tilde{\mu}(i,j)$ for some j_0. It follows that $\tilde{\mu}(i,j) \in \mathbb{Z}$. We conclude that $\tilde{c}_i(h) \in \mathbb{Z}$ for all $1 \leq i \leq r$ and all $h \in \mathbb{N}$. □

Lemma 2.2. *Let $c : \mathbb{N} \to \mathbb{Q}$ be a nonnegative periodic function of period $\delta \in \mathbb{N}$. Then for any $\beta \in \mathbb{N}$ there exists a polynomial $q \in \mathbb{Q}[t]$ with nonnegative coefficients, such that the coefficient function of the series expansion of*

$$\frac{q(t)}{(1 - t^\delta)^\beta}$$

is a quasipolynomial of degree $\beta - 1$ whose leading coefficient equals c.

Proof. Write $c = \sum_i c_i e_{\delta,i}$ with $c_i \in \mathbb{Q}$ nonnegative. Recall that the coefficient function of

$$\frac{t^i}{(1 - t^\delta)^\beta} = \sum_{h \geq 0} \binom{h + \beta - 1}{\beta - 1} t^{\delta h + i}$$

is a quasipolynomial of degree $\beta - 1$ with leading coefficient function

$$\frac{1}{\delta^{\beta - 1} (\beta - 1)!} e_{\delta,i}.$$

So the polynomial $q(t) := \delta^{\beta-1}(\beta-1)! \sum_{i=0}^{\delta-1} c_i t^i$ satisfies the claim. □

Lemma 2.3. *Let p_1, p_2 be two quasipolynomials of the same period and the same degree. Assume moreover that the leading coefficient function of p_1 is nonnegative and greater than or equal to the leading coefficient function of p_2. Then there exists a $k \in \mathbb{N}$ such that $p_1(h) - p_2(h-k) \geq 0$ for all $h \geq k$.*

Proof. Let $\delta \in \mathbb{N}$ be the common period of p_1 and p_2. We only consider values of k that are multiples of δ, so we set $k = \tilde{k}\delta$. Let

$$p_1(h) = \sum_{i=0}^{\ell} a_i(h)h^i \qquad \text{and} \qquad p_2(h) = \sum_{i=0}^{\ell} b_i(h)h^i.$$

Let $\tilde{h} := h - \tilde{k}\delta$. We compute

$$\begin{aligned} p_1(h) - p_2(h - \tilde{k}\delta) &= p_1(\tilde{h} + \tilde{k}\delta) - p_2(\tilde{h}) \\ &= \sum_{i=0}^{\ell} a_i(\tilde{h} + \tilde{k}\delta)(\tilde{h} + \tilde{k}\delta)^i - b_i(\tilde{h})\tilde{h}^i \\ &= \sum_{i=0}^{\ell} a_i(\tilde{h})(\tilde{h} + \tilde{k}\delta)^i - b_i(\tilde{h})\tilde{h}^i \\ &= (a_\ell(\tilde{h}) - b_\ell(\tilde{h}))\tilde{h}^\ell + \sum_{i=0}^{\ell-1} \left(\sum_{j=i}^{\ell} \binom{j}{i} \tilde{k}^{j-i}\delta^{j-i} a_j(\tilde{h}) - b_i(\tilde{h}) \right) \tilde{h}^i. \end{aligned}$$

By assumption we have that $a_\ell(\tilde{h}) - b_\ell(\tilde{h}) \geq 0$. Further, we see that all other coefficient functions of $p_1(\tilde{h} + \tilde{k}\delta) - p_2(\tilde{h})$ are non-constant polynomials in $\tilde{k}$ with leading coefficient $\binom{\ell}{i}\delta^{\ell-i}a_\ell(\tilde{h}) > 0$. Therefore all coefficient functions of $p_1(\tilde{h} + \tilde{k}\delta) - p_1(\tilde{h})$ are nonnegative for $\tilde{k} \gg 0$. It follows that for a sufficiently large $\tilde{k}$, it holds that $p_1(\tilde{h} + \tilde{k}\delta) - p_2(\tilde{h}) \geq 0$ for all $\tilde{h} \geq 0$, and consequently $p_1(h) - p_2(h - \tilde{k}\delta) \geq 0$ for all $h \geq \tilde{k}\delta$. □

Now we are ready to present and prove our main theorem. It shows that a decomposition as in Theorem 1.1 is always possible if one allows *rational* coefficients.

Theorem 2.4. *Let $d_1, \ldots, d_n$ be pairwise coprime or equal positive integer numbers. Let $P \in \mathbb{Z}[[t]][t^{-1}]$ be a nonnegative formal Laurent series which is rational with denominator $(1-t^{d_1})\cdots(1-t^{d_n})$. Then it can be written in the form*

$$P(t) = \sum_{I \subseteq \{1,\ldots,n\}} \frac{Q_I(t)}{\prod_{i \in I}\left(1 - t^{d_i}\right)}$$

with nonnegative $Q_I \in \mathbb{Q}[t, t^{-1}]$.

Let us introduce some more notation to simplify the presentation of the proof. Let $\delta_1, \ldots, \delta_r \in \mathbb{N}$ denote the different values of the d_i, and let $\alpha_i := |\left\{ j \,\middle|\, d_j = \delta_i \right\}|$ be the multiplicity of δ_i. Then $P(t)$ is a rational function with denominator $\prod_i (1 - t^{\delta_i})^{\alpha_i}$. From some power of t on, the coefficients of P are given by a quasipolynomial which we denote by $\mathscr{N}(P)$ (cf. [3, Prop. 4.4.1]).

Proof. We proceed by induction on $\beta := \deg \mathscr{N}(P) + 1$. If $\mathscr{N}(P) = 0$, then P is a polynomial and there is nothing to be proven. So from now on we assume that $\mathscr{N}(P) \neq 0$. Using that the δ_i are pairwise coprime, we compute a partial fraction decomposition of P

as follows:

$$P(t) = \frac{p(t)}{(1-t^{\delta_1})^{\alpha_1}\cdots(1-t^{\delta_r})^{\alpha_r}} = \frac{p(t)}{(1-t)^n\prod_{i=1}^r(\sum_{j=0}^{\delta_i-1}t^j)^{\alpha_i}}$$
$$= \frac{p_0(t)}{(1-t)^n} + \sum_{i=1}^r \frac{p_i(t)}{(\sum_{j=0}^{\delta_i-1}t^j)^{\alpha_i}} = \frac{p_0(t)}{(1-t)^n} + \sum_{i=1}^r \frac{p_i(t)(1-t)^{\alpha_i}}{(1-t^{\delta_i})^{\alpha_i}};$$

here, $p, p_0, p_1, \ldots, p_r \in \mathbb{Q}[t,t^{-1}]$. Expanding the last expression into a series yields a decomposition

$$\mathscr{N}(P) = q_0 + q_1 + \ldots + q_r \tag{2.1}$$

of $\mathscr{N}(P)$, where $q_0 \in \mathbb{Q}[t]$ is a polynomial and q_i is a quasipolynomial of period δ_i and degree at most $\alpha_i - 1$ for $1 \le i \le r$. Note that this decomposition is not necessarily unique.

Because $\mathscr{N}(P)(h)$ is nonnegative for all $h \gg 0$, its leading coefficient c is a nonnegative periodic function. There are two cases to distinguish:

(1) If $\beta > \max\{\,\alpha_i \mid 1 \le i \le r\,\}$, then c is determined by the first summand in (2.1). In particular, c is a constant function. In this case, choose numbers $0 \le \beta_i \le \alpha_i$ for $1 \le i \le r$ such that $\beta = \beta_1 + \cdots + \beta_r$. Then the coefficient function of the series expansion of $1/\prod_i(1-t^{\delta_i})^{\beta_i}$ is a quasipolynomial of degree $\beta - 1$, and its leading coefficient function is constant. Thus there exists a nonnegative $\lambda \in \mathbb{Q}$ such that c equals the leading coefficient of $\mathscr{N}(G)$ for

$$G(t) := \frac{\lambda}{\prod_{i=1}^r(1-t^{\delta_i})^{\beta_i}}.$$

(2) If $\beta \le \max\{\,\alpha_i \mid 1 \le i \le r\,\}$, then c is a sum of periodic functions of the periods δ_i for those i where $\beta \le \alpha_i$. By Lemma 2.1, we can write c as a sum of nonnegative functions $\tilde{c}_1, \ldots, \tilde{c}_r : \mathbb{N} \to \mathbb{Q}$ of periods $\delta_1, \ldots, \delta_r$, where $\tilde{c}_i = 0$ if $\beta > \alpha_i$. By Lemma 2.2, there are nonnegative polynomials $\tilde{q}_1, \ldots, \tilde{q}_r \in \mathbb{Q}[t]$, such that c is the leading coefficient of $\mathscr{N}(G)$ for

$$G(t) := \sum_{i=1}^r \frac{\tilde{q}_i(t)}{(1-t^{\delta_i})^{\beta}}.$$

In both cases, $\mathscr{N}(P)$ and $\mathscr{N}(G)$ satisfy the hypotheses of Lemma 2.3. Hence, there exists a $k \in \mathbb{N}$, such that $\mathscr{N}(P)(h) - \mathscr{N}(G)(h-k) \ge 0$ for all $h \ge k$. By enlarging k, we may also assume that the coefficient of t^h in P is given by $\mathscr{N}(P)(h)$ for all $h \ge k$. On the other hand, the coefficients of G are given by $\mathscr{N}(G)(h)$ for all $h \ge 0$.

Thus, by construction the coefficient of t^h in the series $P' := P - t^k G$ is given by the corresponding coefficient in P for $h < k$ and by $\mathscr{N}(P)(h) - \mathscr{N}(G)(h-k)$ for $h \ge k$. In particular, P' has nonnegative coefficients. But $\deg \mathscr{N}(P') < \deg \mathscr{N}(P)$, so the claim follows by induction. □

Corollary 2.5. *Let $P \in \mathbb{Z}[[t]][t^{-1}]$ be a formal Laurent series satisfying the assumptions of Theorem 2.4. Then there exist a $\lambda \in \mathbb{N}$ and a finitely generated R-module M, such that λP is the Hilbert series of M.*

Proof. This follows from Theorem 2.4 and Theorem 1.1. □

Theorem 2.6. *Assume that in the situation of Theorem 2.4 we have $n = 2$. Then the numerator polynomials Q_I can be chosen to have nonnegative* integral *coefficients. In particular, P can be realized as a Hilbert series of a finitely generated graded R-module.*

Proof. As a notation, we write $c_i(P)$ for the i-th coefficient of $\mathcal{N}(P)$. If the degrees are equal the problem can be reduced to the standard graded case, so the claim follows from [4, Thm. 2.1]. Therefore we may assume that $d_1 \neq d_2$. Since $n = 2$, $\mathcal{N}(P)$ has degree at most 1. If $\mathcal{N}(P) = 0$, then P is a polynomial, so nothing is to be proven. Next we assume that $\deg \mathcal{N}(P) = 0$. By a partial fraction decomposition of P we see that it can be written in the form

$$P(t) = \frac{p_1(t)}{1-t^{d_1}} + \frac{p_2(t)}{1-t^{d_2}}.$$

From this we read off that $c_0(P)$ is the sum of two periodic functions of period d_1 resp. d_2. By Lemma 2.1, we can choose these functions to be nonnegative and integer valued. In other words, there exist two polynomials $\tilde{p}_1, \tilde{p}_2 \in \mathbb{Z}[t]$ with nonnegative coefficients such that

$$c_0(P) = c_0\left(\frac{\tilde{p}_1(t)}{1-t^{d_1}} + \frac{\tilde{p}_2(t)}{1-t^{d_2}}\right),$$

so by subtracting a suitable shift of this rational function from $P(t)$ we reduce to the case of a polynomial.

Finally we consider the case of $\deg \mathcal{N}(P) = 1$. Let us write

$$P(t) = \frac{p(t)}{(1-t^{d_1})(1-t^{d_2})} \tag{2.2}$$

with $p(t) \in \mathbb{Q}[t, t^{-1}]$. First, we show that the coefficients of $p(t)$ are integers. For this, let $p(t) = \sum_i a_i t^i$ and write $P(t) = \sum_{j \geq 0} f_j t^j$. It follows from (2.2) that

$$a_i = f_i - f_{i-d_1} - f_{i-d_2} + f_{i-d_1 d_2} \in \mathbb{Z}.$$

It is not difficult to see that

$$c_1\left(\frac{t^i}{(1-t^{d_1})(1-t^{d_2})}\right) = \frac{1}{d_1 d_2}$$

for all i, and in particular this coefficient function is constant. As the coefficients of $p(t)$ are integers, it follows that $c_1(P)$ is an integral multiple of $1/d_1 d_2$. Hence there exists $\lambda \in \mathbb{N}$ such that

$$P'(t) := P(t) - \frac{\lambda t^k}{(1-t^{d_1})(1-t^{d_2})}$$

satisfies $\deg \mathcal{N}(P') = 0$. Moreover, Lemma 2.3 implies that the coefficients of the series expansion of P' are nonnegative for $k \gg 0$. Thus we have reduced the claim to the previous case. □

3. COUNTEREXAMPLES

The decomposition is not always possible with integral coefficients. We describe a general construction of counterexamples. For this we consider pairwise coprime numbers $\delta_1,\dots,\delta_r \in \mathbb{N}$ and exponents $\alpha_1,\dots,\alpha_r \in \mathbb{N}$. Consider two rational functions P_1, P_2 of the form

$$\frac{1}{\prod_i (1-t^{\delta_i})^{\beta_i}}$$

with $0 \leq \beta_i \leq \alpha_i$. Assume P_1 and P_2 have the following properties:

(i) $\deg \mathcal{N}(P_1) = \deg \mathcal{N}(P_2)$. Let us call this number d.
(ii) $d+1 > \max\{\alpha_1,\dots,\alpha_r\}$. This ensures that the leading coefficients $c_d(P_1)$ and $c_d(P_2)$ are constant.
(iii) $c_d(P_1) > c_d(P_2)$, and the former should not be a multiple of the latter.

Under these assumptions, it is easy to see that there exists a $\lambda \in \mathbb{N}$, such that $\tilde{P} := P_1 - \lambda P_2$ is a series, so that $c_d(\tilde{P})$ is smaller than $c_d(P_2)$. This series may have negative coefficients. But by Lemma 2.3 we may instead consider $P := P_1 - \lambda t^k P_2$ for a sufficiently large $k \in \mathbb{N}$, and this series has nonnegative coefficients.

Now assume additionally that $c_d(P_2)$ is the minimal leading coefficient of all series of the given type and dimension. Then it is immediate that P cannot be written as a nonnegative integral linear combination of such series. We give two explicit examples of this behaviour.

Example 3.1. Consider the rational function

$$\begin{aligned} P(t) &:= \frac{1}{(1-t^2)(1-t^5)} - \frac{t^4}{(1-t^3)(1-t^5)} \\ &= \frac{1}{2}\left(1+t^2+\frac{t^6}{1-t^2}+\frac{t^2}{1-t^3}+\frac{1+t^6}{1-t^5}+\frac{t^{12}}{(1-t^3)(1-t^5)}\right). \end{aligned}$$

One can read off from the first line that the leading coefficient of $\mathcal{N}(P)$ is $1/10 - 1/15 = 1/30$, and thus smaller than $1/15$. So by the argument given above, $P(t)$ cannot be written as a nonnegative integral linear combination. On the other hand, the second line gives a rational decomposition. This shows in particular that the coefficients of the series of P are nonnegative.

Example 3.2. The same phenomenon occurs in the case that there are only two different degrees, say 2 and 3, but $\alpha_1, \alpha_2 > 1$. As an explicit example consider the following rational function:

$$\begin{aligned} P &:= \frac{1}{(1-t^2)^2(1-t^3)} - \frac{t^2}{(1-t^2)(1-t^3)^2} \\ &= \frac{1}{2}\left(\frac{1}{1-t^3}+\frac{1}{(1-t^2)^2}+\frac{t^3}{(1-t^3)^2}+\frac{t^4}{(1-t^2)(1-t^3)^2}\right). \end{aligned}$$

Example 3.3. The condition that the degrees $\delta_1, \ldots, \delta_r$ are pairwise coprime is essential, as the following example shows. Consider the rational function

$$P(t) := \frac{1+t-t^6-t^{10}-t^{11}-t^{15}+t^{20}+t^{21}}{(1-t^6)(1-t^{10})(1-t^{15})} = \frac{1+t+t^7+t^{13}+t^{19}+t^{20}}{1-t^{30}}.$$

One can read off from the second line that $P(t)$ cannot be written as a sum with positive coefficients and the required denominator: The coefficient of t^0 is 1, but the terms t^6, t^{10} and t^{15} all have coefficient zero.

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