An improved algorithm to develop semi-analytical planetary theories using Sundman generalized variables

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Abstract

One of the main problems in celestial mechanics is the construction of the analytical theories of planetary motion. The most common solution of this problem is arranged by means of Poisson series developments. These developments depend on the selection of the anomaly to be used as temporal variable. In this paper we develop an improved alghoritm in order to use of an arbitrary anomaly included in the family of the generalized Sundman anomalies as temporal variables.

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1. Introduction

One of the main problems in celestial mechanics is the study of the motion of the main bodies of the solar system. Its solutions are the so-called planetary theories. To obtain these solutions there are two main ways:

- the numerical methods, based on the integration by the appropriate numerical methods of the differential equation of the motion.
- the analytical and semi-analytical theories, based on the integration of the differential equations through solution of the well known two-body problem and using the perturbation theory.

Let $OXYZ$ be the mean heliocentric ecliptic coordinate system for the epoch J2000. Let $\vec{\sigma} = (a, e, i, \omega, \Omega, M)$ be the third set of elements of Brower [3]defined by the semi-major axis a and eccentricity e of the ellipse, the Euler angles of the orbital plane, it is, the argument of the ascending node Ω , the argument of the perihelion ω and the mean anomaly $M = n(t - t_0) + \varepsilon$ where n is the mean motion, t_0 the osculating epoch and ε the mean anomaly in the osculating epoch. To study the two-body problem it is convenient to use the true anomaly V and the eccentric anomaly E [11].

The coordinates of the secondary with respect to the primary in the orbital plane are given by [11]

$$\vec{r}_{orb} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} r\cos V \\ r\sin V \\ 0 \end{bmatrix} = \begin{bmatrix} a(1-e\cos E) \\ a\sqrt{1-e^2} \\ 0 \end{bmatrix} \tag{1}$$

where the radius vector r is given by [11]

$$r = \frac{a(1-e^2)}{1+e\cos V} = a(\cos E - e) \tag{2}$$

The eccentric anomaly is connected to the mean anomaly through the Kepler equation [19]

$$E - e\sin E = M \tag{3}$$

and the true anomaly V is connected to the mean anomaly M by the center equation [19]

$$V = M + \sum_{k=1}^{\infty} C_k(e)\sin kM \tag{4}$$

where the coefficients $C_k(e)$ are defined in [19].

The spatial coordinates $(x, y, x)^t$ of the secondary with respect to the primary are given by

$$\vec{r} = R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{r}_{orb} \tag{5}$$

where $R_k(\theta)$ is the matrix rotation of angle θ around the k axis.

In the two-body problem the elements a, e, i, Ω, ω and the mean motion n are constant. The solution of the perturbed motion is the same but replacing the constant elements by the osculating elements $a(t)$,. . . ,$\omega(t)$, and $\sigma(t)$. The value of the osculating elements can be obtained by the integration of the Lagrange planetary equations [11]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \sigma} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2e}\frac{\partial R}{\partial \sigma} \\
\frac{di}{dt} &= -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\sigma}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}
\end{aligned} \tag{6}$$

σ is a new variable defined by means of the equation:

$$M = \sigma + \int_{T_0}^{t} n\, dt \tag{7}$$

and it coincides with ε in the case of the unperturbed motion. R is the disturbing potential $R = \sum R_i$ due to the disturbing bodies $i = 1, ..., N$. This one is defined as [11]

$$R = \sum_{k=1}^{N} Gm_k \left[\left(\frac{1}{\Delta_k} \right) - \frac{x \cdot x_k + y \cdot y_k + z \cdot z_k}{r_k^3} \right] \tag{8}$$

where $\vec{r} = (x, y, z)$ and $\vec{r}_k = (x_k, y_k, z_k)$ are the coordinates of the secondary and the disturbing body with respect to the primary, Δ_k is the distance between the secondary and the disturbing body k, and m_k the mass of the k body.

The Lagrange planetary equations are appropriate to integrate the perturbed motion by means of analytical or semianalytical methods. To use analytical methods it is necessary to develop the second member of the Lagrange planetary equations as Fourier series of the selected anomalies with literal developments of the coefficients [19], [8], [3], [1]. The semianalytical methods use numerical values for the amplitudes of the Fourier series.

By integrating these developments we obtain the Poisson series [19], [4]. One of the main problems of the analytical and the semi-analytical methods is the slow convergence rate of the development of the inverse of the distance between the bodies (i, j), that implies the use of very long developments.

In the year (1856) Hansen, in order to improve the convergence rate of the series to describe the motion of the comet Encke ($e \approx 0.84$), introduced the concept of partial anomalies. This method improves its convergence using two new anomalies Ψ_1 and Ψ_2 depending of the region of the orbit that is occupied by the secondary [16].

In the year (1870) Gylden suggested that if we used the elliptical anomaly [4] as temporal variable we could improve the properties of the integration methods.

Based on a temporal transformation $dt = Crd\tau$ introduced by Sundman in order to regularize the origin in the three-body problem, Nacozy [17] introduced a new family of transformations $dt = Cr^{\alpha} d\tau$ called generalized Sundman transformation. This family includes the mean anomaly M ($\alpha = 0, C = n = \sqrt{a^3/\mu}$), the eccentric anomaly E, ($\alpha = 1$, $C = n = \sqrt{a^3/\mu}$) the true anomaly V ($\alpha = 2$, $C = 1/\sqrt{\mu a(1 - e^2)}$) and the Nacozy intermediate anomaly u for $\alpha = 3/2$. The use of these variables improves the convergence properties of the numerical methods.

In this paper we extend the algorithm used by Chapront in order to use the generalized Sundman anomalies as temporal varialbe in the semianalytical methods of integration. This algorithm involves the development of the most common quantities of the two-body problem as Fourier series of the new anomaly, the development of the inverse of the distance between every couple of planets (i, j), the expansion of the second member of the planetary equations of Lagrange and the integration of the Lagrange planetary equations through an appropriate iterative technique.

In section 2 we define the family of Sundman generalized anomalies ψ_α and we obtain an analytical equation to connect the anomaly Ψ_α to the eccentric anomaly E. In this section we study the development of the main quantities of the two bodies problem as Fourier series of Ψ_α.

In section 3 we apply the previous results to develop the inverse of the distance between two bodies using an iterative algorithm based on the Kovalesky method and subsequently, we obtain the development of the second member of the Lagrange Planetary equations according to an arbitrary anomaly in the genralized Sundman family.

In section 4 an iterative integration formula to integrate the second member of the Lagrange planetary equations is developed.

In section 5 a set of numerical examples, using generalized Sundman anomalies, are developed.

In the section 6 the main conclusions of this paper are showed.

2. The Sundmand generalized anomaly

Let us define $dM = K(e,\alpha)r^\alpha d\Psi_\alpha$ as a generalized Sundman transformation where $\Psi_\alpha(M)$ is a 2π periodic function in M satisfying $\Psi_\alpha = M$ when $M = k\pi,\ k \in Z$ and $\Psi(-M) = -\Psi(M)$, and $\frac{dM}{d\Psi_\alpha} > 0$. The value of $K(e,\alpha)$ is given by

$$K(e,\alpha)\int_0^{2\pi} d\Psi_\alpha = \int_0^{2\pi} r^{-\alpha} dM = E = a^{-\alpha}\int_0^{2\pi}(1-e\cos E)^{1-\alpha}dE \tag{9}$$

$$K(e,\alpha) = a^{-\alpha}\left\{(1-e)^p F(\frac{1}{2},-p,1;\frac{2e}{e-1}) + (1+e)^p F(\frac{1}{2},-p,1;\frac{2e}{1+e})\right\} \tag{10}$$

where $p = 1-\alpha$ and $F(a,b,c;z)$ is the hypergeometric function.

The generalized Sundman anomaly is connected to the eccentric anomaly by

$$\Psi_\alpha = G_0(e,\alpha)E + \sum_{s=1}^{\infty}\frac{2}{s}G_s(e,\alpha)\sin sE \tag{11}$$

for details see [14]. The eccentric anomaly and the functions $\sin sE$ and $\cos sE$ can be develop according to the mean anomaly through the use of the Bessel series [11], [19] an from them we can obtain the development

$$\Psi_\alpha = M + \sum_{s=0}^{\infty} H_s(e,\alpha)\sin sM \tag{12}$$

the value of the functions $H_s(e,\alpha)$ are specified in [14].

To manage the most common quantities involved in the two-body problem it is necessary to obtain the development of E, M (generalized kepler equation),

$\cos E$, $\sin E$, r and $\frac{1}{r}$ as Fourier series developments according to the variable Ψ_α. For this pourpose we can rewrite (11), and (12) as power series of eccentricity e and then we apply the Deprit alghoritmh; this algorithm extends the Lagrange series inversion method. From these developments it is suitable to obtain the orbital and spatial coordinates of the secondary as Fouries series according to Ψ_α.

Using this method we obtain from (12) the generalized Kepler equation

$$M = \Psi_\alpha + \sum_{s=0}^{\infty} T_s(e,\alpha)\sin\Psi_\alpha \tag{13}$$

and from (11) the developments of $\sin kE$, $\cos kE$, r, $\frac{1}{r}$ as Fourier series with respec to Ψ_α. For details we can see [14].

From these developments we can obtains the developments of

$$\xi = a(\cos E - e), \quad \eta = \sqrt{1-e^2}, \quad r = a(1 - e\cos E), \quad \frac{1}{r} = \frac{1}{a(1-e\cos E)} \tag{14}$$

and so the orbital coordinates of the scondary (x, y, z).

An alternative way to obtain these developments for an arbitrary function $f(E) \in \mathcal{C}^1[0, 2\pi]$ is the direct computation of the coefficients of the Fourier series through a numerical quadrature method.

3. Development of the disturbing potential and its derivatives

Let $\vec{r}$ and $\vec{r}'$ be the vector radii of the body and the perturbing body. To develop the second member of the planetary Lagrange equations as double Fourier series of the anomalies it is necessary to develop the partial derivatives $\frac{\partial R}{\partial \sigma}$. For this purpose we proceed [11],[2], [8] as

$$\frac{\partial R}{\partial \sigma} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial \sigma} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial \sigma} \tag{15}$$

For the third set of elements of Brower the partial derivatives of the coordinates (x, y, z) with respect to the elements are given in [11]. The values $\frac{\partial R}{\partial x_i}$ are given by

$$\frac{\partial R}{\partial x_i} = GM(1+m')\left[\frac{x_i' - x_i}{\triangle^3} - \frac{x_i'}{r'^3}\right] \tag{16}$$

The main difficulty to obtain these quantities is to obtain the development of the inverse of the distance $\frac{1}{\delta}$ between the two planets. To evaluate this distance can be proceed using the Kovalevsky algorithm [10], [5].

$$\left(\frac{1}{\triangle_k}\right)_{m+1} = \frac{3}{2}\left(\frac{1}{\triangle_k}\right)_m - \frac{1}{2}\left(\frac{1}{\triangle_k}\right)_m^3 \triangle^2 \tag{17}$$

where the m index denotes the number of the iteration. An appropriate first approximation [19] can be

$$\left(\frac{1}{\triangle_k}\right)_0 = \frac{1}{a'}\left[b_{1/2}^{(0)}(\alpha) + \sum_{j=1}^{\infty} b_{1/2}^{(j)}(\alpha)\cos jS\right] \tag{18}$$

where $\alpha = \frac{a}{a'}$, and $b_{p/2}^{(j)}$ are the Laplace coefficients [19], and S the angle between the vector radii $\vec{r}$ and $\vec{r}'$.

$$b_{p/2}^{(j)} = \frac{(p/2)_j}{(1)_j} F\left(\frac{p}{2}, \frac{p}{2} + j, j+1; \alpha^2\right) \tag{19}$$

where F is the Hypergeometric function and $(s)_j$ is the Pochhammer symbol.

The values of $\cos jS$ can be computed from the iteration formula

$$\cos nS = 2\cos((n-1)S)\cos S - \cos((n-2)S) \tag{20}$$

where

$$\cos\, S = \frac{\overrightarrow{r} \cdot \overrightarrow{r'}}{r\, r'} = \frac{x \cdot x' + y \cdot y' + z \cdot z'}{r\, r'} \tag{21}$$

The quantity $\cos\, S$ can be developed as Fourier series from the previous developments of the spatial vector radii $\vec{r}$ and $\vec{r}'$ according to Ψ_α. In the next developments we assume that the parameter α has been selected and the subindex i denotes the number of planet and α will be omitted.

The use of Kovalesky iteration formula requires a very high precise development for the quantity $\triangle^2 = (x-x')^2 + (y+y')^2 + (z-z')^2$ [5].

Using these techniques we can develop the second member of the Lagrange planetary equations (6) for a generic element σ_i in the form

$$\frac{d\sigma_i}{dt} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} A_{k_1,k_2} t^m \cos(k_1\Psi_1 + k_2\Psi_2 + B_{k_1,k_2}) \tag{22}$$

where $A_{i,j}$ and $B_{i,j}$ are real quantities and k_1, k_2, m are integers $m \leq 0$. The second member of the previous equation is a Poisson series, were each term of the series is called Poisson term. To guarantee the uniqueness of the representation of each term we assume for a general term of Poisson

$$A_{k_1,k_2,\ldots,k_M} t^m \cos(k_1\Psi_1 + k_2\Psi_2 + \ldots + k_M\Psi_M + B_{k_1,k_2,\ldots,k_M}) \tag{23}$$

in the case $|k_1| + \ldots + |k_M| \neq 0$ that $A_{k_1,k_2,\ldots,k_M} > 0$, the first $k_i \neq 0$ is positive and $0 \leq B_{k_1,k_2,\ldots,k_M} < 2\pi$. In the case of $k_1 = k_2 = \cdots = k_M = 0$, we include in the amplitude $A_{k_1,k_2,\ldots,k_M}$ the value of the $\cos B_{k_1,k_2,\ldots,k_M}$ so that $B_{k_1,k_2,\ldots,k_M} = 0$.

In the first order of perturbation the exponent m is 0. To evaluate the planetary equation corresponding to $\frac{da}{dt}$ it is necessary to take into account the Chapront considerations on the initial values of a and n [5].

4. Integration algorithms

To Integrate the Lagrange planetary equation in its developed form (22) it is necessary to evaluate the integrals

$$\int_{t_0}^{t} \cos(k_1\Psi_+ k_2\Psi_2 + B_{k_1,k_2})dt \tag{24}$$

for this purpose we have for $i = 1, 2$ the developments of the Kepler equation

$$M_i = \Psi_1 + \sum_{s=1}^{\infty} T_s(e_i, \alpha) \sin s\Psi_i \tag{25}$$

where the functions $T_s(e_i, \alpha)$ can be evaluated by analytical methods [14].

To integrate the generic term $\cos(k_1\Psi_{\alpha_1} + k_2\Psi_{\alpha_2} + B_{k_1,k_2})$ we can proceed derivating (25)

$$n_i dt = dM_i = d\Psi_i + \left[\sum_{s=1}^{\infty} sT_s(e_i, \alpha) \cos s\Psi_i\right] d\Psi_i \tag{26}$$

and from them

$$dt = \frac{d(k_1\Psi_1 + k_2\Psi_2)}{(k_1n_1 + k_2n_2)} + \frac{k_1}{(k_1n_1 + k_2n_2)} \left[\sum_{s=1}^{\infty} sT_s(e_1, \alpha) \cos s\Psi_1\right] d\Psi_1 + \\ + \frac{k_2}{(k_1n_1 + k_2n_2)} \left[\sum_{s=1}^{\infty} sT_s(e_2, \alpha) \cos s\Psi_2\right] d\Psi_2 \tag{27}$$

From (26) we obtain

$$d\Psi_i = n_i dt \left[\sum_{p=0}^{\infty} (-1)^p S_i^p\right] = n_i \left[\sum_{s=0}^{\infty} P_s(e_i, \alpha) \cos s\Psi_i]\right] dt \tag{28}$$

where $S_i = \sum\limits_{s=1}^{\infty} sT_s(e_i, \alpha_i) \cos s\Psi_i$.

Replacing in (27) we obtain.

$$dt = \frac{d(k_1\Psi_1 + k_2\Psi_2)}{(k_1n_1 + k_2n_2)} + \frac{k_1n_1h_1(e_1, \Psi_1) + k_2n_2h_2(e_2, \Psi_2)}{(k_1n_1 + k_2n_2)} dt \tag{29}$$

where

$$h_i(e_i, \Psi_i) = \left[\sum_{s=1}^{\infty} sT_s(e_i, \alpha) \cos s\Psi_i\right] \left[\sum_{s=0}^{\infty} P_s(e_i, \alpha) \cos \Psi_i\right], \quad i = 1, 2 \tag{30}$$

Functions $T_s(e_i, \alpha)$ satisfy the d'Alembert propiety, by this reason (29) can be used as an iterative formula, increasing the order in the eccentricities for the residual term by one in each iteration [13].

From (29) we have

$$\int \cos(k_1\Psi_1+k_2\Psi_2+B_{k_1,k_2})dt = \frac{1}{k_1n_1+k2n_2}\cos(k_1\Psi_1+k_2\Psi_2+B_{k_1,k_2}-\frac{pi}{2})+$$
$$+\int \frac{k_1n_1h_1(e_1,\Psi_1)+k_2n_2h_2(e_2,\Psi_2)}{k_1n_1+k2n_2}\cos(k_1\Psi_1+k_2\Psi_2+B_{k_1,k_2})dt \quad (31)$$

in the case of the integration of a Poisson term with $m > 0$ we can procced through integration by parts

$$\int t^m \cos(k_1\Psi_{\alpha_1}+k_2\Psi_{\alpha_2}+B_{k_1,k_2})dt = t^m\int \cos(k_1\Psi_1+k_2\Psi_2+B_{k_1,k_2})dt-$$
$$-m\int t^{m-1}\left[\int \cos(k_1\Psi_1+k_2\Psi_2+B_{k_1,k_2})dt\right]dt \quad (32)$$

Note that the two integrals icluded in the second member are the same and it is a Poisson series with $m = 0$.

5. Numerical examples

To test the method a set of numerical examples in the first order of perturbation has been computed. For this purpose we select the couple Jupiter-Saturn to test the algorithm.

The orbital elements (Table 1) were taken from Simon [18] in order to compare our values for $\alpha = 0$ with the respective ones given by Chapront [6]. The initial osculation epoch is J2000 and the planetary masses were taken according to the IAU 1976 constants.

Table 1: Planetary elements for Jupiter and Saturn

Planet	a	k	h
Jupiter	5.2042662908	0.0469877116	0.0130817658
Saturn	9.5820161867	0.0003336009	0.0557224686
	q	p	λ
Jupiter	-0.0086968779	0.0198660071	0.8727430950
Saturn	-0.0020729462	0.0111943279	0.5999772955

where $\overline{\omega} = \Omega+\omega$, $k = e\cos\overline{\omega}$, $h = e\cos\overline{\omega}$, $q = \gamma\cos\Omega$, $p = \gamma\sin\Omega$, $\lambda = M+\overline{\omega}$ and $\gamma = \sin\frac{i}{2}$. The managment of common developments used in Celestial mechanics is a very hard task, so it is convenient to use an appropriate special software package called Poisson Series Processor (PSP) [9], [4],[15]. In this paper the PSP used was the C++ class poison.h developed by the authors. This processor series contains the most common arithmetic operations $+, -, *, \ldots,$ function evaluation sin, cos, exp,. . . , etc.

Table 2: Coefficients c_i of Kepler equation

α	$\sin\Psi$	$\sin 2\Psi$	$\sin 3\Psi$	$\sin 4\Psi$	$\sin 5\Psi$
0.5	-0.0242409359	-2.204541e-4	-3.8613e-6	-8.87e-8	-2.4e-9
1.0	-0.0484979255	0.000000e-4	0.0000e-6	0.00e-8	0.0e-9
1.5	-0.0727549189	6.619681e-4	-5.6518e-6	4.47e-8	-3.e-10
2.0	-0.0969958510	1.7647287e-3	-3.80567e-5	8.656e-7	-2.02e-8

Table 2 shows the five first terms c_i of the development of the Kepler $M = \Psi + \sum c_i \sin\Psi_i$ equation of Jupiter for several values of α.

Table 3 shows the length of the series and the diference between to iterations of the inverse of the distance for couple Jupiter-Saturn for several values of the parameter α. The error of each iteration err_k has been computed by bounding with respect to the $|| \quad ||_1$, it is $err_k = || \left(\frac{1}{\triangle}\right)_k - \left(\frac{1}{\triangle}\right)_{k-1} ||_1$.

Table 3: Number of terms of the inverse of the distance developments

k	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.0$	err
1	1064	911	754	613	504	$3.2e-2$
2	1196	1051	912	775	772	$6.0e-3$
3	1234	1114	992	911	912	$2.2e-4$
4	1140	988	862	851	852	$3.1e-7$
5	1151	988	884	858	862	$3.4e-12$

Tables 4 and 5 show, in arcsec, the main amplitude terms, A and A', of characteristic $|k_1 - k_2| = 0$ and 3 in the developoment of the major-semi axis a of Jupiter and Saturn for the couple Jupiter-Saturn for the values of $\alpha = 0.5, 1.0, 1.5, 2.0$. Values of A and A', for $\alpha = 0.0$ coincide with the ones obtained by Chapront [6].

Tables 6 and 7 show the convergence of the integrator applied to the planetary equations of a_J and a_S for the copule Jupiter-Saturn.

6. Concluding Remarks

So as to test the algorithm, the problem of the calculus of the first order perturbations of the semi axes for the couple Jupiter-Saturn has been used.

The use of appropriate anomalies in the generalized Sundman family can be applied to improve the efficiency of semi-analytical algorithms.

The lenght of the series depends of the anomalies used as temporal variables. An appropriate choice of the anomaly in the generalized Sundman family of anomalies can be allowed to have more compact developments in the inverse of

Table 4: Amplitude of terms of characteristic $|k_1 - k_2| = 0$ for Jupiter and Saturn semiaxis in arcsec

k_1	$\alpha = 0.5$		$\alpha = 1.0$		$\alpha = 1.5$		$\alpha = 2.0$	
	A	A'	A	A'	A	A'	A	A'
1	42.734	6960.092	43.064	6971.697	43.444	6978.382	43.852	6980.115
2	144.434	659.008	146.500	676.374	148.201	691.597	149.526	704.492
3	65.652	299.688	66.920	307.842	67.851	314.429	68.424	319.222
4	31.213	142.577	32.141	148.088	32.809	152.505	33.211	155.793
5	15.202	69.503	15.855	73.173	16.312	76.011	16.563	77.968
6	7.515	34.393	7.959	36.802	8.264	38.599	8.417	39.713
7	3.752	17.187	4.046	18.744	4.247	19.892	4.344	20.570
8	1.885	8.646	2.076	9.635	2.206	10.362	2.267	10.775
9	0.952	4.369	1.073	4.989	1.156	5.445	1.194	5.695
10	0.482	2.214	0.557	2.598	0.610	2.881	0.634	3.033
11	0.244	1.124	0.291	1.359	0.324	1.534	0.338	1.626
12	0.124	0.572	0.153	0.714	0.173	0.821	0.182	0.876
13	0.063	0.291	0.080	0.376	0.093	0.441	0.098	0.475
14	0.032	0.148	0.042	0.199	0.050	0.238	0.053	0.258
15	0.016	0.075	0.022	0.105	0.027	0.129	0.029	0.141
16	0.008	0.038	0.012	0.056	0.015	0.070	0.016	0.077
17	0.004	0.019	0.006	0.030	0.008	0.038	0.009	0.043
18	0.002	0.010	0.003	0.016	0.004	0.021	0.005	0.023
19	0.001	0.005	0.002	0.008	0.002	0.011	0.003	0.013
20	0.001	0.003	0.001	0.005	0.001	0.006	0.001	0.007
21	0.000	0.001	0.001	0.002	0.001	0.003	0.001	0.004
22	0.000	0.001	0.000	0.001	0.000	0.002	0.000	0.002
23	0.000	0.000	0.000	0.001	0.000	0.001	0.000	0.001
24	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.001

the distance, and so it allows to simplify the evaluation of the second member of the Lagrangre planetary equations.

The coefficients of the terms containing small divisors, as shown in the $2n_1 - 5n_2$ case for the couple Jupiter-Saturn, can be determinated with a higher level of precission for each vaue of α.

The management of these developments can be obtainable by using a Poisson series processor. The performance of the algorithm is good for the interesting values of α, it is, the ones contained in the interval $[0, 2]$; for values of $\alpha \geq 2.5$ the convergence rate decreases. The kernel of processor poisson.h is avaible under certain conditions if it is requested.

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Table 5: Amplitude terms of characteristic $|k_1 - k_2| = 3$ for Jupiter and Saturn semiaxis in arcsec

k_1	k_2	$\alpha = 0.5$		$\alpha = 1.0$		$\alpha = 1.5$		$\alpha = 2.0$	
		A	A'	A	A'	A	A'	A	A'
2	-5	50.746	572.892	50.528	569.362	49.671	559.056	48.209	542.256
3	-6	1.333	12.767	1.002	10.173	0.862	9.451	1.131	12.810
4	-7	0.710	6.017	0.486	4.450	0.326	3.237	0.223	2.384
5	-8	0.455	3.520	0.290	2.426	0.174	1.597	0.100	1.012
1	-4	0.376	4.853	0.466	4.522	1.002	10.820	1.788	19.958
6	-9	0.306	2.210	0.187	1.461	0.105	0.904	0.054	0.526
7	-10	0.206	1.415	0.123	0.911	0.066	0.538	0.031	0.288
0	3	0.185	2.382	0.411	4.546	0.677	7.432	0.978	10.830
8	-11	0.138	0.908	0.081	0.575	0.042	0.328	0.019	0.165
9	-12	0.091	0.581	0.053	0.363	0.027	0.203	0.011	0.097
10	-13	0.060	0.369	0.035	0.229	0.017	0.125	0.007	0.058
11	-14	0.039	0.233	0.022	0.144	0.011	0.078	0.004	0.035
12	-15	0.025	0.145	0.014	0.090	0.007	0.048	0.003	0.021
13	-16	0.016	0.090	0.009	0.056	0.004	0.030	0.002	0.013
14	-17	0.010	0.056	0.006	0.035	0.003	0.018	0.001	0.008
2	1	0.010	0.052	0.010	0.060	0.010	0.068	0.012	0.072
15	-18	0.006	0.034	0.004	0.021	0.002	0.011	0.001	0.005
1	2	0.006	0.239	0.008	0.286	0.012	0.300	0.014	0.275
3	0	0.005	0.116	0.008	0.105	0.017	0.098	0.031	0.202
16	-19	0.004	0.021	0.002	0.013	0.001	0.007	0.000	0.003
4	-1	0.003	0.311	0.006	0.052	0.017	0.089	0.047	0.296
5	-2	0.002	0.003	0.004	0.017	0.012	0.071	0.038	0.217
17	-20	0.002	0.013	0.001	0.008	0.001	0.004	0.000	0.002
6	-3	0.002	0.003	0.002	0.011	0.009	0.056	0.033	0.191
7	-4	0.001	0.002	0.001	0.007	0.007	0.042	0.026	0.153
18	-21	0.001	0.008	0.001	0.005	0.000	0.003	0.000	0.001
8	-5	0.001	0.002	0.001	0.004	0.005	0.029	0.020	0.116
9	-6	0.001	0.002	0.000	0.002	0.003	0.020	0.015	0.084
19	-22	0.001	0.005	0.000	0.003	0.000	0.002	0.000	0.000
10	-7	0.001	0.001	0.000	0.001	0.002	0.013	0.010	0.059
20	-23	0.000	0.003	0.000	0.002	0.000	0.001	0.000	0.000
11	-8	0.000	0.001	0.000	0.001	0.001	0.009	0.007	0.041
12	-9	0.000	0.001	0.000	0.000	0.001	0.006	0.005	0.027
13	-10	0.000	0.000	0.000	0.000	0.000	0.004	0.003	0.018
14	-11	0.000	0.000	0.000	0.000	0.000	0.002	0.002	0.012
15	-12	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.008
16	-13	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.005
17	-14	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.003
21	-24	0.000	0.002	0.000	0.001	0.000	0.000	0.000	0.000
22	-25	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000
23	-26	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000
18	-15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002
19	-16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001
20	-17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001

Table 6: $||residual||_1$ and n terms of residual for Jupiter semiaxe

n_{it}	$\alpha = 0.5$		$\alpha = 1.0$		$\alpha = 1.50$		$\alpha = 2.0$		$\alpha = 2.5$	
1	2.50e-07	380	4.56e-07	357	6.39e-07	371	8.15e-07	373	1.01e-06	378
2	3.06e-08	298	9.46e-08	315	1.66e-07	330	2.45e-07	345	3.61e-07	358
3	5.49e-09	219	3.30e-08	271	8.06e-08	293	1.36e-07	317	2.03e-07	335
4	7.39e-10	139	9.27e-09	220	3.59e-08	256	8.69e-08	283	1.68e-07	305
5	8.30e-11	82	2.03e-09	158	1.13e-08	213	3.43e-08	250	7.79e-08	274
6			4.59e-10	109	3.67e-09	163	1.40e-08	205	3.66e-08	239
7			1.07e-10	94	1.24e-09	125	6.00e-09	167	1.85e-08	193
8			2.40e-11	72	4.21e-10	122	2.69e-09	142	1.02e-08	157
9					1.38e-10	113	1.11e-09	143	4.96e-09	149
10					4.70e-11	96	4.88e-10	140	2.60e-09	154
11					1.50e-11	66	2.06e-10	134	1.28e-09	161
12							9.20e-11	120	6.89e-10	158
13							3.90e-11	102	3.42e-10	151
14							1.80e-11	87	1.84e-10	140
15									9.20e-11	123
16									5.00e-11	113
17									2.70e-11	101
18									1.80e-11	99
19									1.40e-11	97
20									1.40e-11	98
21									1.30e-11	98
22									1.30e-11	96
23									1.20e-11	98

Table 7: $||residual||_1$ and n terms of residual for Saturn semiaxe

n_{it}	$\alpha = 0.5$		$\alpha = 1.0$		$\alpha = 1.50$		$\alpha = 2.0$		$\alpha = 2.5$	
1	3.27e-06	462	6.18e-06	433	9.06e-06	451	1.21e-05	461	1.53e-05	469
2	3.16e-07	373	1.04e-06	387	1.99e-06	413	3.13e-06	426	4.49e-06	442
3	4.63e-08	283	3.01e-07	337	8.07e-07	376	1.51e-06	395	2.37e-06	414
4	5.20e-09	206	6.71e-08	290	2.66e-07	335	6.53e-07	361	1.25e-06	385
5	5.71e-10	130	1.42e-08	229	8.01e-08	288	2.45e-07	326	5.41e-07	353
6	6.60e-11	89	3.24e-09	168	2.63e-08	236	1.01e-07	288	2.58e-07	316
7			7.58e-10	133	9.01e-09	186	4.43e-08	238	1.35e-07	277
8			1.72e-10	120	2.99e-09	164	1.91e-08	199	7.06e-08	234
9			3.90e-11	92	1.00e-09	156	8.14e-09	182	3.57e-08	205
10					3.37e-10	145	3.49e-09	183	1.81e-08	199
11					1.15e-10	129	1.52e-09	176	9.31e-09	204
12					3.90e-11	94	6.57e-10	169	4.83e-09	205
13					1.40e-11	81	2.87e-10	157	2.48e-09	204
14							1.26e-10	138	1.29e-09	199
15							5.70e-11	138	6.66e-10	188
16							2.70e-11	129	3.51e-10	181
17							1.40e-11	110	1.87e-10	172
18									1.04e-10	170
19									6.20e-11	171