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Autores / Autors

Peñarrocha Alós, Ignacio

Dolz Algaba, Daniel

Sanchis Llopis, Roberto

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RESEARCH ARTICLE

Inferential networked \mathcal{H}_∞ control with accessibility constraints in both the sensor and actuator channels.

I. Peñarrocha, D. Dolz, R. Sanchis

ipenarro@esid.uji.es, ddolz@.uji.es rsanchis@esid.uji.es

Departament d'Enginyeria de Sistemes Industrials i Disseny,

Universitat Jaume I,

Campus de Riu Sec,

12071 Castelló, Spain.

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The predictor and controller design for an inferential control scheme over a network is addressed. A linear plant with disturbances and measurement noise is assumed to be controlled by a controller that communicates with the sensors and the actuators through a constrained network. An algorithm is proposed such that the scarce available outputs are used to make a prediction of the system evolution with an observer that takes into account the amount of lost data between successful measurements transmissions. The state prediction is then used to calculate the control actions sent to the actuator. The possibility of control action drop due to network constraints is taken into account. This networked control scheme is analyzed and both the predictor and controller designs are addressed taking into account the disturbances, the measurement noise, the scarce availability of output samples and the scarce capability of control actions update. The time-varying sampling periods that result for the process inputs and outputs due to network constraints have been determined as a function of the probability of successful transmission on a specified time with a Bernoulli distribution. For both designs \mathcal{H}_∞ performance has been established and LMI design techniques have been used to achieve a numerical solution.

Keywords: Networked control, inferential control, networked estimation, linear matrix inequalities

1. Introduction

Networked based Control Systems (NCS) have been studied extensively in recent years due to the benefits they offer to real control system implementation. However, the use of a network in a control loop introduces some negative effects that can degrade the system performance, as packet dropouts, variable sampling intervals and communications delays, as well as communication constraints due to its shared nature. The works Chow and Tipsuwan (2001), Ling et al. (2007), Hespanha et al. (2007), Heemels et al. (2010), Gupta and Chow (2010) show and classify the different problems that appear when dealing with process control over networks, describing some possible solutions, analysis schemes and open problems. One of the main problems when dealing with control over networks is the delay and dropout between controller and actuator, that leads to a time-varying delay on the closed loop. Different types of models are used in the literature, that can be classified in continuous time Wen and Zeng (2012), Liu and Yang (2012), discrete time or hybrid event-driven ones Souza et al. (2011), for which different design procedures are proposed. Which model is applied depends on the mechanisms used to sample the output, holding or resetting the control input Schenato (2009) and the message sending protocols. When both the measurements and the control actions are sent through a network, the control strategies

presented in the literature can be classified in state feedback, output feedback or observer based state feedback control. In the first and second strategy the control actions are only updated (or, at least, sent) when a measurement of the full state or the output, respectively, is available. In the observer based approach, the control actions are updated (or, at least, sent) independently of the output sampling mechanism. The control goals also differ between some works that only deal with the stability of the closed loop (neglecting the disturbances), and other ones that take into account the presence of state disturbances and some kind of performance optimization is carried out with \mathcal{H}_∞ or guaranteed cost control strategies.

In recent years, a significant attention has been paid to observer based networked control systems. In Yang et al. (2006), Luan et al. (2011), Guo et al. (2012), Liu and You (2012) an observer based control for a NCS was addressed assuming delays on both the input and output channel.

In Wang et al. (2007), Li et al. (2010), Liu et al. (2011) an observer based control approach for NCS is proposed where the received measurements as well as the applied control actions are driven to zero when communication fails. Those approaches make use of the most probable measured output and input in the state estimation procedure assuming independent Bernoulli distributions on the input and output packet dropout phenomena and, with this technique, the enlargement of the state vector is avoided. The design of the observer based controller is carried out off line, while in Hristu-Varsakelis and Zhang (2008) the control action computation is done online through an LQG strategy that was latter generalized in the paper Moayedi et al. (2011). In this work, both the reset and zero order hold strategies are addressed, as well as different approaches related to the acknowledgement on the success on packet transmissions.

In Zhang and Yu (2007) the problem of input packet dropout was dealt with a zero order hold approach, and a polyquadratic approach was used to assure the stability of the closed loop system. In Fang and Wang (2008), Che et al. (2010), Wen et al. (2011) this problem was dealt assuming a stochastic Bernoulli distribution of the packet dropouts and state disturbances on the models, but the measurement noise was assumed to be null. The controller and the observer were obtained simultaneously via an LMI procedure, leading to constant gains on the observer and the controller that are applied at each control period with the available (possibly) delayed outputs. In Yu et al. (2009) the idea of a varying gain depending on the last transmission interval (the time elapsed between the last two successfully sent control actions) was introduced. This has been recently extended in Shi and Yu (2011), Wang et al. (2011) where a Markovian characterization of the time-varying delays is done, and where the disturbances have been considered but again, the measurement noise has been neglected.

The input channel network-induced delay has been addressed recently through different prediction strategies Gonzalez et al. (2011, 2012) trying to compensate the delay. In Ishido et al. (2011), Xia et al. (2011), Yang et al. (2008) the strategy is based on sending a packet with consecutive predicted inputs to a buffer located on the actuator side, that decides which input must apply taking into account the possible input packet dropout. The study of observers for the scenario presented in this work has been initially addressed in Peñarrocha et al. (2012a) with an online approach, and in Peñarrocha et al. (2012b) with an offline approach.

For the knowledge of the authors, the off line design of an observer based control approach for systems with time delay and packet dropout in both channels using a zero order hold approach has not been considered taking into account the measurement noise phenomena.

When dealing with noisy sensors, in the observer based NCS approach, the observer gains must be fitted depending on the availability of output measurements in order to extract and filter more efficiently its information. Furthermore, the effect of the noise over the state estimation error should be taken into account in the predicted state feedback approach in order to smooth the output.

In this work, the design of an observer based control strategy for systems with state disturbances and measurement noise, controlled through a network, is addressed, assuming data drops

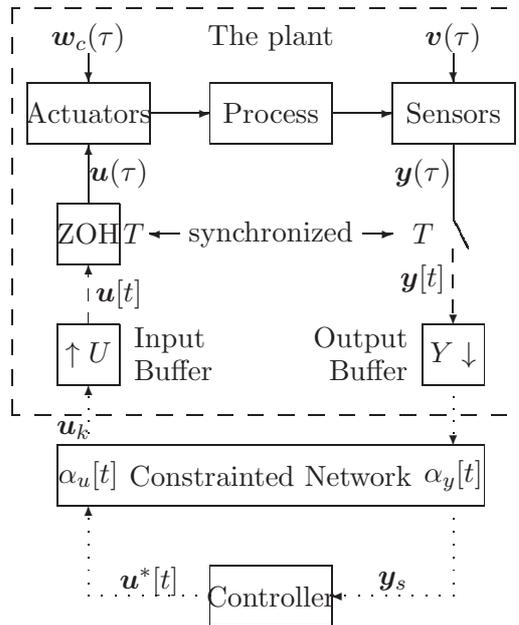


Figure 1. Proposed control scheme using a network with variable accessibility constraints. A solid line denotes continuously transmitted signals, a dashed line denotes a periodically transmitted signal and a dotted line denotes an arbitrarily on time transmitted signal

due to accessibility constraints and transmission delays in both channels, sensors to controller and controller to actuators. The observer has a time varying gain that depends on the consecutive measurement packet dropouts, and the state estimation error is taken into account in the \mathcal{H}_∞ controller design. The sensors, actuators and controller are assumed to be time driven, and the probability of successful access to the network from the nodes is assumed to be defined by a Bernoulli distribution that can also be slowly varying with time. The disturbance attenuation, as well as the probability of dropped data is taken into account in the design of the observer and the controller. As the state is not assumed to be measured, an observer is designed to improve the results of previous works, taking into account disturbances and measurement noise, and the knowledge of new measurements reception. An observed state feedback controller is then designed taking into account the state uncertainty due to the use of an observer. The idea can be easily extended to the design of full state feedback controllers with state measurement noise. The main contributions are the consideration of measurement noise, and the use of an off line calculated time varying gain in the observer, that depends on the elapsed time between consecutive received measurements.

The paper is organized as follows. In Section 2 the problem is presented, including the plant description, a proposal of the network message protocol for control purposes, and the sampling scenario resulting from the network operation. In section 3, the predictor-based (inferential) control algorithm is proposed, and, in section 4, the closed loop dynamics is analyzed. In section 5 the predictor is designed, and the controller design is addressed in section 6. In section 7 several examples are shown and, finally, the main conclusions are summarized in Section 8.

2. Problem statement

Figure 1 shows the proposed networked control problem with a plant with input and output interface devices, a controller, and a network that allows the communication between controller and process signals. The networked control scheme is described as follows.

2.1. Plant description

The plant is assumed to be a continuous-time process with several control inputs ($\mathbf{u}(\tau)$) applied by some actuators and several controlled outputs ($\mathbf{y}(\tau)$) measured by some sensors. The process behaviour is assumed to be defined by the dynamic equations

$$\dot{\mathbf{x}}(\tau) = \mathbf{A}_c \mathbf{x}(\tau) + \mathbf{B}_c \mathbf{u}(\tau) + \mathbf{w}_c(\tau) \quad (1a)$$

$$\mathbf{y}(\tau) = \mathbf{C} \mathbf{x}(\tau) + \mathbf{v}(\tau) \quad (1b)$$

where all the signals are continuous-time, being $\mathbf{u}(\tau) \in \mathbb{R}^{n_u}$ the input of the process, $\mathbf{y}(\tau) \in \mathbb{R}^{n_y}$ the output measured by a noisy sensor, $\mathbf{x}(\tau) \in \mathbb{R}^n$ the state, $\mathbf{w}_c(\tau) \in \mathbb{R}^n$ the process disturbance and $\mathbf{v}(\tau) \in \mathbb{R}^{n_y}$ the sensors measurement noise. \mathbf{A}_c , \mathbf{B}_c and \mathbf{C} are matrices of proper dimensions, where the pair $(\mathbf{A}_c, \mathbf{C})$ is assumed to be detectable and the pair $(\mathbf{A}_c, \mathbf{B}_c)$ is assumed to be controllable.

The input signals of the actuators, $\mathbf{u}(\tau)$, are updated every T seconds through a zero order holder (ZOH) with the value of each correspondent $\mathbf{u}[t]$ contained in a packet that is stored on an unitary buffer (denoted by $\uparrow U$ on the figure).

That buffer can be accessed via a network at any arbitrary instant of time in order to update the packet stored in it. The sensors measurements are sampled every T seconds and all the sampled values ($\mathbf{y}[t]$) are encapsulated in a packet and written on a unitary buffer ($Y \downarrow$ on the figure), that can be accessed by the network at any arbitrary instant of time. It is assumed that both the sampler and the zero order holder are synchronized and, therefore, the input update and the output sample occur at the same time. Based on this assumption, an equivalent sampled data model of (2) at period T can be written as

$$\mathbf{x}[t+1] = \mathbf{A} \mathbf{x}[t] + \mathbf{B} \mathbf{u}[t] + \mathbf{w}[t] \quad (2a)$$

$$\mathbf{y}[t] = \mathbf{C} \mathbf{x}[t] + \mathbf{v}[t] \quad (2b)$$

where all the signals are discrete-time sampled data signals, being $\mathbf{u}[t] = \mathbf{u}(tT)$ the control inputs, $\mathbf{y}[t] = \mathbf{y}(tT)$ the sampled outputs, $\mathbf{x}[t] = \mathbf{x}(tT)$ the discrete state, $\mathbf{w}[t] = \int_{tT}^{(t+1)T} e^{\mathbf{A}_c(T-\tau)} \mathbf{w}_c(\tau) d\tau$ the equivalent discrete process disturbance and $\mathbf{v}[t] = \mathbf{v}(tT)$ the sensor measurement noise at time tT . \mathbf{A} , \mathbf{B} and \mathbf{C} are matrices of proper dimensions that can be obtained from the matrices of the continuous model by

$$\mathbf{A} = e^{\mathbf{A}_c T}, \quad \mathbf{B} = \int_0^T e^{\mathbf{A}_c(T-\tau)} \mathbf{B}_c d\tau. \quad (3)$$

2.2. Network message protocol for control purposes

The controller, sensor nodes and actuators share the information through a network. This network is assumed to be shared with other devices that are not shown in the control scheme, and so, the controller can not always access the network at the desired instants of time (there are accessibility constraints). However, once the network is accessed by a node, it is assumed that the message is sent, without drops or errors, and the transmission delay is varying but bounded by $\tau_m < T$. The following times are defined: T : The basic control period. $\tau_c < T$: The control actions computation time. $\tau_s < T$: The maximum time that takes all the sensors to make a measurement. $\tau_m < T$: The maximum time needed to complete the transmission of a message once the network is accessed. τ_{ca} : The time elapsed since the controller starts trying to access the network to send the message with the packet control inputs, till the message actually starts to be sent to the input buffer. τ_{sc} : The time elapsed since the output buffer starts trying to

access the network to send the message with the packet output measurements, till the message actually starts to be sent to the controller.

When the control actions have been calculated (τ_c seconds after the interrupt start at $\tau = tT$), the controller tries to send a message to the control input buffer through the network with the values of the inputs ($\mathbf{u}^*[t+1]$). The controller keeps trying to send the message until it is sent, or until the time spent is larger than $\tau_{ca,max} = T - \tau_c - \tau_m$. In that case, the control actions are discarded (i.e. $\mathbf{u}[t+1] = \mathbf{u}[t]$), and the controller stops trying to send the message until the next control period. An input availability factor $\alpha_u[t]$ is defined that takes values 0 or 1 depending on the successful inputs transmission. In the case the message is discarded, $\alpha_u[t] = 0$. If the message can be sent, the actuators input buffer is updated and the applied inputs at period $t+1$ are $\mathbf{u}[t+1] = \mathbf{u}^*[t+1]$, and $\alpha_u[t] = 1$. This means that the controller knows which are the applied inputs at every time.

The operation to read a package with the sampled outputs values is different, since it is the input buffer who tries to send the measured values to the controller. Once the sensors have obtained the measurement, and they have been encapsulated in a single packet and stored in the input buffer, the input buffer keeps trying to send the message with the output values to the controller during the interval $T - \tau_s - \tau_m$, while there is time for the message to arrive to the controller before the next sampling instant. If the message has not been sent in this time, the input buffer discards the full packet and stops trying to send the message until a new packet is stored. An output availability factor $\alpha_y[t]$ is defined that takes values 0 or 1 depending on the successful outputs transmission. At instant $t+1$, if the controller has received the measurements, $\mathbf{y}[t]$, it stores them and defines $\alpha_y[t] = 1$, otherwise $\alpha_y[t] = 0$ and the outputs are assumed to be lost (due to drop out or to the impossibility of the input buffer to access the network).

With the previously described operation mode, the access restriction and time-varying network-induced delay effects on the networked control system are transformed into a problem of controlling a process with randomly missing synchronous measurements with constant delay, and with random, sporadic but synchronous inputs updates, avoiding the design of a control system with time-varying delays.

Note that the message transmission delay and network accessibility constraints depend on the network, not on the controller algorithm, but the time dedicated to establish communication with the sensors and actuators is limited by the control period T . This means that if a low control period T is selected, then the probability of having output samples and input updates at the required time will also be low. This implies that the selection of the control period must be a compromise between the closed loop response time of the process and the time needed to transmit the messages (that depends on the network traffic) in order to have a sufficient number of successfully transmitted output samples and input updates.

On the next section, the time-varying sampling period distribution and time-varying input update period distribution that result from the network access constraints are bounded as a function of the probability of having a successful output and input transmission, respectively.

2.3. Sampling scenario that results from the network operation

Let us define β_y as the controller's probability of receiving a packet with the output measurements $\mathbf{y}[t]$, i.e. as the probability that $\tau_{sc} < T - \tau_s - \tau_m$. In terms of the output availability factor this is expressed as:

$$\beta_y = P(\alpha_y[t] = 1). \quad (4)$$

Note that the complementary probability of failing on receiving the output samples is $P(\alpha_y[t] = 0) = 1 - \beta_y$.

Let us also define β_u as the actuators probability of receiving an input update $\mathbf{u}[t]$, i.e., as the

probability that $\tau_{ca} < T - \tau_c - \tau_m$. In terms of the input updating success factor this is expressed as:

$$\beta_u = P(\alpha_u[t] = 1), \quad (5)$$

and the complementary probability $P(\alpha_u[t] = 0) = 1 - \beta_u$.

These probabilities are not equal in general, and do not depend only on the network operation (delays and access restrictions), but also on the selected control period T . Both probabilities are assumed to be slowly time-variant (due to changes in network traffic), but lower bounded by values $\beta_{u,\min}$ and $\beta_{y,\min}$ that represent the worst expected network behaviour.

Note also that those probabilities can be modified by the user if the period T is changed during the controller operation. This can be useful if some degradation on the state estimator or in the output behaviour is detected. In this sense, a strategy to change the period online could be developed in order to maximize the performance of the controlled system. In this work, however, this idea is not addressed and the changes on the probabilities β_u and β_y are assumed to be related only to network changes due to its shared nature.

With the proposed network operation mode, the s -th sampled outputs are available at a certain instant $t = t_s$ (when the transmission can be completed). The number of basic control periods between two consecutive measurements received by the controller at instants t_{s-1} and t_s is denoted with N_s , being $N_s = t_s - t_{s-1}$.

Following the same previous idea, for the inputs update a similar definition is proposed. t_k is the instant when the k -th inputs update takes place, while N_k is defined as the number of basic periods between two consecutive control inputs received by the actuators at instants t_k and t_{k+1} , i.e. $N_k = t_{k+1} - t_k$.

Both values N_s and N_k vary randomly with time. Their distributions are a function of the probabilities β_y and β_u , associated to the availability factors $\alpha_y[t]$ and $\alpha_u[t]$, respectively. These factors are assumed to be binomial independent variables, such that $P\{\alpha_y[t] = 1\} = \beta_y$, $P\{\alpha_u[t] = 1\} = \beta_u$. The probability of N_k to be equal a given value N can be calculated as the probability of having $N - 1$ consecutive values of $\alpha_u = 0$ and then one value $\alpha_u = 1$, leading to the equation

$$\begin{aligned} P\{N_k = N\} &= P\left\{\bigcap_{i=1}^{N-1} (\alpha_u[t+i] = 0) \cap (\alpha_u[t+N] = 1)\right\} \\ &= (1 - \beta_u)^{N-1} \beta_u. \end{aligned} \quad (6)$$

that is also valid changing N_k by N_s and β_u by β_y . The above function is monotonically decreasing in N , i.e., the lower the N , the higher the probability. The next result allows to obtain a probabilistic bound on the values of N_s and N_k based on this equation.

Lemma 2.1: *Let $\alpha[t]$ be a binomial variable with $P\{\alpha[t] = 1\} = \beta$. Let us call $N_j \in \mathbb{N}$ (j can refer either to k or s) the number of periods between two consecutive instants when $\alpha = 1$. For a given $\varepsilon \in (0, 1)$, if \bar{N} is chosen to fulfill*

$$\bar{N} \geq \frac{\ln(\varepsilon)}{\ln(1 - \beta)} + 1, \quad (7)$$

then, $P\{N_j > \bar{N}\} \leq \varepsilon$.

Proof: The probability that N_j is higher than \bar{N} can be expressed by means of (6), leading to

$$P\{N_j > \bar{N}\} = 1 - P\{N_j \in \{1, \dots, \bar{N}\}\} = (1 - \beta)^{\bar{N}-1} \leq \varepsilon.$$

Taking logarithms on the above expression and taking into account that $\ln(1 - \beta) < 0$, expression (7) is obtained. \square

Equation (7) will be used in the design procedure to calculate the maximum value of N_k and N_s that will be taken into account in the design equations. The selection of a sufficiently low value of ε will guarantee that the probability of having a larger N_k or N_s is very small. On the other hand, the equation (6) will be used in the design procedure to calculate the probability of having a control input update N_k periods after the last update.

3. Inferential control algorithm proposal

The controller uses the randomly available sampled outputs values received through the network to estimate the state at the basic control period T , and to predict it one period ahead. With this prediction, it calculates the control actions for the next period using a state feedback scheme, and tries to transmit the value to the actuators zero order hold buffer. The main difficulty is that at the time when the control actions are calculated, the controller does not know if the transmission to the actuators will be possible or not, and hence the controller design must be based on the probability of successful communication.

The controller is programmed to execute a periodical interrupt every T seconds. At the start of the interrupt, the controller has the following information:

- $\alpha_y[t - 1]$ and hence, if $\mathbf{y}[t - 1]$ is available or not.
- $\alpha_u[t - 2]$ and hence the previously applied $\mathbf{u}[t - 1]$.
- $\alpha_u[t - 1]$ and hence the values $\mathbf{u}[t]$ that are being applied in the current period.
- $\hat{\mathbf{x}}[t - 1|t - 2]$, i.e. the estimation of the previous state with the information of 2 periods before.

The tasks that are scheduled in the interrupt code during time $\tau \in [t \cdot T, (t + 1) \cdot T[$ are as follows:

- (1) The values of the sampled outputs of the previous control period (if available) are used to update the estimation of the state at the previous control period using the equation:

$$\hat{\mathbf{x}}[t-1] = \hat{\mathbf{x}}[t-1|t-2] + \mathbf{L}[t-1](\mathbf{y}[t-1] - \mathbf{C}\hat{\mathbf{x}}[t-1|t-2])\alpha_y[t-1], \quad (8a)$$

where $\mathbf{L}[t]$ is a time varying gain that must be designed.

- (2) The current state ($\mathbf{x}[t]$) is estimated running the model in open loop. Let us call this estimation $\hat{\mathbf{x}}[t|t - 1]$:

$$\hat{\mathbf{x}}[t|t - 1] = \mathbf{A}\hat{\mathbf{x}}[t - 1] + \mathbf{B}\mathbf{u}[t - 1]. \quad (8b)$$

- (3) The state at the next period ($\mathbf{x}[t + 1]$) is predicted running again the model in open loop. Let us call this prediction $\hat{\mathbf{x}}[t + 1|t - 1]$:

$$\hat{\mathbf{x}}[t + 1|t - 1] = \mathbf{A}\hat{\mathbf{x}}[t|t - 1] + \mathbf{B}\mathbf{u}[t]. \quad (8c)$$

- (4) The control actions for the next period (let us call it $\mathbf{u}^*[t + 1]$) are calculated as a function

of the predicted state at $t + 1$. A standard discrete state feedback controller is used

$$\mathbf{u}^*[t + 1] = \mathbf{K} \hat{\mathbf{x}}[t + 1|t - 1],$$

where \mathbf{K} is a gain that must be designed to guarantee the stability and an adequate disturbance attenuation.

- (5) The control actions, $\mathbf{u}^*[t + 1]$, are tried to be transmitted to the actuator zero order hold buffer until τ_m seconds before the next periodical interrupt, while the sensor buffer tries to transmit the measured outputs, $\mathbf{y}[t]$ to the controller. If the outputs have been received, the values $\mathbf{y}[t]$ are stored and the output availability factor is set to one, i.e., $\alpha_y[t] = 1$. If they have not been received, then $\alpha_y[t] = 0$. If the control inputs transmission was possible, the input availability factor is set to one and the input values for the next control period are stored as the calculated ones. If the transmission was not possible, the input values for the next control period are stored as the previous ones (due to the use of a zero order hold) and the input availability factor is set to zero, i.e.,

$$\mathbf{u}[t + 1] = \begin{cases} \mathbf{u}[t], & \alpha_u[t] = 0, \\ \mathbf{u}^*[t + 1], & \alpha_u[t] = 1. \end{cases}$$

In summary, the applied control actions can be expressed as a function of the predicted state as

$$\mathbf{u}[t + 1] = \mathbf{K} \hat{\mathbf{x}}[t + 1|t - 1] \alpha_u[t] + \bar{\alpha}_u[t] \mathbf{u}[t], \quad (8d)$$

with $\bar{\alpha}_u[t] = (1 - \alpha_u[t])$.

The time varying gains $\mathbf{L}[t]$ and the constant gain \mathbf{K} must be designed to guarantee the stability of the observer and the closed loop, and an adequate attenuation of the disturbances and measurement noise. The first step is to derive the closed loop dynamics equation that results from combining the process equations with the previous algorithm. This is the purpose of the next section.

4. Closed loop dynamics analysis

In the next theorem, the time varying matrix that defines the global closed-loop dynamics is obtained. For this purpose a relationship between the vectors of inputs, states and states estimation error is established.

Theorem 4.1: *Consider the control scheme shown in figure 1 where the input updates and output measurements are synchronously taken every T seconds. Assume that there exists an equivalent sampled-data model given by (2) that defines the dynamic behaviour at the sampling instants. Then, if a controller is implemented using algorithm (8), the closed loop dynamic behaviour is defined by equations*

$$\begin{bmatrix} \mathbf{x}[t + 1] \\ \mathbf{u}[t + 1] \\ \tilde{\mathbf{x}}[t] \end{bmatrix} = \mathbf{A}_{CL}[t] \begin{bmatrix} \mathbf{x}[t] \\ \mathbf{u}[t] \\ \tilde{\mathbf{x}}[t - 1] \end{bmatrix} + \mathbf{B}_{CL}[t] \begin{bmatrix} \mathbf{w}[t] \\ \mathbf{v}[t - 1] \end{bmatrix}, \quad (9)$$

$$\mathbf{y}[t] = [\mathbf{C} \ 0 \ 0] \begin{bmatrix} \mathbf{x}[t] \\ \mathbf{u}[t] \\ \tilde{\mathbf{x}}[t-1] \end{bmatrix} + \mathbf{v}[t], \quad (10)$$

where $\tilde{\mathbf{x}}[t]$ is the state estimation error defined as

$$\tilde{\mathbf{x}}[t] = \mathbf{x}[t] - \hat{\mathbf{x}}[t], \quad (11)$$

and where matrices $\mathbf{A}_{CL}[t]$ and $\mathbf{B}_{CL}[t]$ are

$$\mathbf{A}_{CL}[t] = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{K}\mathbf{A}\alpha_u[t] \begin{pmatrix} \mathbf{K}\mathbf{B}\alpha_u[t] \\ +\bar{\alpha}_u[t] \end{pmatrix} & \mathbf{0} & -\mathbf{K}\mathbf{A}^2\alpha_u[t] \\ \mathbf{0} & \mathbf{0} & (\mathbf{I} - \mathbf{L}[t]\mathbf{C}\alpha_y[t])\mathbf{A} \end{bmatrix}, \quad (12)$$

and

$$\mathbf{B}_{CL}[t] = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}\mathbf{A}\alpha_u[t] & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{L}[t]\mathbf{C}\alpha_y[t]) & -\mathbf{L}[t]\alpha_y[t] \end{bmatrix}. \quad (13)$$

Proof: To prove the theorem, first, the state estimation error dynamics is obtained and then, the state and control input evolution are used to obtain the global closed loop dynamics.

From the definition of the state estimation error (11), the state estimation can be written as $\hat{\mathbf{x}}[t] = \mathbf{x}[t] - \tilde{\mathbf{x}}[t]$. From equation (2a) the following relationship can be obtained

$$\mathbf{A}\mathbf{x}[t-1] + \mathbf{B}\mathbf{u}[t-1] = \mathbf{x}[t] - \mathbf{w}[t-1].$$

Using these expressions in the state estimation equation (8b) it leads to

$$\hat{\mathbf{x}}[t|t-1] = \mathbf{x}[t] - \mathbf{A}\tilde{\mathbf{x}}[t-1] - \mathbf{w}[t-1]. \quad (14)$$

Advancing one period the equation (8a), and using the above expression and equation (2b), the evolution of the state estimation error can be written as

$$\tilde{\mathbf{x}}[t] = (\mathbf{I} - \mathbf{L}[t]\alpha_y[t]\mathbf{C})(\mathbf{A}\tilde{\mathbf{x}}[t-1] + \mathbf{w}[t-1]) - \mathbf{L}[t]\alpha_y[t]\mathbf{v}[t]. \quad (15)$$

Note that this dynamics only depends on the output availability factor $\alpha_y[t]$, and does not depend on the input update availability.

The state prediction at instant $t+1$ calculated by (8c) can be rewritten using (14) as

$$\hat{\mathbf{x}}[t+1|t-1] = \mathbf{A}(\mathbf{x}[t] - \mathbf{A}\tilde{\mathbf{x}}[t-1] - \mathbf{w}[t-1]) + \mathbf{B}\mathbf{u}[t] \quad (16)$$

Introducing this expression in (8d) to obtain the control action it leads to

$$\begin{aligned} \mathbf{u}[t+1] = & \mathbf{K}(\mathbf{A}(\mathbf{x}[t] - \mathbf{A}\tilde{\mathbf{x}}[t-1] - \mathbf{w}[t-1]) + \mathbf{B}\mathbf{u}[t])\alpha_u[t] \\ & + \bar{\alpha}_u[t]\mathbf{u}[t]. \end{aligned} \quad (17)$$

Note that this dynamics only depends on the input update availability factor $\alpha_u[t]$, and does not depend on the output measurement availability.

Finally, expressions (15), (17) and the model (2) can be expressed in matrix form as in (9). \square

Remark 1: The closed loop dynamics obtained on the previous theorem shows that the closed loop behaves as a linear stochastically time-varying system where the dynamic matrix $\mathbf{A}_{CL}[t]$ presents a triangular structure.

It is important to stress that the estimation error does not depend on the process state or inputs, nor in the input update availability factor, but only depends on the disturbance, measurement noise and output availability factor as shown in the equation

$$\tilde{\mathbf{x}}[t] = (\mathbf{I} - \mathbf{L}[t]\mathbf{C}\alpha_y[t])(\mathbf{A}\tilde{\mathbf{x}}[t-1] + \mathbf{w}[t]) - \mathbf{L}[t]\alpha_y[t]v[t]. \quad (18)$$

This means that the state observer can be designed independently of the controller to achieve a stable behaviour and an appropriate disturbance and noise attenuation level. As a result of the observer design, a bound in the state estimation error can be obtained.

On the other hand, the dynamics of the system state depends on the estimation error of the designed observer but not on the measurement noise. It can also be noticed that does not depend on the output availability factor. The resulting dynamics can be written as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}[t+1] \\ \mathbf{u}[t+1] \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{K}\mathbf{A}\alpha_u[t] & \mathbf{K}\mathbf{B}\alpha_u[t] + \bar{\alpha}_u[t] \end{bmatrix} \begin{bmatrix} \mathbf{x}[t] \\ \mathbf{u}[t] \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}\mathbf{A}\alpha_u[t] - \mathbf{K}\mathbf{A}^2\alpha_u[t] \end{bmatrix} \begin{bmatrix} \mathbf{w}[t] \\ \tilde{\mathbf{x}}[t-1] \end{bmatrix} \end{aligned} \quad (19)$$

In Peñarrocha et al. (2005), Wu et al. (2011), Gawthrop and Wang (2011) the separation principle for observer-based NCS systems that present a triangular structure was demonstrated, leading to an independent design of the observer and the controller gains. Furthermore, the work Wu et al. (2010) demonstrated that the mean square stability of NCS systems with a time varying model that is only updated when successful transmissions occurs is equivalent to the mean square stability at the (fast) control period.

The previous results show that both the controller and the predictor can be designed independently to have a mean square stable behaviour at their updating periods, and this guarantees the mean square stability of the closed loop. They also show that the predictor design must take into account only the problems with irregular outputs sampling, while the controller design needs to take into account only the irregular inputs update. As a result, the design strategy is proposed as follows:

- (1) Obtain some bound on the disturbance and measurement noises, for example, $\|\mathbf{w}[t]\|_{RMS}$ and $\|v_i[t]\|_{RMS}, i = 1, \dots, n_y$, that will be used in the design steps.
- (2) Design a stable observer that minimizes the effect of the disturbance and measurement noise on the state estimation error (18), using the previous bounds and taking into account the irregular output sampling (i.e., the dependency on the availability factor $\alpha_y[t]$ with probability β_y). This design should give as a result a law for the gain $\mathbf{L}[t]$ and a bound on $\tilde{\mathbf{x}}[t]$, for example, $\|\tilde{\mathbf{x}}\|_{RMS}$.
- (3) Design a controller to stabilize the system and to minimize the effect of the state estimation error and disturbances, using the previous norms, and taking into account the non periodic input update (i.e., the dependency on the availability factor $\alpha_u[t]$ with probability β_u). For this design, equation (19), where the disturbance vector includes the state estimation

error, must be used. The result should be a law for \mathbf{K} .

5. Predictor design

In order to design the predictor needed to compute the control actions, the prediction error dynamics is first obtained.

Theorem 5.1 Prediction error dynamics *The prediction error dynamics of the algorithm (8) applied to system (2) when there is no modeling error and the measurements are available every N_s periods (with N_s time variant), is described by the linear time-variant system*

$$\tilde{\mathbf{x}}_s = (\mathbf{I} - \mathbf{L}_s \mathbf{C}) \left(\mathbf{A}^{N_s} \tilde{\mathbf{x}}_{s-1} + \sum_{j=1}^{N_s} \mathbf{A}^{j-1} \mathbf{w}[t_s - j] \right) - \mathbf{L}_s \mathbf{v}_s \quad (20)$$

that is updated every time new measurements are available. The estimation error vector is defined when the measurements are available ($t = t_s$) as $\tilde{\mathbf{x}}_s \equiv \tilde{\mathbf{x}}[t_s] = \mathbf{x}[t_s] - \hat{\mathbf{x}}[t_s]$. The state prediction error between measurements (from $t = t_{s-1}$ to $t = t_s - 1$) is given by ($r \in [0, N_s - 1]$)

$$\tilde{\mathbf{x}}[t_{s-1} + r] = \mathbf{A}^r \tilde{\mathbf{x}}_{s-1} + \sum_{i=0}^{r-1} \mathbf{A}^{r-i-1} \mathbf{w}[t_{s-1} + i]. \quad (21)$$

Proof: At the measuring instant t_s , the state estimation error in (18) can be expressed (with $\alpha_y[t] = 1$) as

$$\tilde{\mathbf{x}}[t_s] = (\mathbf{I} - \mathbf{L}_s \mathbf{C}) (\mathbf{A} \tilde{\mathbf{x}}[t_s - 1] + \mathbf{w}[t_s - 1]) - \mathbf{L}_s \mathbf{v}_s. \quad (22)$$

being $\mathbf{L}_s = \mathbf{L}[t_s]$ and $\mathbf{v}_s = \mathbf{v}[t_s]$. At the instants when no measurements are available, the state estimation error evolution is obtained taking $\alpha_y[t] = 0$ in (18)

$$\tilde{\mathbf{x}}[t] = \mathbf{A} \tilde{\mathbf{x}}[t - 1] + \mathbf{w}[t - 1], \quad t_{s-1} < t < t_s. \quad (23)$$

Introducing recursively equation (23) from $t = t_s - 1$ to $t = t_s - N_s + 1$ into (22), it leads to (20) taking into account that $t_s - N_s = t_{s-1}$. Expression (21) can be obtained using recursively equation (23) from $t = t_{s-1} + 1$ to $t = t_{s-1} + r$. \square

Remark 1: Let us assume that \bar{N}_s is the maximum number of periods between available measurements, i.e., a bound of N_s . Then, if a new vector gathering the disturbances between measurements is defined as

$$\mathbf{W}_s = [\mathbf{v}_s^\top \mathbf{w}[t_s - 1]^\top \cdots \mathbf{w}[t_s - \bar{N}_s]^\top]^\top,$$

the prediction error dynamics at measuring instants can be written in a compact way as

$$\tilde{\mathbf{x}}_s = \mathcal{A}_s \tilde{\mathbf{x}}_{s-1} + \mathcal{B}_s \mathbf{W}_s \quad (24)$$

where

$$\mathcal{A}_s = (\mathbf{I} - \mathbf{L}_s \mathbf{C}) \mathbf{A}^{N_s}, \quad (25)$$

$$\mathcal{B}_s = [-\mathbf{L}_s (\mathbf{I} - \mathbf{L}_s \mathbf{C}) \mathbf{\Lambda}(N_s)]_{n \times (1 + \bar{N}_s n)} \quad (26)$$

being $\Lambda(N_s)$ the matrix defined as

$$\Lambda(N_s) = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{N_s-1} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}}_{\bar{N}_s} \quad (27)$$

The state estimation error dynamics at the inter-sampling periods (21) can also be compacted as

$$\tilde{\mathbf{x}}[t_{s-1} + r] = \mathbf{A}^r \tilde{\mathbf{x}}_{s-1} + \mathbf{F}(N_s, r) \mathbf{W}_s, \quad r \in [0, N_s - 1] \quad (28)$$

where $\mathbf{F}(N_s, r)$ is a $n \times (1 + \bar{N}_s n)$ matrix defined by

$$\mathbf{F}(N_s, r) = [\mathbf{0}_{n \times 1} \underbrace{\mathbf{0} \cdots \mathbf{0} \mathbf{I} \mathbf{A} \cdots \mathbf{A}^{r-1}}_{\bar{N}_s} \mathbf{0} \cdots \mathbf{0}]. \quad (29)$$

The observer dynamics depends on the vector gain \mathbf{L}_s defined at measuring instants ($t = t_s$), that must be designed to assure: the predictor stability, robustness to the irregular data availability and a proper attenuation of the disturbances and measurement noises. One possible approach is online gain calculation (Kalman filter approach), but this would not lead to any a priori information about achievable bounds on the state estimation error (as expected in the proposed strategy in Remark 1). Furthermore, a high online computational effort is needed because a stationary state can not be reached. Therefore, an off line gain scheduling is proposed, trying to find a priori state estimation error bounds, and reducing the online computational cost needed. As the number of periods between consecutive measurements (N_s) can be known online, the design of gain \mathbf{L}_s in this work is addressed defining a different gain for each possible value of N_s . In order to calculate the set of gains \mathbf{L}_s off line, an upper bound (\bar{N}_s) in N_s must be assumed. To calculate that bound, equation

$$\bar{N}_s = \left\lceil \frac{\lg(\varepsilon)}{\lg(1 - \beta_y)} + 1 \right\rceil. \quad (30)$$

is used, where ε is chosen to be sufficiently low to guarantee that the probability of having $N_s > \bar{N}_s$ is very small. The result of the off line calculation is a finite set of gains

$$\mathbf{L}_s = \mathbf{L}(N_s) \in \mathcal{L} = \{\mathbf{L}_{(1)}, \mathbf{L}_{(2)}, \dots, \mathbf{L}_{(\bar{N}_s)}\}, \quad (31)$$

and a bound on the norm of the state estimation error (calculated at the basic control period). Every time a new measurement is available, a different gain \mathbf{L}_s is applied, depending on the number of periods (N_s) that the predictor has been waiting for new measurements.

With this approach, matrices \mathcal{A}_s and \mathcal{B}_s in (24) can be defined as a function of N_s :

$$\mathcal{A}_s = \mathcal{A}(N_s) = (\mathbf{I} - \mathbf{L}(N_s) \mathbf{C}) \mathbf{A}^{N_s}, \quad (32a)$$

$$\mathcal{B}_s = \mathcal{B}(N_s) = [-\mathbf{L}(N_s) (\mathbf{I} - \mathbf{L}(N_s) \mathbf{C}) \Lambda(N_s)] \quad (32b)$$

Theorem 5.2 \mathcal{H}_∞ observer design *Consider the predictor algorithm defined by equations (8b) and (8a) applied to system (2) and assume that the outputs are available every $N_s \leq \bar{N}_s$ periods.*

For given $\gamma_{v_1}, \dots, \gamma_{v_{n_y}}, \gamma_w \in \mathbb{R}^+$, assume that there exist \bar{N}_s matrices $\mathbf{Q}_N \in \mathbb{R}^{n \times n}$, a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$, and matrices $\mathbf{X}_N \in \mathbb{R}^{n \times n_y}$ (with $N = \{1, \dots, \bar{N}_s\}$) such that the following LMI fulfills

$$\begin{bmatrix} \vdots & \mathbf{M}_A(1) & \mathbf{M}_B(1) \\ \oplus_{N=1}^{\bar{N}_s} \mathbf{Q}_N + \mathbf{Q}_N^T - \mathbf{P} & \vdots & \vdots \\ \hline \vdots & \mathbf{M}_A(\bar{N}_s) & \mathbf{M}_B(\bar{N}_s) \\ \vdots & \mathbf{P} - \mathbf{A}_1 & -\mathbf{A}_2 \\ \star & \vdots & \star & \Gamma - \mathbf{A}_3 \end{bmatrix} \succ 0, \quad (33)$$

with \star denoting symmetric terms and

$$\mathbf{M}_A(N) = \sqrt{p_N} (\mathbf{Q}_N - \mathbf{X}_N \mathbf{C}) \mathbf{A}^N, \quad (34a)$$

$$\mathbf{M}_B(N) = \sqrt{p_N} [-\mathbf{X}_N \quad (\mathbf{Q}_N - \mathbf{X}_N \mathbf{C}) \mathbf{\Lambda}(N)] \quad (34b)$$

$$\mathbf{A}_1 = \sum_{N=0}^{\bar{N}_s-1} p_{N+1} \mathbf{A}^{N\top} \mathbf{A}^N; \quad \mathbf{A}_2 = \sum_{N=0}^{\bar{N}_s-1} p_{N+1} \mathbf{A}^{N\top} \mathbf{F}(\bar{N}_s, N),$$

$$\mathbf{A}_3 = \sum_{N=0}^{\bar{N}_s-1} p_{N+1} \mathbf{F}(\bar{N}_s, N)^\top \mathbf{F}(\bar{N}_s, N),$$

$$\Gamma = \text{diag}\{\gamma_{v_1}, \dots, \gamma_{v_{n_y}}, \frac{1}{\bar{N}_s} \gamma_w \mathbf{I}\},$$

where p_N is the probability of having an outputs sampling period of N , given by $p_N = (1 - \beta_y)^{N-1} \beta_y$, $N = 1, \dots, \bar{N}_s - 1$, and $p_{\bar{N}_s} = 1 - \sum_{N=1}^{\bar{N}_s-1} p_N$, and matrices $\mathbf{\Lambda}(N)$ and $\mathbf{F}(\bar{N}_s, N)$ are given by (27) and (29).

Then, defining the predictor gain as $\mathbf{L}_N = \mathbf{Q}_N^{-1} \mathbf{X}_N$, the state estimation algorithm defined by (8b) and (8a) converges asymptotically to zero in the absence of disturbances and, under zero initial condition, the RMS norm of the state prediction error, computed at the basic control period, is bounded by

$$\mathcal{E}\{\|\tilde{\mathbf{x}}[t]\|_{RMS}^2\} < \sum_{i=1}^{n_y} \gamma_{v_i} \|v_i[t]\|_{RMS}^2 + \gamma_w \|\mathbf{w}[t]\|_{RMS}^2 \quad (35)$$

Proof: The conditions $\mathbf{Q}_N^\top + \mathbf{Q}_N - \mathbf{P} \succ 0$ imply that \mathbf{Q}_N are of full rank and \mathbf{P} is strictly positive definite. Therefore the following matrix inequality is fulfilled

$$(\mathbf{Q}_N - \mathbf{P})^\top \mathbf{P}^{-1} (\mathbf{Q}_N - \mathbf{P}) \succeq 0$$

which is equivalent to

$$\mathbf{Q}_N^\top \mathbf{P}^{-1} \mathbf{Q}_N \succeq \mathbf{Q}_N^\top + \mathbf{Q}_N - \mathbf{P}. \quad (36)$$

Introducing $\mathbf{X}_N = \mathbf{Q}_N \mathbf{L}_N$ in (33), taking into account (36) and applying Schur complements

one obtains

$$\begin{bmatrix} \left(\sum_{N=1}^{\bar{N}_s} p_N \mathbf{A}(N)^\top \mathbf{P} \mathbf{A}(N) \right) & \sum_{N=1}^{\bar{N}_s} p_N \mathbf{A}(N)^\top \mathbf{P} \mathbf{B}(N) + \mathbf{A}_2 \\ -\mathbf{P} + \mathbf{A}_1 & \\ \star & \left(\sum_{N=1}^{\bar{N}_s} p_N \mathbf{B}(N)^\top \mathbf{P} \mathbf{B}(N) \right) \\ & -\mathbf{\Gamma} + \mathbf{A}_3 \end{bmatrix} \prec \mathbf{0} \quad (37)$$

As $\mathbf{A}_1 \succeq \mathbf{0}$, inequality (37) implies

$$\tilde{\mathbf{x}}_{s-1}^\top \left(\sum_{N=1}^{\bar{N}_s} p_N \mathbf{A}(N)^\top \mathbf{P} \mathbf{A}(N) - \mathbf{P} \right) \tilde{\mathbf{x}}_{s-1} < 0.$$

Assuming that there are no disturbances or measurement noises, using (24) and (32a), the above expression leads to

$$\mathcal{E}\{\tilde{\mathbf{x}}_s^\top \mathbf{P}_s \tilde{\mathbf{x}}_s\} - \tilde{\mathbf{x}}_{s-1}^\top \mathbf{P}_{s-1} \tilde{\mathbf{x}}_{s-1} \prec 0,$$

which assures mean square stability of the prediction error if the Lyapunov function $\mathcal{V}_s = \tilde{\mathbf{x}}_s^\top \mathbf{P} \tilde{\mathbf{x}}_s$ is defined.

Now, multiplying inequality (37) by $[\tilde{\mathbf{x}}_{s-1}^\top \mathbf{W}_s^\top]$ on the left, and by its transpose on the right, it leads to

$$\mathcal{E}\{\mathcal{V}_s\} - \mathcal{V}_{s-1} + \sum_{N=0}^{N_s-1} p_{N+1} \tilde{\mathbf{x}}[t_{s-1}+N]^\top \tilde{\mathbf{x}}[t_{s-1}+N] - \mathbf{W}_s^\top \mathbf{\Gamma} \mathbf{W}_s < 0,$$

where the predictor dynamic error (24) and open loop prediction error (28) have been taken into account. Assuming a null initial prediction error ($\tilde{\mathbf{x}}_0 = \mathbf{0}$) and adding from $s = 1$ to $s = S$ it leads to

$$\mathcal{V}_S + \sum_{s=1}^S \left(\sum_{N=0}^{\bar{N}_s-1} p_{N+1} \tilde{\mathbf{x}}[t_{s-1}+N]^\top \tilde{\mathbf{x}}[t_{s-1}+N] - \mathbf{W}_s^\top \mathbf{\Gamma} \mathbf{W}_s \right) < 0. \quad (38)$$

As $\mathbf{P} \succ 0$, then $\mathcal{V}_S > 0$, leading to

$$\sum_{s=1}^S \left(\sum_{N=0}^{\bar{N}_s-1} p_{N+1} \tilde{\mathbf{x}}[t_{s-1}+N]^\top \tilde{\mathbf{x}}[t_{s-1}+N] - \mathbf{W}_s^\top \mathbf{\Gamma} \mathbf{W}_s \right) < 0. \quad (39)$$

Introducing the definitions of $\mathbf{\Gamma}$ and \mathbf{W}_s it can be written that

$$\begin{aligned} & \sum_{s=1}^S \left(\sum_{N=0}^{\bar{N}_s-1} p_{N+1} \tilde{\mathbf{x}}[t_{s-1}+N]^\top \tilde{\mathbf{x}}[t_{s-1}+N] - \sum_{i=1}^{n_y} \gamma_{v_i} v_i[t_s]^2 \right. \\ & \left. - \frac{1}{\bar{N}_s} \gamma_w \sum_{N=0}^{\bar{N}_s-1} \mathbf{w}[t_{s-1}+N]^\top \mathbf{w}[t_{s-1}+N] \right) < 0. \end{aligned} \quad (40)$$

Dividing by S this expression, and taking the limit when S tends to infinite, the *RMS* norm of the signals at period T is obtained as in (35). \square

The previous result needs the assumption that N_s can not take values above \bar{N}_s . This is only true if the network can guarantee the output measurements transmission when $\bar{N}_s - 1$ periods have elapsed without any output receptions. If the network can not assure this, but \bar{N}_s is selected according equation (30), N_s can take larger values, but with a very low probability ε . In those sporadic cases, the use of the gain calculated for \bar{N}_s is proposed, i.e. $\mathbf{L}(N_s) = \mathbf{L}(\bar{N}_s) \forall N_s \geq \bar{N}_s$. Therefore, the predictor gains can be designed following the previous procedure, but a test for stability that takes into account the unboundedness of N_s is needed. The next result expresses this stability test.

Theorem 5.3: *Assume that for a given set of matrices $\mathbf{L}(N_s)$, the gains $\mathbf{L}(1), \dots, \mathbf{L}(\bar{N}_s - 1)$ are used for the values $N_s = 1$ to $\bar{N}_s - 1$, and the gain $\mathbf{L}(\bar{N}_s)$ is used when $N_s \geq \bar{N}_s$. Assume also that there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ such that the following conditions hold*

$$\mathbf{P} \sum_{i=1}^{\bar{N}_s-1} \beta_y (1 - \beta_y)^{i-1} \mathbf{A}^{iT} \mathbf{X}(i) \mathbf{A}^i + \beta_y (1 - \beta_y)^{\bar{N}_s-1} \mathbf{A}^{\bar{N}_s T} \mathbf{Q} \mathbf{A}^{\bar{N}_s} \succ 0, \quad (41)$$

$$\sqrt{1 - \beta_y} |\bar{\lambda}(\mathbf{A})| < 1, \quad (42)$$

with

$$\mathbf{Q} = \text{vec}^{-1} \left\{ \left(\mathbf{I} - (1 - \beta_y)(\mathbf{A}^\top \otimes \mathbf{A}^\top) \right)^{-1} \text{vec} \{ \mathbf{X}(\bar{N}_s) \} \right\} \quad (43)$$

$$\mathbf{X}(i) = (\mathbf{I} - \mathbf{L}(i)\mathbf{C})^\top \mathbf{P} (\mathbf{I} - \mathbf{L}(i)\mathbf{C}) \quad (44)$$

then, the observer is stable in the average sense for all the possible sampling scenarios described by the probability β_y (from $N_s = 1$ to ∞).

Proof: Let us first define the Lyapunov function $\mathcal{V}(\tilde{\mathbf{x}}_s) = \tilde{\mathbf{x}}_s^\top \mathbf{P} \tilde{\mathbf{x}}_s$. The expected value of the Lyapunov function at each instant when new measurements are available depends on the time when the previous measurements were available, and is given by

$$\mathcal{E}\{\mathcal{V}(\tilde{\mathbf{x}}_s)\} = \tilde{\mathbf{x}}_{s-1}^\top \left[\sum_{i=1}^{\bar{N}_s-1} \beta_y (1 - \beta_y)^{i-1} \mathbf{A}^{iT} \mathbf{X}(i) \mathbf{A}^i + \sum_{i=\bar{N}_s}^{\infty} \beta_y (1 - \beta_y)^{i-1} \mathbf{A}^{iT} \mathbf{X}(\bar{N}_s) \mathbf{A}^i \right] \tilde{\mathbf{x}}_{s-1}$$

if null disturbances are assumed. The second addend can be written as $\beta_y (1 - \beta_y)^{\bar{N}_s-1} (\mathbf{A}^{\bar{N}_s})^\top \mathbf{Q} \mathbf{A}^{\bar{N}_s}$, with $\mathbf{Q} = \sum_{i=0}^{\infty} (1 - \beta_y)^i \mathbf{A}^{iT} \mathbf{X}(\bar{N}_s) \mathbf{A}^i$. The infinity sum \mathbf{Q} will only result in a finite value if the eigenvalues of the powered matrices are lower than one, i.e., if $\sqrt{1 - \beta_y} |\bar{\lambda}(\mathbf{A})| < 1$. If this condition is fulfilled, using the encapsulating sum formula, the previous sum can be written as expressed on equation (43). The stability in the average sense will be satisfied if $\mathcal{E}\{\mathcal{V}(\tilde{\mathbf{x}}_s)\} < \mathcal{V}(\tilde{\mathbf{x}}_{s-1})$, that is an equivalent condition to that expressed in (41). Note

that the expression is an LMI as the $\text{vec}\{\cdot\}$ and $\text{vec}^{-1}\{\cdot\}$ operators keeps the linearity over the decision variables on matrix \mathbf{P} . \square

Remark 2: The condition $\sqrt{1 - \beta_y} |\bar{\lambda}(A)| < 1$ is always fulfilled if the process is open loop stable. However, if the system is unstable, this condition imposes a restriction in the relation between the network accessibility and the maximum unstable eigenvalue in order to guarantee the stabilizability of the closed loop (the lower the network accessibility, the lower the maximum eigenvalue allowed).

In the case that LMI (41) is not feasible, then a lower value of ε , i.e. a larger value of \bar{N}_s , should be selected to recalculate the predictor gains.

Remark 3 Design procedure If the norms of the disturbance and noises are known, then minimizing the sum

$$\sum_{i=1}^{n_y} \gamma_{v_i} \|v_i[t]\|_{RMS}^2 + \gamma_w \|\mathbf{w}[t]\|_{RMS}^2$$

along LMI (33) will lead to the observer gains set, $\mathbf{L}(N_s)$, that minimize $\|\tilde{\mathbf{x}}[t]\|_{RMS}$. This minimization can be solved with standard convex optimization tools.

6. Controller design

Theorem 6.1 :

Consider the controller algorithm defined by equations (8) applied to system (2). Assume that during period t_k the controller can transmit to the input buffer the value $\mathbf{u}^[t+1]$. Then, the state dynamics of the control system with control algorithm (8) when there is no modelling error and there is one input update every N_k control periods (with N_k time variant), is described by the linear time-variant system*

$$\begin{aligned} \mathbf{x}[t_k+1+N_k] &= \left(\mathbf{A}^{N_k} + \sum_{i=0}^{N_k-1} \mathbf{A}^i \mathbf{B} \mathbf{K} \right) \mathbf{x}[t_k+1] \\ &+ \sum_{i=0}^{N_k-1} \mathbf{A}^i (\mathbf{w}[t_k+N_k-i] - \mathbf{B} \mathbf{K} \boldsymbol{\xi}_k) \end{aligned} \quad (45)$$

that is updated every time a new input update is available, and being $\boldsymbol{\xi}_k$ the disturbance contribution of the 2-step prediction used on the control action calculation that is given by

$$\boldsymbol{\xi}_k = \mathbf{w}[t_k] + \mathbf{A} \mathbf{w}[t_k - 1] + \mathbf{A}^2 \tilde{\mathbf{x}}[t_k - 1]. \quad (46)$$

Proof: If during the $t = t_k$ -th period the input buffer is accessed ($\alpha_u[t_k] = 1$), the control actions that will be applied at period $t_k + 1$ can be written as a function of the state, state estimation error and disturbances (using the second row on matrices \mathbf{A}_{CL} and \mathbf{B}_{CL} in (9)) as

$$\begin{aligned} \mathbf{u}[t_k + 1] &= \mathbf{K} \mathbf{A} \mathbf{x}[t_k] + \mathbf{K} \mathbf{B} \mathbf{u}[t_k] \\ &- \mathbf{K} \mathbf{A}^2 \tilde{\mathbf{x}}[t_k - 1] - \mathbf{K} \mathbf{A} \mathbf{w}[t_k - 1], \end{aligned} \quad (47)$$

with $\mathbf{u}[t_k]$ the control action that has been applied since the last input update was available at

$t = t_k - N_{k-1}$. Combining (2a) with (47), it leads to

$$\mathbf{u}[t_k+1] = \mathbf{K}\mathbf{x}[t_k+1] - \mathbf{K}(\mathbf{w}[t_k] + \mathbf{A}\mathbf{w}[t_k-1] + \mathbf{A}^2\tilde{\mathbf{x}}[t_k-1]). \quad (48)$$

The evolution of the state from the instant in which the input update is done ($t_k + 1$) until the next input update occurs (at instant $t_k + N_k + 1$) is given by

$$\mathbf{x}[t_k+i] = \mathbf{A}\mathbf{x}[t_k+i-1] + \mathbf{B}u[t_k+1] + \mathbf{w}[t_k+i-1] \quad (49)$$

where $i = 2, \dots, N_k + 1$ and the control action is the value written on the buffer during period t_k ($u[t_k + 1]$). Introducing expression (48) in (49) and applying it recursively from $i = 2$ to $i = N_k$, it finally leads to (45). \square

Remark 1: If a new vector gathering the disturbances between input updates is defined as

$$\mathbf{W}_k = [\mathbf{w}[t_k + \bar{N}_k]^\top \cdots \mathbf{w}[t_k + 1]^\top \boldsymbol{\xi}_k]^\top,$$

with \bar{N}_k the maximum input update period considered in the control system, then, the system evolution between input update instants can be written in a compact way as

$$\mathbf{x}[t_k + 1 + N_k] = \mathcal{A}_k \mathbf{x}[t_k + 1] + \mathcal{B}_k \mathbf{W}_k \quad (50)$$

$$(51)$$

where

$$\mathcal{A}_k = \mathbf{A}^{N_k} + \sum_{i=0}^{N_k-1} \mathbf{A}^i \mathbf{B} \mathbf{K}, \quad (52)$$

$$\mathcal{B}_k = [\boldsymbol{\Lambda}'(N_k) - \sum_{i=0}^{N_k-1} \mathbf{A}^i \mathbf{B} \mathbf{K}] \quad (53)$$

being $\boldsymbol{\Lambda}'(N_k)$ the matrix defined as

$$\boldsymbol{\Lambda}'(j) = [\underbrace{\mathbf{0} \cdots \mathbf{0} \quad \mathbf{I} \quad \mathbf{A} \quad \mathbf{A}^2 \quad \cdots \quad \mathbf{A}^{j-1}}_{\bar{N}_k}]_{n \times \bar{N}_k n}, \quad (54)$$

The closed loop system dynamics depends on the matrix gain \mathbf{K} that is applied at input updating instants ($t = t_k$), and must be designed to assure: the system stability, robustness to the irregular data availability and a proper attenuation of the state estimation error and the disturbances.

The difference on the controller design with respect to the predictor design is that the controller does not know in advance when the control inputs will be again updated, or, in other words, how many periods an updated control input will be applied until the next input update is possible. For this reason, the design of a controller gain that depends on each input updating period N_k is not possible, and then a constant gain \mathbf{K} is proposed¹. Taking this into account, matrices \mathcal{A}_k

¹The dependence on N_k would have been possible if a Markov chain were considered, as in the recent works Yu et al. (2009), Shi and Yu (2011), Wang et al. (2011). This idea will be developed in future work.

and \mathcal{B}_k can be written as a function of the input update period as

$$\mathcal{A}_k = \mathcal{A}(N_k) = \mathbf{A}^{N_k} + \sum_{i=0}^{N_k-1} \mathbf{A}^i \mathbf{B} \mathbf{K}, \quad (55)$$

$$\mathcal{B}_k = \mathcal{B}(N_k) = [\mathbf{A}'(N_k) - \sum_{i=0}^{N_k-1} \mathbf{A}^i \mathbf{B} \mathbf{K}]. \quad (56)$$

In the controller design, a stochastic strategy that assures the convergence of the control system in an average sense, depending on the actual value of β_u , is proposed.

Theorem 6.2 *Stochastic \mathcal{H}_∞ controller design* Consider the control algorithm (8) applied to system (2). Assume that the probability of accessing the input buffer in each control period is β_u , but the resulting input updating period is bounded by $N_k \leq \bar{N}_k$, i.e., the priorities of the network are assumed to be changed to guarantee the control input successful transmission in the case that $\bar{N}_k - 1$ periods have elapsed since the last update. For a given $\gamma_w, \gamma_\xi \in \mathbb{R}^+$, assume that there exist symmetric positive definite matrices $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n \times n}$ and matrices $\mathbf{X} \in \mathbb{R}^{n_u \times n}$ such that

$$\begin{bmatrix} \vdots & M_A(1) & M_B(1) \\ \oplus_{N=1}^{\bar{N}_k} \mathbf{Q} & \vdots & \vdots \\ \vdots & M_A(\bar{N}_k) & M_B(\bar{N}_k) \\ \hline \star & \mathbf{P} - \mathbf{I} & \mathbf{0} \\ \vdots & \vdots & \mathbf{\Gamma} \end{bmatrix} \succ \mathbf{0}, \quad (57)$$

$$\mathbf{P} \mathbf{Q} = \mathbf{I}, \quad (58)$$

where p_N is the probability of having an input updating period of N , given by $p_N = (1 - \beta_u)^{N-1} \beta_u$, $N = 1, \dots, \bar{N}_k - 1$, and $p_{\bar{N}_k} = 1 - \sum_{i=1}^{\bar{N}_k-1} p_N$. The matrices are defined as follows

$$M_A(N) = \sqrt{p_N} \left(\mathbf{A}^N + \sum_{i=0}^{N-1} \mathbf{A}^i \mathbf{B} \mathbf{K} \right), \quad M_B(N) = \sqrt{p_N} [\mathbf{A}'(N) - \sum_{i=0}^{N-1} \mathbf{A}^i \mathbf{B} \mathbf{K}] \quad (59)$$

with

$$\mathbf{\Gamma} = \text{diag} \left\{ \frac{1}{\bar{N}_k} \gamma_w \mathbf{I}, \gamma_\xi \mathbf{I} \right\}. \quad (60)$$

Then, the system is mean square stable in the absence of disturbances or estimation error, and, under zero initial conditions, the state is bounded by

$$\mathcal{E} \{ \|\mathbf{x}[t]\|_{RMS}^2 \} < \gamma_w \|\mathbf{w}[t]\|_{RMS}^2 + \gamma_\xi \|\boldsymbol{\xi}[t]\|_{RMS}^2. \quad (61)$$

Proof: Applying Schur complements to matrix inequality (57) and multiplying by

$[\mathbf{x}[t_k + 1]^\top, \mathbf{W}_k^\top]$ on the left and by its transpose on the right it finally leads to

$$\begin{aligned} & \sum_{N=1}^{\bar{N}_k} \left(p_N \underbrace{(\mathcal{A}(N) \mathbf{x}[t_k + 1] + \mathcal{B}(N) \mathbf{W}_k)^\top \mathbf{P}(\star)}_{\star} \right) \\ & - \mathbf{x}[t_k + 1]^\top (\mathbf{P} - \mathbf{I}) \mathbf{x}[t_k + 1] \\ & - \frac{\gamma_w}{\bar{N}_k} \sum_{N=1}^{\bar{N}_k} p_N \mathbf{w}[t_k + N]^\top \mathbf{w}[t_k + N] - \gamma_\xi \boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k < 0. \end{aligned} \quad (62)$$

If a Lyapunov function is defined as $\mathcal{V}[t] = \mathbf{x}[t]^\top \mathbf{P} \mathbf{x}[t]$, the first addend of the previous expression represents $\mathcal{E}\{\mathcal{V}[t_{k+1} + 1]\}$, that is the expected value of the Lyapunov function for the next control period. Then, under zero disturbance and zero state estimation error condition, the convergence in average is assured ($\mathcal{E}\{\mathcal{V}[t_{k+1} + 1]\} < \mathcal{V}[t_k + 1]$). Adding the expression (62) from $k = 0$ to $k = K > 0$ it leads

$$\begin{aligned} & \sum_{k=0}^K \mathcal{E}\{\mathcal{V}[t_{k+1} + 1]\} - \mathcal{V}[t_k + 1] + \mathbf{x}[t_k + 1]^\top \mathbf{x}[t_k + 1] \\ & < \frac{\gamma_w}{\bar{N}_k} \sum_{k=0}^K \sum_{N=1}^{\bar{N}_k} \mathbf{w}[t_k + N]^\top \mathbf{w}[t_k + N] + \gamma_\xi \boldsymbol{\xi}_k^\top \boldsymbol{\xi}_k. \end{aligned}$$

Assuming null initial state ($\mathbf{x}[t_k + 1] = 0$), dividing by K and taking the limit when K tends to infinite, then (61) is obtained. \square

The previous result needs the assumption that N_k can not take values above \bar{N}_k . This is only true if the network can guarantee the control inputs transmission when $\bar{N}_k - 1$ periods have elapsed without any input update. If the network can not assure this, but \bar{N}_k is selected according equation (7), N_k can take larger values, but with a very low probability ε . Therefore, the gain can be designed following the previous procedure, but a test for stability that takes into account the unboundedness of N_k is needed. The next result expresses this stability test.

Theorem 6.3: *For a given controller matrix gain \mathbf{K} , if the probability of successful transmission of the control input in each period is β_u , a sufficient condition to assure that the closed loop is stable in the average sense in the absence of disturbances and for all the possible sampling scenarios (from $N_k = 1$ to ∞) is the existence of a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ such that the following LMI condition hold*

$$\begin{aligned} & \beta_u (\mathbf{I} - \bar{\mathbf{K}})^\top \text{vec}^{-1} \{ (\mathbf{I} - \bar{\beta}_u \mathbf{A}^\top \otimes \mathbf{A}^\top)^{-1} \text{vec} \{ \mathbf{A}^\top \mathbf{P} \mathbf{A} \} \} (\mathbf{I} - \bar{\mathbf{K}}) \\ & + \bar{\mathbf{K}}^\top \mathbf{P} \bar{\mathbf{K}} + \beta_u \text{Sym} \{ \bar{\mathbf{K}}^\top \mathbf{P} (\mathbf{I} - \bar{\beta}_u \mathbf{A})^{-1} \mathbf{A} (\mathbf{I} - \bar{\mathbf{K}}) \} - \mathbf{P} \prec 0 \end{aligned} \quad (63)$$

being $\bar{\mathbf{K}} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \mathbf{K}$, $\bar{\beta}_u = 1 - \beta_u$.

Proof: Let us define the Lyapunov function at the instant in which a new control action is updated as $\mathcal{V}_k = \mathbf{x}_k^\top \mathbf{P} \mathbf{x}_k$. In the absence of disturbances, the expected value of the Lyapunov

function in the next instant when a new control action is updated is given by

$$\mathcal{E}\{\mathcal{V}_{k+1}\} = \sum_{N=1}^{\infty} \beta_u (1-\beta_u)^{N-1} \mathbf{x}_k^T \underbrace{\left(\mathbf{A}^N + \sum_{j=0}^{N-1} \mathbf{A}^j \mathbf{B} \mathbf{K} \right)}_{\star} \mathbf{P}(\star) \mathbf{x}_k$$

Taking into account that $\sum_{j=0}^{N-1} \mathbf{A}^j = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A}^N)$ it follows

$$\mathcal{E}\{\mathcal{V}_{k+1}\} = \mathbf{x}_k^T \sum_{N=1}^{\infty} \beta_u \bar{\beta}_u^{N-1} \underbrace{(\mathbf{A}^N (\mathbf{I} - \bar{\mathbf{K}}) + \bar{\mathbf{K}})}_{\star} \mathbf{P}(\star) \mathbf{x}_k.$$

If the infinite summation is developed, using the encapsulating sum formula, the following compact form can be obtained

$$\begin{aligned} \mathcal{E}\{\mathcal{V}_{k+1}\} &= \mathbf{x}_k^T (\beta_u (\mathbf{I} - \bar{\mathbf{K}})^T \text{vec}^{-1} \{ (\mathbf{I} - (1 - \beta_u) \mathbf{A}^T \otimes \mathbf{A}^T)^{-1} \text{vec} \{ \mathbf{A}^T \mathbf{P} \mathbf{A} \} \} (\mathbf{I} - \bar{\mathbf{K}}) \\ &\quad + \bar{\mathbf{K}}^T \mathbf{P} \bar{\mathbf{K}} + \beta_u \text{Sym} \{ \bar{\mathbf{K}}^T \mathbf{P} (\mathbf{I} - (1 - \beta_u) \mathbf{A})^{-1} \mathbf{A} (\mathbf{I} - \bar{\mathbf{K}}) \}) \mathbf{x}_k. \end{aligned} \quad (64)$$

From this expression it is straightforward to show that (63) is equivalent to condition $\mathcal{E}\{\mathcal{V}_{k+1}\} < \mathcal{V}_k$. \square

In the case that LMI (63) is not feasible, then a lower value of ε , i.e. a larger value of \bar{N}_k , should be selected to recalculate the controller gain.

Remark 2: If the norms of the disturbance and the estimation error of the two-step prediction $\boldsymbol{\xi}_k$ are known, then, minimizing the sum $\gamma_w \|\mathbf{w}[t]\|_{RMS}^2 + \gamma_\xi \|\boldsymbol{\xi}[t]\|_{RMS}^2$ subject to LMI (57) and (58) will minimize $\|\mathbf{x}[t]\|_{RMS}^2$.

The norm of $\|\boldsymbol{\xi}_k\|_{RMS}^2$ is not previously known but it can be bounded by

$$\|\boldsymbol{\xi}[t]\|_{RMS}^2 < \|\mathbf{I} + \mathbf{A}\|_2^2 \|\mathbf{w}[t]\|_{RMS}^2 + \|\mathbf{A}^2\|_2^2 \|\tilde{\mathbf{x}}[t]\|_{RMS}^2, \quad (65)$$

where the norm of $\mathbf{w}[t]$ is assumed to be known, and where the norm of $\tilde{\mathbf{x}}[t]$ is the resulting bound on the predictor design.

The optimization problem presented in Remark 2 is not an strict LMI problem due to bilinear equality constraint (58). As stated in El Ghaoui et al. (1997), this kind of nonconvex feasibility can be solved using the complementarity linearization algorithm following the next procedure. First, the optimization problem presented in Remark 2 is rewritten as

$$\min \text{trace}(\mathbf{P}\mathbf{Q})$$

subject to (57),

$$\gamma_w \|\mathbf{w}[t]\|_{RMS}^2 + \gamma_\xi \|\boldsymbol{\xi}[t]\|_{RMS}^2 < \bar{\gamma} \quad (66)$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{I} \\ \mathbf{I} & \mathbf{Q} \end{bmatrix} \succeq 0 \quad (67)$$

where $\bar{\gamma}$ is a real positive value. The above nonlinear minimization problem, where $\bar{\gamma}$ must be minimized, can be solved using the following bisection algorithm over a cone complementarity iterative algorithm.

- Step 1 Choose a large enough initial $\bar{\gamma}_u$ such that there exists a feasible solution to LMI conditions (57), (66) and (67) with $\bar{\gamma} = \bar{\gamma}_u$. Set $\bar{\gamma}_l = 0$, and set initially $\bar{\gamma} = \frac{1}{2}(\bar{\gamma}_l + \bar{\gamma}_u)$.
- Step 2 Set $k = 0$ and find a feasible solution set $[\mathbf{P}^k, \mathbf{Q}^k, \mathbf{K}^k, \gamma_w^k, \gamma_\xi^k]$, satisfying (57), (66) and (67).
- Step 3 Solve the following LMI problem for the decision variables $\mathbf{P}, \mathbf{Q}, \mathbf{K}, \gamma_w$ and γ_ξ :

$$\begin{aligned} & \min \text{trace}(\mathbf{P}^k \mathbf{Q} + \mathbf{P} \mathbf{Q}^k) \\ & \text{subject to (57), (66), (67)} \\ & \text{set } k = k + 1, \mathbf{P}^k = \mathbf{P}, \mathbf{Q}^k = \mathbf{Q}. \end{aligned}$$

- Step 4 If $k < k_{\max}$, for a given prescribed maximum number of iterations k_{\max} , and (57) is not satisfied after replacing \mathbf{Q} by \mathbf{P}^{-1} , then return to Step 3. If $k < k_{\max}$ and (57) are satisfied, update the upper bound on $\bar{\gamma}$ as $\bar{\gamma}_u = \bar{\gamma}$, store the actual controller gain \mathbf{K} , and go to Step 5. If $k = k_{\max}$, update the lower bound on $\bar{\gamma}$ as $\bar{\gamma}_l = \bar{\gamma}$ and go to step 5.
- Step 5 If $\bar{\gamma}_u - \bar{\gamma}_l > \delta$, for a given small δ , update $\bar{\gamma}$ with $\bar{\gamma} = \frac{1}{2}(\bar{\gamma}_l + \bar{\gamma}_u)$ and go to Step 2. If $\bar{\gamma}_u - \bar{\gamma}_l \leq \delta$ exit with the last stored solution \mathbf{K} in Step 4.

7. Example

Consider the 3-input, 3-output, 6th order unstable LTI plant from Hristu-Varsakelis and Zhang (2008) defined by matrices

$$A = \begin{bmatrix} 1.1 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 0 & -0.75 & -1.5 & -0.75 & -0.75 \\ -1.1 & 0 & 0 & -1.1 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 & 0 \\ 1.1 & 0.75 & 0 & 1.1 & 0 & -0.75 \\ -0.75 & 0 & -0.75 & -0.75 & 0 & -0.75 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Assume that there is a network where the probabilities of successful transmission of packets containing either all data measurements from sensors or all control actions from the controller are $\beta_y = \beta_u = \beta = 0.6$. Assume a disturbance bounded by the norm $\|\mathbf{w}\|_{RMS} = 0.01$, and two different measurement noise levels bounded by $\|v_l\|_{RMS} = 0.01$ (low) and $\|v_h\|_{RMS} = 0.1$ (high), affecting the three measurements. The maximum number of periods between samples, with a probability of $1 - \varepsilon = 0.999$ results to be $\bar{N}_s = \bar{N}_k = 9$. Two strategies are compared for both noise levels, a constant predictor gain \mathbf{L} and a varying one depending on the sampling period $\mathbf{L}_s = \mathbf{L}(N_s)$ ($N_s = 1, \dots, 9$). Following the results in section V and VI, the bounds on the prediction error $\|\tilde{\mathbf{x}}\|_{RMS}$, on the two-step prediction error $\|\boldsymbol{\xi}\|_{RMS}$ and on the state tracking error $\|\mathbf{x}\|_{RMS}$ are obtained, as well as the predictor and controller gains. Both the LMI (41) and (63) are feasible with the computed gains and, therefore, stability is guaranteed. Table 1 shows the errors RMS norm. It can be appreciated how the use of a varying predictor gain depending on the sampling period, allows to decrease the state tracking error $\|\mathbf{x}\|_{RMS}$ in approximately 15% for both low measurement noise and high measurement noise. Figure 2 shows the norms of the predictor gains as a function of the number of periods between samples, N_s for

Table 1. Comparative results of the four situations

Strategy	$\ \tilde{x}\ _{RMS}$	$\ \xi\ _{RMS}$	$\ x\ _{RMS}$
$L(N_s), \ v_l\ _{RMS}$	0.4581	1.3514	2.5374
$L, \ v_l\ _{RMS}$	0.5497	1.6204	2.9430
$L(N_s), \ v_h\ _{RMS}$	0.7722	2.2745	3.9284
$L, \ v_h\ _{RMS}$	0.9054	2.6662	4.5183

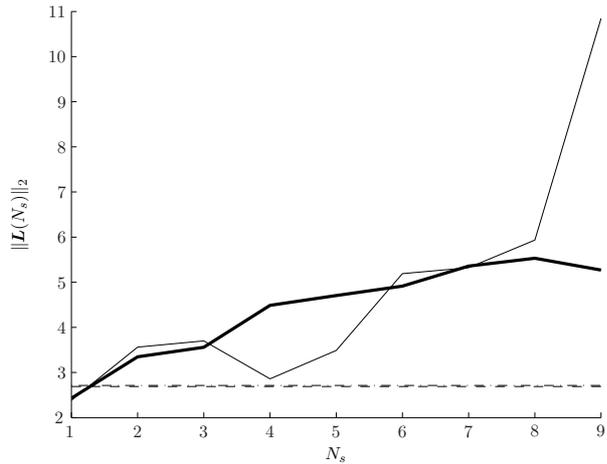


Figure 2. Norm of the predictor gain as a function of N_s . '—': varying $L(N_s)$, high noise. '- -': varying $L(N_s)$, low noise. '- · -': constant L , high noise. '· · ·': constant L , low noise.

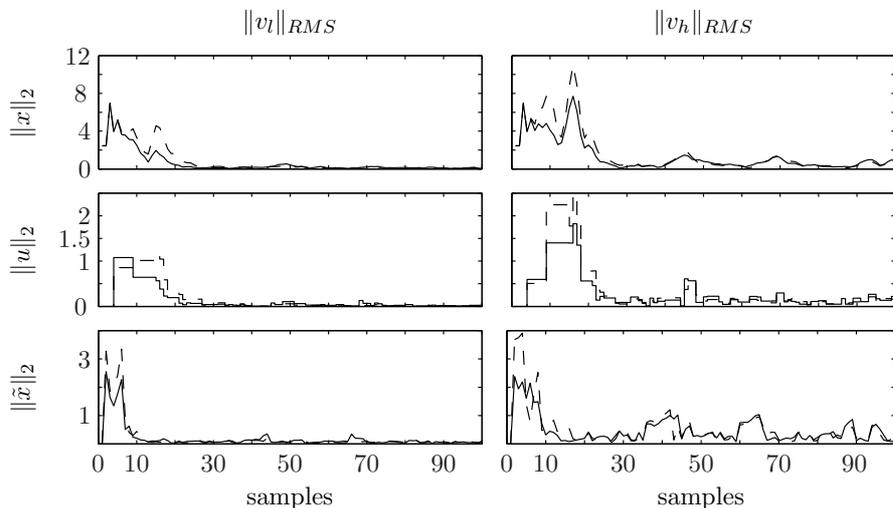


Figure 3. State stabilization transient. States (x), control input (u) and prediction error (\tilde{x}) vector norms for $\|v_l\|_{RMS} = 0.01$ and $\|v_h\|_{RMS} = 0.1$. '—': varying $L(N_s)$, '- -': constant L

each of the different proposed scenarios (low and high noise measurement level, and constant vs. varying gain).

In figures 3 and 4 a simulation of the proposed controller and the sampling and input updating instants are shown. It can be appreciated that the proposed approach with a varying predictor gain has a lower prediction error, and, therefore, the performance improves the one obtained with a constant gain observer.

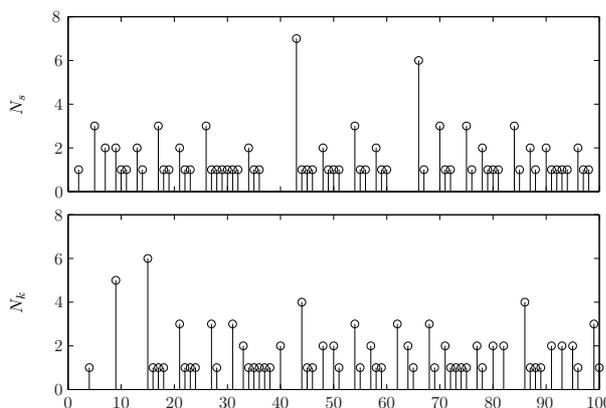


Figure 4. Measurement and control input updating periods N_s , N_k respectively

8. Conclusions

In this work, an inferential control scheme for networked control systems has been proposed. The plant has a unitary input buffer where the inputs updates are written by the controller through the network, and a zero order hold that reads every T seconds the value stored in the buffer to apply it to the process. There is also an output unitary buffer where the sensor measurements are written every T seconds synchronously with the inputs update. This buffer sends the outputs sampled measurements to the controller through the network. The network is assumed to have restricted accessibility and to induce delays. An observer-controller algorithm has been proposed where the network access and transmission related problems (modeled by the probability of successful transmission at every period) are transformed into a problem of random missing measurements and sporadic input updates (or time-varying sampling and control periods).

A model based predictor that attenuates disturbances and measurement noise, and takes into account the variability on the sampling period has been designed assuring \mathcal{H}_∞ performance. It predicts the state in future periods to calculate the control action as a state feedback. The controller is designed to attenuate the effect of the disturbances and the state prediction error, taking into account the variability on the input update period, and assuring an \mathcal{H}_∞ performance. The predictor is designed first, and the bound on the estimation error obtained from the predictor design is used in the controller design.

The presented results could be extended to adapt the controller and observer gains to a slowly time varying network accessibility constrains. The idea would be to estimate the probability of successful transmission and to define the gains as a function of that probability. This idea will be developed in a future work.

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