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FINITE GROUPS WITH TWO p-REGULAR CONJUGACY CLASS LENGTHS II

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Abstract

Let G be a finite group. We prove that if the set of p-regular conjugacy class sizes of G has exactly two elements, then G has Abelian p-complement or $G = PQ \times A$, with $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and A Abelian.

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1. Introduction

Itô proved in [5] that if G is a finite group such that all its noncentral conjugacy classes have equal size, then $G = Q \times A$, where Q is a Sylow q-subgroup of G, for some prime q, and A lies in $\mathbf{Z}(G)$. In [1], Beltrán and Felipe proved a generalization of this result for p-regular conjugacy class sizes and some prime p, with the assumption that the group G is p-solvable. In the present paper, we improve this result by showing that the p-solvability condition is not necessary.

THEOREM A. *Let G be a finite group. If the set of p-regular conjugacy class sizes of G has exactly two elements, for some prime p, then G has Abelian p-complement or $G = PQ \times A$, with $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and $A \subseteq \mathbf{Z}(G)$, with q a prime distinct from p. As a consequence, if $\{1, m\}$ are the p-regular conjugacy class sizes of G, then $m = p^a q^b$. In particular, if $b = 0$ then G has Abelian p-complements and if $a = 0$ then $G = P \times Q \times A$ with $A \subseteq \mathbf{Z}(G)$.*

The proof given in [1] for p-solvable groups is divided into two cases, when the centralizers of noncentral p-regular elements are all G-conjugated and when they are not. In the second case, it is easy to check that the hypothesis of p-solvability is not needed, so our study reduces then to the case in which all the centralizers of noncentral p-regular elements are conjugated. In order to solve this case, we are going to base

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our arguments on the proof of a theorem of Camina [2, Theorem 1]. We stress that while Camina used the classification obtained by Gorenstein and Walter [3] of those groups whose Sylow 2-subgroups are dihedral (this having been used to complete the classification of the simple finite groups), we present a more simple proof by making use of a well-known theorem of Kazarin which asserts that in any finite group the subgroup generated by an element of prime power class size is always solvable [4, Theorem 15.7].

Furthermore, we remark that it is not feasible that all the centralizers of noncentral elements of a group G are conjugate, but it is easy to find examples where all the centralizers of noncentral p-regular elements are conjugate (consequently G has exactly two p-regular conjugacy class sizes) for some prime p. For instance, the centralizers of all noncentral 2-elements of SL(2, 3) are conjugate and the 3-regular class sizes are $\{1, 6\}$. Another example is Alt(4), whose 2-regular class sizes are $\{1, 4\}$.

We shall assume that every group is finite and we shall denote by $G_{p'}$ the set of p-regular elements of G.

2. Preliminary results

We shall need some results on conjugacy classes of p-regular elements.

LEMMA 1. *Let G be a finite group. Then all the conjugacy class sizes in $G_{p'}$ are p-numbers if and only if G has Abelian p-complements.*

PROOF. See [1, Lemma 2]. □

The following is exactly [2, Lemma 1], but we present an easier proof. It generalizes [1, Lemma 3] by eliminating the hypothesis of p-solvability.

LEMMA 2. *Suppose that G is a finite group and that p is not a divisor of the sizes of p-regular conjugacy classes. Then $G = P \times H$ where P is a Sylow p-subgroup and H is a p-complement of G.*

PROOF. Let $g \in G$ and consider its $\{p, p'\}$-decomposition as $g = g_p g_{p'}$. Suppose that $g_{p'}$ is noncentral. As the class size of $g_{p'}$ is a p'-number, if we fix a Sylow p-subgroup P of G, then there exists some $t \in G$ such that $g_p \in P^t \subseteq C_G(g_{p'})$. Therefore,

$$G = \bigcup_{t \in G} P^t C_G(P^t).$$

Then $G = PC_G(P)$ and so, $G = P \times H$ where H is a p-complement of G. □

LEMMA 3. *Let P be an Abelian p-group, with p a prime and let K be a group of automorphisms of P such that $|K|$ is divisible by p. Suppose that $C_P(x) = C_P(y)$ for all $x, y \in K - \{1\}$. Then $\mathbf{O}_{p'}(K) = 1$.*

PROOF. Assume that $H = \mathbf{O}_{p'}(K) > 1$ and we shall get a contradiction. Suppose first that $C_P(H) = 1$ and take some nontrivial $x \in H$. If there exists some element

$w \in C_P(x) - \{1\}$, then clearly $w \in C_P(H)$ and so, necessarily, $C_P(x) = 1$ and hence, $C_P(y) = 1$ for all $y \in K - \{1\}$. But if we count the orbit sizes this cannot happen because p divides $|K|$.

As a result, $C_P(H) \neq 1$. Now, as P is Abelian, by coprime action we can write $P = C_P(H) \times [P, H]$, and since $C_P(K) = C_P(H)$ and K is a group of automorphisms of P, it follows that $[P, H] \neq 1$. Thus, if $x \in K - \{1\}$, then $C_P(x) = C_P(K) \times C_{[P,H]}(x)$. Now, if $w \in C_{[P,H]}(x) - \{1\}$, then $C_P(w) = C_P(K)$, so $w \in C_P(K) \cap [P, H] = 1$. This is not possible, so $C_{[P,H]}(x) = 1$ for all $x \in K - \{1\}$. But this contradicts again the fact that p is a divisor of $|K|$. □

LEMMA 4. *Let G be a finite group such that all its Sylow subgroups are cyclic. If r and s are two distinct primes dividing $|G|$, then there exists a subgroup U of G such that $|U| = rs$.*

PROOF. We work by induction on the order of G. First, it is known that any finite group whose Sylow subgroups are all cyclic is solvable (see for instance [6, 10.1.10]). Let M be a maximal normal subgroup of G, so $|G : M| = p$ for some prime p. We can assume that M is a p'-subgroup, otherwise we can apply the inductive hypothesis to M and the result is obtained. Also, we only have to show that there exists a subgroup of order pq for any prime $q \neq p$ dividing $|M|$, since the other cases are obtained by the inductive hypothesis as well. If P is a Sylow p-subgroup of G, then P acts coprimely on M, so if we fix a prime q dividing $|M|$, we know (see for example [4, 14.3]) that there exists some P-invariant Sylow q-subgroup Q of G, which is cyclic. Hence, if $x \in Q$ has order q, then $U = \langle x \rangle P$ has order pq, as required. □

3. Proof of Theorem A

We shall prove by induction on the order of G that either G has Abelian p-complements or G is a $\{p, q\}$-group for some prime $q \neq p$ without considering central factors. Likewise, we notice that when G is solvable then the theorem is already proved by [1, Theorem A]. We shall assume then that the p-complements of G are not Abelian and that there exist at least two prime divisors of the order of $G/\mathbf{Z}(G)$ different from p, in order to get a contradiction.

As we have already pointed out in the introduction, we are also going to assume that all the centralizers of noncentral elements in $G_{p'}$ are conjugated in G. In the other case the theorem can be proved exactly the same as case 2 of [1, Theorem A], where the condition of p-solvability is not necessary. More precisely, the conjugation of the centralizers of all noncentral elements in $G_{p'}$ will be used from Step 4.

The first two steps are exactly Steps 1 and 4 of [1, Theorem A], so we shall omit their proofs.

STEP 1. We can assume that $C_G(x) = P_x \times L_x$, with P_x a Sylow p-subgroup of $C_G(x)$ and $L_x \leq Z(C_G(x))$, for any noncentral $x \in G_{p'}$.

STEP 2. $C_G(x) < N_G(C_G(x))$ for every noncentral $x \in G_{p'}$.

STEP 3. If $x \in G_{p'}$ is noncentral, then every Sylow subgroup of $N_G(C_G(x))/C_G(x)$ is cyclic or generalized quaternion. Furthermore, if $q \neq p$ is a prime divisor of the order of this group, then the Sylow q-subgroup has order q.

We fix some $x \in G_{p'}$ and write $W = N_G(C_G(x))/C_G(x)$. Let Q be a Sylow q-subgroup of W for some prime q dividing $|W|$ (possibly $q = p$). By the assumptions we have made at the beginning of the proof there exists some prime r, divisor of $|G/\mathbf{Z}(G)|$, distinct from q and p. Clearly r divides $|C_G(x)|$ since all these centralizers have the same size. Let R_x be a Sylow r-subgroup of $C_G(x)$ and notice that Q acts as a permutation group on R_x since if $g \in Q$, then $C_{R_x}(g) = R_x \cap \mathbf{Z}(G)$. Moreover, since this is a coprime action and R_x is Abelian, we can write $R_x = [R_x, Q] \times C_{R_x}(Q)$. Also, observe that Q acts fixed-point-freely on $[R_x, Q]$, for if $t \in [R_x, Q] - \{1\}$, then $C_G(t) = C_G(x)$ by Step 1, so no element of $Q - \{1\}$ may fix t. Consequently, we can apply a well known result ([4, Theorem 16.12] for instance) to obtain that Q is cyclic or generalized quaternion.

Assume now that $q \neq p$ and take Q_x a Sylow q-subgroup of $C_G(x)$, which is normal by Step 1. Accordingly, Q acts on $\overline{Q}_x = Q_x/\mathbf{Z}(G)_q$. If M is the semidirect product defined by this action, we can take some element in $\mathbf{Z}(M) \cap \overline{Q}_x$ which has exactly order q. If $\bar{t} \in \overline{Q}_x$, with $t \in Q_x$ is such an element, we can construct the subgroup $T = \langle t \rangle \mathbf{Z}(G)_q \leq C_G(x)$. Observe that Q acts faithfully on T, that is, $C_Q(T) = 1$, since $C_G(t) = C_G(x)$ by Step 1. Furthermore, notice that $[T, Q] \subseteq Z(G)_q$. We claim now that Q is a q-elementary subgroup. Let $v \in Q$. As $t^q \in \mathbf{Z}(G)$, then $1 = [t^q, v] = [t, v]^q$, where the last equality follows because T is Abelian. Also, since $[t, v] \in \mathbf{Z}(G)$ we have $[t, v]^q = [t, v^q]$, so we conclude that $v^q \in C_Q(T) = 1$ and thus Q is elementary, as claimed. But this implies that Q is cyclic of order q by the above paragraph, and hence the step is proved.

STEP 4. For any $x \in G_{p'}$, we have $|N_G(C_G(x))/C_G(x)| = q$ for some fixed prime $q \neq p$.

First we are going to prove that $W = N_G(C_G(x))/C_G(x)$ is q-group for some prime q (including the possibility $q = p$). Suppose that $|W|$ is divisible by at least two distinct primes and we shall prove that there exists a subgroup U of W such that $|U|$ is the product of two prime numbers. By Step 3, if every Sylow subgroup of W is cyclic then there exists such subgroup U by Lemma 4. We can assume then that 2 divides $|W|$ and that the Sylow 2-subgroups of W are generalized quaternion, so we can apply a classic theorem of Brauer and Suzuki (see [4, 45.1]) to obtain that $\mathbf{O}_{2'}(W)\langle\tau\rangle \trianglelefteq W$, where τ is an involution of W. Again by Step 3, the Sylow subgroups of $\mathbf{O}_{2'}(W)$ are cyclic, so if $|\mathbf{O}_{2'}(W)|$ is divisible by at least two distinct primes then the subgroup U exists by Lemma 4 as well. So we can suppose that $\mathbf{O}_{2'}(W)$ is a cyclic r-group for some prime $r \neq 2$. Hence we can take $\alpha \in \mathbf{O}_{2'}(W)$ of order r and we may construct the subgroup $U = \langle\alpha\rangle\langle\tau\rangle$ of order $2r$. As a result, in all the cases we have a subgroup $U \leq W$ such that $|U| = rs$, for some primes r and s, as we wanted to prove. We shall see now that this leads to a contradiction. If both primes are distinct from p, then either U has a normal r-complement or has a normal s-complement, and we shall assume without loss that the r-complement is normal. In the other case, that is, if $|U| = pr$, with $r \neq p$ then, arguing as in the first paragraph of Step 3, we get that U operates as a permutation group and fixed-point-freely on $[S_x, U] - 1$, where S_x is the Sylow s-subgroup of $C_G(x)$ for some prime $s \notin \{p, r\}$. Notice that such s exists by the assumption we have made at the beginning. Furthermore, in this second case (by applying for instance [4, Lemma 16.12]) we get that U is cyclic, so in particular, U has nontrivial normal r-complement. Thus, in both cases, U has a normal r-complement for some prime $r \neq p$. However, U is an automorphism group of R_x, where R_x is the Abelian Sylow r-subgroup of $C_G(x)$. Moreover, if $u, v \in U - \{1\}$, then $C_{R_x}(u) = C_{R_x}(v) = \mathbf{Z}(G)_r$, so by Lemma 3, we get $\mathbf{O}_{r'}(U) = 1$, which is a contradiction.

Take now a noncentral Sylow r-subgroup R_x of $C_G(x)$, for some prime $r \neq p$. If $t \in R_x$ is noncentral, then by applying Step 1, we have $C_G(x) = C_G(t)$. If $w \in N_G(R_x)$, then by the same reason, $C_G(t^w) = C_G(t)$. Therefore, $C_G(x) = C_G(t)^w = C_G(x)^w$ and $w \in N_G(C_G(x))$. Thus $N_G(R_x) \leq N_G(C_G(x))$. Nevertheless, notice that if R_x is not a Sylow r-subgroup of G, then $R_x < N_G(R_x)$, so r divides $|N_G(R_x)/R_x|$, and this implies that $|W|$ is divisible by r, so W cannot be a p-group. By taking into account Step 3, the step is proved.

The fact that all the centralizers are conjugated implies that we can assume for the rest of the proof that $|N_G(C_G(x))/C_G(x)| = q$, for a fixed prime $q \neq p$ and for any noncentral $x \in G_{p'}$.

STEP 5. We can assume that $\mathbf{O}_p(G) = 1$ and that $|G : N_G(C_G(x))|$ is a p-number for any noncentral $x \in G_{p'}$.

We fix a noncentral $x \in G_{p'}$ and for any prime $r \neq p$ we take R a Sylow r-subgroup of G. If R is Abelian, as all the centralizers of noncentral elements in $G_{p'}$ have the same order, then the Sylow r-subgroup of $C_G(x)$, R_x, is a Sylow r-subgroup of G and R is conjugated to R_x. Thus, r does not divide $|G : N_G(C_G(x))|$. If R is not Abelian, then it is an elementary fact that there exists some $t \in R - \mathbf{Z}(R)$ such that $C_R(t) \trianglelefteq R$. As the centralizers of all noncentral p-regular elements are conjugate, we can assume without loss that $C_G(t) = C_G(x)$. In particular, $C_R(t) \subseteq C_G(x)$. On the other hand, is $g \in N_G(C_R(t))$, then $t^g \in C_R(t)$ and $C_G(t) = C_G(t^g)$ by Step 1. Consequently, $C_G(x) = C_G(t) = C_G(t^g) = C_G(x)^g$ and so $g \in N_G(C_G(x))$. Thus $R \leq N_G(C_R(t)) \leq N_G(C_G(x))$, and so $|G : N_G(C_G(x))|$ is an r'-number too. Accordingly, in both cases we have proved that $|G : N_G(C_G(x))|$ is a p-number.

Now we assume that $\mathbf{O}_p(G) \neq 1$ and we are going to see that $\overline{G} = G/\mathbf{O}_p(G)$ satisfies the hypotheses of the theorem. We fix some noncentral element $x \in G_{p'}$. Let $\overline{y} \in C_{\overline{G}}(\overline{x})$ and notice that $[x, y] \in \mathbf{O}_p(G)$. Hence, we can write $x^y = xa$, with $a \in \mathbf{O}_p(G)$, so x^y is a p'-element of $C_G(x)\mathbf{O}_p(G)$, and then $x^y \in L_x^t$, for some $t \in \mathbf{O}_p(G)$, where L_x is the p'-subgroup appearing in Step 1. Therefore $x^{yt^{-1}} \in L_x$ and $C_G(x) = C_G(x^{yt^{-1}})$. As a consequence, $yt^{-1} \in N_G(C_G(x))$, so $y = wt$ with $w \in N_G(C_G(x))$. Thus, $\overline{y} = \overline{w}$ and $\overline{wx} = \overline{xw}$, that is, $[w, x] \in \mathbf{O}_p(G)$. On the other hand, as $w \in N_G(C_G(x))$ and x is a p-regular element, this forces $[w, x]$ to be a p-regular element, so $[x, w] = 1$. Therefore, $C_{\overline{G}}(\overline{x}) = \overline{C_G(x)}$ and we conclude that $\overline{G}$ has two class sizes of p-regular elements. By the inductive hypothesis, either $\overline{G}$ has an Abelian p-complement or $\overline{G} = \overline{PQ} \times \overline{A}$, with $\overline{P} \in Syl_p(\overline{G})$, $\overline{Q} \in Syl_q(\overline{G})$ and $\overline{A} \leq \mathbf{Z}(\overline{G})$. In the first case, G has an Abelian p-complement, contradicting our first assumptions and in the second one, G is a solvable group, so the proof would be finished.

STEP 6. $\mathbf{O}_r(G) \subseteq \mathbf{Z}(G)$, for every prime $r \neq p$.

Let r be any prime distinct from p and suppose that $K = \mathbf{O}_r(G)$ is noncentral. By Step 5, we have $K \subseteq N_G(C_G(x))$, for all $x \in G_{p'}$. The hypothesis and Step 1 imply that there exists an Abelian noncentral normal Sylow s-subgroup of $C_G(x)$, say S_x, for some prime $s \neq p, r$. Notice that S_x is normalized by K and thus $[S_x, K] \subseteq S_x \cap K = 1$, so $K \subseteq C_G(S_x) = C_G(x)$, where the last equality follows as a consequence of Step 1. On the other hand, if $t \in K - \mathbf{Z}(G)$, then $C_G(t) = C_G(x)$ again by Step 1. Moreover, if $w \in N_G(K)$, then $C_G(t^w) = C_G(x)$, hence $C_G(x)^w = C_G(t)^w = C_G(t^w) = C_G(x)$. Thus, $G = N_G(K) \subseteq N_G(C_G(x))$ and $C_G(x) \trianglelefteq G$. By Step 4, we have $|G : C_G(x)| = q$. This means that $m = q$, so by applying Lemma 2 and Itô's theorem on groups with two conjugacy class sizes (see for instance [4, Theorem 33.6]) , we obtain $G = P \times Q \times A$, with $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and A Abelian, against our initial assumption.

STEP 7. We can now derive the conclusion.

First, we notice that $\mathbf{Z}(G)_q \neq 1$, since any element lying in the centre of a Sylow q-subgroup of G must be central in G too because q divides m by Step 4. We write $\overline{G} = G/\mathbf{Z}(G)_q$ and we shall prove that $\overline{G}$ has two p-regular conjugacy class sizes.

We can trivially assume that $\overline{G}$ is not Abelian, otherwise G would be solvable and the proof is finished. If $\overline{a} \in \overline{G} - \mathbf{Z}(\overline{G})$, we observe that $\overline{C_G(a)} \subseteq C_{\overline{G}}(\overline{a})$. If $\overline{C_G(a)} = C_{\overline{G}}(\overline{a})$ for all $\overline{a} \in \overline{G} - \mathbf{Z}(\overline{G})$, it certainly follows that $\overline{G}$ has two p-regular conjugacy class sizes, as we wanted. Suppose then that there is a p-regular element $\overline{a} \in \overline{G}$ such that $\overline{C_G(a)} \neq C_{\overline{G}}(\overline{a})$. It is easy to see that if $\overline{w} \in C_{\overline{G}}(\overline{a})$ then $w \in N_G(C_G(a))$, that is, $C_{\overline{G}}(\overline{a}) \subseteq \overline{N_G(C_G(a))}$. As $|\overline{N_G(C_G(a))} : \overline{C_G(a)}| = q$ by Step 4, this implies that $\overline{N_G(C_G(a))} = C_{\overline{G}}(\overline{a})$ and so, by Step 5 we conclude that $|\overline{G} : C_{\overline{G}}(\overline{a})|$ is a p-number. Now, by a renowned theorem due to Kazarin (see for example [4, 15.7]), the subgroup $\langle \overline{a}^{\overline{G}} \rangle$ is a solvable normal subgroup of $\overline{G}$. It is easy to see then that this implies that $\langle a^G \rangle$ is a noncentral solvable normal subgroup of G too, but this is not possible in view of Steps 5 and 6.

Therefore, we have proved that $\overline{G}$ has two p-regular conjugacy class sizes, and by induction we obtain that $\overline{G}$ has an Abelian p-complement or $\overline{G} = \overline{PQ} \times \overline{A}$, with $\overline{P} \in Syl_p(\overline{G})$, $\overline{Q} \in Syl_q(\overline{G})$ and $\overline{A} \subseteq \mathbf{Z}(\overline{G})$. Both cases lead to the solvability of G, so the proof is finished.

The last assertions in the statement of the theorem will follow then by immediate application of Lemmas 1 and 2.

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