GROMOV-HAUSDORFF CONVERGENCE OF NON-ARCHIMEDIAN FUZZY METRIC SPACES

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ABSTRACT. We introduce the notion of the Gromov-Hausdorff fuzzy distance between two non-Archimedean fuzzy metric spaces (in the sense of Kramosil and Michalek). Basic properties involving convergence and the fuzzy version of the completeness theorem are presented. We show that the topological properties induced by the classic Gromov-Hausdorff distance on metric spaces can be deduced from our approach.

1. INTRODUCTION

In his celebrated paper [14], Gromov introduced the so-called Gromov-Hausdorff convergence as a procedure to study the convergence of metric spaces. Not longer after Gromov published his paper, this subject became a useful tool of much wider applicability. Among other interesting applications, this convergence can be exploited to understand how geometric constraints on metric spaces give rise to topological constraints. For example, it was used [3] to show the existence of a solution to an abstract Steiner (i.e., minimal connection) problem, even in some ambient spaces which are not locally compact (such is the case, for instance, for a Hilbert space). Adopting this strategy, devised by Gromov [15], one can also tackle the higher-dimensional geodesic problem, i.e., *the Plateau problem* (see [2]). The interesting reader might consult the survey [24] on the Gromov-Hausdorff convergence of compact metric spaces. Further interesting applications are presented in the papers [4, 7, 10, 23]. Our main goal in this paper is to introduce and to discuss an appropriate notion of Gromov-Hausdorff convergence for compact non-Archimedean fuzzy metric spaces (in the sense of Kramosil and Michalek [19]). We aim at fitting this concept into the schema of a Hausdorff fuzzy metric as defined in [26]. Among other things, we shall give a fuzzy version of Gromov's theorem on completeness. It is worth noting that our results on non-Archimedean compact fuzzy metric spaces permit us to obtain the corresponding ones for metric spaces by means of the standard fuzzy metric space $(X, M_d, \cdot)$ associated to a metric space (X, d). Taking into account the relevance of the Gromov-Hausdorff convergence in general topology, geometry, functional analysis, etc...., one might hope that a deeper understanding of

the Gromov-Hausdorff convergence for (compact) fuzzy metric spaces will help to establish stronger results in applications.

The paper is organized as follows. Section 2 is devoted to present the basic notions and facts that will be of use later. We shall focus special attention on the concept of a Hausdorff fuzzy metric. In Section 3 we present our approach to Gromov-Hausdorff fuzzy distance for non-Archimedean fuzzy metric spaces. Section 4 is devoted to basic properties involving Gromov-Hausdorff fuzzy convergence. We present the completeness theorem and the relationship with the case of metric spaces is analyzed. The conclusions are laid in the last section.

2. Preliminaries and Basic Facts

We begin by reminding the reader some notions on fuzzy metric spaces used in this paper. Following [30], by a continuous t-norm it is understood a binary operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (i) $*$ is associative and commutative, (ii) $*$ is continuous, (iii) $a * 1 = a$ for every $a \in [0,1]$, and (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

It is a well-known fact, and easy to check, that for each continuous t-norm $*$ one has $* \leq \wedge$, where $\wedge$ is the continuous t-norm given by $a \wedge b = \min\{a, b\}$.

Definition 2.1. ([19]) A *fuzzy metric* (in the sense of Kramosil and Michalek) on a set X is a pair $(M, *)$ such that M is a fuzzy set in $X \times X \times [0, \infty)$ and $*$ is a continuous t-norm satisfying for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, 0) = 0$;
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$; and
(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is a left continuous function.

By a *fuzzy metric space* (in the sense of Kramosil and Michalek) we mean a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X. Recall that every fuzzy metric $(M, *)$ on X induces a topology τ_M on X, which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$ (the subscript M will be omitted if no confusion arises).

If in Definition 2.1 the triangular inequality (iv) is replaced by

$$M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s) \qquad \text{(NA)}$$

for all $x, y, z \in X$ and all $t, s > 0$, then $(X, M, *)$ is called a *non-Archimedean fuzzy metric space* (see [18]). It is an easy matter to show that condition (NA) implies condition (iv), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space. Non-Archimedean fuzzy metric spaces appear related to some coincidence point theorems in fuzzy normed spaces and also in fixed point theorems on fuzzy metric spaces (see for instance [1, 22]). It is routine to verify that condition (NA) is equivalent to the two following conditions: (NA1) $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$, and (NA2) $M(x, y, \_)$ is nondecreasing for all $x, y \in X$.

One major fact related to non-Archimedean fuzzy metrics is the following essentially well-known construction:

Construction 2.2. (Compare [8, 11]) Let (X, d) be a metric space. Define a fuzzy set M_d in $X \times X \times [0, \infty)$ by

$$M_d(x, y, t) = \begin{cases} \dfrac{t}{t + d(x, y)} & \text{for all } x, y \in X \text{ and } t > 0; \\ 0 & \text{for all } x, y \in X \text{ and } t = 0. \end{cases}$$

Then $(M_d, \wedge)$ is a fuzzy metric on X, and thus $(M_d, *)$ is a fuzzy metric on X for all continuous t-norm $*$, the so-called fuzzy metric induced by (X, d), or the standard fuzzy metric induced by (X, d). Moreover, it is easy to show that if the t-norm $*$ coincides with the usual product $a \cdot b$ of two real numbers $a, b \in [0, 1]$, then $(M_d, \cdot)$ is non-Archimedean (this fact itself is a particular case of a more general result stated in [20, Example 4.2.3]).

An important consequence of the previous construction is that the topology induced by a non-Archimedean fuzzy metric can fail to be non-Archimedean (in the usual sense). The reason is apparent: the topology τ_d induced by a metric d coincides with the topology τ_{M_d} induced by M_d (see [8]). This fact justifies that a non-Archimedean fuzzy metric is called *strong* for several authors (see [13, 28] for a brief discussion of this technical point).

We now turn to another concept which is relevant to Gromov-Hausdorff fuzzy convergence: the Hausdorff fuzzy metric.

The definition of the Hausdorff probabilistic metric of a probabilistic metric space ([5, 31, 32, 34]) was well adapted for dealing with fuzzy metric spaces by Rodríguez-López et al. [26]. Their approach is the following. Let A be a (nonempty) subset of a fuzzy metric space $(X, M, *)$. For $x \in X$ and $t > 0$, let $M(x, A, t) = \sup\{M(x, a, t) : a \in A\}$ (see [35, Definition 2.4]). Now, for each couple A and B of nonempty subsets of X, define

$$H_M^-(A, B, 0) = H_M^+(A, B, 0) = 0 \, ;$$

and, for $t > 0$,

$$H_M^+(A, B, t) = \sup_{0 < s < t} \inf_{a \in A} M(a, B, s) \, ;$$

$$H_M^-(A, B, t) = \sup_{0 < s < t} \inf_{b \in B} M(A, b, s) \, .$$

Let

$$H_M(A, B, t) = \min\{H_M^+(A, B, t), H_M^-(A, B, t)\}$$

for all $t \geq 0$. Then $H_M(A, B, t)$ is a fuzzy metric on the set $C_0(X)$ of all the nonempty closed subsets of X, called the *Hausdorff fuzzy metric* of $(X, M, *)$. In general, it is not possible to define the Hausdorff fuzzy metric in a more simplified way as

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\} \quad \text{for all} \quad t \geq 0$$

(see, for instance, [26, Example 4]). However, this simplification is possible in the case of compact subsets. First we need a lemma that follows from an easy adaptation of the proof of Proposition 1 in [25]. It is important to notice that, fixed $x, y \in X$, the function $M(x, y, \_)$ has only countably many points of discontinuity: indeed, the function $M(x, y, \_)$ is nondecreasing, i.e., a monotone function. Throughout what follows, we shall freely use this fact without explicit mention.

Lemma 2.3. *Let $(X, M, *)$ be a fuzzy metric space. If a sequence $\{(x_n, y_n, t_n)\}_{n \in \mathbb{N}}$ in $X \times X \times [0, +\infty)$ converges to (x, y, t) and $\lim_n M(x_n, y_n, t_n)$ exists, then the following assertions hold:*

(a) $M(x, y, t) \leq \lim_n M(x_n, y_n, t_n)$;

(b) $M(x, y, t) = \lim_n M(x_n, y_n, t_n)$, *in the case that t is a point of continuity of $M(x, y, \_)$.*

The following result was presented (without proof) in [21].

Proposition 2.4. *Let $(X, M, *)$ be a fuzzy metric space. If A and B are nonempty compact subsets of X, then, for each $t > 0$, the following equalities hold:*

(a) $H_M^+(A, B, t) = \inf_{a \in A} M(a, B, t)$;

(b) $H_M^-(A, B, t) = \inf_{b \in B} M(A, b, t)$.

Proof. (a) Since $M(x, y, \_)$ is nondecreasing for all $x, y \in X$, we have

$$H_M^+(A, B, t) \leq \inf_{a \in A} M(a, B, t).$$

Suppose now that $H_M^+(A, B, t) \lneqq \inf_{a \in A} M(a, B, t)$. Choose a real number l such that

$$H_M^+(A, B, t) \lneqq l \lneqq \inf_{a \in A} M(a, B, t).$$

By the definition of $H_M^+(A, B, t)$, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of A and an increasing sequence of positive real numbers $\{s_n\}_{n \in \mathbb{N}}$ converging to t such that $M(a_n, B, s_n) \lneqq l$. Thus, $M(a_n, b, s_n) \lneqq l$ for every $b \in B$.
Since both $[0, 1]$ and A are compact, we may assume (and we do) that $\{a_n\}_{n \in \mathbb{N}}$ converges to an element $a_0 \in A$ and that $\lim_n M(a_n, b, s_n)$ exists for every $b \in B$. Then, by Lemma 2.3 (a), we have

$$M(a_0, b, t) \leq \lim_n M(a_n, b, s_n) \leq l$$

for all $b \in B$ so that $M(a_0, B, t) \leq l \lneqq \inf_{a \in A} M(a, B, t)$, a contradiction. Thus, the equality (a) holds. The proof of (b) runs along similar lines. □

Corollary 2.5. *Let $(X, M, *)$ be a fuzzy metric space. If A and B are (nonempty) compact subsets of X, then*

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\}$$

for all $t \geq 0$.

We close this section with a straightforward (and probably well-known) but useful result.

Proposition 2.6. *Let $\{f_i\}_{i \in I}$ be a family of left-continuous nondecreasing functions from an interval $\mathbb{I}$ into the reals. If $\sup\{f_i(x) : i \in I\}$ exists for all $x \in \mathbb{I}$, then the function f defined as*

$$f(x) = \sup\{f_i(x) : i \in I\}, \quad x \in \mathbb{I},$$

is left-continuous on $\mathbb{I}$.

3. The non-Archimedean Hausdorff-Gromov fuzzy distance

In this section we define and study a distance between non-Archimedean fuzzy metric spaces, using the notion of Hausdorff fuzzy metric as constructed by Rodríguez-López et al. [26] and as discussed in the previous section. Although we deal mainly with non-Archimedean fuzzy metrics, most proofs are so constructed that they apply to more general situations.

Our definition is in the spirit to that of Gromov's approach [14]: if (X, d_1) and (Y, d_2) are two nonempty compact metric spaces, then the Gromov-Hausdorff distance $d_{GH}(X, Y)$ is defined as the infimum of all $\varepsilon > 0$ such that there exist a compact metric space Z and isometric embeddings $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$ such that $d_H(f(X), g(Y)) < \varepsilon$ where d_H stands for the Hausdorff distance.

Recall that given two fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, \circ)$, a function $f : X \to Y$ is called *a fuzzy isometry* [12] if $M_X(x, y, t) = M_Y(f(x), f(y), t)$ for every $x, y \in X$ and every $t > 0$. Two fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, \circ)$ are said to be *isometric* if there exists an isometry from X onto Y.

Definition 3.1. The non-Archimedean Hausdorff-Gromov fuzzy distance between two non-Archimedean fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, *)$, denoted by M_{GH}, is defined as

$$M_{GH}(X, Y, 0) = 0,$$

and, for all $t > 0$, as

$$M_{GH}(X, Y, t) = \sup\{H_{M_Z}(f(X), g(Y), t)\}$$

where the supremum is taken over all non-Archimedean fuzzy metric spaces $(Z, M_Z, *)$ and all fuzzy isometries embeddings $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$.

As in the case of metric spaces, in order to see how the non-Archimedean Hausdorff-Gromov fuzzy distance works in practice, we shall introduce an alternative formulation based upon the notion of *admissible fuzzy metric*: given two fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, *)$, a fuzzy metric M on the disjoint union $X \sqcup Y$ is said to be *admissible* if restricts to the given metric on X and Y, respectively. The two following lemmas provide two useful ways of constructing admissible non-Archimedean fuzzy metrics.

Lemma 3.2. *Let $(X, M_X, *)$, $(Y, M_Y, *)$ and $(Z, M_Z, *)$ be three non-Archimedean fuzzy metric spaces. If $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$ are two* fuzzy *isometries, then, for every $\delta \in (0, 1)$, $(M_\delta, *)$ is an admissible non-Archimedean fuzzy metric on $X \sqcup Y$ where M_δ is the function on $(X \sqcup Y) \times (X \sqcup Y) \times [0, +\infty)$ defined as*

$$M_\delta(x, y, 0) = 0, \ \ \text{for each } x, y \in X \sqcup Y,$$

and, for each $t > 0$,

$$M_\delta(x, y, t) = M_X(x, y, t) \ \ \text{if } x, y \in X,$$
$$M_\delta(x, y, t) = M_Y(x, y, t) \ \ \text{if } x, y \in Y,$$
$$M_\delta(x, y, t) = M_\delta(y, x, t) = M_Z(f(x), g(y), t) * \delta \ \ \text{if } x \in X \ \text{and } y \in Y.$$

Proof. Conditions (i)-(ii)-(iii) and (v) in Definition 2.1 are straightforward. Thus, we only need to show condition (NA), that is,

$$M_\delta(x, z, \max\{t, s\}) \geq M_\delta(x, y, t) * M_\delta(y, z, s)$$

for all $x, y, z \in X \sqcup Y$ and all $t, s > 0$. Notice that (NA) is clear if x, y, z belong either to X or to Y. Therefore, since the roles of X and Y can be interchanged, to establish condition (NA) it suffices to consider two cases:

Case 1. $x, z \in X$ and $y \in Y$. Then we have

$$\begin{aligned}
M_\delta(x, z, \max\{t, s\}) = M_X(x, z, \max\{t, s\}) &= M_Z(f(x), f(z), \max\{t, s\}) \\
&\geq M_Z(f(x), g(y), t) * M_Z(g(y), f(z), s) \\
&\geq M_Z(f(x), g(y), t) * \delta * M_Z(g(y), f(z), s) * \delta \\
&= M_\delta(x, y, t) * M_\delta(y, z, s),
\end{aligned}$$

which shows condition (NA) in Case 1.

Case 2. $x, y \in X$ and $z \in Y$. In this situation we have

$$\begin{aligned}
M_\delta(x, z, \max\{t, s\}) = M_Z(f(x), g(z), \max\{t, s\}) * \delta &\\
\geq M_Z(f(x), f(y), t) * M_Z(f(y), g(z), s) * \delta &\\
= M_X(x, y, t) * M_Z(f(y), g(z), s) * \delta &\\
= M_\delta(x, y, t) * M_\delta(y, z, s), &
\end{aligned}$$

and the proof of Case 2 is complete. Thus, $(M_\delta, *)$ is an admissible non-Archimedean fuzzy metric on $X \sqcup Y$. This finishes the proof.   □

Notice that the previous lemma fails to be true for $\delta = 1$.

Lemma 3.3. *Let $(X, M_X, *)$, $(Y, M_Y, *)$ and $(Z, M_Z, *)$ be three non-Archimedean fuzzy metric spaces and let M_1 and M_2 be two admissible fuzzy metrics on $X \sqcup Y$ and $Y \sqcup Z$, respectively. If, for every $\delta \in (0, 1)$, we define*

$$M_\delta(x, y, 0) = 0 \ \text{for every } x, y \in X \sqcup Y \sqcup Z,$$

and, for all $t > 0$,

$$\begin{aligned}
M_\delta(x, z, t) &= M_1(x, z, t) \ \text{ if } x, z \in X \sqcup Y, \\
M_\delta(x, z, t) &= M_2(x, z, t) \ \text{ if } x, z \in Y \sqcup Z, \\
M_\delta(x, z, t) &= M_\delta(z, x, t) = \sup_{y \in Y}\{M_1(x, y, t) * M_2(y, z, t)\} * \delta \ \text{ if } (x, z) \in X \times Z.
\end{aligned}$$

then, $(M_\delta, *)$ is an admissible non-Archimedean fuzzy metric on $X \sqcup Y \sqcup Z$.

Proof. By definition of M_δ we only need to check that it is a non-Archimedean fuzzy metric. Notice that Conditions (i), (ii) and (iii) in Definition 2.1 are straightforward, and that Condition (v) follows easily from Proposition 2.6. We shall prove Conditions (NA1) and (NA2). For this purpose, we first consider Condition (NA2), that is, we shall prove that $M_\delta(x, y, \_)$ is nondecreasing for all $x, y \in X \sqcup Y \sqcup Z$. By the definition of M_δ, we only need to address the case $x \in X$ and $z \in Z$. In this situation, if $t > s$, then we have

$$M_\delta(x, z, t) \geq M_1(x, y, t) * M_2(y, z, t) * \delta \geq M_1(x, y, s) * M_2(y, z, s) * \delta,$$

for all $y \in Y$ so that

$$M_\delta(x, z, t) \geq \sup_{y \in Y} \{M_1(x, y, s) * M_2(y, z, s)\} * \delta = M_\delta(x, z, s).$$

We now move on to Condition (NA1), i.e., the inequality

$$M_\delta(x, y, t) \geq M_\delta(x, z, t) * M_\delta(z, y, t)$$

for all $x, y, z \in X \sqcup Y \sqcup Z$ and all $t > 0$.

For, if one analyses the definition of M_δ, it is apparent that it suffices to take up the following two cases:

Case 1. $x, z \in X$ and $y \in Z$.

If $M_\delta(z, y, t) = 0$, the result is trivial. Assume now that $M_\delta(z, y, t) > 0$ and, for each $0 < \epsilon < M_\delta(z, y, t)$, choose $y_0 \in Y$ such that

$$M_1(z, y_0, t) * M_2(y_0, y, t) * \delta \geq M_\delta(z, y, t) - \epsilon.$$

Then,

$$\begin{aligned}
M_\delta(x, y, t) &\geq M_1(x, y_0, t) * M_2(y_0, y, t) * \delta \\
&\geq M_1(x, z, t) * M_1(z, y_0, t) * M_2(y_0, y, t) * \delta \\
&\geq M_\delta(x, z, t) * (M_\delta(z, y, t) - \epsilon),
\end{aligned}$$

and the continuity of the t-norm gives the desired result.

Case 2. $x, y \in X$ and $z \in Z$.

Assume with no loss of generality that $\min\{M_\delta(x, z, t), M_\delta(y, z, t)\} > 0$ and, for each $\epsilon > 0$ with $\epsilon < \min\{M_\delta(x, z, t), M_\delta(y, z, t)\}$, choose $y_0, y_1 \in Y$ such that

$$M_1(y, y_1, t) * M_2(y_1, z, t) * \delta \geq M_\delta(y, z, t) - \epsilon,$$

and

$$M_1(x, y_0, t) * M_2(y_0, z, t) * \delta \geq M_\delta(x, z, t) - \epsilon.$$

Then,

$$\begin{aligned}
M_\delta(x, y, t) = M_1(x, y, t) &\geq M_1(x, y_1, t) * M_1(y, y_1, t) \\
&\geq M_1(x, y_0, t) * M_1(y_0, y_1, t) * M_1(y, y_1, t) \\
&= M_1(x, y_0, t) * M_2(y_0, y_1, t) * M_1(y, y_1, t) \\
&\geq M_1(x, y_0, t) * M_2(y_0, z, t) * M_2(z, y_1, t) * M_1(y, y_1, t) \\
&\geq M_1(x, y_0, t) * M_2(y_0, z, t) * \delta * M_2(z, y_1, t) * M_1(y, y_1, t) * \delta \\
&\geq (M_\delta(x, z, t) - \epsilon) * (M_\delta(y, z, t) - \epsilon)
\end{aligned}$$

and the continuity of the t-norm yields the desired conclusion. □

The following theorem provides an alternative definition for a non-Archimedean Hausdorff-Gromov distance.

Theorem 3.4. *Given two non-Archimedean fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, *)$, then*

$$M_{GH}(X, Y, t) = \sup\{H_M(X, Y, t) : M \in \mathcal{A}(X \sqcup Y)\} \quad (t \geq 0)$$

where $\mathcal{A}(X \sqcup Y)$ stands for the set of all admissible non-Archimedean fuzzy metrics on $X \sqcup Y$.

Proof. Clearly,

$$\sup\{H_M(X, Y, t) : M \in \mathcal{A}(X \sqcup Y)\} \leq M_{GH}(X, Y, t)$$

since a smaller class of fuzzy metric spaces $(Z, M_Z, *)$ and of isometric embeddings are considered. On the other hand, for a non-Archimedean fuzzy metric space $(Z, M_Z, *)$ and embeddings $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$, we have, for all $\delta \in (0, 1)$, an admissible non-Archimedean fuzzy metric $(M_\delta, *)$ on $X \sqcup Y$ defined as in Lemma 3.2. We claim that $H_{M_\delta}(X, Y, t) = H_{M_Z}(f(X), g(Y), t) * \delta$ for all $t > 0$. Indeed, given $t > 0$, since g is an isometry, the equality

$$M_\delta(x, Y, s) = \sup_{y \in Y}\{M_Z(f(x), g(y), s) * \delta\}$$

$$= \sup_{g(y) \in g(Y)}\{M_Z(f(x), g(y), s)\} * \delta = M_Z(f(x), g(Y), s) * \delta$$

holds for every $x \in X$ and every $0 < s < t$. Therefore, as f is also an isometry, we have

$$\inf_{x \in X}\{M_\delta(x, Y, s)\} = \inf_{f(x) \in f(X)}\{M_Z(f(x), g(Y), s)\} * \delta \quad (0 < s < t).$$

We have just showed that $H_{M_\delta}^+(X, Y, t) = H_{M_Z}^+(f(X), g(Y), t) * \delta$ for all $t > 0$. In a similar way, we can prove $H_{M_\delta}^-(X, Y, t) = H_{M_Z}^-(f(X), g(Y), t) * \delta$ and, by the definition of the Hausdorff fuzzy metric, we obtain $H_{M_\delta}(X, Y, t) = H_{M_Z}(f(X), f(Y), t) * \delta$ for all $t > 0$.

If we now take a sequence δ_n converging to 1, then the sequence $H_{M_{\delta_n}}(X, Y, t)$ converges to $H_{M_Z}(f(X), g(Y), t)$, for each $t > 0$. Since $(Z, M_Z, *)$ is an arbitrary non-Archimedean fuzzy metric space, we obtain

$$\sup\{H_M(X, Y, t) : M \in \mathcal{A}(X \sqcup Y)\} \geq M_{GH}(X, Y, t)$$

which completes the proof. □

Our next aim is to study properties involving the Hausdorff-Gromov fuzzy distance.

Theorem 3.5. *If $(X, M_X, *)$ and $(Y, M_Y, *)$ are two non-Archimedean fuzzy metric spaces, then the following assertions hold:*

(i) $M_{GH}(X, Y, t) = M_{GH}(Y, X, t)$, *for all $t \geq 0$;*

(ii) $M_{GH}(X, Y, \_) : [0, +\infty) \to [0, 1]$ *is left-continuous.*

Proof. (i) is clear. To see (ii), notice that, by Theorem 3.4, $M_{GH}(X, Y, \cdot)$ is the supremum of a family of nondecreasing left-continuous functions. Thus, Proposition 2.6 applies. □

Theorem 3.6. *Let $(X, M_X, *)$, $(Y, M_Y, *)$ and $(Z, M_Z, *)$ be three non-Archimedean fuzzy metric spaces. Then,*

$$M_{GH}(X, Z, \max\{t, s\}) \geq M_{GH}(X, Y, t) * M_{GH}(Y, Z, s) \text{ for all } t, s > 0.$$

Proof. M_{GH} as a function of t is nondecreasing. Thus, in order to prove the theorem, it suffices to show

$$M_{GH}(X, Z, t) \geq M_{GH}(X, Y, t) * M_{GH}(Y, Z, t) \text{ for all } t > 0.$$

Now fix $t > 0$. We can clearly assume that both $M_{GH}(X, Y, t)$ and $M_{GH}(Y, Z, t)$ are different from zero. Take $0 < \delta_1 < M_{GH}(X, Y, t)$ and $0 < \delta_2 < M_{GH}(Y, Z, t)$. Then there exist admissible non-Archimedean fuzzy metrics $(M_1, *)$ and $(M_2, *)$ on $X \sqcup Y$ and on $Y \sqcup Z$, respectively, such that

$$H_{M_1}(X, Y, t) > \delta_1, \quad H_{M_2}(Y, Z, t) > \delta_2.$$

Taking account of the definition of the Hausdorff fuzzy metric $(H_{M_1}, *)$,

$$H_{M_1}(X, Y, t) = \min\{\sup_{0<s<t} \inf_{x \in X} M_1(x, Y, s), \ \sup_{0<s<t} \inf_{y \in Y} M_1(X, y, s)\},$$

given $x \in X$ we can find y_x and $0 < s_0 < t$ such that $M_1(x, y_x, s_0) > \delta_1$. Now, by definition of $H_{M_2}(Y, Z, t)$, we can choose $z_x \in Z$ and $0 < s_1 < t$ with $M_2(y_x, z_x, s_1) > \delta_2$.

For each $\delta \in (0, 1)$ consider the admissible non-Archimedean fuzzy metric $(M_\delta, *)$ on $X \sqcup Z$ as defined in Lemma 3.3. If $s > \max\{s_0, s_1\}$, the definition of $(M_\delta, *)$ applies in order to obtain

$$M_\delta(x, z_x, s) \geq M_1(x, y_x, s) * M_2(y_x, z_x, s) * \delta$$

$$\geq M_1(x, y_x, s_0) * M_2(y_x, z_x, s_1) * \delta > \delta_1 * \delta_2 * \delta.$$

We have just showed:

$$M_\delta(x, Z, s) \geq \delta_1 * \delta_2 * \delta, \text{ for all } x \in X, \text{ all } \delta \in (0, 1) \text{ and all } s > \max\{s_0, s_1\}.$$

Thus, $\inf_{x \in X} M_\delta(x, Z, s) \geq \delta_1 * \delta_2 * \delta$. In a similar way, there exists $s^\star$ such that the inequality $\inf_{z \in Z} M_\delta(X, z, s) \geq \delta_1 * \delta_2 * \delta$ holds for all $s > s^\star$. Thus,

$$H_{M_\delta}(X, Z, t) \geq \delta_1 * \delta_2 * \delta, \quad \text{ for all } \delta \in (0, 1).$$

Then, by definition of the non-Archimedean Gromov-Hausdorff fuzzy distance,

$$M_{GH}(X, Z, t) \geq \delta_1 * \delta_2 * \delta$$

for every $\delta \in (0, 1)$, what implies, by the continuity of the t-norm, that

$$M_{GH}(X, Z, t) \geq \delta_1 * \delta_2$$

for every $0 < \delta_1 < M_{GH}(X, Y, t)$ and $0 < \delta_2 < M_{GH}(Y, Z, t)$. Invoking again the continuity of the t-norm, we have

$$M_{GH}(X, Z, t) \geq M_{GH}(X, Y, t) * M_{GH}(Y, Z, t) \text{ for all } t > 0$$

which completes the proof. □

The previous results tell us that the non-Archimedean Gromov-Hausdorff fuzzy distance satisfies conditions (i), (iii), (NA) and (v) in definition of a non-Archimedean fuzzy metric. Notice that condition (ii) can fail to be true: there are two different non-Archimedean fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, *)$ such that $M_{GH}(X, Y, t)$ takes the value one for all $t > 0$ as, for example, in the case of $([0, 1], M_d, \cdot)$ and $(\mathbb{Q} \cap [0, 1], M_d, \cdot)$ where d denotes the Euclidean metric. In the case of compact non-Archimedean fuzzy metric spaces, we shall prove that condition (ii) is satisfied *modulo isometry*. For this, we need some previous results.

Recall (see [8, 16]) that a function f from a fuzzy metric space $(X, M_X, *)$ into a fuzzy metric space $(Y, M_Y, \star)$ is said to be *uniformly continuous* if for each $\varepsilon \in (0, 1)$ and $t > 0$ there exist $\delta \in (0, 1)$ and $s > 0$ such that $M_Y(f(x), f(y), t) > 1 - \varepsilon$ whenever $M_X(x, y, s) > 1 - \delta$. Note that every isometry is a uniformly continuous function.

Every fuzzy metric space $(Z, M_Z, *)$ has an admissible uniformity, say $\mathcal{U}_{M_Z}$, which has as a base the sets of the form

$$\{(x,y) \in Z \times Z : M_Z(x,y,1/n) > 1 - 1/n\}$$

for all $n \in \mathbb{N}$ (see [11, Theorem 1]). Manifestly, a function $f\colon (X, M_X, *) \to (Y, M_Y, \star)$ is uniformly continuous if and only if f is uniformly continuous as a function from the uniform space $(X, \mathcal{U}_{M_X})$ into the uniform space $(Y, \mathcal{U}_{M_Y})$. By means of this fact, the following lemma is a fairly direct consequence of [6, Proposition 8.3.10]. Even though it can be stated in a more general setting by exploiting the proof of [12, Lemma 2], this version is enough for our purposes.

Lemma 3.7. *If D is a dense subset of a compact fuzzy metric space $(X, M_X, *)$, then every uniformly continuous function f from $(D, M_X|_D, *)$ into a compact fuzzy metric space $(Y, M_Y, \star)$ has a uniformly continuous extension to the whole $(X, M_X, *)$.*

The two following lemmas provide some useful properties of isometries.

Lemma 3.8. *Let D be a dense subset of a fuzzy metric space $(X, M_X, *)$ and let $f\colon (D, M_X|_D, *) \to (Y, M_Y, \star)$ be an isometry. If f has a continuous extension $\widehat{f}$ to the whole $(X, M_X, *)$, then $\widehat{f}$ is an isometry.*

Proof. Fix $x, y \in X$ and let C be the set of points of $(0, +\infty)$ where both $M_X(x, y, \_)$ and $M_Y(\widehat{f}(x), \widehat{f}(y), \_)$ are continuous. Now take two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in D converging, respectively, to x and y. If $t \in C$, Lemma 2.3 (b) and the continuity of $\widehat{f}$ imply

$$M_X(x, y, t) = \lim_n M_X(x_n, y_n, t)$$

$$= \lim_n M_Y(f(x_n), f(y_n), t) = M_Y(\widehat{f}(x), \widehat{f}(y), t) \qquad \text{(I)}$$

We conclude the proof by showing that $M_X(x, y, t) = M_Y(\widehat{f}(x), \widehat{f}(y), t)$ for all $t \in (0, +\infty) \backslash C$. Indeed, since $(0, +\infty) \backslash C$ is countable, given $t \notin C$, we can choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ in C converging to t with $t_n < t$ for every $n \in \mathbb{N}$. By the equality (I), the left-continuity of $M_X(x, y, \_)$ and $M_Y(\widehat{f}(x), \widehat{f}(y), \_)$ yields

$$M_X(x, y, t) = \lim_n M_X(x, y, t_n)$$

$$= \lim_n M_Y(\widehat{f}(x), \widehat{f}(y), t_n) = M_Y(\widehat{f}(x), \widehat{f}(y), t)$$

which completes the proof. $\square$

Lemma 3.9. *If f is an isometry from a compact fuzzy metric space $(X, M_X, *)$ into itself, then f is onto.*

Proof. Suppose, on the contrary, that there is $x_0 \in X \backslash f(X)$. Since $f(X)$ is closed, there exists a ball $B(x_0, 1/n, t)$ such that $B(x_0, 1/n, t) \cap f(X) = \emptyset$. Now, take an integer k with $(1 - 1/k) * (1 - 1/k) > (1 - 1/n)$ and consider the cover $\mathcal{C} = \{B(x, 1/k, t/2)\}_{x\in X}$ of X. By compactness, there is a finite subcover $\{B(x_i, 1/k, t/2)\}_{i=1}^p$ of $\mathcal{C}$ of smallest cardinality. Since X and $f(X)$ are isometric, every finite cover of $f(X)$ by balls of the form $B(y, 1/k, t/2)$ has more than $p - 1$ elements. We shall see that this leads us to a contradiction. In fact, select a ball $B(x_j, 1/k, t/2)$ ($1 \le j \le p$) containing x_0. If there exists $z \in B(x_j, 1/k, t/2) \cap f(X)$, then

$$M_X(x_0, z, t) \ge M_X(x_0, x_j, t/2) * M_X(x_j, z, t/2)$$

$$\ge (1 - 1/k) * (1 - 1/k) > 1 - 1/n.$$

Hence $z \in B(x_0, 1/n, t) \cap f(X)$, contrary to the fact that $B(x_0, 1/n, t) \cap f(X) = \emptyset$. We have just shown that $\{B(x_i, 1/k, t/2) : 1 \leq i \leq p, i \neq j\}$ is a cover of $f(X)$ which provides the promised contradiction. $\square$

It is plain that if f is an isometry from a metric space (X, d) into itself (i.e., $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$), then f is an isometry considered as a function from the fuzzy metric space $(X, M_d, \cdot)$ into itself. Thus, the following classic result arises.

Corollary 3.10. *If f is an isometry from a compact metric space (X, d) into itself, then f is onto.*

We now apply the previous results to the non-Archimedean Gromov-Hausdorff fuzzy distance.

Theorem 3.11. *Two compact non-Archimedean fuzzy metric spaces $(X, M_X, *)$ and $(Y, M_Y, *)$ are isometric if and only if $M_{GH}(X, Y, t) = 1$ for all $t > 0$.*

Proof. Sufficiency. Let g be an isometry from $(Y, M_Y, *)$ onto $(X, M_X, *)$. If i is the identity mapping on X, then $H_{M_X}(i(X), g(Y), t) = 1$ for all $t > 0$. Thus, by Definition 3.1, $M_{GH}(X, Y, t) = 1$ for all $t > 0$.

Necessity. Suppose that $M_{GH}(X, Y, t) = 1$ for all $t > 0$ and fix a sequence $\delta_n \in (0, 1)$ converging to zero. Then, by Theorem 3.4, for each $n \in \mathbb{N}$ there exists an admissible non-Archimedean fuzzy metric $(M_n, *)$ on $X \sqcup Y$ with $H_{M_n}(X, Y, \delta_n) > 1 - \delta_n$.

Claim. If $x_0 \in X$, then, for each $n \in \mathbb{N}$, there exists $y(n) \in Y$ such that $M_n(x_0, y(n), \delta_n) > 1 - \delta_n$. Indeed, the definition of the Hausdorff fuzzy metric tells us that

$$\sup_{0 < s < \delta_n} \inf_{x \in X} M_n(x, Y, s) > 1 - \delta_n$$

which implies that we can choose $s_n < \delta_n$ such that $\inf_{x \in X} M_n(x, Y, s_n) > 1 - \delta_n$. By the choice of s_n, we have

$$\inf_{x \in X} M_n(x, Y, \delta_n) \geq \inf_{x \in X} M_n(x, Y, s_n) > 1 - \delta_n.$$

In particular, for $x_0 \in X$,

$$M_n(x_0, Y, \delta_n) = \sup_{y \in Y} M_n(x_0, y.\delta_n) > 1 - \delta_n$$

so that there must exist an element $y(n) \in Y$ with $M_n(x_0, y(n), \delta_n) > 1 - \delta_n$. This proves the claim. Now notice that $(X, M_X, *)$ is separable because it is a compact fuzzy metric space. Let D be a dense countable subset of X. With each $n \in \mathbb{N}$, associate a function j_n from D into Y as follows: for each $d \in D$, $j_n(d)$ is a point $y(n) \in Y$ such that $M_n(d, y(n), \delta_n) > 1 - \delta_n$. By compactness of $(Y, M_Y, *)$ and a standard diagonal argument, we can assume that $\{j_n\}_{n \in \mathbb{N}}$ pointwise converges to a function j on D.

Next we shall prove that j is an isometry. For this, given $d_i, d_j \in D$, let D_{ij}^X (respectively, D_{ij}^Y) denote the set of points of discontinuity of $M_X(d_i, d_j, \_)$ (respectively, of $M_Y(j(d_i), j(d_j), \_)$). Let C denote the set

$$C = (0, +\infty) \setminus \left(\bigcup_{i,j} D_{ij}^X \cup \bigcup_{i,j} D_{ij}^Y \right).$$

Now, fix $t \in C$ and take n_0 such that $2\delta_{n_0} < t$. Then, for every $n \geq n_0$ and every $d_1, d_2 \in D$, we have

$$M_Y(j_n(d_1), j_n(d_2), t) = M_n(j_n(d_1), j_n(d_2), t)$$
$$\geq M_n(j_n(d_1), d_1, \delta_n) * M_n(d_1, d_2, t - 2\delta_n) * M_n(d_2, j_n(d_2), \delta_n)$$
$$> (1 - \delta_n) * M_X(d_1, d_2, t - 2\delta_n) * (1 - \delta_n).$$

Since $[0,1]$ is compact, we can assume that $\lim_n M_Y(j_n(d_1), j_n(d_2), t)$ exists. By Lemma 2.3 (b) and the left continuity of $M_X(d_1, d_2, \_)$, taking limits when n goes to infinity, we get

$$M_Y(j(d_1), j(d_2), t) \geq M_X(d_1, d_2, t).$$

The same argument, but taking $M_X(d_1, d_2, t)$ at the starting, proves

$$M_X(d_1, d_2, t) \geq M_Y(j(d_1), j(d_2), t).$$

We have just shown that, for every $t \in C$ and every $d_1, d_2 \in D$, the equality $M_X(d_1, d_2, t) = M_Y(j(d_1), j(d_2), t)$ holds. To see that the previous equality is valid for every $t > 0$, i.e., that j is an isometry, consider $t \notin C$ and an increasing sequence $t_n \in C$ which converges to t. Then, for every $d_1, d_2 \in D$,

$$M_X(d_1, d_2, t)$$
$$= \lim_n M_X(d_1, d_2, t_n) = \lim_n M_Y(j(d_1), j(d_2), t_n)$$
$$= M_Y(j(d_1), j(d_2), t),$$

because $M_X(d_1, d_2, \_)$ and $M_Y(j(d_1), j(d_2), \_)$ are left-continuous functions.

Now, by Lemmas 3.7 and 3.8, there exists an isometry $g \colon X \to Y$ with $g|_D = j$. To finish the proof we shall show that g is onto. For this, repeat the construction of g but interchanging the roles of $(X, M_X, *)$ and $(Y, M_Y, *)$ obtaining an isometry $h \colon (Y, M_Y, *) \to (X, M_X, *)$. Consider now the isometry $h \circ g \colon (X, M_X, *) \to (X, M_X, *)$. By Lemma 3.9, $h \circ g$ is onto and so is g. $\square$

Remark 3.12. Compactness was only used in the proof of *necessity*. Thus, if $(X, M_X, *)$ and $(Y, M_Y, *)$ are isometric, then $M_{GH}(X, Y, t) = 1$ for all $t > 0$.

Corollary 3.13. *For each $i = 1, 2$, let $(X_i, M_{X_i}, *)$ be a compact non-Archimedean fuzzy metric space isometric to $(Y_i, M_{Y_i}, *)$. Then*

$$M_{GH}(X_1, X_2, t) = M_{GH}(Y_1, Y_2, t)$$

for all $t \geq 0$.

Proof. The case $t = 0$ is obvious. If $t > 0$, then, by Theorems 3.6 and 3.11, we have

$$M_{GH}(X_1, X_2, t) \geq M_{GH}(X_1, Y_1, t) * M_{GH}(Y_1, Y_2, t) * M_{GH}(Y_2, X_2, t)$$
$$\geq 1 * M_{GH}(Y_1, Y_2, t) * 1 = M_{GH}(Y_1, Y_2, t).$$

The result now follows by interchanging the roles of $(X_i, M_{X_i}, *)$ and $(Y_i, M_{Y_i}, *)$ $(i = 1, 2)$. $\square$

Let $\mathcal{M} = \{(X_j, M_{X_j}, *) : j \in J\}$ be a set of compact non-Archimedean fuzzy metric spaces and let $\mathcal{I}(\mathcal{M})$ denote the set of all isometry classes of elements of $\mathcal{M}$. The previous results show

Theorem 3.14. $(\mathcal{I}(\mathcal{M}), M_{GH}, *)$ *is a non-Archimedean fuzzy metric space.*

4. Gromov-Hausdorff fuzzy convergence

This section is devoted to study criteria for convergence of sequences of non-Archimedean fuzzy metric spaces. Taking into account the definition of a convergent sequence in a fuzzy metric space, the last theorem in Section 3 suggests the following

Definition 4.1. A sequence $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ of non-Archimedean fuzzy metric spaces converges to a non-Archimedean fuzzy metric space $(X, M_X, *)$ if $\lim_n M_{GH}(X_n, X, t) = 1$ for all $t > 0$.

We need some basic properties which will be helpful. First some lemmas that are of interest in themselves.

Lemma 4.2. ([17]) For each subset A of a non-Archimedean fuzzy metric space $(X, M_X, *)$, the equality

$$M_X(x, A, t) = M_X(x, \mathrm{cl}_X A, t)$$

holds for any $x \in X$ and $t > 0$.

Lemma 4.3. If A and B are two subsets of a fuzzy metric space $(X, M_X, *)$, then the equality

$$\inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) = \inf_{b \in B} M_X(A, b, t)$$

holds for all $t > 0$.

Proof. Let $t > 0$. We only need to prove that $\inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) \geq \inf_{b \in B} M_X(A, b, t)$. For this, suppose, contrary we claim, that $\inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) < \inf_{b \in B} M_X(A, b, t)$ and choose $\varepsilon > 0$ such that

$$\inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) + \varepsilon < \inf_{b \in B} M_X(A, b, t).$$

Let $b_0 \in \mathrm{cl}_X B$ with $\inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) \leq M_X(A, b_0, t) < \inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) + \varepsilon$ and let $n \in \mathbb{N}$ such that $\inf_{b \in B} M_X(A, b, t) * (1 - 1/n) > \inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) + \varepsilon$. Since B is dense in $\mathrm{cl}_X B$, there is $b_1 \in B$ such that $M_X(b_0, b_1, t) > (1 - 1/n)$. Therefore, if $a \in A$, then

$$M_X(a, b_0, t) \geq M_X(a, b_1, t) * M_X(b_0, b_1, t)$$
$$\geq M_X(a, b_1, t) * (1 - 1/n)$$

which yields

$$M_X(A, b_0, t) \geq M_X(A, b_1, t) * (1 - 1/n).$$

But $M_X(A, b_1, t) \geq \inf_{b \in B} M_X(A, b, t)$ so that

$$M_X(A, b_0, t) \geq M_X(A, b_1, t) * (1 - 1/n) \geq \inf_{b \in B} M_X(A, b, t) * (1 - 1/n)$$
$$\geq \inf_{b \in \mathrm{cl}_X B} M_X(A, b, t) + \varepsilon,$$

a contradiction. This completes the proof. $\qquad \square$

Sherwood showed in [33] that every Menger space having a continuous t-norm has a completion which is unique up to isometry. Since every fuzzy metric space $(Z, M_Z, *)$ (in the sense of Kramosil and Michalek) is *equivalent* to a Menger space belonging to this class (see [19]), one can easily deduce that every fuzzy metric space has a metric completion $(\widehat{Z}, \widehat{M_Z}, *)$ which is unique up to isometry (if no confusion can arise, we simply write $\widehat{Z}$). It is apparent that if $(Z, M_Z, *)$ is non-Archimedean, then so is $(\widehat{Z}, \widehat{M_Z}, *)$.

Lemma 4.4. *Let $(X, M_X, *)$ and $(Y, M_Y, *)$ be two non-Archimedean fuzzy metric spaces. Then every admissible non-Archimedean fuzzy metric M on $X \sqcup Y$ has an extension $\widetilde{M}$ to $X \sqcup \widehat{Y}$.*

Proof. Let M be an admissible non-Archimedean fuzzy metric on $X \sqcup Y$. Consider now the completion $(\widehat{X \sqcup Y}, \widehat{M}, *)$ of $(X \sqcup Y, M, *)$. As noted before, $(\widehat{X \sqcup Y}, \widehat{M}, *)$ is a non-Archimedean fuzzy metric space. For each $x \in X$, $\mathrm{cl}_{\widehat{X \sqcup Y}}(\{x\} \times Y)$ is a complete non-Archimedean fuzzy metric space where $\{x\} \times Y$ is dense so that it coincides with the completion of $\{x\} \times Y$. Thus, $\mathrm{cl}_{\widehat{X \sqcup Y}}(\{x\} \times Y) = \{x\} \times \widehat{Y}$. This fact allows us to define an admissible non-Archimedean fuzzy metric $\widetilde{M}$ on $X \sqcup \widehat{Y}$ as follows:

$$\widetilde{M}(x, y, 0) = 0 \quad \text{for all } x, y \in X \sqcup \widehat{Y},$$

and, for all $t > 0$,

$$\widetilde{M}(x, y, t) = \begin{cases} M_X(x, y, t) & \text{if } x, y \in X, \\ M_Y(x, y, t) & \text{if } x, y \in Y, \\ \widehat{M}(x, y, t) & \text{otherwise.} \end{cases}$$

Notice that $\widetilde{M}$ is the restriction to $X \times \widehat{Y}$ of the fuzzy metric $\widehat{M}$ which implies that it is a non-Archimedean fuzzy metric. $\qquad\square$

The following result is a straightforward consequence of Lemmas 4.2, 4.3, 4.4 and Theorem 3.4.

Corollary 4.5. *Let $(X, M_X, *)$, $(Y, M_Y, *)$ be two non-Archimedean fuzzy metric spaces. If A is dense in X, then $M_{GH}(X, Y, t) = M_{GH}(A, Y, t)$ for all $t > 0$. In particular, if $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ converges to $(X, M_X, *)$, then $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ converges to the completion $(\widehat{X}, \widehat{M_{\widehat{X}}}, *)$ of $(X, M_X, *)$.*

Definition 4.6. [27, Definition 1] A fuzzy metric space $(X, M_X, *)$ is called *precompact* if for each $r > 0$, with $0 < r < 1$ and $t > 0$, there is a finite subset A of X, such that $X = \bigcup_{a \in A} B(a, r, t)$. In this case, we say that M is a precompact fuzzy metric.

Let $(X, M_X, *)$ be a fuzzy metric space. Given δ_1, δ_2 with $\delta_1 \in (0, 1)$ and $\delta_2 > 0$, we define the (δ_1, δ_2)-*cover number* of $(X, M_X, *)$ as

$$\mathrm{Cov}(X, \delta_1, \delta_2) = \min \left\{ |C| : X = \bigcup_{c \in C} B(c, \delta_1, \delta_2) \right\}.$$

where $|C|$ stands for the cardinality of C. Notice that $\mathrm{Cov}(X, \delta_1, \delta_2)$ is finite for all δ_1, δ_2 if and only if $(X, M_X, *)$ is precompact.

If n, m are two natural numbers such that $(1 - 1/n) * (1 - 1/n) > (1 - 1/m)$, then it is an easy matter to show that $1/n \leq 1/m$. We feel free of using this fact without explicit mention.

Lemma 4.7. *Let $(X, M_X, *)$ be a precompact non-Archimedean fuzzy metric space. Given a non-Archimedean fuzzy metric space $(Y, M_Y, *)$, if there exist natural numbers s, p, m such that*

$$\left(1 - \frac{1}{m}\right) * \left(1 - \frac{1}{m}\right) > \left(1 - \frac{1}{p}\right),$$

$$\left(1 - \frac{1}{p}\right) * \left(1 - \frac{1}{p}\right) > \left(1 - \frac{1}{s}\right)$$

and $M_{GH}\left(X, Y, \frac{1}{m}\right) \geq \left(1 - \frac{1}{m}\right)$, then

$$\mathrm{Cov}\left(X, \frac{1}{m}, \frac{1}{m}\right) \geq \mathrm{Cov}\left(Y, \frac{1}{s}, \frac{1}{s}\right).$$

Proof. Since $M_{GH}\left(X, Y, \frac{1}{m}\right) \geq \left(1 - \frac{1}{m}\right)$, we can choose an admissible non-Archimedean fuzzy metric M on $X \sqcup Y$ such that

$$H_M\left(X, Y, \frac{1}{m}\right) > \left(1 - \frac{1}{m}\right).$$

Fix $y_0 \in Y$. Taking into account the definition of H_M, there is $x_0 \in X$ such that $M(x_0, y_0, 1/m) > (1 - \frac{1}{m})$. Let now be $\{x_1, x_2, \ldots, x_n\}$ with $n = \mathrm{Cov}(X, 1/m, 1/m)$ and $X = \bigcup_{i=1}^{n} B(x_i, 1/m, 1/m)$. If now we choose x_i such that $M(x_0, x_i, 1/m) > (1 - 1/m)$, we have

$$
\begin{aligned}
M(y_0, x_i, 1/p) &\geq M(y_0, x_i, 1/m) \\
&\geq M(y_0, x_0, 1/m) * M(x_0, x_i, 1/m) \\
&\geq (1 - 1/m) * (1 - 1/m) > (1 - 1/p).
\end{aligned}
$$

As above, since $H_M(X, Y, 1/m) > (1 - 1/m)$, we can pick, for each $i = 1, 2, \ldots, n$, a point $y_i \in Y$ with $M(x_i, y_i, 1/m) > (1 - 1/m)$. Notice that

$$M(x_i, y_i, 1/p) \geq M(x_i, y_i, 1/m) > (1 - 1/m) \geq (1 - 1/m) * (1 - 1/m) > (1 - 1/p).$$

We complete the proof by showing that $Y = \bigcup_{i=1}^{n} B(y_i, 1/s, 1/s)$. Indeed, if $y \in Y$, pick $x_i \in \{x_1, \ldots, x_n\}$ such that $M(y, x_i, 1/p) > (1 - 1/p)$. Then

$$
\begin{aligned}
M(y, y_i, 1/s) &\geq M(y, y_i, 1/p) \\
&\geq M(y, x_i, 1/p) * M(x_i, y_i, 1/p) \\
&(1 - 1/p) * (1 - 1/p) > (1 - 1/s).
\end{aligned}
$$

$\square$

Lemma 4.8. *If a sequence $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ of precompact non-Archimedean fuzzy metric spaces converges to $(X, M_X, *)$, then $(X, M_X, *)$ is precompact.*

Proof. It suffices to prove that, for each $s \in \mathbb{N}$, $\mathrm{Cov}(X, 1/s, 1/s)$ is finite. By the continuity of the t-norm $*$, we can find natural numbers m, p such that m, p, s satisfy conditions of Lemma 4.7. Now, since $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ converges to $(X, M, *)$, there exists $n \in \mathbb{N}$ such that $M_{GH}(X_n, X, 1/m) >$

$(1 - 1/m)$. By Lemma 4.7, $\mathrm{Cov}(X, 1/s, 1/s) \leq \mathrm{Cov}(X_n, 1/m, 1/m)$. Since $(X_n, M_n, *)$ is precompact, $\mathrm{Cov}(X_n, 1/m, 1/m)$ is finite and the proof is complete.   □

Theorem 4.9. *If* $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ *is a sequence of non-Archimedean fuzzy metric spaces converging to* $(X, M_X, *)$, *then the following hold:*

(a) $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ *converges to every non-Archimedean fuzzy metric space* $(Y, M_Y, *)$ *isometric to* $(X, M_X, *)$.

(b) *If* $(X, M_X, *)$ *is compact and* $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ *converges to a compact non-Archimedean fuzzy metric space* $(Y, M_Y, *)$, *then* $(X, M_X, *)$ *and* $(Y, M_Y, *)$ *are isometric.*

(c) *If* $(X_n, M_{X_n}, *)$ *is precompact for all* $n \in \mathbb{N}$, *then* $\{(X_n, M_{X_n}, *)\}_{n \in \mathbb{N}}$ *converges to a compact non-Archimedean fuzzy metric space* $(Y, M_Y, *)$.

Proof. (a) is a consequence of Theorem 3.6 and Remark 3.12 and (b) easily follows from Theorem 3.6 and Theorem 3.11. To obtain (c) it suffices to apply Corollary 4.5, Lemma 4.8 and the fact that the completion of a precompact non-Archimedean fuzzy metric space is compact (see [17, Corollary 3.8]).   □

We now work toward the establishment of Gromov's theorem on completeness in the realm of non-Archimedean fuzzy metric spaces, one of the most important tools in Gromov's convergence. First we need two lemmas.

Lemma 4.10. *Let* $*$ *be a t-norm. Then there is a sequence* $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset [0, 1]$ *such that, for all* $n \in \mathbb{N}$, $0 < \varepsilon_n < \frac{1}{2^n}$ *and the following holds: if* $1 \leq i < n$, *then*

$$(1 - \varepsilon_i) < (1 - \varepsilon_{i+1}) * (1 - \varepsilon_{i+2}) * \cdots * (1 - \varepsilon_n).$$

Proof. We proceed by induction on n. Suppose we have $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ satisfying

(1) $0 < \varepsilon_i < \frac{1}{2^i}$, $i = 1, \ldots, n$.

(2) $(1 - \varepsilon_i) < (1 - \varepsilon_{i+1}) * \cdots * (1 - \varepsilon_n)$ for all $1 \leq i < n$.

Take now a sequence $(\delta_k)_{k \in \mathbb{N}}$ converging to 1. Since each product $(1 - \varepsilon_{i+1}) * \cdots * (1 - \varepsilon_n) * \delta_k$ converges to $(1 - \varepsilon_{i+1}) * \cdots * (1 - \varepsilon_n)$, for each $1 \leq i < n$ we can choose δ_i such that

(a) $(1 - \varepsilon_i) < (1 - \varepsilon_{i+1}) * \cdots * (1 - \varepsilon_n) * \delta_i$ for all $1 \leq i < n$,

(b) $(1 - \varepsilon_n) < \delta_n$.

If we now choose $0 < \varepsilon_{n+1} < \frac{1}{2^{n+1}}$ with $(1 - \varepsilon_{n+1}) > \delta_i$ for all $1 \leq i \leq n$, then $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon_{n+1}\}$ verifies the desired properties.   □

Suppose we are given a t-norm $*$. Then the real number $a_1 * a_2 * \cdots * a_n$ $(a_i \in [0, 1])$ will be denoted by $\prod_{k=1}^{n}{}^{*} a_k$. As usual, if the t-norm $*$ is the product $\cdot$, we simply write $\prod_{k=1}^{n} a_k$.

Lemma 4.11. *Let* $\{(X_i, M_i, *)\}_{i \in \mathbb{N}}$ *be a sequence of compact non-Archimedean fuzzy metric spaces and let* $N^{i,i+1}$ *be an admissible fuzzy metric on* $X_i \sqcup X_{i+1}$. *If* $i < j$, *then let* $M^{ij}(x, y, t)$ *be the function on* $(X_i \sqcup X_j) \times (X_i \sqcup X_j) \times [0, \infty[$ *defined as: if* $x \in X_i$ *and* $y \in X_j$, *then*

$$M^{ij}(x, y, t) = \sup\{\prod_{k=i}^{j-1}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) \ : \ x_k \in X_k, \ x_i = x, \ x_j = y\}$$

for all $t > 0$, and

$M^{ij}(x, y, 0) = 0$, *for all $x \in X_i$ and $y \in X_j$,*
$M^{ij}(y, x, t) = M^{ij}(x, y, t)$ *for all $x \in X_i$, $y \in X_j$ and all $t \geq 0$,*
$M^{ij}(x, z, t) = M_i(x, z, t)$, *for all $x, z \in X_i$ and all $t \geq 0$,*
$M^{ij}(y, z, t) = M_j(y, z, t)$, *for all $y, z \in X_j$ and all $t \geq 0$.*

Then M^{ij} is an admissible non-Archimedean fuzzy metric on $X_i \sqcup X_j$ satisfying the property

$$M^{ik}(x_i, x_k, t) \geq M^{ij}(x_i, x_j, t) * M^{jk}(x_j, x_k, t)$$

for all $i < j < k$, $x_r \in X_r$, $r = i, j, k$ and all $t > 0$.

Proof. We have to check that M^{ij} satisfies the conditions of Definition 2.1. Conditions (i) and (iii) are obvious. Condition (NA2) is clear and Condition (v) follows from Proposition 2.6. To see Condition (ii), it suffices to consider the case $x \in X_i$ and $y \in X_j$. Since X_i is closed in $(X_i \sqcup X_{i+1}, N^{i,i+1}, *)$, there exists $t > 0$ such that $N^{i,i+1}(x, X_{i+1}, t) < 1$ and then

$$M^{ij}(x, y, t) \leq \sup_{z \in X_{i+1}} N^{i,i+1}(x, z, t) = N^{i,i+1}(x, X_{i+1}, t) < 1$$

which shows (ii). We close the proof by showing Condition (NA1), that is, $M^{ij}(x, y, t) \geq M^{ij}(x, z, t) * M^{ij}(z, y, t)$ for all $x, y, z \in X_i \sqcup X_j$ and all $t > 0$. We need to consider only two cases:

Case 1: $x, z \in X_i$, $y \in X_j$.
Assume, with no loss of generality, that $0 < M^{ij}(z, y, t)$. Then, for each $\varepsilon > 0$, there exists a chain $x_{i+1}, \ldots, x_{j-1}$ such that

$$N^{i,i+1}(z, x_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-1,j}(x_{j-1}, y, t) \geq M^{ij}(z, y, t) - \epsilon$$

which implies that

$$M^{ij}(x, y, t) \geq N^{i,i+1}(x, x_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-1,j}(x_{j-1}, y, t)$$

$$\geq N^{i,i+1}(x, z, t) * N^{i,i+1}(z, x_{i+1}, t)$$

$$* \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-1,j}(x_{j-1}, y, t)$$

$$\geq M^{ij}(x, z, t) * \left(M^{ij}(z, y, t) - \epsilon\right)$$

and, taking limits when ϵ goes to zero, we obtain

$$M^{ij}(x, y, t) \geq M^{ij}(x, z, t) * M^{ij}(z, y, t).$$

Case 2: $x, y \in X_i$, $z \in X_j$.
Assume, without loss of generality, that $0 < \min\{M^{ij}(x, z, t), M^{ij}(y, z, t)\}$. Given $\varepsilon > 0$, choose a chain $x_{i+1}, \ldots, x_{j-1}$ with

$$N^{i,i+1}(x, x_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-1,j}(x_{j-1}, z, t) \geq M^{ij}(x, z, t) - \epsilon$$

and a chain $y_{i+1}, \ldots, y_{j-1}$ with

$$N^{i,i+1}(y, y_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(y_k, y_{k+1}, t) * N^{j-1,j}(y_{j-1}, z, t) \geq M^{ij}(y, z, t) - \epsilon.$$

Then

$$M^{ij}(x, y, t) = M_i(x, y, t) = N^{i,i+1}(x, y, t)$$

$$\geq N^{i,i+1}(x, x_{i+1}, t) * N^{i,i+1}(x_{i+1}, y_{i+1}, t) * N^{i,i+1}(y_{i+1}, y, t)$$

$$= N^{i,i+1}(x, x_{i+1}, t) * N^{i+1,i+2}(x_{i+1}, y_{i+1}, t) * N^{i,i+1}(y_{i+1}, y, t)$$

$$\geq N^{i,i+1}(x, x_{i+1}, t) * N^{i+1,i+2}(x_{i+1}, x_{i+2}, t) * N^{i+1,i+2}(x_{i+2}, y_{i+2}, t)$$

$$* N^{i+1,i+2}(y_{i+2}, y_{i+1}, t) * N^{i,i+1}(y_{i+1}, y, t) \geq \ldots$$

$$\geq N^{i,i+1}(x, x_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-2,j-1}(x_{j-1}, y_{j-1}, t)$$

$$* N^{j-1,j}(x_{j-1}, y, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(y_k, y_{k+1}, t) * N^{i,i+1}(y, y_{i+1}, t)$$

$$\geq N^{i,i+1}(x, x_{i+1}, t) * \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(x_k, x_{k+1}, t) * N^{j-1,j}(x_{j-1}, z, t) * N^{j-1,j}(y_{j-1}, z, t)$$

$$* \prod_{k=i+1}^{j-2}{}^{*} N^{k,k+1}(y_k, y_{k+1}, t) * N^{i,i+1}(y, y_{i+1}, t)$$

$$\geq (M^{ij}(x, z, t) - \epsilon) * (M^{ij}(y, z, t) - \epsilon)$$

and the continuity of the t-norm applies to obtain that

$$M^{ij}(x, y, t) \geq M^{ij}(x, z, t) * M^{ij}(y, z, t).$$

This completes the proof of (NA1). We close the proof by showing that

$$M^{ik}(x_i, x_k, t) \geq M^{ij}(x_i, x_j, t) * M^{jk}(x_j, x_k, t)$$

whenever $i < j < k$, $x_r \in X_r$, $r = i, j, k$ and $t > 0$.
We can assume with no loss of generality that there exists $\varepsilon > 0$ such that

$$0 < \epsilon < \min\{M^{ij}(x_i, x_j, t), M^{jk}(x_j, x_k, t)\}.$$

Consider a chain $x_{i+1}, \ldots, x_{j-1}$ such that

$$\prod_{p=i}^{j-1}{}^{*} N^{p,p+1}(x_p, x_{p+1}, t) \geq M^{ij}(x_i, x_j, t) - \epsilon$$

and a chain $x_{j+1}, \ldots, x_{k-1}$ with

$$\prod_{p=j}^{k-1}{}^{*} N^{p,p+1}(x_p, x_{p+1}, t) \geq M^{jk}(x_j, x_k, t) - \epsilon.$$

Then,

$$M^{ik}(x_i, x_k, t) \geq \prod_{p=i}^{k-1}{}^{*} N^{p,p+1}(x_p, x_{p+1}, t)$$

$$\geq (M^{ij}(x_i, x_j, t) - \epsilon) * (M^{jk}(x_j, x_k, t) - \epsilon).$$

Since the t-norm $*$ is continuous, the result follows by taking limits when ε approaches to zero. $\qquad \square$

Now, we prove a crucial property of the Hausdorff non-Archimedean fuzzy metric associated with these fuzzy metrics M^{ij}.

Lemma 4.12. *If the fuzzy metric M^{ij} is defined as in Lemma 4.11, then we have*

$$H_{M^{ij}}(X_i, X_j, t) \geq H_{N^{i,i+1}}(X_i, X_{i+1}, t) * H_{N^{i+1,i+2}}(X_{i+1}, X_{i+2}, t) * \ldots \ldots * H_{N^{j-1,j}}(X_{j-1}, X_j, t)$$

for all $t > 0$, $i < j$.

Proof. Fix $t > 0$ and assume with no loss of generality that we can find positive real numbers $\delta_{i+1}, \delta_{i+2}, \ldots, \delta_j$ such that, for $r = 0, 1, \ldots, j - i - 1$, we have

$$H_{N^{i+r,i+r+1}}(X_{i+r}, X_{i+r+1}, t) > \delta_{i+r+1}.$$

Then, by definition of the Hausdorff fuzzy metric, given $x_i \in X_i$, we can find a chain $\{x_i, x_{i+1}, \ldots, x_j\}$ such that

$$N^{i+r,i+r+1}(x_{i+r}, x_{i+r+1}, t) > \delta_{i+r+1}$$

$(r = 0, 1, \ldots, j - i - 1)$.
According to the definition of M^{ij}, these inequalities imply

$$M^{ij}(x_i, x_j, t) \geq \delta_{i+1} * \delta_{i+2} * \ldots * \delta j.$$

We have just shown that, for every $x_i \in X_i$,

$$M^{ij}(x_i, X_j, t) \geq \delta_{i+1} * \delta_{i+2} * \ldots * \delta j.$$

Thus, $\inf_{x \in X_i} M^{ij}(x, X_j, t) \geq \delta_{i+1} * \delta_{i+2} * \ldots * \delta j$. A similar argument to the previous one proves that $\inf_{x \in X_j} M^{ij}(X_i, x, t) \geq \delta_{i+1} * \delta_{i+2} * \ldots * \delta j$ so that

$$H_{M^{ij}}(X_i, X_j, t) \geq \delta_{i+1} * \delta_{i+2} * \ldots * \delta j.$$

Choosing δ_r converging to $H_{N^{i+r,i+r+1}}(X_{i+r}, X_{i+r+1}, t)$ (for all $r = 0, 1, \ldots, j - i - 1$) and taking into account that the t-norm $*$ is continuous, we obtain

$$H_{M^{ij}}(X_i, X_j, t) \geq H_{N^{i,i+1}}(X_i, X_{i+1}, t) * H_{N^{i+1,i+2}}(X_{i+1}, X_{i+2}, t) * \ldots \ldots * H_{N^{j-1,j}}(X_{j-1}, X_j, t).$$

This completes the proof. $\qquad \square$

Following [9], a sequence $\{(X_n, M_n, *)\}_{n\in\mathbb{N}}$ of non-Archimedean fuzzy metric spaces is called an M_{GH}-Cauchy sequence if for each $\varepsilon \in (0, 1)$ and each $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M_{GH}(X_n, X_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$. Recall that the notion of a fuzzy pseudometric arises when we replace condition (ii) of Definition 2.1 by (ii'): $M(x, x, t) = 1$ for all $x \in X$ and all $t > 0$. The definition of a non-Archimedean pseudometric is apparent. The completeness theorem states the following.

Theorem 4.13. *If $\{(X_i, M_i, *)\}_{i\in\mathbb{N}}$ is an M_{GH}-Cauchy sequence of compact non-Archimedean fuzzy metric spaces, then $(X_i, M_i, *)$ M_{GH}-converges to a compact non-Archimedean fuzzy metric space $(X, M, *)$.*

Proof. The proof proceeds via the construction of the limit space $(X, M, *)$. To do this, first notice that, since the sequence $\{(X_i, M_i, *)\}_{i\in\mathbb{N}}$ is M_{GH}-Cauchy, it suffices to obtain a convergent subsequence. Hence we can assume with no loss of generality that there exists a sequence (ε_i) of real numbers as in Lemma 4.10 such that

$$M_{GH}(X_i, X_{i+1}, \varepsilon_i) > 1 - \varepsilon_i, \text{ for every } i = 1, 2, \ldots.$$

The definition of the Gromov-Hausdorff non-Archimedean fuzzy metric permits us to choose admissible non-Archimedean fuzzy metrics $N^{i,i+1}$ on $X_i \sqcup X_{i+1}$ with

$$H_{N^{i,i+1}}(X_i, X_{i+1}, \varepsilon_i) > 1 - \varepsilon_i, \text{ for every } i = 1, 2, \ldots.$$

Let M^{ij} be the admissible non-Archimedean fuzzy metric on $X_i \sqcup X_j$, defined as in Lemma 4.11 and consider the set

$$\hat{X} = \{(x_j) \; : \; \lim_{i,j} M^{ij}(x_i, x_j, t) = 1, \text{ for all } t > 0 \}.$$

where, for sake of simplicity, (x_j) stands for the sequence $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \in X_j$ for all $j \in \mathbb{N}$. As a first step we state some useful facts about $\hat{X}$.

Fact 1. $\hat{X} \neq \emptyset$.

Since $H_{N^{i,i+1}}(X_i, X_{i+1}, \varepsilon_i) > 1 - \varepsilon_i$ for every $i = 1, 2, \ldots$, the definition of the Hausdorff fuzzy metric allows us to obtain, for all $x \in X_i$,

$$N^{i,i+1}(x, X_{i+1}, \varepsilon_i) > 1 - \varepsilon_i \quad i = 1, 2, \ldots$$

With this in mind, it is straightforward to construct a sequence (x_k) such that $x_k \in X_k$ and

$$N^{k,k+1}(x_k, x_{k+1}, \varepsilon_k) > 1 - \varepsilon_k, \quad k = 1, 2, \ldots$$

We shall prove that (x_k) belongs to $\hat{X}$. For, fix $t > 0$ and choose $\varepsilon_i < t$. Acoording to Lemma 4.10 and Lemma 4.11, we have

$$
\begin{aligned}
M^{ij}(x_i, x_j, t) &\geq M^{ij}(x_i, x_j, \varepsilon_i) \\
&\geq N^{i,i+1}(x_i, x_{i+1}, \varepsilon_i) * N^{i+1,i+2}(x_{i+1}, x_{i+2}, \varepsilon_i) * \cdots * N^{j-1,j}(x_{j-1}, x_j, \varepsilon_i) \\
&\geq N^{i,i+1}(x_i, x_{i+1}, \varepsilon_i) * N^{i+1,i+2}(x_{i+1}, x_{i+2}, \varepsilon_{i+1}) * \cdots * N^{j-1,j}(x_{j-1}, x_j, \varepsilon_{j-1}) \\
&> (1 - \varepsilon_i) * (1 - \varepsilon_{i+1}) * \cdots * (1 - \varepsilon_{j-1}) \geq 1 - \varepsilon_{i-1}
\end{aligned}
$$

for all $j > i$. Then $\lim_{i,j} M^{ij}(x_i, x_j, t) = 1$, that is, $(x_k) \in \hat{X}$.

Fact 2. If (x_j) and (y_j) belong to $\hat{X}$, then there exists $\lim_j M_j(x_j, y_j, t)$ for all $t > 0$.

To see this, fix $t > 0$ and suppose that we have two convergent subsequences $\{M_{n_k}(x_{n_k}, y_{n_k}, t)\}_{k \in \mathbb{N}}$ and $\{M_{p_k}(x_{p_k}, y_{p_k}, t)\}_{k \in \mathbb{N}}$. Then, by Lemma 4.11,

$$M_{n_k}(x_{n_k}, y_{n_k}, t) = M^{n_k p_k}(x_{n_k}, y_{n_k}, t)$$
$$\geq M^{n_k p_k}(x_{n_k}, x_{p_k}, t) * M^{n_k p_k}(x_{p_k}, y_{p_k}, t) * M^{n_k p_k}(y_{p_k}, y_{n_k}, t).$$

Then, taking limits when k goes to infinity, we obtain

$$\lim_k M_{n_k}(x_{n_k}, y_{n_k}, t) \geq \lim_k M_{p_k}(x_{p_k}, y_{p_k}, t).$$

Analogously by exchanging the roles of $M_{n_k}(x_{n_k}, y_{n_k}, t)$ and $M_{p_k}(x_{p_k}, y_{p_k}, t)$, we have

$$\lim_k M_{p_k}(x_{p_k}, y_{p_k}, t) \geq \lim_k M_{n_k}(x_{n_k}, y_{n_k}, t).$$

Thus, we have proved that any convergent subsequence of $\{M_j(x_j, y_j, t)\}_{j \in \mathbb{N}}$ converges to the same limit and, consequently, there exists $\lim_j M_j(x_j, y_j, t)$, for every $t > 0$.

Fact 3. If we define

$$M((x_j), (y_j), t) = \sup_{s < t} \lim_j M_j(x_j, y_j, s), \text{ for every } t > 0$$

and

$$M((x_j), (y_j), 0) = 0,$$

for all $(x_j), (y_j) \in \hat{X}$, then $(M, *)$ is a non-Archimedean fuzzy pseudometric on $\hat{X}$. Indeed, it is easy to show that, for every $(x_j), (y_j) \in \hat{X}$, $\lim_j M_j(x_j, y_j, \_)$ is a non-decreasing function so that there exists $M((x_j), (y_j), t) = \sup_{s < t} \lim_j M_j(x_j, y_j, s)$. Now it is straightforward to show that M satisfies all the properties to be a non-Archimedean fuzzy pseudometric on $\hat{X}$.

Define next an equivalence relation on $\hat{X}$ as follows

$$(x_j) \sim (y_j) \text{ if, and only if, } M((x_j), (y_j), t) = 1, \text{ for all } t > 0$$

and let $X = \hat{X}/\sim$ denote the quotient space. Notice that the non-Archimedean fuzzy pseudometric M defines a non-Archimedean fuzzy metric on X. For the sake of simplicity, we denote this non-Archimedean fuzzy metric by M.

The next step is to consider an adequate admissible non-Archimedean fuzzy metric on $X_i \sqcup X$ for each i. To do this, for all $y \in X_i$, all class $[(x_j)] \in X$ and all $t > 0$, define

$$M^i(y, [(x_j)], t) := \sup_{s < t} \liminf_j M^{ij}(y, x_j, s),$$

$$M^i(y, [(x_j)], 0) = 0,$$

$$M^i(y, x, t) = M_i(y, x, t), \text{ for } y, x \in X_i, \text{ and}$$

$$M^i([(x_j)], [(y_j)], t) = M((x_j), (y_j), t), \text{ for } [(x_j)], [(y_j)] \in X.$$

Next we shall prove that the previous definition does not depend on the class representative selected. To see this, if $(x_j) \sim (y_j)$ and $y \in X_i$ ($i \in \mathbb{N}$), then $M^{ij}(y, x_j, t) \geq M^{ij}(y, y_j, t) * M^{ij}(y_j, x_j, t)$ for all $j > i$. By the definition of M^i and taking into account that every subsequence of $\{M^{ij}(x_j, y_j, t)\}_{j > i}$ converges to one, we have

$$M^i(y, [(x_j)], t) \geq M^i(y, [(y_j)], t).$$

In a similar way, we obtain the inequality

$$M^i(y, [(y_j)], t) \geq M^i(y, [(x_j)], t)$$

and, consequently, the equality $M^i(y, [(x_j)], t) = M^i(y, [(y_j)], t)$ holds. Thus, M^i is well defined. A straightforward argument shows that M^i is a non-Archimedean fuzzy metric on $X_i \sqcup X$.

We close the proof by proving that the sequence $\{(X_i, M_i, *)\}_{n \in \mathbb{N}}$ converges to the space $(X, M, *)$. For, we have to prove that there exists, for all $t > 0$, $i_0 \in \mathbb{N}$ such that,

$$M_{GH}(X_i, X, t) > (1 - \varepsilon_{i-2}), \text{ for all } i \geq i_0.$$

Notice that, by definition of the non-Archimedean Gromov-Hausdorff fuzzy metric, it suffices to prove that

$$H_{M^i}(X_i, X, t) > (1 - \varepsilon_{i-2}), \text{ for all } i \geq i_0.$$

For this, fix $t > 0$ and choose i_0 such that, for all $i \geq i_0$, $\varepsilon_i < t$. Let $[(x_j)] \in X$. For all ε_i with $i \geq i_0$, we know that $\lim_{n,j} M^{nj}(x_n, x_j, \varepsilon_i) = 1$, so that we can find $n \geq i$ such that

$$\liminf_j M^{nj}(x_n, x_j, \varepsilon_i) > 1 - \varepsilon_i.$$

Then,

$$M^n(x_n, [(x_j)], t) = \sup_{s<t} \liminf_j M^{nj}(x_n, x_j, s) = \sup_{\varepsilon_i < s < t} \liminf_j M^{nj}(x_n, x_j, s) > 1 - \varepsilon_i.$$

Claim: Given $x_n \in X_n$, there exists $y \in X_i$ such that $M^{in}(y, x_n, t) \geq (1 - \varepsilon_{i-1})$.
Indeed, from Lemma 4.10 and Lemma 4.12, $H_{M^{in}}(X_i, X_n, \varepsilon_i) > (1 - \varepsilon_{i-1})$ which implies

$$M^{in}(X_i, x_n, \varepsilon_i) = \sup_{y \in X_i} \{M^{in}(y, x_n, \varepsilon_i)\} > (1 - \varepsilon_{i-1}).$$

Thus, there exists $y \in X_i$ with

$$M^{in}(y, x_n, t) \geq M^{in}(y, x_n, \varepsilon_i) > (1 - \varepsilon_{i-1})$$

which proves our claim.
Now, from

$$M^{ij}(y, x_j, s) \geq M^{in}(y, x_n, s) * M^{nj}(x_n, x_j, s) \text{ for all } s > 0,$$

we obtain

$$\begin{aligned} M^i(y, [(x_j)], t) &= \sup_{s<t} \liminf_j M^{ij}(y, x_j, s) \\ &\geq \sup_{s<t} \liminf_j \left(M^{in}(y, x_n, s) * M^{nj}(x_n, x_j, s) \right) \\ &= M^{in}(y, x_n, t) * M^n(x_n, [(x_j)], t) \\ &\geq (1 - \varepsilon_i) * (1 - \varepsilon_{i-1}) > (1 - \varepsilon_{i-2}) \end{aligned}$$

for $i \leq n < j$. We have just proved that $M^i(X_i, [(x_j)], t) = \sup_{y \in X_i} M^i(y, [(x_j)], t) > (1 - \varepsilon_{i-2})$ for every class $[(x_j)] \in X$. Therefore

$$(4.1) \quad \inf_{[(x_j)] \in X} M^i(X_i, [(x_j)], t) > (1 - \varepsilon_{i-2}).$$

Put $y = x_i \in X_i$. In a similar way, for each $j \geq i$ we can successively find $x_j \in X_j$ with $N^{j,j+1}(x_j, x_{j+1}, t) \geq (1 - \varepsilon_j)$. Then the sequence (x_j) defines a class in X and

$$M^i(y, [(x_j)], t) = \sup_{s<t} \liminf_j M^{ij}(y, x_j, s)$$

$$\geq \sup_{s<t} \liminf_j \prod_{k=i}^{j-1}{}^* N^{k,k+1}(x_k, x_{k+1}, s)$$

$$\geq \sup_{\varepsilon_i < s < t} \liminf_j \prod_{k=i}^{j-1}{}^* N^{k,k+1}(x_k, x_{k+1}, \varepsilon_k)$$

$$\geq (1 - \varepsilon_i) * (1 - \varepsilon_{i+1}) * \ldots * (1 - \varepsilon_{j-1})$$

$$> (1 - \varepsilon_{i-1}) > (1 - \varepsilon_{i-2}).$$

We have just obtained $M^i(y, X, t) = \sup_{(x_j) \in X} M^i(y, (x_j), t) > (1 - \varepsilon_{i-2})$, for every $y \in X_i$, so that

$$(4.2) \quad \inf_{y \in X_i} M^i(y, X, t) > (1 - \varepsilon_{i-2}).$$

Equations (4.1) and (4.2) imply

$$H_{M^i}(X_i, X, t) > (1 - \varepsilon_{i-2}) \text{ for all } i \geq i_0.$$

Thus,

$$M_{GH}(X_i, X, t) > (1 - \varepsilon_{i-2}) \text{ for all } i \geq i_0,$$

and the sequence $\{(X_i, M_i, *)\}_{i \in \mathbb{N}}$ converges to $(X, M, *)$. By Theorem 4.9 (c), the proof is completed.

$\square$

Our next theorem provides a link between the Gromov-Hausdorff distance on the realm of metric spaces and the non-Archimedean Gromov-Hausdorff fuzzy distance. A well-known equivalent definition of d_{GH} (see, for example, [3, p. 79]), which will always be used in the sequel, can be given as follows: d_{GH} is the infimum of all $\varepsilon > 0$ such that there exists a metric d in the disjoint union $X \sqcup Y$ of X and Y, extending the metrics of X and Y (that is, an *admissible* metric), such that $d_H(X, Y) < \varepsilon$. It is a well-known fact that d_{GH} is a metric in the isometric class $\mathcal{M}$ of nonempty compact metric spaces. A sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ of (compact) metric spaces converges to a (compact) metric space (X, d) if $d_{GH}(X_n, X)$ approaches to zero when n tends to infinity.

We will prove that the classical Gromov-Hausdorff convergence of a sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ of metric spaces is equivalent to the convergence of the sequence $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ in the non-Archimedean Gromov-Hausdorff fuzzy metric. First a lemma which follows from a straightforward calculation and it is left to the reader.

Lemma 4.14. *If (X, d) is a metric space, then the following conditions hold:*

  (i) *For all $p > 1$, $d(x, y) < 1/(p^2 - p)$ if and only if $M_d(x, y, 1/p) > 1 - (1/p)$.*

  (ii) *For all $p > 1$, $M_d(x, y, 1/p) \leq 1 - (1/p)$ if and only if $d(x, y) \geq 1/(p^2 - p)$.*

Theorem 4.15. *A sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ of compact metric spaces is d_{GH}-convergent if and only if the sequence $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ is M_{GH}-convergent.*

Proof. <u>Sufficiency.</u> Suppose that $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ converges to (X, d), that is,

$$\lim_n d_{GH}(X_n, X) = 0.$$

Then, given $\varepsilon > 0$, there exists n_0 such that, for every $n \geq n_0$,

$$d_{GH}(X_n, X) < \varepsilon,$$

so that there exists an admissible metric $\widetilde{d}_n$ on $X_n \sqcup X$ such that $(\widetilde{d}_n)_H(X_n, X) < \varepsilon$ $(n \geq n_0)$. By [25, Proposition 3], the non-Archimedean fuzzy metric

$$(M_{(\widetilde{d}_n)_H}, \cdot) = \left(\frac{t}{t + (\widetilde{d}_n)_H}, \cdot \right) \quad (t > 0)$$

coincides with the fuzzy metric $(H_{M_{\widetilde{d}_n}}, \cdot)$. Therefore $H_{M_{\widetilde{d}_n}}(X_n, X, t) > \frac{t}{t + \varepsilon}$ for all $n \geq n_0$ and all $t > 0$. Since the fuzzy metric $(M_{(\widetilde{d}_n)_H}, \cdot)$ is admissible on $X_n \sqcup X$ for all $n \geq n_0$, we have just showed that

$$\lim_n M_{GH}(X_n, X, t) = 1,$$

for all $t > 0$, that is, the sequence $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ converges to $(X, M_d, \cdot)$.

<u>Necessity.</u> It suffices to show that every subsequence of $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ has a subsequence converging to the same metric space (X, d). Since $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ is a convergent sequence, we have that $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ is a precompact set. Then Lemma 4.14 tells us that $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ is precompact so that every subsequence has a d_{GH}-Cauchy subsequence.

Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence as in Lemma 4.10. It simplifies the argument, and causes no loss of generality, to assume that $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ is d_{GH}-Cauchy sequence satisfying the property

$$d_{GH}(X_n, X_{n+1}) < \frac{1}{p_n^2 - p_n}$$

with $\lim_n (1/p_n) = 0$ and $\varepsilon_n < \frac{1}{p_n}$ for $n = 1, 2, \ldots$.

Now, by definition of the Gromov-Hausdorff metric, we can choose admissible metrics $d^{n,n+1}$ on $X_n \sqcup X_{n+1}$ such that

$$d_H^{n,n+1}(X_n, X_{n+1}) < \frac{1}{p_n^2 - p_n}$$

for $n = 1, 2, \ldots$.

Now, by Lemma 4.14 and [25, Proposition 3], we have

$$H_{M_{d^{n,n+1}}}(X_n, X_{n+1}, 1/p_n) > 1 - \frac{1}{p_n}, \text{ for every } n = 1, 2, \ldots$$

As in Lemma 4.11, we can consider an admissible non-Archimedean fuzzy metric on $X_i \sqcup X_j$ defined as follows: for $x \in X_i$ and $y \in X_j$,

$$M^{i,j}(x, y, t) = \sup\{\prod_{s=i}^{j-1} M_{d^{s,s+1}}(x_s, x_{s+1}, t) \ : \ x_s \in X_s \ (x_i = x, \ x_j = y)\}.$$

Consider now the space

$$\hat{X} = \{(x_j) \ : \ \lim_{k,j} M^{k,j}(x_k, x_j, t) = 1, \ \text{for every } t > 0 \}$$

constructed as in Theorem 4.13 (remember that (x_j) is a sequence with $x_j \in X_j$ for all $j \in \mathbb{N}$). We know that $(\hat{X}, M, \cdot)$ is a non-Archimedean pseudometric fuzzy space, where

$$M((x_j), (y_j), t) = \sup_{s<t} \lim_{j} M_{d_j}(x_j, y_j, s), \ \text{for all } t > 0$$

and

$$M((x_j), (y_j), 0) = 0.$$

Notice that the existence of $M((x_j), (y_j), t)$ $(t > 0)$ above implies the existence of

$$\hat{d}((x_j), (y_j)) := \lim_{j} d_j(x_j, y_j)$$

and, consequently, $\hat{d}$ is a pseudometric on $\hat{X}$.

Now, define an equivalence relation on $\hat{X}$ as follows:

$$(x_j) \sim (y_j) \ \text{if, and only if,} \ \hat{d}((x_j), (y_j)) = 0$$

and take the quotient space $X = \hat{X}/\sim$. In the usual way, the pseudometric $\hat{d}$ defines a metric, say d, on X. The construction of the limit space of Theorem 4.13 tells us that the sequence $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ M_{GH}-converge to $(X, M_d, \cdot)$. We close the proof by showing that the sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ converges to (X, d) in the Gromov-Hausdorff distance. For see this, we first construct an admissible metric on $X_i \sqcup X_j$ by putting

$$d^{i,j}(x_i, x_j) = \inf\{\sum_{k=i}^{j-1} d^{k \ k+1}(x_k, x_k + 1) \ : \ x_k \in X_k, \ k = i+1, \ldots, j-1\}$$

for every $x_i \in X_i$ and $x_j \in X_j$ and every $i < j$. It is easy to show that

$$M_{d^{i,j}}(x, y, t) \geq M^{i,j}(x, y, t).$$

Define now, for every $y \in X_i$ and $[(x_j)] \in X$,

$$d^i(y, [(x_j)]) := \limsup_{j} d^{i,j}(y, x_j)$$

which is an admissible metric on $X_i \sqcup X$. Taking the standard fuzzy metric M_{d^i} associated to d^i, it is easy to show that, for all $t > 0$,

$$
\begin{aligned}
M_{d^i}(y, [(x_j)], t) &= \frac{t}{t + d^i(y, [(x_j)])} = \sup_{s<t} \frac{s}{s + d^i(y, [(x_j)])} \\
&= \sup_{s<t} \liminf_{j} M_{d^{i,j}}(y, x_j, s) \geq \sup_{s<t} \liminf_{j} M^{i,j}(y, x_j, s).
\end{aligned}
$$
(∗)

As in the proof of Theorem 4.13, $M^i(y, [(x_j)], t) := \sup_{s<t} \liminf_{j} M^{i,j}(y, x_j, s)$ is an admissible non-Archimedean fuzzy metric on $X_i \sqcup X$ verifying that, for all $t > 0$, there exists $i_0(t)$ such that, if $i \geq i_0(t)$, then $\varepsilon_i < t$ and

$$H_{M^i}(X_i, X, t) > 1 - \epsilon_{i-2}.$$

Now, put $t_i = \frac{1}{p_i}$ for all $i \in \mathbb{N}$. By the above property, we can find $j_i \geq i$ such that $\varepsilon_{j_i} \leq (1/p_i)$ and

$$H_{M^{j_i}}(X_{j_i}, X, 1/p_i) > 1 - \varepsilon_{j_i-2} > 1 - \varepsilon_{j_i} > 1 - (1/p_i).$$

By equality (∗),

$$H_{M_{d^{j_i}}}(X_{j_i}, X, 1/p_i) > H_{M^{j_i}}(X_{j_i}, X, 1/p_i) > 1 - (1/p_i),$$

and Lemma 4.14 tells us that

$$d_H^{j_i}(X_{j_i}, X) < \frac{1}{p_i^2 - p_i}.$$

Therefore $d_{GH}(X_{j_i}, X) < \frac{1}{p_i^2 - p_i}$. We have just proved that the subsequence $\{(X_{j_i}, d_{j_i})\}$ d_{GH}-converges to (X, d). Since the sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ is d_{GH}-Cauchy, this completes the proof. $\quad\square$

The previous result tells us that we can obtain the topological properties induced by the classical Gromov-Hausdorff convergence by means of the fuzzy non-Archimedean Gromov-Hausdorff distance. In addition, the following result is a straightforward consequence of Lemma 4.14 and permits us to obtain the completeness Gromov-Hausdorff theorem as an implication of our outcomes.

Lemma 4.16. *A sequence $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ of compact metric spaces is d_{GH}-Cauchy if and only if the sequence $\{(X_n, M_{d_n}, \cdot)\}_{n \in \mathbb{N}}$ is M_{GH}-Cauchy.*

Now, Theorem 4.13 and Theorem 4.15 imply

Theorem 4.17. (Gromov-Hausdorff completeness theorem) *If $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ is a d_{GH}-Cauchy sequence of compact metric spaces, then $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ is convergent.*

5. Conclusion

We have introduced the notion of the non-Archimedean fuzzy Gromov-Hausdorff convergence and we have established the connection between this kind of convergence and the classical Gromov-Hausdorff convergence. We have also shown its basic properties. Considering that Gromov's theory is widely used in a variety of ways, our results provide a new area for future research and also for further applications of the fuzzy theory.

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