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<td><strong>Autores / Autors</strong></td>
<td>Ferrer González, María Vicenta; Hernández Muñoz, Salvador; Uspenskij, Vladimir</td>
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THE DUAL SPACE OF PRECOMPACT GROUPS

M. FERRER, S. HERNÁNDEZ, AND V. USPENSKIJ

Abstract. For any topological group $G$ the dual object $\hat{G}$ is defined as the set of equivalence classes of irreducible unitary representations of $G$ equipped with the Fell topology. If $G$ is compact, $\hat{G}$ is discrete. In an earlier paper we proved that $\hat{G}$ is discrete for every metrizable precompact group, i.e. a dense subgroup of a compact metrizable group. We generalize this result to the case when $G$ is an almost metrizable precompact group.

1. Introduction

For a topological group $G$ let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$. The set $\hat{G}$ can be equipped with a natural topology, the so-called Fell topology (see Section 2 for a definition).

A topological group $G$ is precompact if it is isomorphic (as a topological group) to a subgroup of a compact group $H$ (we may assume that $G$ is dense in $H$). If $H$ is compact, then $\hat{H}$ is discrete. If $G$ is a dense subgroup of $H$, the natural mapping $\hat{H} \to \hat{G}$ is a bijection but in general need not be a homeomorphism. Moreover, for every countable non-metrizable precompact group $G$ the space $\hat{G}$ is not discrete [12, Theorem 5.1], and every non-metrizable compact group $H$ has a dense subgroup $G$ such that $\hat{G}$ is not discrete [12, Theorem 5.2]. (The Abelian case was considered in
[5, 6, 14]). On the other hand, if $G$ is a precompact metrizable group, then $\hat{G}$ is discrete [12, Theorem 4.1]. (The Abelian case was considered in [2, 4]). The aim of the present paper is to generalize this result to the almost metrizable case: $\hat{G}$ is discrete for every almost metrizable precompact topological group $G$. A topological group $G$ is almost metrizable if it has a compact subgroup $K$ such that the quotient space $G/K$ is metrizable. According to Pasynkov’s theorem [1, 4.3.20], a topological group is almost metrizable if and only if it is feathered in the sense of Arhangel’skii.

We reduce the almost metrizable case to the metrizable case considered in [12, Theorem 4.1].

2. Preliminaries: Fell topologies

All topological spaces and groups that we consider are assumed to be Hausdorff. For a (complex) Hilbert space $\mathcal{H}$ the unitary group $U(\mathcal{H})$ of all linear isometries of $\mathcal{H}$ is equipped with the strong operator topology (this is the topology of pointwise convergence). With this topology, $U(\mathcal{H})$ is a topological group.

A unitary representation $\rho$ of the topological group $G$ is a continuous homomorphism $G \to U(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. A closed linear subspace $E \subseteq \mathcal{H}$ is an invariant subspace for $S \subseteq U(\mathcal{H})$ if $ME \subseteq E$ for all $M \in S$. If there is a closed subspace $E$ with $\{0\} \subsetneq E \subseteq \mathcal{H}$ which is invariant for $S$, then $S$ is called reducible; otherwise $S$ is irreducible. An irreducible representation of $G$ is a unitary representation $\rho$ such that $\rho(G)$ is irreducible.

If $\mathcal{H} = \mathbb{C}^n$, we identify $U(\mathcal{H})$ with the unitary group of order $n$, that is, the compact Lie group of all complex $n \times n$ matrices $M$ for which $M^{-1} = M^*$. We denote this group by $\mathbb{U}(n)$. 
Two unitary representations \( \rho : G \to U(\mathcal{H}_1) \) and \( \psi : G \to U(\mathcal{H}_2) \) are equivalent if there exists a Hilbert space isomorphism \( M : \mathcal{H}_1 \to \mathcal{H}_2 \) such that \( \rho(x) = M^{-1}\psi(x)M \) for all \( x \in G \). The dual object of a topological group \( G \) is the set \( \hat{G} \) of equivalence classes of irreducible unitary representations of \( G \).

If \( G \) is a precompact group, the Peter-Weyl Theorem (see [15]) implies that all irreducible unitary representation of \( G \) are finite-dimensional and determine an embedding of \( G \) into the product of unitary groups \( U(n) \).

If \( \rho : G \to U(\mathcal{H}) \) is a unitary representation, a complex-valued function \( f \) on \( G \) is called a function of positive type (or positive-definite function) associated with \( \rho \) if there exists a vector \( v \in \mathcal{H} \) such that \( f(g) = (\rho(g)v, v) \) (here \((\cdot, \cdot)\) denotes the inner product in \( \mathcal{H} \)). We denote by \( P'_{\rho} \) the set of all functions of positive type associated with \( \rho \). Let \( P_{\rho} \) be the convex cone generated by \( P'_{\rho} \), that is, the set of sums of elements of \( P'_{\rho} \).

Let \( G \) be a topological group, \( \mathcal{R} \) a set of equivalence classes of unitary representations of \( G \). The Fell topology on \( \mathcal{R} \) is defined as follows: a typical neighborhood of \([\rho] \in \mathcal{R}\) has the form

\[
W(f_1, \cdots, f_n, C, \epsilon) = \{ [\sigma] \in \mathcal{R} : \exists g_1, \cdots, g_n \in P_{\sigma} \forall x \in C \ |f_i(x) - g_i(x)| < \epsilon\},
\]

where \( f_1, \cdots, f_n \in P_{\rho} \) (or \( \in P'_{\rho} \)), \( C \) is a compact subspace of \( G \), and \( \epsilon > 0 \). In particular, the Fell topology is defined on the dual object \( \hat{G} \). If \( G \) is locally compact, the Fell topology on \( \hat{G} \) can be derived from the Jacobson topology on the primitive ideal space of \( C^*(G) \), the \( C^* \)-algebra of \( G \) [7, section 18], [3, Remark F.4.5].

Every onto homomorphism \( f : G \to H \) of topological groups gives rise to a continuous injective dual map \( \hat{f} : \hat{H} \to \hat{G} \). A mapping \( h : X \to Y \) between topological
spaces is \textit{compact-covering} if for every compact set \( L \subset Y \) there exists a compact set \( K \subset X \) such that \( h(K) = L \).

**Lemma 2.1.** If \( f : G \to H \) is a compact-covering onto homomorphism of topological groups, the dual map \( \hat{f} : \widehat{H} \to \widehat{G} \) is a homeomorphic embedding.

**Proof.** This easily follows from the definition of Fell topology. \( \square \)

Let \( \pi \) be a unitary representation of a topological group \( G \) on a Hilbert space \( \mathcal{H} \). Let \( F \subseteq G \) and \( \epsilon > 0 \). A unit vector \( v \in \mathcal{H} \) is called \((F, \epsilon)\)-invariant if \( \| \pi(g)v - v \| < \epsilon \) for every \( g \in F \).

A topological group \( G \) has property \((T)\) if and only if there exists a pair \((Q, \epsilon)\) (called a \textit{Kazhdan pair}), where \( Q \) is a compact subset of \( G \) and \( \epsilon > 0 \), such that for every unitary representation \( \rho \) having a unit \((Q, \epsilon)\)-invariant vector there exists a non-zero invariant vector. Equivalently, \( G \) has property \((T)\) if and only if the trivial representation \( 1_G \) is isolated in \( \mathcal{R} \cup \{1_G\} \) for every set \( \mathcal{R} \) of equivalence classes of unitary representations of \( G \) without non-zero invariant vectors \[3, Proposition 1.2.3\].

Compact groups have property \((T)\) \[3, Proposition 1.1.5\], but countable Abelian precompact groups do not have property \((T)\) \[12, Theorem 6.1\].

We refer to Fell’s papers \[9, 10\], the classical text by Dixmier \[7\] and the recent monographs by de la Harpe and Valette \[13\], and Bekka, de la Harpe and Valette \[3\] for basic definitions and results concerning Fell topologies and property \((T)\).

3. \textbf{Almost metrizable groups}

If \( A \) is a subset of a topological space \( X \), the \textit{character} \( \chi(A, X) \) of \( A \) in \( X \) is the least cardinality of a base of neighborhoods of \( A \) in \( X \). (If this definition leads to a finite value of \( \chi(A, X) \), we replace it by \( \omega \), the first infinite cardinal, and similarly for
other cardinal invariants.) If $A$ is a closed subset of a compact space $X$, the character $\chi(A, X)$ equals the pseudocharacter $\psi(A, X)$ – the least cardinality of a family $\gamma$ of open subsets of $X$ such that $\cap \gamma = A$. In particular, if $A$ is a closed $G_\delta$-subset of a compact space $X$, then $\chi(A, X) = \omega$.

If $K$ is a compact subgroup of a topological group, then $G/K$ is metrizable if and only if $\chi(K, G) = \omega$ [1, Lemma 4.3.19]. Let $G$ be an almost metrizable topological group, $\mathcal{K}$ the collection of all compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then for every neighborhood $O$ of the neutral element there is $K \in \mathcal{K}$ such that $K \subset O$ [1, Proposition 4.3.11]. We now show that if $G$ is additionally $\omega$-narrow, then $K$ can be chosen normal (in the algebraic sense). Recall that a topological group $G$ is $\omega$-narrow [1] if for every neighborhood $U$ of the neutral element there exists a countable set $A \subset G$ such that $AU = G$.

**Lemma 3.1.** Let $G$ be an $\omega$-narrow almost metrizable group, $\mathcal{N}$ the collection of all normal (= invariant under inner automorphisms) compact subgroups $K$ of $G$ such that the quotient group $G/K$ is metrizable (equivalently, $\chi(K, G) = \omega$). Then for every neighborhood $O$ of the neutral element there exists $K \in \mathcal{N}$ such that $K \subset O$.

**Proof.** Let $L \subset O$ be a compact subgroup of $G$ such that the quotient space $G/L = \{xL : x \in G\}$ is metrizable. It suffices to prove that $K = \cap \{gLg^{-1} : g \in G\}$, the largest normal subgroup of $G$ contained in $L$, belongs to $\mathcal{N}$.

There exists a compatible metric on $G/L$ which is invariant under the action of $G$ by left translations. To construct such a metric, consider a countable base $U_1, U_2, \ldots$ of neighborhoods of $L$ in $G$. We may assume that for each $n$ we have $U_n = U_n^{-1} = U_n L$ and $U_{n+1}^2 \subset U_n$. Let $\gamma_n = \{gU_n : g \in G\}$. The open cover $\gamma_n$ of $G$ is invariant under left $G$-translations and under right $L$-translations, and $\gamma_{n+1}$ is a barycentric refinement
of $\gamma_n$. The pseudometric on $G$ that can be constructed in a canonical way from the sequence $(\gamma_n)$ of open covers (see [8, Theorem 8.1.10]) gives rise to a compatible $G$-invariant metric on $G/L$. A similar construction was used in [1, Lemma 4.3.19].

If an $\omega$-narrow group transitively acts on a metric space $X$ by isometries, then $X$ is separable [1, 10.3.2]. Thus $X = G/L$ is separable. Let $Y$ be a dense countable subset of $X$. Then $K = \{g \in G : gx = x \text{ for every } x \in X\} = \{g \in G : gx = x \text{ for every } x \in Y\}$ is a $G_\delta$-subset of $L$, hence $\chi(K, L) = \omega$. It follows that $\chi(K, G) \leq \chi(K, L)\chi(L, G) = \omega$ ([8, Exercise 3.1.E]).

□

4. MAIN THEOREM

**Theorem 4.1.** If $G$ is a precompact almost metrizable group, then $\hat{G}$ is discrete.

**Proof.** Let $\rho$ be an irreducible unitary representation of $G$. We must prove that $[\rho]$ is isolated in $\hat{G}$. It suffices to find a discrete open subset $D \subset \hat{G}$ such that $[\rho] \in D$.

Precompact groups are $\omega$-narrow, so Proposition 3.1 applies to $G$. Let $\mathcal{N}$, as above, be the collection of all normal compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then $\mathcal{N}$ is closed under countable intersections, and it follows from Proposition 3.1 that for every $G_\delta$-subset $A$ of $G$ containing the neutral element there exists $K \in \mathcal{N}$ such that $K \subset A$. In particular, there exists $K \in \mathcal{N}$ such that $K$ lies in the kernel of $\rho$. Let $D \subset \hat{G}$ be the set of all classes $[\sigma] \in \hat{G}$ such that $K$ is contained in the kernel of $\sigma$. Then $[\rho] \in D$. It suffices to verify that $D$ is open and discrete.

Step 1. We verify that $D$ is open. Let $\mathcal{R}$ be the set of equivalence classes of all finite-dimensional unitary representations (which may be reducible) of $K$ without non-zero invariant vectors. Let $\tau_n$ be the trivial $n$-dimensional representation $1_K \oplus \cdots \oplus 1_K$ ($n$ summands) of $K$, $n = 1, 2, \ldots$. In the notation of section 2, $P_{\tau_n}$ does not depend
on $n$ and is the set of non-negative constant functions on $K$. It follows that in the space $S = \mathcal{R} \cup \{[\tau_n] : n = 1, 2, \ldots \}$, equipped with the Fell topology, the points $[\tau_n]$ are indistinguishable: any open set containing one of these points contains all the others. Since $K$ has property (T), $[\tau_1] = [1_K]$ is not in the closure of $\mathcal{R}$. Therefore $\mathcal{R}$ is closed in $S$ and $S \setminus \mathcal{R}$ is open in $S$.

We claim that for every irreducible unitary representation $\sigma$ of $G$ the class of the restriction $\sigma|_K$ belongs to $S$. In other words, the claim is that $\sigma|_K$ is trivial if it admits a non-zero invariant vector. Let $V$ be the (finite-dimensional) space of the representation $\sigma$. For $g \in G$ and $x \in V$ we write $gx$ instead of $\sigma(g)x$. The space $V' = \{x \in V : gx = x$ for all $g \in K\}$ of all $K$-invariant vectors is $G$-invariant. Indeed, if $x \in V'$, $g \in G$ and $h \in K$, then $g^{-1}hx = x$ because $g^{-1}hg \in K$ and $x$ is $K$-invariant. It follows that $hx = gx$ which proves that $gx \in V'$. Since $\sigma$ is irreducible, either $V' = \{0\}$ or $V' = V$. Accordingly, either $\sigma|_K$ admits no non-zero invariant vectors or else is trivial.

We have just proved that the restriction map $r : \hat{G} \to S$ is well-defined. Clearly $r$ is continuous, and therefore $D = r^{-1}(S \setminus \mathcal{R})$ is open in $\hat{G}$.

Step 2. We verify that $D$ is discrete. Let $p : G \to G/K$ be the quotient map. Then $D$ is the image of the dual map $\hat{p} : \hat{G/K} \to \hat{G}$. According to [12, Theorem 4.1], the dual space of a metrizable precompact group is discrete. Thus $\hat{G/K}$ is discrete. Since $p$ is a perfect map, it is compact-covering, and Lemma 2.1 implies that $\hat{p} : \hat{G/K} \to \hat{G}$ is a homeomorphic embedding. Therefore, $D = \hat{p}(G/K)$ is discrete. □

References


Universitat Jaume I, Instituto de Matemáticas de Castellón, Campus de Riu Sec, 12071 Castellón, Spain.

E-mail address: mferrer@mat.uji.es

Universitat Jaume I, INIT and Departamento de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain.

E-mail address: hernande@mat.uji.es

Department of mathematics, 321 Morton Hall, Ohio University, Athens, Ohio 45701, USA

E-mail address: uspenski@ohio.edu