

On the Darboux integrability of a cubic CRN model in \mathbb{R}^5

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Abstract

We study the Darboux integrability of a differential system in \mathbb{R}^5 with parameters coming from a chemical reaction model. In particular, we find all its Darboux polynomials and exponential factors and we prove that it is not Darboux integrable.

Keywords. Darboux polynomial; exponential factor; Darboux integrability; chemical reaction network

1 Introduction and statement of the main result

Consider an n -dimensional polynomial differential system of degree $d \in \mathbb{N}$

$$\dot{\mathbf{x}} = P(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad (1.1)$$

where $P(\mathbf{x}) = (P_1(\mathbf{x}), \dots, P_n(\mathbf{x}))$, $P_i \in \mathbb{C}[\mathbf{x}]$, and the dot denotes derivative with respect to the independent variable t .

A function $H(\mathbf{x})$ is a *first integral* of system (1.1) if it is continuous and defined in a full Lebesgue measure subset $\Omega \subseteq \mathbb{R}^n$, is not locally constant on any positive Lebesgue measure subset of Ω and moreover is constant along each orbit in Ω of system (1.1). If H is \mathcal{C}^1 and we name \mathcal{X} the vector field associated to system (1.1), then we have

$$\mathcal{X}(H) = P_1 \frac{\partial H}{\partial x_1} + \dots + P_n \frac{\partial H}{\partial x_n} = 0.$$

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Mathematics Subject Classification (2010): 34C05, 34A34, 34C14

System (1.1) is C^k -completely integrable in Ω if it has $n - 1$ functionally independent C^k first integrals in Ω . Recall that k functions $H_1(\mathbf{x}), \dots, H_k(\mathbf{x})$ are *functionally independent* in Ω if the matrix of gradients $(\nabla H_1, \dots, \nabla H_k)$ has rank k in a full Lebesgue measure subset of Ω .

For an n -dimensional system of differential equations the existence of some first integrals reduces the complexity of its dynamics and the existence of $n - 1$ functionally independent first integrals solves completely the problem (at least theoretically) of determining its phase portrait. In general for a given differential system it is a difficult problem to determine the existence or non-existence of first integrals.

During recent years the interest in the study of the integrability of differential equations has attracted much attention from the mathematical community. Darboux theory of integrability plays a central role in the integrability of the polynomial differential systems since it gives a sufficient condition for the integrability inside a wide family of functions. We highlight that it works for real or complex polynomial differential systems and that the study of complex algebraic solutions is necessary for obtaining all the real first integrals of a real polynomial differential system.

A *Darboux polynomial* of (1.1) is a polynomial $f \in \mathbb{C}[\mathbf{x}]$ such that

$$\mathcal{X}(f) = P_1 \frac{\partial f}{\partial x_1} + \dots + P_n \frac{\partial f}{\partial x_n} = kf,$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $k \in \mathbb{C}[\mathbf{x}]$, which is called the *cofactor* of f , has degree at most $d - 1$, where $d = \max\{\deg P_1, \dots, \deg P_n\}$ is the degree of system (1.1). An *invariant algebraic surface* is a surface given by $f = 0$. Note that it is invariant by the dynamics in the sense that if a trajectory starts on the surface it does not leave it.

An *exponential factor* of (1.1) is a function $F = \exp(g/f)$, with $f, g \in \mathbb{C}[\mathbf{x}]$, such that

$$\mathcal{X}(F) = P_1 \frac{\partial F}{\partial x_1} + \dots + P_n \frac{\partial F}{\partial x_n} = LF,$$

where $L \in \mathbb{C}[\mathbf{x}]$, which is called the *cofactor* of F , has degree at most $d - 1$. It is widely known that in this case f is a Darboux polynomial of (1.1) and that $\mathcal{X}(g) = kg + Lf$, where k is the cofactor of f .

A Darboux first integral H has the form

$$H = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}, \quad (1.2)$$

where f_1, \dots, f_p are Darboux polynomials, F_1, \dots, F_q are exponential factors and λ_i, μ_j are complex numbers, for $i = 1, \dots, p$ and $j = 1, \dots, q$.

The following result, proved in [7] explains how to find Darboux first integrals.

Proposition 1.1. *Assume that a polynomial differential system of degree m admits p Darboux polynomials f_i with cofactors k_i , $i = 1, \dots, p$, and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j , $j = 1, \dots, q$. Then, there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^p \lambda_i k_i + \sum_{j=1}^q \mu_j L_j = 0$$

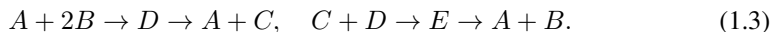
if and only if the function given in (1.2) is a Darboux first integral of the polynomial differential system.

The Darboux theory of integrability relates the number of Darboux polynomials and exponential factors of the differential system with the existence of a Darboux first integral, see for example [11]. We recall that a Darboux first integral is a product of complex powers of Darboux polynomials and exponential factors.

The main aim in this paper is to study the Darboux integrability of a cubic differential system that belongs to \mathbb{R}^5 and has an important contribution in Chemical Reaction Network Theory (CRNT). A chemical reaction network $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ is defined as a set of species \mathcal{S} , a set of complexes \mathcal{C} and a set of reactions \mathcal{R} between complexes; each complex is a combination of species. It is here assumed that a reaction occurs according to mass-action kinetics, that is, at a rate proportional to the product of the species concentrations in the reactant or source complex. The set of reactions together with a rate vector give rise to a polynomial system of ordinary differential equations. We refer the reader to [8, 9, 10] for more information about CRNT. For a concrete system of chemical reactions the parameter and state spaces are typically high-dimensional and one uses numerical methods to analyze the solutions. Due to high computational complexity this can be done only for a small set of values of system's parameters. Thus instead of studying quantitative aspects of the dynamics, recently there has been an increasing interest in studying *qualitative* properties of the CRN. For example in [1, 2, 3, 4, 5, 6] the authors considered the question of existence of single versus multiple steady states (also referred to as multistationary). The existence of first integrals of a polynomial differential system describing a CRN often provides essential

qualitative information (the level sets are invariant under the flow) about the solution or can be used to reduce the dimension of the total state space. Since the computation of nonlinear conservation laws (first integrals) is highly nontrivial, most of the known results related to the CRN dynamics provide only linear first integrals.

In this paper, our purpose is to show, by following an example (see system (1.4) below), how to apply the Darboux theory of integrability to obtain nontrivial and nonlinear algebraic and Darboux first integrals. Indeed, we consider the following reaction network appearing in [10]:



By employing the common assumption that reaction rates are of mass-action type, the concentrations change with time according to the following system of ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= -c_1 x_1 x_2^2 + c_2 x_4 + c_4 x_5, \\ \dot{x}_2 &= -2c_1 x_1 x_2^2 + c_4 x_5, \\ \dot{x}_3 &= c_2 x_4 - c_3 x_3 x_4, \\ \dot{x}_4 &= c_1 x_1 x_2^2 - c_2 x_4 - c_3 x_3 x_4, \\ \dot{x}_5 &= c_3 x_3 x_4 - c_4 x_5, \end{aligned} \quad (1.4)$$

where c_1, c_2, c_3, c_4 are positive constants. This is a good example to work with since the system belongs to \mathbb{R}^5 , it is cubic and some nice objects are found, such as six exponential factors and a Darboux first integral. In [10] the positive steady-state solutions of this system are studied. Here we go deeper in the study of this system by studying its Darboux integrability. We believe that the techniques used in this paper, such as reduction of one dimension of the system, can be used for studying other CRN systems and, in general, polynomial differential systems of high dimension.

We shall deal in this paper with this differential system. We shall study its Darboux integrability by characterizing its Darboux polynomials and exponential factors. In the following theorem, which is our main result, we prove that there just exist two Darboux first integrals (one polynomial and one Darboux), one invariant algebraic surface of degree one and six exponential factors. As far as we know in all papers dealing with CRN models, the authors search only linear first integrals. More complicated first integrals demand a

deeper study, and in general, are very difficult to detect if the system is not Hamiltonian, which is the case. Note that we are finding a first integral (which is not linear) and one invariant algebraic surface, i.e., two new invariant objects which provide some light in the reduction of the dimension of the system and on its qualitative behavior.

Theorem 1.2. *The following results hold for system (1.4).*

- (a) *The unique irreducible polynomial first integral is $H_1 = x_1 + x_4 + x_5$. Any other polynomial first integral is a polynomial function of H_1 .*
- (b) *The unique irreducible Darboux polynomial is $F = c_2 - c_3x_3$. It has cofactor $k = -c_3x_4$.*
- (c) *It has six exponential factors: $F_1 = e^{x_3}$, $F_2 = e^{x_2-2x_1}$, $F_3 = e^{x_1+x_4}$, $F_4 = e^{(x_2-2x_1)^2}$, $F_5 = e^{(2x_1-x_2)(x_1-x_3+x_4)}$ and $F_6 = e^{(x_1-x_3+x_4)^2}$. If $e^{g/h}$ is another exponential factor, then $h \in \mathbb{C}[H_1]$ and*

$$g(x_1, x_2, x_3, x_4) = a_1x_3 + a_2(x_2 - 2x_1) + a_3(x_1 + x_4) + a_4(x_2 - 2x_1)^2 + a_5(x_2 - 2x_1)(x_1 - x_3 + x_4) + a_6(x_1 - x_3 + x_4)^2, \quad (1.5)$$

with $a_i \in \mathbb{C}$, $i = 1, \dots, 6$.

- (d) *It has the (non-rational) Darboux first integral*

$$H_2 = F^{3c_2/c_3} e^{-(x_1+x_4)} e^{-(x_2-2x_1)} e^{x_3} = F^{3c_2/c_3} e^{x_1-x_2+x_3-x_4}.$$

- (e) *It is not Darboux completely integrable.*

As far as we know, in the papers about CRN only linear first integrals are searched, since more complicated first integrals demand a deeper study. Our aim in this paper is to provide an example where we find additional (not linear) first integrals in order to give some light in the reduction of the dimension of the system.

In order for the reader to follow the proofs of the paper that sometimes demand large expressions, we have added an appendix with a worksheet of mathematica with the computations that solve the PDE appearing in the paper.

2 Proof of the Theorem 1.2

Statement (e) follows immediately from statements (a)-(d), since there is no way to construct two additional Darboux first integrals functionally independent of H_1, H_2 . This assertion follows because there is no non-zero linear combination of the cofactors of the Darboux polynomials and exponential factors of the system equal to zero. In particular, it is clear that the system has not rational first integrals. Hence, we need to prove only statements (a), (b), (c) and (d). We shall prove them separately.

2.1 Proof of statement (a)

Straightforward computations show that H_1 is a first integral of (1.4). The restriction of system (1.4) to $H_1 = h$ is the differential system

$$\begin{aligned}\dot{x}_1 &= -c_1x_1x_2^2 - c_4x_1 + (c_2 - c_4)x_4 + c_4h, \\ \dot{x}_2 &= -2c_1x_1x_2^2 - c_4x_1 - c_4x_4 + c_4h, \\ \dot{x}_3 &= (c_2 - c_3x_3)x_4, \\ \dot{x}_4 &= c_1x_1x_2^2 - c_2x_4 - c_3x_3x_4.\end{aligned}\tag{2.6}$$

Let \mathcal{Y}_h be the corresponding vector field. For simplicity we shall write \mathcal{Y} instead of \mathcal{Y}_h because there is no possible confusion. Next lemma shows that (2.6) has no polynomial first integrals.

Lemma 2.1. *System (2.6) has no polynomial first integrals.*

Proof. Let $g(x_1, x_2, x_3, x_4)$ be a polynomial first integral of degree $m \in \mathbb{N}$ of system (2.6). We write $g = \sum_{i=1}^m g_i(x_1, x_2, x_3, x_4)$, where g_i is a homogeneous polynomial of degree i , with $g_m \neq 0$. The differential equation corresponding to the terms of degree $m + 2$ of $\mathcal{Y}(g) = 0$ is

$$-c_1x_1x_2^2 \left(\frac{\partial g_m}{\partial x_1} + 2\frac{\partial g_m}{\partial x_2} - \frac{\partial g_m}{\partial x_4} \right) = 0,$$

from which we obtain $g_m(x_1, x_2, x_3, x_4) = g_m(x_3, X_1, X_2)$, where we have introduced the variables $(X_1, X_2) = (x_2 - 2x_1, x_1 + x_4)$. Indeed we shall prove in a while that $g_m = g_m(X_1, X_3)$, where $X_3 = X_2 - x_3$.

Concerning the terms of degree $m + 1$ we have the differential equation

$$-c_1x_1x_2^2 \left(\frac{\partial g_{m-1}}{\partial x_1} + 2\frac{\partial g_{m-1}}{\partial x_2} - \frac{\partial g_{m-1}}{\partial x_4} \right) - c_3x_3x_4 \left(\frac{\partial g_m}{\partial x_3} + \frac{\partial g_m}{\partial x_4} \right) = 0,$$

from which we get

$$g_{m-1} = \frac{c_3 x_3}{c_1(x_2 - 2x_1)} \left(\frac{\partial g_m}{\partial x_3}(x_3, X_1, X_2) + \frac{\partial g_m}{\partial X_2}(x_3, X_1, X_2) \right) \times \\ \left(\frac{x_1 + x_4}{x_2 - 2x_1} \log \frac{x_2}{4x_1} + \frac{x_2 + 2x_4}{2x_2} \right) + \bar{g}_{m-1}(x_3, x_2 - 2x_1, x_1 + x_4),$$

where \bar{g}_{m-1} is an arbitrary function. Since the logarithm must be removed, we have

$$g_m = g_m(X_1, X_3),$$

where we have introduced $X_3 = X_2 - x_3$, as we stated above. Hence we get $g_{m-1} = g_{m-1}(x_3, X_1, X_2)$. Next we deal with the differential equation corresponding to the terms of degree m . We obtain

$$g_{m-2} = \frac{X_2 \log \frac{x_2}{4x_1}}{c_1 X_1^2} \left((2c_2 - c_4) \frac{\partial g_m}{\partial X_1} + (c_2 + c_4) \frac{\partial g_m}{\partial X_2} + c_3 x_3 \left(\frac{\partial g_{m-1}}{\partial x_3} + \frac{\partial g_{m-1}}{\partial X_2} \right) \right) \\ + G_{m-2} + \bar{g}_{m-2}(x_3, X_1, X_2),$$

where G_{m-2} is a rational function and \bar{g}_{m-2} is an arbitrary function. We must remove the logarithm, hence a PDE must be solved. We obtain

$$g_{m-1}(x_3, X_1, X_2) = - \frac{(c_2 + c_4) \frac{\partial g_m}{\partial X_3} + (2c_2 - c_4) \frac{\partial g_m}{\partial X_1}}{c_3} \log x_3 + \bar{g}_{m-1}(X_1, X_3).$$

A new logarithm appears. To remove it we must take

$$g_m(X_1, X_3) = ((c_2 + c_4)X_1 + (c_4 - 2c_2)X_3)^m,$$

and therefore $g_{m-1}(x_3, X_1, X_2) = g_{m-1}(X_1, X_3)$. Now back to the expression of g_{m-2} we have

$$g_{m-2} = \frac{3c_2 c_4 m}{2c_1} \frac{((c_2 + c_4)X_1 + (c_4 - 2c_2)X_3)^{m-1}}{x_2} + \bar{g}_{m-2}(x_3, X_1, X_2).$$

Since g_{m-2} is to be a polynomial, $x_2 \nmid ((c_2 + c_4)X_1 + (c_4 - 2c_2)X_3)$ and $c_i > 0$ for all i , we have $m = 0$. Then g is a constant and the lemma follows. \square

Remark 2.2. The sequence of resolution in the proof of Lemma 2.1 will be used later on for other purposes.

After Lemma 2.1 we can prove statement (a) of Theorem 1.2. Let f be a polynomial first integral of (1.4) which is not a function of H_1 . Note that in view of Lemma 2.1,

when $H_1 = h$, the restricted system has no polynomial first integrals, so a polynomial first integral f if it exists, when $H_1 = h$ must be zero. By the Nunstelletstaz theorem, there exists a power j (which may be zero) so that $H_1 - h$ must divide f , so we write $f = (H_1 - h)^j \bar{H}$, where $j \in \mathbb{N} \cup \{0\}$ and $(H_1 - h) \nmid \bar{H}$, with \bar{H} a polynomial. Since $X(f) = 0$, we have $X(\bar{H}) = 0$. Let $g = \bar{H}|_{H_1=h} \neq 0$. Then $\mathcal{Y}(g) = 0$. By Lemma 2.1 we have $g \equiv 0$, which is a contradiction with the assumption $(H_1 - h) \nmid \bar{H}$. Hence such f cannot exist and therefore statement (a) of Theorem 1.2 follows.

2.2 Proof of statement (b)

We start the study of the Darboux polynomials of system (1.4) by simplifying the general expression of the cofactor of any Darboux polynomial.

Proposition 2.3. *Let f be a Darboux polynomial of degree $m \in \mathbb{N}$ of system (1.4) with cofactor k . Then $k = k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 + k_6x_1^2 + k_7x_1x_2$, where $k_i \in \mathbb{C}$. Moreover, $-k_6/c_1, -k_7/(2c_1) \in \mathbb{N} \cup \{0\}$.*

Proof. We write the general (quadratic) cofactor $k \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ as

$$\begin{aligned} k = & k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 + k_6x_1^2 + k_7x_1x_2 + k_8x_1x_3 \\ & + k_9x_1x_4 + k_{10}x_1x_5 + k_{11}x_2^2 + k_{12}x_2x_3 + k_{13}x_2x_4 + k_{14}x_2x_5 \\ & + k_{15}x_3^2 + k_{16}x_3x_4 + k_{17}x_3x_5 + k_{18}x_4^2 + k_{19}x_4x_5 + k_{20}x_5^2. \end{aligned}$$

Taking the homogeneous part of degree $m + 1$ of the equation $\mathcal{X}(f) = kf$ and using the Euler theorem of homogeneous functions for f_m we get the equation

$$\begin{aligned} - & \left[k_6x_1^2 + k_7x_1x_2 + k_8x_1x_3 + k_9x_1x_4 + k_{10}x_1x_5 + (c_1m + k_{11})x_2^2 + k_{12}x_2x_3 \right. \\ & + k_{13}x_2x_4 + k_{14}x_2x_5 + k_{15}x_3^2 + k_{16}x_3x_4 + k_{17}x_3x_5 + k_{18}x_4^2 + k_{19}x_4x_5 \\ & \left. + k_{20}x_5^2 \right] f_m + c_1x_2^2 \left[(x_2 - 2x_1) \frac{\partial f_m}{\partial x_2} + x_3 \frac{\partial f_m}{\partial x_3} + (x_1 + x_4) \frac{\partial f_m}{\partial x_4} + x_5 \frac{\partial f_m}{\partial x_5} \right] \\ = & 0. \end{aligned}$$

The general solution of this equation is

$$\begin{aligned} f_m(x_1, x_2, x_3, x_4, x_5) = & e^{\frac{x_1 P_1}{2c_1 x_2 (2x_1 - x_2)^2}} x_2^{\frac{P_2}{4c_1 (2x_1 - x_2)^2}} (2x_1 - x_2)^{m + \frac{P_3}{4c_1}} \\ & \times C_m \left(x_1, \frac{x_3}{2x_1 - x_2}, \frac{x_1 + x_4}{2x_1 - x_2}, \frac{x_5}{2x_1 - x_2} \right), \end{aligned}$$

where

$$\begin{aligned} P_1 = & 4k_{20}x_5^2 + 4k_{19}x_4x_5 + 4k_{18}x_4^2 + 4k_{17}x_3x_5 + 4k_{16}x_3x_4 + 4k_{15}x_3^2 \\ & + 2(k_{19} - k_{10})x_2x_5 + 2(2k_{18} - k_9)x_2x_4 + 2(k_{16} - k_8)x_2x_3 \\ & + 4k_{10}x_1x_5 + 4k_9x_1x_4 + 4k_8x_1x_3 + 2(k_9 - 2k_6)x_1x_2 + 4k_6x_1^2 \\ & + (k_{18} - k_9 + k_6)x_2^2; \end{aligned}$$

$$\begin{aligned} P_2 = & 4k_{20}x_5^2 + 4k_{19}x_4x_5 + 4k_{18}x_4^2 + 4k_{17}x_3x_5 + 4k_{16}x_3x_4 + 4k_{15}x_3^2 + 4k_{14}x_2x_5 \\ & + 4k_{13}x_2x_4 + 4k_{12}x_2x_3 + 4(k_9 - 2k_7 - k_6)x_1^2 \\ & + (-k_{18} + 2k_{13} + k_9 - 2k_7 - k_6)x_2^2 + 4(k_{19} - 2k_{14})x_1x_5 + 8(k_{18} - k_{13})x_1x_4 \\ & + 4(k_{16} - 2k_{12})x_1x_3 + 4(k_{18} - k_{13} - k_9 + 2k_7 + k_6)x_1x_2; \end{aligned}$$

$$P_3 = k_{18} - 2k_{13} + 4k_{11} - k_9 + 2k_7 + k_6;$$

and C_m is an arbitrary function. In order to get a polynomial the exponent of the exponential must be a constant and the exponents of x_2 and $2x_1 - x_2$ must be non-negative integers. Therefore we must take $k_{11} = -c_1n_1$ and $k_7 = -2c_1n_2$, where $n_1, n_2 \in \mathbb{N} \cup \{0\}$, and $k_6 = k_8 = k_9 = k_{10} = 0, k_{12} = \dots = k_{20} = 0$. We get

$$\begin{aligned} f_m(x_1, x_2, x_3, x_4, x_5) \\ = x_2^{n_2}(x_2 - 2x_1)^{m-n_1-n_2} C_m \left(x_1, \frac{x_3}{x_2 - 2x_1}, \frac{x_1 + x_4}{x_2 - 2x_1}, \frac{x_5}{x_2 - 2x_1} \right). \end{aligned}$$

Since this is to be a polynomial of degree m , we take

$$f_m(x_1, x_2, x_3, x_4, x_5) = x_1^{n_1} x_2^{n_2} (x_2 - 2x_1)^{m-n_1-n_2-n} P_n(x_3, x_1 + x_4, x_5),$$

where P_n is a homogeneous polynomial of degree $n \in \mathbb{N} \cup \{0\}$, $n \leq m - n_1 - n_2$.

Renaming the coefficients of k , the proposition follows. \square

Lemma 2.4. *The unique Darboux polynomial of degree one of system (1.4) is $c_2 - c_3x_3$. Its cofactor is $k = -c_3x_4$.*

Proof. It follows after easy computations. \square

Next lemma shows that there are no more Darboux polynomials, and thus it finishes the proof of statement (b) of Theorem 1.2.

Lemma 2.5. *System (1.4) has no irreducible Darboux polynomials of degree greater than one.*

Proof. Let \mathcal{F} be an irreducible Darboux polynomial of system (1.4) and let $f = \mathcal{F}_{H_1=h}$. We recall that $F = c_2 - c_3x_3$ is a Darboux polynomial of system (2.6). Consider system (2.6) restricted to $F = 0$:

$$\begin{aligned}\dot{x}_1 &= -c_1x_1x_2^2 - c_4x_1 + (c_2 - c_4)x_4 + c_4h, \\ \dot{x}_2 &= -2c_1x_1x_2^2 - c_4x_1 - c_4x_4 + c_4h, \\ \dot{x}_4 &= c_1x_1x_2^2 - 2c_2x_4.\end{aligned}\tag{2.7}$$

Note that f is an irreducible Darboux polynomial of system (2.6). Let g be the Darboux polynomial of (2.7) corresponding to f restricted to $F = 0$; that is, $g = f|_{F=0}$. Let $m \in \mathbb{N} \cup \{0\}$ be the degree of g . We have:

$$\begin{aligned}(-c_1x_1x_2^2 - c_4x_1 + (c_2 - c_4)x_4 + c_4h)\frac{\partial g}{\partial x_1} \\ + (-2c_1x_1x_2^2 - c_4x_1 - c_4x_4 + c_4h)\frac{\partial g}{\partial x_2} + (c_1x_1x_2^2 - 2c_2x_4)\frac{\partial g}{\partial x_4} \\ - (k_0 + k_1x_1 + k_2x_2 + k_4x_4 - c_1n_1x_2^2 - 2c_1n_2x_1x_2)g = 0.\end{aligned}\tag{2.8}$$

The expression of the cofactor of g can be deduced from Proposition 2.3. We write $g = \sum_{i=0}^m g_i(x, y)$, with g_i a homogeneous polynomial of degree i , $g_m \neq 0$. We shall prove that $m = 0$.

From (2.8), the equation of degree $m + 2$ becomes, after canceling a common factor c_1x_2 ,

$$-x_1x_2 \left(\frac{\partial g_m}{\partial x_1} + 2\frac{\partial g_m}{\partial x_2} - \frac{\partial g_m}{\partial x_4} \right) + (n_1x_2 + 2n_2x_1)g_m = 0.$$

Then $g_m = x_1^{n_1}x_2^{n_2}\bar{g}_m(x_2 - 2x_1, x_1 + x_4)$, with \bar{g}_m a homogeneous polynomial of degree $m - n_1 - n_2$. The equation of degree $m + 1$ of (2.8) is

$$\begin{aligned}-c_1x_1x_2^2 \left(\frac{\partial g_{m-1}}{\partial x_1} + 2\frac{\partial g_{m-1}}{\partial x_2} - \frac{\partial g_{m-1}}{\partial x_4} \right) + c_1x_2(n_1x_2 + 2n_2x_1)g_{m-1} \\ - (k_1x_1 + k_2x_2 + k_4x_4)x_1^{n_1}x_2^{n_2}\bar{g}_m(x_2 - 2x_1, x_1 + x_4) = 0,\end{aligned}$$

from which we obtain

$$\begin{aligned}g_{m-1} &= -\frac{2k_1x_1 - k_1x_2 + k_4x_2 + 2k_4x_4}{2c_1(x_2 - 2x_1)}x_1^{n_1}x_2^{n_2-1}\bar{g}_m(x_2 - 2x_1, x_1 + x_4) \\ &\quad - \frac{(2k_2 - k_4)x_1 - k_2x_2 - k_4x_4}{c_1(x_2 - 2x_1)^2}x_1^{n_1}x_2^{n_2}\bar{g}_m(x_2 - 2x_1, x_1 + x_4) \log \frac{x_2}{4x_1} \\ &\quad + x_1^{n_1}x_2^{n_2}\bar{g}_{m-1}(x_2 - 2x_1, x_1 + x_4),\end{aligned}$$

where \bar{g}_{m-1} is a homogeneous polynomial of degree $m - n_1 - n_2 - 1$. Since the logarithm must be removed, we have $k_2 = k_4 = 0$. Hence

$$g_{m-1}(x_1, x_2, x_4) = \frac{k_1}{2c_1} x_1^{n_1} x_2^{n_2-1} \bar{g}_m(x_2 - 2x_1, x_1 + x_4) \\ + x_1^{n_1} x_2^{n_2} \bar{g}_{m-1}(x_2 - 2x_1, x_1 + x_4).$$

The equation of degree m of (2.8) is

$$-c_1 x_1 x_2^2 \left(\frac{\partial g_{m-2}}{\partial x_1} + 2 \frac{\partial g_{m-2}}{\partial x_2} - \frac{\partial g_{m-2}}{\partial x_4} \right) + c_1 x_2 (n_1 x_2 + 2n_2 x_1) g_{m-2} \\ + ((c_2 - c_4)x_4 - c_4 x_1) \frac{\partial g_m}{\partial x_1} - c_4 (x_1 + x_4) \frac{\partial g_m}{\partial x_2} - 2c_2 x_4 \frac{\partial g_m}{\partial x_4} - k_0 g_m - k_1 x_1 g_{m-1} = 0.$$

We obtain

$$g_{m-2}(x_1, x_2, x_4) = \frac{x_1^{n_1} x_2^{n_2}}{c_1 (x_2 - 2x_1)^3} E_{m-2}(\bar{g}_m) \log \frac{x_2}{x_1} \\ + \frac{x_1^{n_1-1} x_2^{n_2-2}}{c_1^2 (x_2 - 2x_1)^2} P_{m-2} + x_1^{n_1} x_2^{n_2} \bar{g}_{m-2}(x_2 - 2x_1, x_1 + x_4),$$

where \bar{g}_{m-2} is a homogeneous polynomial of degree $m - n_1 - n_2 - 2$, P_{m-2} is a homogeneous polynomial and $E_{m-2}(\bar{g}_m) = 0$ is the following PDE with unknown \bar{g}_m :

$$-(c_4(-4n_1 + n_2)X_1 + k_0 X_2 + c_2 n_1(4X_1 + X_2))\bar{g}_m \\ - (c_2 + c_4)X_1 X_2 \frac{\partial \bar{g}_m}{\partial X_1} + (2c_2 - c_4)X_1 X_2 \frac{\partial \bar{g}_m}{\partial X_2} = 0,$$

where we have written $X_1 = x_2 - 2x_1$ and $X_2 = x_1 + x_4$ for simplicity. This (homogeneous) equation has the solution

$$\bar{g}_m = X_1^{n_4} X_2^{n_3} (c_2(2X_1 - X_2) - c_4(X_1 + X_2))^{m-n_1-n_2-n_3-n_4},$$

where $n_3, n_4 \in \mathbb{N} \cup \{0\}$ are such that $m - \sum_{i=1}^4 n_i \geq 0$, $k_0 + c_2 n_1 + (c_2 + c_4)n_4 = 0$ and

$$c_4 n_2 + 4(c_2 - c_4)n_1 + (2c_2 - c_4)n_3 = 0. \quad (2.9)$$

Since the logarithm in the expression of g_{m-2} above must be removed, we fix this expression for \bar{g}_m to cancel E_{m-2} .

We note that this expression holds for $c_4 \neq 2c_2$. The case $c_4 = 2c_2$ will be considered later on in this proof. The equation of degree $m - 1$ of (2.8) is

$$\begin{aligned} & -c_1x_1x_2^2 \left(\frac{\partial g_{m-3}}{\partial x_1} + 2\frac{\partial g_{m-3}}{\partial x_2} - \frac{\partial g_{m-3}}{\partial x_4} \right) + c_1x_2(n_1x_2 + 2n_2x_1)g_{m-3} \\ & + ((c_2 - c_4)x_4 - c_4x_1) \frac{\partial g_{m-1}}{\partial x_1} - c_4(x_1 + x_4) \frac{\partial g_{m-1}}{\partial x_2} - 2c_2x_4 \frac{\partial g_{m-1}}{\partial x_4} \\ & + (c_2n_1 + (c_2 + c_4)n_4)g_{m-1} - k_1x_1g_{m-2} + c_4h \left(\frac{\partial g_m}{\partial x_1} + \frac{\partial g_m}{\partial x_2} \right) = 0. \end{aligned}$$

We do not write the expression of g_{m-3} because it is too long. In this expression there is a logarithm that must be removed. Its coefficient provides a PDE with unknown \bar{g}_{m-1} :

$$\begin{aligned} & -\frac{1}{4c_1}X_1^{n_4-1}X_2^{n_3-1}(c_2(2X_1 - X_2) - c_4(X_1 + X_2))^{m-n_1-n_2-n_3-n_4-1} \times \\ & \left[6c_2^2k_1(2n_1 + n_3)X_1^2(2X_1 - X_2) + c_4^2(X_1 + X_2)(k_1(2 + 12n_1 - 3n_2 + 3n_3)X_1^2 \right. \\ & + 4c_1hX_2(-4n_1X_1 + n_2X_1 - n_3X_1 + n_4X_2)) + c_2c_4(-k_1X_1^2(2(2 + 18n_1 - 3n_2 + 6n_3)X_1 \\ & + (-2 + 3n_2 + 3n_3)X_2) + 4c_1hX_2(2n_3X_1^2 - 2n_2X_1(X_1 - 2X_2) + n_1X_1(8X_1 - X_2) \\ & \left. - 3mX_1X_2 + 2n_3X_1X_2 + n_4X_1X_2 + n_4X_2^2)) \right] - (c_2 + c_4)X_1X_2 \frac{\partial \bar{g}_{m-1}}{\partial X_1} \\ & + (c_4 - 2c_2)X_1X_2 \frac{\partial \bar{g}_{m-1}}{\partial X_2} + ((-4c_2n_1 + 4c_4n_1 - c_4n_2)X_1 + (c_2 + c_4)n_4X_2)\bar{g}_{m-1} = 0. \end{aligned}$$

From this new differential equation we can obtain the expression of \bar{g}_{m-1} . To facilitate the computations we use the Euler Theorem of homogeneous functions:

$$\begin{aligned} & X_1^{n_4-1}X_2^{n_3-1}(c_2(2X_1 - X_2) - c_4(X_1 + X_2))^{m-n_1-n_2-n_3-n_4-1} \left[6c_2^2k_1(2n_1 + n_3)X_1^2(2X_1 - X_2) \right. \\ & + c_4^2(X_1 + X_2)(k_1(2 + 12n_1 - 3n_2 + 3n_3)X_1^2 + 4c_1hX_2(-4n_1X_1 + n_2X_1 \\ & - n_3X_1 + n_4X_2)) + c_2c_4(-k_1X_1^2(2(2 + 18n_1 - 3n_2 + 6n_3)X_1 + (-2 + 3n_2 + 3n_3)X_2) \\ & + 4c_1hX_2(2n_3X_1^2 - 2n_2X_1(X_1 - 2X_2) + n_1X_1(8X_1 - X_2) - 3mX_1X_2 + 2n_3X_1X_2 \\ & \left. + n_4X_1X_2 + n_4X_2^2)) \right] + 4c_1(c_4(-(-1 + m + 3n_1 - 2n_2)X_1 - n_4X_2) \\ & + c_2(2(-1 + m + n_1 - n_2)X_1 - n_4X_2))\bar{g}_{m-1}(X_1, X_2) \\ & + 4c_1X_1(c_2(-2X_1 + X_2) + c_4(X_1 + X_2)) \frac{\partial \bar{g}_{m-1}}{\partial X_1} = 0. \end{aligned}$$

We can solve the homogeneous part of this PDE to obtain

$$\begin{aligned} \bar{g}_{m-1}(X_1, X_2) &= X_1^{n_4} \\ & (c_2(2X_1 - X_2) - c_4(X_1 + X_2))^{\frac{c_4 - c_4(m+3n_1-2n_2)+2c_2(m+n_1-n_2-1)}{2c_2-c_4}} - n_4 \bar{G}_{m-1}(X_2), \end{aligned}$$

where \bar{G}_{m-1} is an arbitrary function. Now replacing $\bar{G}_{m-1}(X_2)$ by $\bar{G}_{m-1}(X_1, X_2)$ and replacing n_3 by its value, that can be obtained from (2.9), we can compute the expression

of \bar{G}_{m-1} and therefore of \bar{g}_{m-1} . We obtain:

$$\begin{aligned} \bar{g}_{m-1}(X_1, X_2) = & X_1^{n_4-1} (c_2(2X_1 - X_2) - c_4(X_1 + X_2))^{c_4 - \frac{c_4(m+3n_1-2n_2)+2c_2(m+n_1-n_2-1)}{2c_2-c_4} - n_4} \times \\ & \left[c_4 h n_4 X_2^{\frac{c_2(2-4n_1)+c_4(-1+4n_1-n_2)}{2c_2-c_4}} - \frac{c_4 k_1}{2c_1} X_1 X_2^{\frac{-2c_2+c_4-4c_2 n_1+4c_4 n_1-c_4 n_2}{2c_2-c_4}} \right. \\ & + \frac{c_2 c_4 h}{c_2 + c_4} [3m + 5n_1 - 4n_2 - 3n_4] X_2^{\frac{-4c_2 n_1 - 4c_4 n_1 + c_4 n_2}{2c_2 - c_4}} \log(X_1) \\ & - \frac{3c_2 c_4 h}{2c_2^2 + c_2 c_4 - c_4^2} [2c_2(m + n_1 - n_2 - n_4) + c_4(-m - 3n_1 + 2n_2 + n_4)] \\ & X_2^{\frac{-4c_2 n_1 - 4c_4 n_1 + c_4 n_2}{2c_2 - c_4}} \log(c_2(2X_1 - X_2) - c_4(X_1 + X_2)) \\ & \left. + C_{m-1} X_2^{\frac{-4c_2 n_1 - 4c_4 n_1 + c_4 n_2}{2c_2 - c_4}} \right], \end{aligned}$$

where C_{m-1} is a constant. The coefficients of the logarithms must vanish in order to have a polynomial, so we have some conditions on the coefficients of the system and on the exponents: either $3(m - n_4) + 5n_1 - 4n_2 = 0$ and $c_2 + c_4 = 0$, or $m = n_2/2 + n_4$ and $n_1 = n_2/2$. The first case does not hold since $c_2, c_4 > 0$. Hence the second case is the only one to be considered. From (2.9) we get $(c_4 - 2c_2)(n_2 + n_3) = 0$. Since we are assuming $c_4 \neq 2c_2$, we have $n_2 = n_3 = 0$, and thus $n_1 = 0$ and $n_4 = m$. After this, we have $g_m = (x_1 + x_4)^m$ and

$$\begin{aligned} g_{m-1} = & \frac{X_1^{m-1}}{2c_1 X_2 (2c_2 X_1 - c_4 X_1 - c_2 X_2 - c_4 X_2) (2X_1 + X_2 - 2x_4)} \left(-2c_4 k_1 X_1^3 + 4c_1 C_{m-1} X_1^2 X_2 \right. \\ & + 2c_2 k_1 X_1^2 X_2 - 2c_4 k_1 X_1^2 X_2 + 2c_1 C_{m-1} X_1 X_2^2 - c_2 k_1 X_1 X_2^2 - c_4 k_1 X_1 X_2^2 \\ & \left. + 4c_1 c_4 h m X_1 X_2^2 + 2c_1 c_4 h m X_2^3 + 2c_4 k_1 X_1^2 x_4 - 4c_1 C_{m-1} X_1 X_2 x_4 - 4c_1 c_4 h m X_2^2 x_4 \right). \end{aligned}$$

To ensure that this is a polynomial, we must take $k_1 = 0$ (just dividing and equaling the remainder to zero). Now

$$g_{m-1} = -X_1^{m-1} \frac{C_{m-1} X_1 + c_4 h m X_2}{c_2(-2X_1 + X_2) + c_4(X_1 + X_2)},$$

so we fix

$$C_{m-1} = \frac{c_4 h m (c_4 - 2c_2)}{c_2 + c_4}$$

to finally obtain a polynomial. Now we have, for g_{m-2} ,

$$g_{m-2} = -\frac{c_2 m}{2c_1 x_2} (x_1 + x_4)^{m-1} + \bar{g}_{m-2}(x_2 - 2x_1, x_1 + x_4).$$

To obtain a polynomial we must take $m = 0$.

Next we consider the case $c_4 = 2c_2$. We start over and, with similar arguments, we obtain $k_2 = k_4 = 0$, $k_0 = -c_2(n_1 + 3n_4)$, $n_2 = 2n_1$ and $n_1 = m - n_4$ (there is no n_3 but we are keeping the previous notation). We have

$$g_m = x_1^{m-n_4} x_2^{2(m-n_4)} (x_2 - 2x_1)^{-2(m-n_4)} (x_1 + x_4)^{n_4}.$$

Then $n_4 = m$. We also obtain

$$g_{m-1} = -\frac{X_1^{m-1}}{6c_1X_2^2(2X_1 + X_2 - 2x_4)} \left[-4k_1X_1^3 - 12c_1C_{m-1}X_1^2X_2 - 2k_1X_1^2X_2 \right. \\ \left. - 6c_1C_{m-1}X_1X_2^2 - 3k_1X_1X_2^2 + 8c_1hmX_1X_2^2 + 4c_1hmX_2^3 + 4k_1X_1^2x_4 \right. \\ \left. + 12c_1C_{m-1}X_1X_2x_4 - 8c_1hmX_2^2x_4 \right].$$

We need again $k_1 = 0$, and then

$$g_{m-1} = -\frac{2}{3}hmX_1^{m-1} + \frac{C_{m-1}}{X_2}X_1^m.$$

Hence $C_{m-1} = 0$. Finally we have

$$g_{m-2} = -\frac{c_2m}{2c_1x_2}(x_1 + x_4)^{m-1} + \bar{g}_{m-2}(x_2 - 2x_1, x_1 + x_4).$$

Again we must take $m = 0$.

So we have proved that $m = 0$; that is, $\deg g = 0$. Thus f restricted to $F = 0$ is a constant. Write $f = c + F^j \bar{f}$, where c is that constant, $j \in \mathbb{N}$ and \bar{f} is a polynomial such that $F \nmid \bar{f}$. We note that $c \neq 0$ since f is irreducible.

Taking into account this expression of f , we check the equation $\mathcal{Y}(f) = kf$, where k is the cofactor of f :

$$F^j (\mathcal{Y}(\bar{f}) - (jc_3x_4 + k)\bar{f}) = ck. \quad (2.10)$$

If $\Delta := \mathcal{Y}(\bar{f}) - (jc_3x_4 + k)\bar{f} \equiv 0$ then, since $k \neq 0$ from Lemma 2.1, we have $c = 0$, which is a contradiction. Thus $\Delta \neq 0$ and hence $F \mid k$, which implies that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $k = \alpha F$. Moreover equation (2.10) becomes $F^{j-1}\Delta = c\alpha$; hence $j = 1$ and

$$\mathcal{Y}(\bar{f}) - (\alpha F + c_3x_4)\bar{f} = c\alpha.$$

Applying the arguments in the proof of Lemma 2.1 and taking into account that k is linear (from Proposition 2.3 and because $n_1 = n_2 = 0$), we get that \bar{f} is a (nonzero) constant, say $\bar{f} = \beta$. Hence $f = c + \beta F$. Since $c \neq 0$, this is a contradiction with Lemma 2.4. \square

2.3 Proof of statement (c)

We consider system (1.4) restricted to $H_1 = h$; that is, we consider system (2.6). The following result characterizes the exponential factors of system (2.6) of the form $\exp(g)$, with $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$.

Lemma 2.6. *Let $\exp(g)$, with $g \in \mathbb{C}[x_1, x_2, x_3, x_4]$, be an exponential factor of system (2.6). Then g is a linear combination of x_3 , $x_2 - 2x_1$, $x_1 + x_4$, $(x_2 - 2x_1)^2$, $(x_2 - 2x_1)(x_1 - x_3 + x_4)$ and $(x_1 - x_3 + x_4)^2$.*

Proof. Since $\exp(g)$ is an exponential factor of system (2.6), g satisfies

$$\begin{aligned} \mathcal{Y}(g) = k = & k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_1^2 + k_6x_1x_2 + k_7x_1x_3 + k_8x_1x_4 \\ & + k_9x_2^2 + k_{10}x_2x_3 + k_{11}x_2x_4 + k_{12}x_3^2 + k_{13}x_3x_4 + k_{14}x_4^2, \end{aligned} \quad (2.11)$$

where k is its cofactor and the k_i are complex numbers. Assume that g is a polynomial of degree $m \in \mathbb{N}$, with $m \geq 3$. We write it as sum of its homogeneous parts $g = \sum_{i=1}^m g_i(x_1, x_2, x_3, x_4)$, where g_i is a homogeneous polynomial of degree i and $g_m \neq 0$. The right hand side of (2.11) has degree two, hence its left hand side must also have degree two. Since $m \geq 3$, the computation of g_m, g_{m-1} and g_{m-2} follows in the same way as the proof of Lemma 2.1. Therefore we get $m = 0$, which is a contradiction. Hence $m \leq 2$. Easy computations show that g is a linear combination of $x_3, x_2 - 2x_1, x_1 + x_4, (x_2 - 2x_1)^2, (x_2 - 2x_1)(x_1 - x_3 + x_4)$ and $(x_1 - x_3 + x_4)^2$. \square

Remark 2.7. In particular, the functions appearing in statement (c) of Theorem 1.2 are exponential factors.

In view of Lemma 2.6, if $E = \exp(g)$ is an exponential factor of system (2.6), then g writes as (1.5) and the cofactor of E has the form

$$L = L_0 + c_4L_1H_1, \quad (2.12)$$

where

$$\begin{aligned} L_0 = & (a_2 - a_3)c_4x_1 + ((a_1 - 2a_2)c_2 + (a_2 - a_3)c_4)x_4 - (4a_4 - 3a_5 + 2a_6)c_4x_1^2 \\ & + (2a_4 - a_5)c_4x_1x_2 - (a_5 - 2a_6)c_4x_1x_3 + 2((4a_4 - a_6)c_2 \\ & - 2(a_4 - a_5 + a_6)c_4)x_1x_4 + ((2a_4 - a_5)c_4 - (4a_4 + a_5)c_2)x_2x_4 \\ & + (2(a_5 + a_6) - (a_1 + a_3)c_3 + (2a_6 - a_5)c_4)x_3x_4 \\ & + ((a_5 - 2a_6)c_4 - 2(a_5 + a_6)c_2)x_4^2 \end{aligned}$$

and

$$L_1 = -(a_2 - a_3) + (4a_4 - 3a_5 + 2a_6)x_1 - (2a_4 - a_5)x_2 + (a_5 - 2a_6)x_3 - (a_5 - 2a_6)x_4.$$

We shall use these expressions later on in the proof of Lemma 2.9.

We go back now to system (1.4). Since it has only one Darboux polynomial and one polynomial first integral, if it has an exponential factor, then it must be of the form

$\exp(f/(F^n Q(H_1)))$, with $n \in \mathbb{N} \cup \{0\}$ and $Q \in \mathbb{C}[H_1]$. Next we prove that the expression of an exponential factor of this form cannot contain a power of F in the denominator of the exponent.

Lemma 2.8. *Suppose that system (1.4) has an exponential factor $E = \exp(f/(F^n Q(H_1)))$, with $f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$, $n \in \mathbb{N} \cup \{0\}$, $F \nmid f$ and Q a polynomial. Then $n = 0$.*

Proof. Suppose that $n > 0$. Let L be the cofactor of E . Since $\mathcal{X}(Q(H_1)) = 0$, we have

$$LE = \mathcal{X}(E) = E \frac{\mathcal{X}(f) \cdot F^n - f \cdot \mathcal{X}(F^n)}{F^{2n} Q(H_1)}.$$

Hence

$$\mathcal{X}(f)F^n + nc_3x_4fF^n = LF^{2n}Q(H_1),$$

see Lemma 2.4. Therefore

$$\mathcal{X}(f) + nc_3x_4f = LF^nQ(H_1). \quad (2.13)$$

Since $n > 0$, equation (2.13) on $H_1 = h$ and $F = 0$ becomes

$$\begin{aligned} & (-c_1x_1x_2^2 - c_4x_1 + (c_2 - c_4)x_4 + c_4h) \frac{\partial \bar{f}}{\partial x_1} + (-2c_1x_1x_2^2 - c_4x_1 - c_4x_4 + c_4h) \frac{\partial \bar{f}}{\partial x_2} \\ & + (c_1x_1x_2^2 - 2c_2x_4) \frac{\partial \bar{f}}{\partial x_4} = -nc_3x_4\bar{f}. \end{aligned}$$

where \bar{f} is the restriction of f to $H_1 = h$ and $F = 0$. This means that \bar{f} is a Darboux polynomial of system (2.7) with cofactor $-nc_3x_4 \neq 0$. In view of the proof of Lemma 2.5 this is a contradiction, which comes from the assumption $n \neq 0$. Therefore $n = 0$ and the lemma follows. \square

The following result completes the proof of statement (c).

Lemma 2.9. *Let $E = \exp(f/Q(H_1))$ be an exponential factor of system (1.4), with $Q \in \mathbb{C}[H_1]$ and $f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$. Then $f - gQ(H_1)$, with g as in (1.5), is a polynomial function of H_1 .*

Proof. Set $x_5 = H_1 - x_1 - x_4$. We write the cofactor k of $\exp(f/Q(H_1))$ in the variables x_1, x_2, x_3, x_4, H_1 as follows:

$$\begin{aligned} k &= k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_1^2 + k_6x_1x_2 + k_7x_1x_3 + k_8x_1x_4 \\ &+ k_9x_2^2 + k_{10}x_2x_3 + k_{11}x_2x_4 + k_{12}x_3^2 + k_{13}x_3x_4 + k_{14}x_4^2 \\ &+ (k_{15} + k_{16}x_1 + k_{17}x_2 + k_{18}x_3 + k_{19}x_4)H_1 + k_{20}H_1^2, \end{aligned}$$

where $k_i \in \mathbb{C}$ for all i . We also write Q and f as polynomials in H_1 :

$$Q(H_1) = \sum_{j=0}^n d_j H_1^j \quad \text{and} \quad f = \sum_{j=0}^n f_j(x_1, x_2, x_3, x_4) H_1^j,$$

where $d_j \in \mathbb{C}$ and $f_j \in \mathbb{C}[x_1, x_2, x_3, x_4]$. Since E is an exponential factor, f satisfies

$$\mathcal{X}(f) = kQ(H_1). \quad (2.14)$$

Evaluating (2.14) on $H_1 = 0$, we have that $\exp(f_0)$, with $f_0 = f|_{H_1=0}$, is an exponential factor of system (2.6) with $h = 0$ with the cofactor $d_0 \bar{k} = d_0 k|_{H_1=0}$. In view of Lemma 2.6, we have $f_0 = f_0^0 + d_0 g$, with g as in (1.5). Moreover, $\bar{k} = L_0$. Now computing the coefficient of H_1 in (2.14) we get

$$\begin{aligned} & c_4 \frac{\partial f_0}{\partial x_1} + c_4 \frac{\partial f_0}{\partial x_2} + (-c_1 x_1 x_2^2 - c_4 x_1 + (c_2 - c_4) x_4) \frac{\partial f_1}{\partial x_1} \\ & + (-2c_1 x_1 x_2^2 - c_4 x_1 - c_4 x_4) \frac{\partial f_1}{\partial x_2} + (c_2 - c_3 x_3) x_4 \frac{\partial f_1}{\partial x_3} \\ & + (c_1 x_1 x_2^2 - c_2 x_4 - c_3 x_3 x_4) \frac{\partial f_1}{\partial x_4} \\ & = d_1 L_0 + d_0 (k_{15} + k_{16} x_1 + k_{17} x_2 + k_{18} x_3 + k_{19} x_4). \end{aligned}$$

Proceeding as in the proof of Lemma 2.6, we obtain $f_1 = f_1^0 + d_1 g$ and $k_{15} + k_{16} x_1 + k_{17} x_2 + k_{18} x_3 + k_{19} x_4 = c_4 L_1$. Now computing the coefficient of H_1^2 in (2.14) we get

$$\begin{aligned} & c_4 \frac{\partial f_1}{\partial x_1} + c_4 \frac{\partial f_1}{\partial x_2} + (-c_1 x_1 x_2^2 - c_4 x_1 + (c_2 - c_4) x_4) \frac{\partial f_2}{\partial x_1} \\ & + (-2c_1 x_1 x_2^2 - c_4 x_1 - c_4 x_4) \frac{\partial f_2}{\partial x_2} + (c_2 - c_3 x_3) x_4 \frac{\partial f_2}{\partial x_3} \\ & + (c_1 x_1 x_2^2 - c_2 x_4 - c_3 x_3 x_4) \frac{\partial f_2}{\partial x_4} \\ & = d_2 L_0 + d_1 L_1 + d_0 k_{20}, \end{aligned}$$

or equivalently

$$\begin{aligned} & (-c_1 x_1 x_2^2 - c_4 x_1 + (c_2 - c_4) x_4) \frac{\partial f_2}{\partial x_1} + (-2c_1 x_1 x_2^2 - c_4 x_1 - c_4 x_4) \frac{\partial f_2}{\partial x_2} \\ & + (c_2 - c_3 x_3) x_4 \frac{\partial f_2}{\partial x_3} + (c_1 x_1 x_2^2 - c_2 x_4 - c_3 x_3 x_4) \frac{\partial f_2}{\partial x_4} = d_2 L_0 + d_0 k_{20}. \end{aligned}$$

Proceeding again as in the proof of Lemma 2.6 we get $f_2 = f_2^0 + d_2 g$ and $k_{20} = 0$. Therefore $k = L$, see (2.12). Proceeding inductively with $k = L$ we get $f_j = f_j^0 + d_j g$,

for $j \geq 2$. In short,

$$f = \sum_{j=0}^n d_j (f_j^0 + g) H_1^j = P(H_1) + gQ(H_1),$$

with $P(H_1) = \sum_{j=0}^n d_j f_j^0 H_1^j$ and g as in (1.5). Then the lemma follows. \square

After Lemma 2.9, if $\exp(f/Q(H_1))$ is an exponential factor, then

$$e^{f/Q(H_1)} = e^g e^{P(H_1)/Q(H_1)},$$

with P a polynomial in H_1 . Then statement (c) follows.

Remark 2.10. The cofactors of F_1, \dots, F_6 are, respectively,

$$\begin{aligned} k_1 &= (c_2 - c_3 x_3) x_4; \\ k_2 &= -2c_2 x_4 - c_4 x_5; \\ k_3 &= -c_3 x_3 x_4 + c_4 x_5; \\ k_4 &= -2(-2x_1 + x_2)(2c_2 x_4 + c_4 x_5); \\ k_5 &= -c_2 x_2 x_4 + 2c_2 x_3 x_4 - 2c_2 x_4^2 - 3c_4 x_1 x_5 + c_4 x_2 x_5 + c_4 x_3 x_5 - c_4 x_4 x_5; \\ k_6 &= -2(x_1 - x_3 + x_4)(c_2 x_4 - c_4 x_5). \end{aligned}$$

2.4 Proof of statement (d)

Let H be a Darboux first integral of system (1.4). Then it must be of the form $H = F^{\lambda_1} \exp(g)$ where g is given in (1.5). In view of Proposition 1.1, we must have

$$\begin{aligned} 0 &= -\lambda_1 c_3 x_4 + L \\ &= ((a_1 - 2a_2)c_2 - \lambda_1 c_3)x_4 + (a_3 - a_2)c_4 x_5 + 2(4a_4 - a_6)c_2 x_1 x_4 \\ &\quad + (4a_4 + 3a_5 + 2a_6)c_4 x_1 x_5 + (a_5 - 4a_4)c_2 x_2 x_4 - (2a_4 + a_5)c_4 x_2 x_5 \\ &\quad + (-2a_5 c_2 + 2a_6 c_2 - a_1 c_3 - a_3 c_3)x_3 x_4 - (a_5 + 2a_6)c_4 x_3 x_5 \\ &\quad + 2(a_5 - a_6)c_2 x_4^2 + (a_5 + 2a_6)c_4 x_4 x_5, \end{aligned} \quad (2.15)$$

where L is the cofactor of $\exp(g)$, see (2.12). Solving (2.15) we get $\lambda = 3a_1 c_2 / c_3$, $a_2 = a_3 = -a_1$ and $a_4 = a_5 = a_6 = 0$. Therefore statement (d) follows.

Acknowledgments. A. Ferragut is partially supported by the Spanish Government grant MTM2013-40998-P and C. Valls is supported by Portuguese National Funds through FCT - Fundação para a Ciência e a Tecnologia within projects PTDC/MAT/117106/2010 and PEst-OE/EEI/LA0009/2013 (CAMGSD).

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