FINITE CODIMENSIONAL ISOMETRIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS*

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Abstract

Based on the vector-valued generalization of Holsztyński's theorem by M. Cambern, we provide a complete description of the linear isometries of $C(X,E)$ into $C(Y,F)$ whose range has finite codimension.

1 Introduction.

Throughout this paper, X and Y will stand for compact Hausdorff spaces, and E and F for Banach spaces over the field $\mathbb{K}$ of real or complex numbers. $C(X,E)$ and $C(Y,F)$ will be the Banach spaces of continuous E-valued and F-valued functions defined on X and Y, respectively, endowed with the supremum norm $\|\cdot\|_\infty$. If $E = F = \mathbb{K}$, then we will write $C(X)$ and $C(Y)$ instead of $C(X,E)$ and $C(Y,F)$.

The classical Banach-Stone theorem states that if there exists a linear isometry T of $C(X)$ onto $C(Y)$, then there are a homeomorphism ψ of Y onto X and a continuous map $a : Y \longrightarrow \mathbb{K}$, $|a| \equiv 1$, such that T can be written as a weighted composition map, that is,

$$(Tf)(y) = a(y)f(\psi(y)) \text{ for all } y \in Y \text{ and all } f \in C(X).$$

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An important generalization of the Banach-Stone theorem was given by W. Holsztyński in [13] (see also [3]) by considering non-surjective isometries. Namely, he proved that, in this case, there is a closed subset Y_0 of Y where the isometry can still be represented as a weighted composition map.

This result of Holsztyński was used in [11] (see also [2, 4, 9, 10, 12, 14, 16]) to classify linear isometries on $C(X)$ whose range has codimension 1 as follows: Let $T : C(X) \longrightarrow C(X)$ be a codimension 1 linear isometry. Then there exists a closed subset X_0 of X such that either

(1) $X_0 = X \setminus \{p\}$

where p is an isolated point of X, or

(2) $X_0 = X$,

and such that there exists a continuous map h of X_0 onto X and a function $a \in C(X_0)$, $|a| \equiv 1$, such that $(Tf)(x) = a(x) \cdot f(h(x))$ for all $x \in X_0$ and all $f \in C(X)$.

In the context of continuous vector-valued functions, M. Jerison ([18]) investigated the vector analogue of the Banach-Stone theorem: If X and Y are compact Hausdorff spaces and E is a strictly convex Banach space, then every linear isometry T of $C(X, E)$ onto $C(Y, E)$ can be written as a weighted composition map; namely, $(Tf)(y) = \omega(y)(f(\psi(y)))$, for all $f \in C(X, E)$ and all $y \in Y$, where ω is a continuous map from Y into the space of continuous linear operators from E to E (taking values in the subset of surjective isometries) endowed with the strong operator topology. Furthermore, ψ is a homeomorphism of Y onto X. As in the scalar-valued case, Jerison's results have been extended in many directions (see e.g., [5], [1], [15] or [6]). In particular, M. Cambern obtained in [8] the following formulation of Holsztyński's theorem for spaces of continuous vector-valued functions.

Theorem 1.1 *If F is a strictly convex Banach space, then every linear isometry T of $C(X, E)$ into $C(Y, F)$ can be written as a weighted composition map; namely,*

$$(Tf)(y) = J_y(f(h(y))),$$

for all $f \in C(X, E)$ and all $y \in Y_0 \subset Y$, where J is a continuous map from Y into the space $L(E, F)$ of bounded operators from E into F endowed with the strong operator topology, with $\|J_y\| \leq 1$ for all $y \in Y$ and $\|J_y\| = 1$ for $y \in Y_0$. Furthermore, h is a continuous function of Y_0 onto X. If E is finite-dimensional, then Y_0 is a closed subset of Y.

Let us recall that there are counter-examples (see [7] or [18]) which show that all the above results may not hold if the assumption of strict convexity is not observed.

In this paper we provide, based on this theorem of Cambern, a complete description of the linear isometries of $C(X,E)$ into $C(Y,F)$, E and F strictly convex, whose range has finite codimension n_0.

2 Preliminaries and main results.

Given a continuous linear operator $T : C(X,E) \longrightarrow C(Y,F)$, the map

$$\begin{array}{rcl} J : Y & \longrightarrow & L(E,F) \\ y & \mapsto & J_y \end{array}$$

given by $J_y(\mathbf{e}) := (T\widehat{\mathbf{e}})(y)$ for all $\mathbf{e} \in E$ (being $\widehat{\mathbf{e}}$ the function constantly equal to $\mathbf{e}$) is well defined and continuous when, as usual, $L(E,F)$ is endowed with the strong operator topology. Furthermore, $\|J_y\| \leq \|T\|$ for all $y \in Y$.

On the other hand, we can define three subsets of Y as follows:

$$\begin{array}{rcl} Y_3 & := & \{y \in Y : (Tf)(y) = \mathbf{0} \ \ \forall f \in C(X,E)\}; \\ Y_1 & := & \{y \in Y \setminus Y_3 : \exists x_y \in X \text{ such that } (Tf)(y) = \mathbf{0} \text{ if } f(x_y) = \mathbf{0}, f \in C(X,E)\}; \\ Y_2 & := & Y \setminus (Y_1 \cup Y_3). \end{array}$$

It is easy to see that the point $x_y \in X$ corresponding to each $y \in Y_1$ is uniquely determined, so if we define $\overline{h} : Y_1 \longrightarrow X$ by $\overline{h}(y) := x_y$, then

$$(Tf)(y) = J_y\left(f\left(\overline{h}\,(y)\right)\right)$$

for every $f \in C(X,E)$ and $y \in Y_1$. Summing up, Y_1 coincides with the subset of Y where T can be written as a (nontrivial) weighted composition map. This implies that, given any $y_0 \in Y_1$ and a neighborhood U of $\overline{h}(y_0)$ in X, there exists $f \in C(X,E)$ such that $f \equiv 0$ outside U and $(Tf)(y_0) \neq 0$, so the set V of all $y \in Y_1$ with $(Tf)(y) \neq 0$ is an open neighborhood of y_0 in Y_1. Now it is clear that $\overline{h}(V_1) \subset U$, and the fact that $\overline{h}$ is continuous follows easily.

Recall that a Banach space E is said to be *strictly convex* if every element of its unit sphere is an extreme point of the closed unit ball of E. It is well-known that if E is strictly convex and $\mathbf{e}_1, \mathbf{e}_2 \in E \setminus \{\mathbf{0}\}$, then $\|\mathbf{e}_1 + \mathbf{e}_2\| =$

$\|\mathbf{e}_1\| + \|\mathbf{e}_2\|$ implies $\mathbf{e}_1 = r\mathbf{e}_2$ for some positive real r (see [19, pp. 332–336]). From this, it is straightforward to see that

$$\|\mathbf{e}_1\|, \|\mathbf{e}_2\| < \max\{\|\mathbf{e}_1 + \mathbf{e}_2\|, \|\mathbf{e}_1 - \mathbf{e}_2\|\}$$

whenever $\mathbf{e}_1, \mathbf{e}_2 \in E \setminus \{\mathbf{0}\}$.

From now on, E and F will be strictly convex normed spaces (see Remark 2.1 below). Also, T will be a linear isometry of $C(X, E)$ into $C(Y, F)$ whose range has finite codimension $n_0 \geq 1$.

For a function $f \in C(X, E)$, we will write $c(f)$ to denote the cozero set of f, that is, $c(f) := \{x \in X : f(x) \neq 0\}$. If V is a subset of X, we will write $\operatorname{cl} V$ to denote its closure in X.

We rephrase the formulation of Holsztyński's theorem for spaces of continuous vector-valued functions obtained by M. Cambern in [8].

Theorem 2.1 (Cambern) *The restriction of $\overline{h}$ to $Y_0 := \{y \in Y_1 : \|J_y\| = 1\}$ is a continuous function onto X. Also, if E is finite-dimensional, then Y_0 is a closed subset of Y.*

We denote by h the restriction of $\overline{h}$ to Y_0. We then have that $h : Y_0 \longrightarrow X$ is continuous and surjective, and that for $y \in Y_1 \setminus Y_0$, the mapping $J_y : E \longrightarrow F$ defined by

$$J_y(\mathbf{e}) := \widehat{\mathbf{e}}(y)$$

is linear and continuous and its norm is less than 1.

Points in Y_1 can be classified into two disjoint categories:

$$\begin{aligned} Y_{10} &:= \{y \in Y_1 : J_y \text{ is an isometry}\}; \\ Y_{11} &:= \{y \in Y_1 : J_y \text{ is not an isometry}\}. \end{aligned}$$

We shall see that $Y_{11} \cup Y_2 \cup Y_3$ consists of finitely many isolated points of Y. Indeed, if F is assumed to be infinite-dimensional, then it will be proved that $Y_{11} \cup Y_2 \cup Y_3$ is empty, that is, $Y = Y_0 = Y_{10}$.

Related to the subsets Y_0 and Y_1 and the corresponding maps h and $\overline{h}$, we consider, for each $x \in X$, the sets

$$F_x := \{y \in Y_0 : h(y) = x\}$$

and

$$G_x := \{y \in Y_1 : \overline{h}(y) = x\}.$$

It will turn out that G_x (and consequently F_x) is finite for every $x \in X$.

Prior to providing the description of T, we still need to classify the points of X into three not necessarily disjoint classes that will be widely used in the paper:

$$\begin{aligned} A_0 &:= \{x \in X : \exists y \in F_x \text{ with } J_y \text{ not a surjective isometry}\}; \\ A_1 &:= \{x \in X : \operatorname{card} G_x \geq 2\}; \\ A_2 &:= \{x \in X : x \notin A_0, \operatorname{card} G_x = 1\}. \end{aligned}$$

We shall prove that A_0 and A_1 are finite.

Summarizing, there exists $J : Y \longrightarrow L(E, F)$ continuous with respect to the strong operator topology and $\overline{h} : Y_1 \longrightarrow X$ continuous and surjective such that $(Tf)(y) = J_y \left(f\left(\overline{h}\,(y)\right)\right)$ for all $f \in C(X, E)$ and $y \in Y_1$. We next state the main results.

Theorem 2.2 *Assume that F is infinite-dimensional. Then $Y_0 = Y_1 = Y$ and $h = \overline{h} : Y \longrightarrow X$ is a homeomorphism. Moreover, J_y is an isometry for all $y \in Y$, which is surjective for all $y \in Y \setminus Y_N$, where Y_N is a finite subset satisfying*

$$\sum_{y \in Y_N} \operatorname{codim}(\operatorname{ran} J_y) = n_0.$$

The finite-dimensional case turns out to be more intricate. First it is apparent that, since $\overline{h}$ is surjective, if Y is finite, then X is also finite. Consequently, it is clear that $n_0 = (\dim F)(\operatorname{card} Y) - (\dim E)(\operatorname{card} X)$. Next we study the case when Y is infinite.

Theorem 2.3 *Assume $\dim F < \infty$. If Y is infinite, then the set of all $y \in Y$ for which $J_y : E \longrightarrow F$ is a surjective isometry is clopen and its complement is finite. Furthermore,*

$$n_0 = (\dim F)\left(\operatorname{card}(Y_2 \cup Y_3) + \sum_{x_i \in A_1} (\operatorname{card}(G_{x_i}) - 1)\right).$$

Remark 2.1 Theorem 2.3 does not hold in general if E (or F) is not strictly convex. For instance, suppose that, for $F = \mathbb{K}$ and $E = \mathbb{K}^2$ endowed with the sup norm, and Y being the topological sum of two copies $X \times \{1\}$, $X \times \{2\}$ of X and n_0 isolated points p_i. It is easy to see that the map $T : C(X, E) \longrightarrow C(Y, F)$ defined, for each $f \in C(X, E)$, by $(Tf)(x, i) := \langle f(x), \mathbf{e_i} \rangle$ (where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the canonical basis in $\mathbb{K}^2$), and $(Tf)(p_j) := 0$ for all j, is a linear isometry with codimension n_0. As in [17], it can be checked that T is not a weighted composition map.

3 Some previous lemmas.

Lemma 3.1 *The set A_0 is finite.*

Proof. Suppose, contrary to what we claim, that A_0 is infinite. Then we can find pairwise distinct $x_1, x_2, \ldots, x_{n_0+1} \in A_0$. For $i = 1, 2, \ldots, n_0 + 1$, we choose $y_i \in F_{x_i}$ with J_{y_i} not a surjective isometry. Next we divide the set $\{1, 2, \ldots, n_0 + 1\}$ into three mutually disjoint subsets. Namely,

$$\begin{aligned} I_1 &:= \{i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ isometry}\}; \\ I_2 &:= \{i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ not injective}\}; \\ I_3 &:= \{i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ injective but not isometry}\}. \end{aligned}$$

Let $i \in I_2$. Then there is $\mathbf{e}_i \in E$ with $\|\mathbf{e}_i\| = 1$ and $J_{y_i}(\mathbf{e}_i) = \mathbf{0}$. Take $f_i \in C(X)$ such that $0 \leq f_i \leq 1$, $f_i(x_i) = 1$, and $f_i(x_j) = 0$ for $j \neq i$. It is clear that, if we put $k_i := f_i \mathbf{e}_i \in C(X, E)$, then $\|k_i\|_\infty = 1$ and $(Tk_i)(y_i) = \mathbf{0}$. Furthermore, for $j \neq i$, $1 \leq j \leq n_0 + 1$, we have that

$$k_i(x_j) = k_i(h(y_j)) = \mathbf{0}.$$

Hence, $(Tk_i)(y_j) = \mathbf{0}$.

Consequently, for each $i \in I_2$, the set

$$V_i := \left\{ y \in Y : \|(Tk_i)(y)\| < \frac{1}{2} \right\}$$

is open in Y and contains y_j for all j. For the same reason, if we define $V := Y$ if $I_2 = \emptyset$ and

$$V := \bigcap_{i \in I_2} V_i$$

otherwise, then V is an open neighborhood of y_j for all $j \in \{1, 2, \dots, n_0+1\}$.

Next we consider pairwise disjoint open neighborhoods V_i' of y_i in Y for all $i \in \{1, 2, \dots, n_0 + 1\}$, and define

$$W_i := V_i' \cap V.$$

It is clear that $W_i \cap W_j = \emptyset$ if $i \neq j$ and that $y_i \in W_i$ for all i.

Next we consider, for each $i \in \{1, 2, \dots, n_0 + 1\}$, a function $g_i \in C(Y)$ such that $0 \leq g_i \leq 1$, $c(g_i) \subset W_i$ and $g_i(y_i) = 1$, and a vector $\mathbf{f}_i \in F$ given as follows:

1. If $i \in I_1$, then we choose $\mathbf{f}_i \notin \operatorname{ran} J_{y_i}$ with $\|\mathbf{f}_i\| = 1$.
2. If $i \in I_2 \cup I_3$, then we take a norm-one $\mathbf{e}_i' \in E$ with $0 < \|J_{y_i}(\mathbf{e}_i')\| < 1$, and define $\mathbf{f}_i := J_{y_i}(\mathbf{e}_i')$.

As the codimension of the range of T is n_0, there exist $a_1, \dots, a_{n_0+1} \in \mathbb{K}$ such that $g := \sum_{i=1}^{n_0+1} a_i g_i \mathbf{f}_i \neq 0$ belongs to the range of T. Let us choose i_0 such that $\|g\|_\infty = |a_{i_0}| \, \|\mathbf{f}_{i_0}\|$. We claim that $i_0 \in I_2$ (so $I_2 \neq \emptyset$).

Let $f \in C(X, E)$ with $Tf = g$. If we fix $i \in I_1$, then

$$a_i \mathbf{f}_i = (Tf)(y_i) = J_{y_i}(f(h(y_i))).$$

This is to say that $a_i \mathbf{f}_i$ belongs to the range of J_{y_i} and, since $i \in I_1$, we get $a_i = 0$. Hence $i_0 \notin I_1$. Next, if $i \in I_3$, then $g(y_i) = J_{y_i}(f(x_i))$, and also $g(y_i) = a_i \mathbf{f}_i = a_i J_{y_i}(\mathbf{e}_i')$, implying that $|a_i| = |a_i| \, \|\mathbf{e}_i'\| = \|f(x_i)\| \leq \|g\|_\infty$. Hence $|a_i| \, \|\mathbf{f}_i\| < \|g\|_\infty$ and $i_0 \notin I_3$, as we wanted to prove.

Since $\|g\|_\infty = |a_{i_0}| \, \|\mathbf{f}_{i_0}\| = \left\| J_{y_{i_0}}(f(x_{i_0})) \right\|$, we deduce that $f(x_{i_0}) \neq \mathbf{0}$ and, since E is strictly convex, it is now clear that either

$$\|k_{i_0}(x_{i_0}) + f(x_{i_0})\| > 1$$

or

$$\|k_{i_0}(x_{i_0}) - f(x_{i_0})\| > 1,$$

that is, either $\|k_{i_0} + f\|_\infty > 1$ or $\|k_{i_0} - f\|_\infty > 1$.

With no loss of generality, we shall assume that $\|g\|_\infty = \frac{1}{2}$.

We claim that $\|Tk_i \pm g\|_\infty \leq 1$ for all i. To this end, fix $y \in Y$ and assume first that $y \in c(g)$, so $y \in V$. Hence $\|(Tk_i)(y)\| < 1/2$ and, consequently,

$\|(Tk_i \pm g)(y)\| < 1$. Assume next that $y \notin c(g)$, which is to say that $g(y) = \mathbf{0}$. Then, since $\|k_i\|_\infty = 1$, $\|(Tk_i \pm g)(y)\| \leq 1$. Hence

$$\|Tk_i \pm g\|_\infty \leq 1.$$

This contradicts the isometric property of T, and we are done. □

The proof of the following lemma is immediate.

Lemma 3.2 *Let $x \in X$ and let $y_1, y_2 \in G_x$ with J_{y_1} injective. If $g \in C(Y, F)$ satisfies $g(y_1) = 0$ and $g(y_2) \neq 0$, then $g \notin \operatorname{ran} T$.*

Lemma 3.3 *The set A_1 is finite.*

Proof. Suppose, contrary to what we claim, that A_1 is infinite. Then, since A_0 is finite by Lemma 3.1, we can find pairwise distinct $x_1, x_2, \ldots, x_{n_0+1}$ in $A_1 \setminus A_0$. For each $i = 1, 2, \ldots, n_0 + 1$, we choose two distinct elements y_i^1, y_i^2 in G_{x_i}. Since h is onto, we can assume that $y_i^1 \in F_{x_i}$ for all i.

Also for each i, we can choose a function $g_i \in C(Y, F)$ such that

- $g_i(y_i^2) \neq \mathbf{0}$ and $g_i(y_j^2)) = \mathbf{0}$ for $j \neq i$.
- $g_i(y_j^1) = \mathbf{0}$ for all $j = 1, 2, \ldots, n_0 + 1$.

By Lemma 3.2, no nonzero linear combination of the g_i belongs to $\operatorname{ran} T$, which is impossible. □

Lemma 3.4 *For each $x \in X$, the set G_x is finite.*

Proof. Suppose, contrary to what we claim, that there is $x_0 \in X$ such that G_{x_0} is infinite.

First, if there exists $y_0 \in G_{x_0}$ such that J_{y_0} is injective, then we take $y_1, y_2, \ldots, y_{n_0+1} \in G_{x_0}$ pairwise distinct and different from y_0. For each $i \in \{1, 2, \ldots, n_0 + 1\}$ we choose a function $g_i \in C(Y, F)$ such that $g_i(y_i) \neq \mathbf{0}$ and $g_i(y_j) = \mathbf{0} = g_i(y_0)$ for $j \neq i$. Using Lemma 3.2, no nontrivial linear combination of the g_i belongs to $\operatorname{ran} T$. We conclude that, for all $y \in G_{x_0}$, J_y is not injective.

We shall prove that this is also impossible. To this end, let us first see that

$$G_{x_0} \cap \operatorname{cl}\left(h^{-1}\left(X \setminus A_0\right)\right) = \emptyset.$$

If $y \in G_{x_0}$, then there exists $\mathbf{e}_y \in E$, $\|\mathbf{e}_y\| = 1$, such that $J_y(\mathbf{e}_y) = 0$. On the other hand, given $y' \in h^{-1}(X \setminus A_0)$, $J_{y'}$ is an isometry and, consequently, $\|J_{y'}(\mathbf{e}_y)\| = 1$. In other words, we have that $(T\widehat{\mathbf{e}_y})(y) = 0$ and, for all $y' \in h^{-1}(X \setminus A_0)$, $\|(T\widehat{\mathbf{e}_y})(y')\| = 1$. This yields $y \notin \mathrm{cl}\,(h^{-1}(X \setminus A_0))$.

Since we are assuming that G_{x_0} is infinite, we can now consider two subsets of G_{x_0}, $\{y_1^1, \ldots, y_{n_0+1}^1\}$ and $\{y_1^2, \ldots, y_{n_0+1}^2\}$, consisting of $2n_0 + 2$ pairwise distinct elements.

Let us also consider, for each $i \in \{1, 2, \ldots, n_0+1\}$ and each $j \in \{1, 2\}$, an open neighborhood U_i^j of y_i^j such that $U_i^j \cap h^{-1}(X \setminus A_0) = \emptyset$. Clearly, we can assume that these $2n_0 + 2$ sets are pairwise disjoint, and then take functions $g_i^j \in C(Y, F)$ such that $c(g_i^j) \subset U_i^j$ and $\left\|g_i^j(y_i^j)\right\| = 1 = \left\|g_i^j\right\|_\infty$ for all i, j. Then we have two nonzero functions $g_1 := \sum_{i=1}^{n_0+1} \alpha_i g_i^1$ and $g_2 := \sum_{i=1}^{n_0+1} \beta_i g_i^2$ in the range of T, that is, $Tf_1 = g_1$ and $Tf_2 = g_2$ for some $f_1, f_2 \in C(X, E)$. Assume, without loss of generality, that $\|g_1\|_\infty = \|g_2\|_\infty = 1$.

Since $g_i \equiv 0$ on $h^{-1}(X \setminus A_0)$ $(i = 1, 2)$, we infer that $f_i \equiv 0$ on $X \setminus A_0$. However, if $f_i(x_0) = \mathbf{0}$, then $g_i(y) = \mathbf{0}$ for all $y \in G_{x_0}$. Consequently, $f_i(x_0) \neq \mathbf{0}$ for $i = 1, 2$. As A_0 is finite and $x_0 \in A_0$, we deduce that $\{x_0\}$ is an open set. Then we can write the functions f_i as

$$f_i = f_i \chi_{\{x_0\}} + f_i \chi_{A_0 \setminus \{x_0\}}.$$

As $f_i \chi_{A_0 \setminus \{x_0\}}(x_0) = \mathbf{0}$, then $(Tf_i \chi_{A_0 \setminus \{x_0\}})(y) = \mathbf{0}$ for all $y \in G_{x_0}$, so $(Tf_i \chi_{\{x_0\}})(y) = (Tf_i)(y)$ for all $y \in G_{x_0}$.

Hence, since each $\|Tf_i(y)\| = \|g_i(y)\|$ attains its maximum in G_{x_0},

$$\left\|Tf_i \chi_{\{x_0\}}\right\|_\infty \geq \|Tf_i\|_\infty = 1,$$

implying that $\left\|Tf_i \chi_{\{x_0\}}\right\|_\infty = 1$. This yields $\|f_i(x_0)\| = 1$, $i = 1, 2$. As a consequence, either $\|f_1(x_0) + f_2(x_0)\| > 1$ or $\|f_1(x_0) - f_2(x_0)\| > 1$, which implies that either

$$\|Tf_1 + Tf_2\|_\infty > 1$$

or

$$\|Tf_1 - Tf_2\|_\infty > 1.$$

These inequalities contradict the fact that

$$\|g_1 \pm g_2\|_\infty = \max\left(\|g_1\|_\infty, \|g_2\|_\infty\right) = 1.$$

□

Lemma 3.5 *The set Y_3 is finite.*

Proof. Suppose that there exist n_0+1 distinct points $y_1, \ldots, y_{n_0+1}$ in Y_3. Let us choose n_0+1 functions $g_1, \ldots, g_{n_0+1}$ in $C(Y,F)$ such that $g_i(y_j) = \mathbf{0}$ if $i \neq j$ and $g_i(y_i) \neq \mathbf{0}$ for $i \in \{1, \ldots, n_0+1\}$. It is apparent that no nonzero linear combination of $\{g_1, \ldots, g_{n_0+1}\}$ belongs to the range of T, which is impossible. $\square$

Lemma 3.6 *The set Y_2 is finite and each point of Y_2 is isolated in Y.*

Proof. We first check that $Y_2 \cap \operatorname{cl} Y_1 = \emptyset$. Obviously, $Y_2 \cap Y_1 = \emptyset$.

First, by Lemmas 3.1, 3.3 and 3.4, $\overline{h}^{-1}(A_0 \cup A_1)$ is finite. Since $X = A_0 \cup A_1 \cup A_2$, in order to prove that $Y_2 \cap \operatorname{cl} Y_1 = \emptyset$, it suffices to check that

$$Y_2 \cap \operatorname{cl}(\overline{h}^{-1}(A_2)) = \emptyset,$$

which, by the definition of A_2, is the same as proving $Y_2 \cap \operatorname{cl}(h^{-1}(A_2)) = \emptyset$.

Let $y_0 \in \operatorname{cl}(h^{-1}(A_2))$ and consider, for $f \in C(X,E)$ and $\epsilon > 0$, the set

$$K(f,\epsilon) := \{x \in X : |\|f(x)\| - \|(Tf)(y_0)\|| \leq \epsilon\}.$$

Each of these is a closed subset of X, which is also nonempty as a consequence of the fact that, for each $y \in h^{-1}(A_2)$, $\|f(h(y))\| = \|(Tf)(y)\|$. We are going to check that the family of all these sets satisfies the finite intersection property. Indeed, we shall prove that if $f_1, \ldots, f_n \in C(X,E)$ and $\epsilon_1, \ldots, \epsilon_n > 0$, then

$$\bigcap_{i=1}^{n} K(f_i, \epsilon_i) \neq \emptyset.$$

The set

$$U := \bigcap_{i=1}^{n} \{y \in Y : \|(Tf_i)(y) - (Tf_i)(y_0)\| < \epsilon_i\}$$

is an open neighborhood of y_0 and, by assumption, there exists $y_1 \in h^{-1}(A_2) \cap U$. Then

$$|\|(Tf_i)(y_1)\| - \|(Tf_i)(y_0)\|| < \epsilon_i$$

for $i = 1, 2, \ldots, n$. On the other hand, for each i, $(Tf_i)(y_1) = J_{y_1}(f_i(h(y_1)))$ and, as J_{y_1} is a surjective isometry, we have that $\|(Tf_i)(y_1)\| = \|f_i(h(y_1))\|$. Consequently,

$$|\|f_i(h(y_1))\| - \|(Tf_i)(y_0)\|| < \epsilon_i,$$

which implies that, as was to be proved,

$$h(y_1) \in \bigcap_{i=1}^{n} K(f_i, \epsilon_i).$$

Hence, since X is compact, there exists

$$x_0 \in \bigcap_{\substack{\epsilon>0 \\ f\in C(X,E)}} K(f, \epsilon).$$

By definition, we deduce that, for every $f \in C(X,E)$, $\|f(x_0)\| = \|(Tf)(y_0)\|$. In particular, if $f(x_0) = \mathbf{0}$, then $(Tf)(y_0) = \mathbf{0}$, and consequently $y_0 \notin Y_2$. This contradiction yields

$$Y_2 \cap \operatorname{cl} Y_1 = \emptyset.$$

Now, as $Y_2 = Y \setminus (Y_3 \cup \operatorname{cl} Y_1)$ and Y_3 is a finite set, we infer that Y_2 is open.

Next, suppose that Y_2 contains infinitely many elements. Then there exist $n_0 + 1$ pairwise disjoint open subsets $V_1, \ldots, V_{n_0+1}$ contained in Y_2. For each $i \in \{1, 2, \ldots, n_0 + 1\}$, we can take $g_i \in C(Y, F)$, $g_i \neq 0$, with $c(g_i) \subset V_i$. From the finite codimensionality of the range of T, we infer that there exists a nonzero linear combination $g := \sum_{i=1}^{n_0+1} \alpha_i g_i$ in the range of T, that is, there exists $f \in C(X,E)$ such that $Tf = g$. Then, it is apparent that $g(h^{-1}(X)) \equiv 0$ and, in order to get a contradiction, it suffices to check that $f(X) \equiv 0$. To this end, note that, by definition, if $x \notin A_0$, then, given $y \in F_x$, J_y is an isometry. Hence, $\mathbf{0} = (Tf)(y) = J_y(f(x))$ yields $f(x) = \mathbf{0}$, which is to say that $f \equiv 0$ on X except perhaps on a finite set $\{x_1, \ldots, x_n\} \subset A_0$. Then we can write $f = f\chi_{\{x_1\}} + \ldots + f\chi_{\{x_n\}}$. Also for each $y \in Y_1$, there exists at most one i such that $\left(Tf\chi_{\{x_i\}}\right)(y) \neq \mathbf{0}$ because in that case, necessarily, $\overline{h}(y) = x_i$. We then infer that $Tf\chi_{\{x_i\}} \equiv \mathbf{0}$ on Y_1 for all i. Hence there exists $y_1 \in Y_2$ such that $\left\|\left(Tf\chi_{\{x_i\}}\right)(y_1)\right\| = \left\|Tf\chi_{\{x_i\}}\right\|_\infty \neq 0$ for some $i \in \{1, \ldots, n\}$. Since $y_1 \in Y_2$, we can find $k \in C(X,E)$ such that $k(x_i) = \mathbf{0}$ and $(Tk)(y_1) \neq \mathbf{0}$. If we suppose, with no loss of generality, that $\|k\|_\infty = \left\|f\chi_{\{x_i\}}\right\|_\infty = 1$, then $\left\|k \pm f\chi_{\{x_i\}}\right\|_\infty = 1$, but either $\left\|(Tf\chi_{\{x_i\}})(y_1) + (Tk)(y_1)\right\| > 1$ or $\left\|(Tf\chi_{\{x_i\}})(y_1) - (Tk)(y_1)\right\| > 1$, which is impossible. □

Lemma 3.7 *The set $Y_{11} \cup Y_2 \cup Y_3$ is finite, and all of its points are isolated in Y.*

Proof. We already know, by Lemma 3.6, that the result is true for Y_2. On the other hand, it is apparent that

$$Y_{11} \subset \bigcup_{x \in X \setminus A_0} (G_x \setminus F_x) \cup \bigcup_{x \in A_0} G_x.$$

Since A_0, A_1 and G_x are finite sets (see Lemmas 3.1, 3.3 and 3.4), then we deduce that Y_{11} is finite. Also, for any $\mathbf{e} \in E$, $\|\mathbf{e}\| = 1$, the open set $C_{\mathbf{e}} := \{y \in Y : \|(T\widehat{\mathbf{e}})(y)\| < 1\}$ is contained in the finite set $Y_{11} \cup Y_2 \cup Y_3$, which implies that $C_{\mathbf{e}}$ consists of isolated points. If $y_0 \in Y_{11}$, then there exists $\mathbf{e} \in E$ such that $\|\mathbf{e}\| = 1$ and $\|(T\widehat{\mathbf{e}})(y_0)\| = \|J_{y_0}(\mathbf{e})\| < 1$, which is to say that $y_0 \in C_{\mathbf{e}}$, that is, it is isolated.

A similar reasoning shows that every element of Y_3 is isolated in Y. □

Corollary 3.1 *Y_1 is a clopen subset of Y.*

4 The infinite-dimensional case

In this section we shall assume that F is infinite-dimensional. Our first result shows that J_y is an isometry for all $y \in Y$.

Lemma 4.1 $Y_{11} \cup Y_2 \cup Y_3 = \emptyset$.

Proof. Suppose that $y_0 \in Y_{11} \cup Y_2 \cup Y_3$ and consider n_0+1 linearly independent vectors $\mathbf{g}_1, \ldots, \mathbf{g}_{n_0+1} \in F$. Since $\{y_0\}$ is a clopen subset (Lemma 3.7), then $\chi_{\{y_0\}}\mathbf{g}_1, \ldots, \chi_{\{y_0\}}\mathbf{g}_{n_0+1}$ belong to $C(Y, F)$ and are linearly independent. Then, there exists a nonzero linear combination

$$g := \sum_{i=1}^{n_0+1} \alpha_i \chi_{\{y_0\}} \mathbf{g}_i$$

in the range of T.

It is apparent that $g(h^{-1}(X \setminus A_0)) \equiv 0$. Hence, $f := T^{-1}g$ satisfies $f(X \setminus A_0) \equiv 0$ and, if we write $A_0 = \{x_1, \ldots, x_k\}$ (see Lemma 3.1), then $f = f\chi_{\{x_1\}} + \ldots + f\chi_{\{x_k\}}$. As $g(y_0) \neq \mathbf{0}$, we infer that $y_0 \notin Y_3$. Hence we only have two possible cases:

1. $y_0 \in Y_2$

2. $y_0 \in Y_{11}$

Before studying these cases, we need some preparation. With no loss of generality, we can assume that $\|g\|_\infty = \|f\|_\infty = 1$. Hence, there exists $j \in \{1, \dots, k\}$, say $j = 1$, such that $\|f(x_1)\| = 1$. Let us now check that $f(x_2) = \cdots = f(x_k) = \mathbf{0}$. To this end, we define

$$f_1 := f\chi_{\{x_1\}}$$

$$f_2 := f\chi_{\{x_2,\dots,x_k\}}$$

Claim 4.1 *$Tf_1 = g$.*

As $\|f(x_1)\| = 1$, there is $y_1 \in Y$ with $\|(Tf_1)(y_1)\| = 1$. Besides, as $f_1 \equiv 0$ on $X \setminus \{x_1\}$, $y_1 \notin G_x$ for any $x \neq x_1$, which is to say that $y_1 \in G_{x_1} \cup Y_2$. Therefore, if $y_1 \neq y_0$, then we have

$$\|T(f_1 - f_2)(y_1)\| = \|(Tf_1)(y_1) - (Tf)(y_1) + (Tf_1)(y_1)\| =$$
$$= \|2(Tf_1)(y_1) - g(y_1)\| = \|2(Tf_1)(y_1)\| = 2$$

but

$$\|f_1 - f_2\|_\infty = \|f_1(x_1)\| = 1.$$

This contradiction yields $y_1 = y_0$ and, consequently, $\|(Tf_1)(y_0)\| = 1$.

On the other hand, let us check that $(Tf_2)(y_0) = \mathbf{0}$. If this is not the case, then $\|f_1 + f_2\|_\infty = 1 = \|f_1 - f_2\|_\infty$, but as F is strictly convex, then either

$$\|(Tf_1)(y_0) + (Tf_2)(y_0)\| > 1$$

or

$$\|(Tf_1)(y_0) - (Tf_2)(y_0)\| > 1,$$

which is impossible since T is an isometry.

Consequently, for $y_2 \in Y \setminus \{y_0\}$ with $\|(Tf_2)(y_2)\| = \|Tf_2\|_\infty \leq 1$, we have $(Tf_1)(y_2) = -(Tf_2)(y_2)$. Also, if $Tf_2 \neq 0$, then either

$$\left\|(Tf_1)(y_2) + \frac{(Tf_2)(y_2)}{\|Tf_2\|_\infty}\right\| > 1$$

or

$$\left\|(Tf_1)(y_2) - \frac{(Tf_2)(y_2)}{\|Tf_2\|_\infty}\right\| > 1,$$

contrary to the fact that

$$\left\| f_1 \pm \frac{f_2}{\|Tf_2\|_\infty} \right\|_\infty = 1.$$

This contradiction yields $f_2 \equiv 0$, which is to say that $Tf_1 = g$. The proof of the claim is done.

Case 1 If we suppose that $y_0 \in Y_2$, then there exists $f_3 \in C(X, E)$ such that $\|f_3\|_\infty = 1$, $f_3(x_1) = \mathbf{0}$ and $(Tf_3)(y_0) \neq \mathbf{0}$. It is clear that $\|f_3 + f_1\|_\infty = 1 = \|f_3 - f_1\|_\infty$ but either

$$\|(Tf_3 + Tf_1)(y_0)\| > 1$$

or

$$\|(Tf_3 - Tf_1)(y_0)\| > 1.$$

This contradiction shows that $y_0 \notin Y_2$.

Case 2 Assume finally that $y_0 \in Y_{11}$, that is, J_{y_0} is not an isometry. Hence we know that there exists $\mathbf{e} \in E$, $\|\mathbf{e}\| = 1$, such that $\|J_{y_0}(\mathbf{e})\| < 1$. Let us define

$$\alpha = 1 - \|J_{y_0}(\mathbf{e})\|$$

and

$$f_3 := \chi_{\{x_1\}}\mathbf{e}.$$

It is clear that $\|f_3\|_\infty = 1$ and $\|(Tf_3)(y_0)\| = \|J_{y_0}(\mathbf{e})\| < 1$. On the other hand

$$\|(T(\alpha f_1 \pm f_3))(y_0)\| \leq \alpha\|(Tf_1)(y_0)\| + \|(Tf_3)(y_0)\| = 1.$$

Also if $y \neq y_0$, $(Tf_1)(y) = 0$ and $\|(Tf_3)(y)\| \leq \|Tf_3\|_\infty = 1$. Consequently

$$\|(T(\alpha f_1 \pm f_3))\|_\infty \leq 1.$$

However, either

$$\|\alpha f_1(x_1) + f_3(x_1)\| > 1$$

or

$$\|\alpha f_1(x_1) - f_3(x_1)\| > 1$$

which contradicts the isometric condition of T. The lemma is proved. □

Lemma 4.2 *$Y = Y_0$ and $h : Y \longrightarrow X$ is a surjective homeomorphism. Moreover J_y is an isometry for every $y \in Y$. Furthermore, the set $Y_N \subset Y$ of all y such that J_y is not surjective is finite.*

Proof. By Lemma 4.1, $Y = Y_{10}$, so every J_y is an isometry and $Y = Y_0$.

Suppose next that there exists $x_0 \in X$ with $\operatorname{card} G_{x_0} \geq 2$, and take $y_1, y_2 \in G_{x_0}$, $y_1 \neq y_2$. Pick $g = Tf \in C(Y, F)$ with $g(y_1) = \mathbf{0}$. By Lemma 3.2, $g(y_2) = \mathbf{0}$, which is impossible because $\operatorname{codim}(\operatorname{ran} T)$ is finite. We deduce that, for all $x \in X$, $\operatorname{card} G_x = 1$, and consequently $F_x = G_x$. We infer that h is injective and, since it is a continuous surjection and Y is compact, then h is a surjective homeomorphism.

Finally, let us note that, if $h(y) \notin A_0$, then J_y is a surjective isometry. Consequently, as A_0 is finite, so is Y_N. $\square$

Proposition 4.1 *Let $g \in C(Y, F)$ be such that $g(y) \in \operatorname{ran} J_y$ for all $y \in Y$. Then $g \in \operatorname{ran} T$.*

Proof. By Lemma 4.2, given $x \in X$,

$$J_{h^{-1}(x)} : E \longrightarrow F$$

is a linear isometry which is also surjective except for finitely many $x \in h(Y_N)$, being $Y_N := \{y_1, \ldots, y_k\}$.

Fix any $x_0 \in X$ and take an open neighborhood V of $h^{-1}(x_0)$ such that $V \cap Y_N \subset \{h^{-1}(x_0)\}$. Hence, for all $y \in V \setminus \{h^{-1}(x_0)\}$, we have that J_y is a surjective isometry.

Claim 4.2 *Let $\mathbf{f} \in \operatorname{ran} J_{h^{-1}(x_0)}$ and let $\epsilon > 0$. There exists an open neighborhood U_ϵ of x_0 such that, if $x \in U_\epsilon$, then $\mathbf{f} \in \operatorname{ran} J_{h^{-1}(x)}$ and*

$$\|(J_{h^{-1}(x_0)})^{-1}(\mathbf{f}) - (J_{h^{-1}(x)})^{-1}(\mathbf{f})\| < \epsilon.$$

As $\mathbf{f} \in \operatorname{ran} J_{h^{-1}(x_0)}$, there exists $\mathbf{e} \in E$ with $J_{h^{-1}(x_0)}(\mathbf{e}) = \mathbf{f}$. Hence $(T\widehat{\mathbf{e}})(h^{-1}(x_0)) = J_{h^{-1}(x_0)}(\mathbf{e}) = \mathbf{f}$ and there exists an open neighborhood V_ϵ of $h^{-1}(x_0)$ such that $V_\epsilon \subset V$ and

$$\|(T\widehat{\mathbf{e}})(y) - (T\widehat{\mathbf{e}})(h^{-1}(x_0))\| < \epsilon$$

for all $y \in V_\epsilon$, that is,

$$\|J_y(\mathbf{e}) - \mathbf{f}\| < \epsilon.$$

On the other hand, as $\mathbf{f} \in \operatorname{ran} J_y$ for all $y \in V_\epsilon$, there exists $\mathbf{e}'_y \in E$ such that $\mathbf{f} = J_y(\mathbf{e}'_y)$. Hence, if $y \in V_\epsilon$, then $\|J_y(\mathbf{e}) - J_y(\mathbf{e}'_y)\| < \epsilon$, that is,

$$\|J_y(\mathbf{e} - \mathbf{e}'_y)\| < \epsilon,$$

and, since J_y is an isometry, $\|\mathbf{e} - \mathbf{e}'_y\| < \epsilon$. Summarizing, if $x \in U_\epsilon := h(V_\epsilon)$, then

$$\|(J_{h^{-1}(x_0)})^{-1}(\mathbf{f}) - (J_{h^{-1}(x)})^{-1}(\mathbf{f})\| < \epsilon$$

and the proof of the claim is done.

Next, define the function $f : X \longrightarrow E$ by

$$f(x) := (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x)))$$

for all $x \in X$. Hence, if we prove that f is continuous, then for $y = h^{-1}(x)$, we have

$$(Tf)(y) = J_y(f(h(y)) = J_y((J_y)^{-1}(g(y))) = g(y).$$

Thus, it only remains to check the continuity of f at x_0. To this end, fix any $\epsilon > 0$. Since g is continuous, there exists an open neighborhood W of $h^{-1}(x_0)$ in Y such that, if $y \in W$, then

$$\|g(y) - g(h^{-1}(x_0))\| < \frac{\epsilon}{2}.$$

Let us define $U := h(W) \cap U_{\epsilon/2}$, where $U_{\epsilon/2}$ is given by the claim above for $\mathbf{f} := g(h^{-1}(x_0))$. Then, by definition, if $x \in U$,

$$\begin{aligned}
\|f(x_0) - f(x)\| &= \left\|(J_{h^{-1}(x_0)})^{-1}(g(h^{-1}(x_0))) - (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x)))\right\| \\
&\leq \left\|(J_{h^{-1}(x_0)})^{-1}(\mathbf{f}) - (J_{h^{-1}(x)})^{-1}(\mathbf{f})\right\| \\
&+ \left\|(J_{h^{-1}(x)})^{-1}(\mathbf{f}) - (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x)))\right\| \\
&< \frac{\epsilon}{2} + \left\|(J_{h^{-1}(x)})^{-1}\left(\mathbf{f} - g(h^{-1}(x))\right)\right\| \\
&= \frac{\epsilon}{2} + \left\|\mathbf{f} - g(h^{-1}(x))\right\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

and the continuity of f is proved. □

We can now prove the main result in this section.

Proof of Theorem 2.2. Taking into account the previous lemmas, it only remains to check that $\sum_{i=1}^{k} \operatorname{codim}(\operatorname{ran} J_{y_i}) = n_0$, where $Y_N = \{y_1, \ldots, y_k\}$ is the subset introduced in Lemma 4.2.

Notice first that, due to the representation of T,

$$\operatorname{codim}(\operatorname{ran} J_{y_i}) \leq \operatorname{codim}(\operatorname{ran} T)$$

for each i. Then there exist k sets formed by linearly independent vectors

$$\begin{aligned} \mathbf{F}_1 &:= \{\mathbf{f}(1,1), \ldots, \mathbf{f}(1,n_1)\}, \\ \mathbf{F}_2 &:= \{\mathbf{f}(2,1), \ldots, \mathbf{f}(2,n_2)\}, \\ &\vdots \\ \mathbf{F}_k &:= \{\mathbf{f}(k,1), \ldots, \mathbf{f}(k,n_k)\} \end{aligned}$$

such that

$$\operatorname{ran} J_{y_i} + \operatorname{span} \mathbf{F}_i = F$$

and

$$\operatorname{ran} J_{y_i} \cap \operatorname{span} \mathbf{F}_i = \{\mathbf{0}\} \tag{1}$$

for each $i \in \{1, 2, \ldots, k\}$.

Contrary to what we claim, suppose first that

$$\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \operatorname{codim}(\operatorname{ran} J_{y_i}) > n_0.$$

Let us consider, for each $i \in \{1, 2, \ldots, k\}$, an open neighborhood V_i of y_i such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Let $g_i \in C(Y)$ be such that $c(g_i) \subset V_i$ and $g_i(y_i) = 1$. Define also, for each $i \in \{1, 2, \ldots, k\}$ and each $j \in \{1, 2, \ldots, n_i\}$, a function $g(i,j) := g_i \mathbf{f}(i,j)$. Hence we have $\sum_{i=1}^{k} n_i$ linearly independent functions in $C(Y, F)$, so there exists a linear combination

$$g_0 := \sum_{i,j} \alpha(i,j) g(i,j)$$

in the range of T, with some $\alpha(i_0, j_0) \neq 0$. Let $f \in C(X, E)$ satisfy $Tf = g_0$. Then

$$\mathbf{0} \neq \sum_{j=1}^{n_{i_0}} \alpha(i_0, j) \mathbf{f}(i_0, j) = g_0(y_{i_0}) = (Tf)(y_{i_0}) = J_{y_{i_0}}(f(h(y_{i_0}))).$$

We deduce that $\operatorname{ran} J_{y_{i_0}} \cap \operatorname{span} \mathbf{F}_{i_0} \neq \{\mathbf{0}\}$, which contradicts (1) above. Hence $\sum_{n=1}^{k} \operatorname{codim}(\operatorname{ran} J_{y_n}) \leq n_0$.

Suppose now that $\sum_{n=1}^{k} \operatorname{codim}(\operatorname{ran} J_{y_n}) < n_0$. We shall check that, given n_0 linearly independent functions $g_1, \ldots, g_{n_0}$ in $C(Y, F)$, there exists a nonzero linear combination in the range of T. This fact implies that the codimension of the range of T is strictly less than n_0, which is impossible.

Let us define the linear mappings

$$\lambda : \mathbf{K}^{n_0} \longrightarrow \operatorname{span}\{g_1, \ldots, g_{n_0}\}$$

by $\lambda(\gamma_1, \ldots, \gamma_{n_0}) := \sum_{j=1}^{n_0} \gamma_j g_j$ for all $(\gamma_1, \ldots, \gamma_{n_0}) \in \mathbb{K}^{n_0}$. Next, for $i \in \{1, 2, \ldots, k\}$, consider

$$\mu_i : C(Y, F) \longrightarrow F / \operatorname{ran} J_{y_i}$$

where $\mu_i(g) := g(y_i) + \operatorname{ran} J_{y_i}$ for all $g \in C(Y, F)$, and finally let

$$\mu : C(Y, F) \longrightarrow (F / \operatorname{ran} J_{y_1}) \times \cdots \times (F / \operatorname{ran} J_{y_k}),$$

where $\mu(g) := (\mu_1(g), \ldots, \mu_k(g))$ for all g. As a consequence, $\mu \circ \lambda$ turns out to be a linear mapping from a n_0-dimensional space to a space whose dimension is $\sum_{i=1}^{k} n_i < n_0$. It is apparent that $\mu \circ \lambda$ is not injective. Thus there exists $(\gamma_1, \ldots, \gamma_{n_0}) \in \mathbb{K}^{n_0} \setminus \{(0, \ldots, 0)\}$ such that $(\mu \circ \lambda)(\gamma_1, \ldots, \gamma_{n_0}) = \mathbf{0}$. This means that $(\mu_i \circ \lambda)(\gamma_1, \ldots, \gamma_{n_0}) = \mathbf{0} + \operatorname{ran} J_{y_i}$ for each $i \in \{1, \ldots, k\}$, which is to say that $\sum_{j=1}^{n_0} \gamma_j g_j(y_i) \in \operatorname{ran} J_{y_i}$ for all $i \in \{1, \ldots, k\}$. Taking into account the definition of Y_N, we see by Proposition 4.1 that $\sum_{j=1}^{n_0} \gamma_j g_j \in \operatorname{ran} T$, as was to be proved. $\square$

Contrary to what could be expected in principle, the points of Y_N need not be isolated, as the following example shows.

Example 4.1 Let $X = Y := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and let $h : Y \longrightarrow X$ be the identity map. Given $f \in C(X, \ell^2)$, we define

$$(Tf)\left(\frac{1}{n}\right) := (\lambda_n^n, \lambda_1^n, \lambda_2^n, \ldots, \lambda_{n-1}^n, \lambda_{n+1}^n, \ldots),$$

where $f(1/n) := (\lambda_1^n, \lambda_2^n, \ldots, \lambda_{n-1}^n, \lambda_n^n, \lambda_{n+1}^n, \ldots)$. Also, if

$$f(0) = (\lambda_1^0, \lambda_2^0, \ldots, \lambda_{n-1}^0, \lambda_n^0, \lambda_{n+1}^0, \ldots),$$

then define

$$(Tf)(0) := (0, \lambda_1^0, \lambda_2^0, \ldots, \lambda_{n-1}^0, \lambda_n^0, \lambda_{n+1}^0, \ldots),$$

so that Tf belongs to $C(Y, \ell^2)$.

It is clear that T is a linear isometry where $J_{\frac{1}{n}} : \ell^2 \longrightarrow \ell^2$ turns out to be $J_{\frac{1}{n}}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n, \lambda_{n+1}, \ldots) = (\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots)$. On the other hand $J_0(\mathbf{e}_n) = \mathbf{e}_{n+1}$ for all $n \in \mathbb{N}$, and J_0 is a codimension 1 linear isometry on ℓ^2. Consequently T is a codimension 1 linear isometry, where the constant function $\widehat{\mathbf{e}_1}$ does not belong to the range of T. In this case, $Y_N = \{0\} \in Y$, which is not isolated.

5 The finite-dimensional case.

From now on, we shall assume that $m := \dim F < \infty$.

Lemma 5.1 *Suppose that $x \in X$ and $G_x = \{y_1, \ldots, y_{n_x}\}$. Then the mapping $Q_x : E \longrightarrow F^{n_x}$, defined by*

$$Q_x(\mathbf{e}) := ((T\mathbf{e})(y_1), \ldots, (T\mathbf{e})(y_{n_x}))$$

for all $\mathbf{e} \in E$, is a linear isometry if F^{n_x} is endowed with the sup norm $\|(\mathbf{f}_1, \ldots, \mathbf{f}_{n_x})\|_\infty = \max_{1 \leq i \leq n_x} \|\mathbf{f}_i\|$.

Proof. Fix $\mathbf{e} \in E$ with $\|\mathbf{e}\| = 1$. Since T is an isometry, $\|Q_x(\mathbf{e})\| \leq 1$, so we must see that there exists $i \in \{1, \ldots, n_x\}$ with $\|J_{y_i}(\mathbf{e})\| = 1$. Obviously, if some y_i belongs to Y_{10}, then J_{y_i} is an isometry and we are done.

Consequently, we suppose that $G_x \cap Y_{10} = \emptyset$. This implies that $x \notin \overline{h}(Y_{10})$ and, since Y_{10} is compact, x is isolated in X. Hence the characteristic function $f := \chi_{\{x\}}\mathbf{e}$ is continuous. As $f \equiv 0$ on $X \setminus \{x\}$, it is clear that $Tf \equiv 0$ on $\overline{h}^{-1}(X) \setminus \overline{h}^{-1}(x)$, which is to say that there must exist $y \in G_x \cup Y_2$ such that $\|(Tf)(y)\| = \|Tf\|_\infty = 1$. If we suppose that $y \in Y_2$, then there exists $f' \in C(X, E)$ with $f'(x) = 0$ and $(Tf')(y) \neq 0$. Without loss of generality, we shall assume that $\|f'\|_\infty = 1$. Hence $\|f + f'\|_\infty = 1 = \|f - f'\|_\infty$. However, as F is strictly convex, we have $\|(Tf)(y) + (Tf')(y)\| > 1$ or $\|(Tf)(y) - (Tf')(y)\| > 1$, which contradicts the isometric property of T. As a consequence, Tf attains its maximum in G_x, which is to say that there exists $i \in \{1, \ldots, n_x\}$ with $\|J_{y_i}(\mathbf{e})\| = \|(Tf)(y_i)\| = 1$, as we wanted to see.

□

Next we deduce the relationship between the sets A_0 and A_1 introduced in Section 2.

Corollary 5.1 *A_0 is contained in A_1.*

Proof. Let $x_0 \in A_0$ and $y_0 \in F_{x_0}$ with J_{y_0} not a surjective isometry, which, in this finite-dimensional case, means that it is not an isometry. If $x_0 \notin A_1$, then $G_{x_0} = F_{x_0} = \{y_0\}$, and Lemma 5.1 easily leads to a contradiction. $\square$

Proposition 5.1 *Let Y be infinite. Suppose that $g \in C(Y, F)$ satisfies $g(\overline{h}^{-1}(A_1)) \equiv 0$. Then there exists a unique $f \in C(X, E)$ such that $Tf \equiv g$ on Y_1.*

Proof. Define the function $f \in C(X, E)$ as follows:

- $f(x) := \mathbf{0}$ for $x \in A_1$.
- $f(x) := (J_{\overline{h}^{-1}(x)})^{-1}(g(\overline{h}^{-1}(x)))$ if $x \notin A_1$.

We first check that f is well-defined outside A_1, that is, $J_{\overline{h}^{-1}(x)}$ is a surjective isometry. Let $x \notin A_1$. Then $\overline{h}^{-1}(x) = h^{-1}(x)$ because $G_x = F_x$. Also, by Corollary 5.1, $x \notin A_0$, so $J_{h^{-1}(x)} : E \longrightarrow F$ is a surjective isometry.

Next we study the continuity of f. Let $x_0 \in X \setminus A_1$ and $\epsilon > 0$. We consider an open neighborhood V_1 of $h^{-1}(x_0)$ in Y such that, for all $y \in V_1$,

$$\|g(y) - g(h^{-1}(x_0))\| < \frac{\epsilon}{2}.$$

With no loss of generality, we can assume that $V_1 \subset Y_{10}$ because $h^{-1}(x_0) \in Y_{10} \setminus \overline{h}^{-1}(A_1)$ and this set is open being Y_{10} clopen by Lemma 3.7. Also, since $\overline{h}^{-1}(A_1)$ is finite, V_1 can be taken such that $\mathrm{cl}(V_1) \cap \overline{h}^{-1}(A_1) = \emptyset$.

We can rewrite the above inequality as

$$\|J_y(f(h(y))) - J_{h^{-1}(x_0)}(f(x_0))\| < \frac{\epsilon}{2}$$

for all $y \in V_1$.

On the other hand, since $Y_{10} \subset Y_0$ is clopen and $J : Y_0 \longrightarrow L(E, F)$ is continuous with respect to the strong operator topology, we can take an open neighborhood V_2 of $h^{-1}(x_0)$ with $V_2 \subset Y_{10}$ such that

$$\|J_y(f(x_0)) - J_{h^{-1}(x_0)}(f(x_0))\| < \frac{\epsilon}{2}$$

for all $y \in V_2$. We thus deduce that if $y \in V_1 \cap V_2$, then

$$\|J_y(f(h(y))) - J_y(f(x_0))\| < \epsilon$$

that is,

$$\|J_y[f(h(y)) - f(x_0)]\| < \epsilon.$$

But as $y \in Y_{10}$, J_y is an isometry, and consequently,

$$\|f(h(y)) - f(x_0)\| < \epsilon \tag{2}$$

for all $y \in V_1 \cap V_2$. Hence, in order to obtain the continuity of f at $x_0 \in X \setminus A_1$, it suffices to notice that sets of the form $h(V_1 \cap V_2)$ are open neighborhoods of x_0.

Let us now study the continuity of f on A_1. To this end, fix $x_0 \in A_1$. Since A_1 is a finite set, there exists an open neighborhood U of x_0 such that $U \cap A_1 = \{x_0\}$.

Suppose that f is not continuous at x_0. Then there exist $\epsilon > 0$ and a net (x_α) in U which converges to x_0 such that $\|f(x_\alpha)\| \geq \epsilon$ for all α. Since each element of the net x_α belongs to $X \setminus A_1$, we infer that $\overline{h}^{-1}(x_\alpha)$ is a singleton in Y_{10}. Furthermore, as Y_{10} is compact, there exists a subnet $\overline{h}^{-1}(x_\beta)$ convergent to a certain $y_0 \in Y_{10}$. Since $\overline{h}$ is continuous, we deduce that (x_β) converges to $\overline{h}(y_0)$ and, as a consequence, that $\overline{h}(y_0) = x_0$. This fact yields $y_0 \in \overline{h}^{-1}(A_1)$. By hypothesis, $g(y_0) = \mathbf{0}$. However, each $J_{\overline{h}^{-1}(x_\beta)}$ is an isometry and, by the definition of f,

$$g(\overline{h}^{-1}(x_\beta)) = J_{\overline{h}^{-1}(x_\beta)}(f(x_\beta)).$$

Hence $\|g(\overline{h}^{-1}(x_\beta))\| \geq \epsilon$ for all β. This implies that g is not continuous at y_0, a contradiction, which completes the proof of the continuity of f. The rest of the proof is apparent. □

Proof of Theorem 2.3. Put $A_1 = \{x_1, x_2, \ldots, x_k\}$ and, for each $x_i \in A_1$ (see Lemmas 3.3 and 3.4), let

$$G_{x_i} = \{y(x_i, 1), \ldots, y(x_i, n_i)\}.$$

By Corollary 3.1, for each $i \in \{1, 2, \ldots, k\}$ and each $j \in \{1, 2, \ldots, n_i\}$ we can consider an open neighborhood $U(i, j)$ of $y(x_i, j)$ such that $U(i, j) \subset Y_1$ and

$U(i,j) \cap U(i',j') = \emptyset$ if $(i,j) \neq (i',j')$. For each pair (i,j) we choose a function $g_{(i,j)} \in C(Y)$ such that $g_{(i,j)}(y(x_i,j)) = 1 = \|g_{(i,j)}\|_\infty$ and $c(g_{(i,j)}) \subset U(i,j)$.

Note that, since Y is infinite, the set $Y_{10} \setminus \overline{h}^{-1}(A_0)$ is nonempty, which easily leads to $\dim E = \dim F$. Now, by Lemma 5.1, each mapping $Q_{x_i} : E \longrightarrow F^{n_i}$ is an isometry, so $m := \dim F = \dim Q_{x_i}(E)$. Hence we can find $m(n_i - 1)$ linearly independent vectors in F^{n_i} of the form

$$\Im(i,l) := (\mathbf{f}(i,l,1), \mathbf{f}(i,l,2), \ldots, \mathbf{f}(i,l,n_i))$$

for $l = 1, \ldots, m(n_i - 1)$ such that

$$F^{n_i} = \operatorname{ran} Q_{x_i} \bigoplus \operatorname{span}\{\Im(i,1), \ldots, \Im(i, m(n_i - 1))\}. \tag{3}$$

Next we define, for each $i \in \{1, 2, \ldots, k\}$, $m(n_i - 1)$ functions in $C(Y,F)$ related to $\Im(i,j)$ and $g_{(i,j)}$ of the form

$$\aleph_{[i,l]} := \sum_{j=1}^{n_i} g_{(i,j)} \mathbf{f}(i,l,j)$$

for $l = 1, \ldots, m(n_i - 1)$.

Note that, for $i \in \{1, 2, \ldots, k\}$ and each $l \in \{1, 2, \ldots, m(n_i - 1)\}$, we have $\aleph_{[i,l]}(Y_2 \cup Y_3) \equiv \mathbf{0}$, and if $i' \neq i$, $i' \in \{1, 2, \ldots, k\}$, then $\aleph_{[i,l]}(G_{x_{i'}}) \equiv \mathbf{0}$, and, for $j \in \{1, 2, \ldots, n_i\}$,

$$\aleph_{[i,l]}(y(x_i,j)) = \mathbf{f}(i,l,j). \tag{4}$$

Now assume that $Y_2 := \{z_1, \ldots, z_t\}$ and $Y_3 := \{w_1, \ldots, w_s\}$ (see Lemmas 3.5, 3.6 and 3.7). For every $i \in \{1, 2, \ldots, t\}$ and every $l \in \{1, 2, \ldots, m\}$ we can consider $\Xi_{[i,l]} := \chi_{\{z_i\}} \mathbf{b}_l \in C(Y,F)$ where $B := \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ is a basis of F. In like manner, we can define, for every $i \in \{1, 2, \ldots, s\}$ and every $l \in \{1, 2, \ldots, m\}$, $\Upsilon_{[i,l]} := \chi_{\{w_i\}} \mathbf{b}_l \in C(Y,F)$.

We now claim that the functions we have just introduced are linearly independent. To this end, suppose that

$$\sum_{i,l} \alpha(i,l) \aleph_{[i,l]} + \sum_{i,l} \beta(i,l) \Xi_{[i,l]} + \sum_{i,l} \gamma(i,l) \Upsilon_{[i,l]} \equiv 0 \in C(Y,F).$$

If we evaluate this sum at the point $z_i \in Y_2$, then we get

$$\sum_{l=1}^{m} \beta(i,l) \mathbf{b}_l = \mathbf{0} \in F.$$

As $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is a basis of F, we infer that each $\beta(i,l) = 0$. Similarly, by evaluating the above sum at each point of Y_3, we conclude that $\gamma(i,l) = 0$ for each $i \in \{1, 2, \ldots, s\}$ and $l \in \{1, 2, \ldots, m\}$.

On G_{x_i} the above sum turns out to be

$$\sum_{l=1}^{m(n_i-1)} \alpha(i,l)\aleph_{[i,l]} \equiv 0 \in C(Y, F).$$

Taking into account equality (4), this means that for each $y(x_i, j)$, $1 \leq j \leq n_i$,

$$\sum_{l=1}^{m(n_i-1)} \alpha(i,l)\mathbf{f}(i,l,j) \equiv \mathbf{0}, \tag{5}$$

so $\sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Im(i,l) = \mathbf{0} \in F^{n_i}$. As a consequence, all the $\alpha(i,l)$ are zero because all vectors $\Im(i,l)$ are linearly independent.

Claim 5.1 *The function*

$$g := \sum_{i,l} \alpha(i,l)\aleph_{[i,l]} + \sum_{i,l} \beta(i,l)\Xi_{[i,l]} + \sum_{i,l} \gamma(i,l)\Upsilon_{[i,l]}$$

does not belong to the range of T, except when $g \equiv 0$.

Suppose that there exists $f \in C(X, E)$ with $Tf = g$. This yields, by the definition of Y_3, that each $\gamma(i,l)$ is zero. We shall check that all $\alpha(i,l)$ are zero. Fix $i \in \{1, \ldots, k\}$. Given $j \in \{1, 2, \ldots, n_i\}$, we have

$$g(y(x_i, j)) = J_{y(x_i,j)}(f(x_i)).$$

On the other hand, by equality (4),

$$\begin{aligned} g(y(x_i,j)) &= \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\aleph_{[i,l]}(y(x_i,j)) \\ &= \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\mathbf{f}(i,l,j) \in F, \end{aligned}$$

which implies that

$$Q_{x_i}(f(x_i)) = \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Im(i,l) \in F^{n_i}.$$

Since

$$\operatorname{ran} Q_{x_i} \cap \operatorname{span}\{\Im(i,1),\dots,\Im(i,m(n_i-1))\} = \{\mathbf{0}\},$$

we have $Q_{x_i}(f(x_i)) = \mathbf{0} \in F^{n_i}$, and consequently $\alpha(i,l)$ is zero for all l. Summarizing, $g \equiv 0$ on Y_1, implying that $g \equiv 0$ on Y_2. This completes the proof of the claim.

Gathering the information obtained so far, we deduce that the vectors

$$\aleph_{[i,l]} + \operatorname{ran} T, \Xi_{[i,l]} + \operatorname{ran} T, \Upsilon_{[i,l]} + \operatorname{ran} T,$$

are linearly independent in the space $C(Y,F)/\operatorname{ran} T$. In order to finish the proof, it suffices to check that, given $g \in C(Y,F)$, there exist scalars $\alpha(i,j), \beta(i,j), \gamma(i,j)$ such that

$$g - \sum_{i,l} \alpha(i,l)\aleph_{[i,l]} + \sum_{i,l} \beta(i,l)\Xi_{[i,l]} + \sum_{i,l} \gamma(i,l)\Upsilon_{[i,l]}$$

belongs to the range of T.

For each $i \in \{1,2,\dots,k\}$ we consider the vector

$$N_i := (g(y(x_i,1)), g(y(x_i,2)), \dots, g(y(x_i,n_i))) \in F^{n_i}.$$

Then, by equality (3), there exist $\mathbf{e}_i \in E$ and constants $\alpha(i,1),\dots,\alpha(i,m(n_i-1))$ such that

$$N_i = Q_{x_i}(\mathbf{e}_i) + \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Im(i,l).$$

Hence, if we fix $j \in \{1,2,\dots,n_i\}$, then, by equality (4),

$$\begin{aligned} g(y(x_i,j)) &= (T\mathbf{e}_i)(y(x_i,j)) + \sum_{l=1}^{m(n_i-1)} \alpha(i.l)\mathbf{f}(i,l,j) \\ &= (Tf_i)(y(x_i,j)) + \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\aleph_{[i,l]}(y(x_i,j)) \in F, \end{aligned}$$

where $f_i \in C(X,E)$ with $f_i(x_i) = \mathbf{e}_i$ and $f_i(x_{i'}) = \mathbf{0}$ for $i \neq i'$. If we do so for each $i \in \{1,2,\dots,k\}$ and each $j \in \{1,2,\dots,n_i\}$, we obtain k functions $f_i \in C(X,E)$ such that, for $i_0 \in \{1,2,\dots,k\}$ and $j_0 \in \{1,2,\dots,n_{i_0}\}$,

$$g(y(x_{i_0},j_0)) = \sum_{i=1}^{k} (Tf_i)(y(x_{i_0},j_0)) + \sum_{i,l} \alpha(i,l)\aleph_{[i,l]}(y(x_{i_0},j_0)).$$

Therefore, the function

$$g_0 := g - \sum_{i=1}^{k} Tf_i - \sum_{i,l} \alpha(i,l) \aleph_{[i,l]}$$

vanishes on each $y(x_i, j)$, which is to say, on $\overline{h}^{-1}(A_1)$. By Proposition 5.1, there exists $f_0 \in C(X, E)$ such that $Tf_0 \equiv g_0$ on Y_1. Hence there exist certain constants $\beta(i,l)$ and $\gamma(i,l)$ such that

$$g_0 - Tf_0 - \sum_{i,l} \beta(i,l) \Xi_{[i,l]} - \sum_{i,l} \gamma(i,l) \Upsilon_{[i,l]} \equiv 0$$

on $Y_2 \cup Y_3$ and, consequently, on Y. That is,

$$g - \sum_{i=1}^{k} Tf_i - Tf_0 - \sum_{i,l} \alpha(i,l) \aleph_{[i,l]} - \sum_{i,l} \beta(i,l) \Xi_{[i,l]} - \sum_{i,l} \gamma(i,l) \Upsilon_{[i,l]} \equiv 0$$

on Y. We now easily complete the proof of the theorem. □

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