

Feebly compact paratopological groups and real-valued functions

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Abstract We present several examples of feebly compact Hausdorff paratopological groups (i.e., groups with continuous multiplication) which provide answers to a number of questions posed in the literature. It turns out that a 2-pseudocompact, feebly compact Hausdorff paratopological group G can fail to be a topological group. Our group G has the Baire property, is Fréchet–Urysohn, but it is not precompact.

It is well known that every infinite pseudocompact topological group contains a countable non-closed subset. We construct an infinite feebly compact Hausdorff paratopological group G all countable subsets of which are closed. Another peculiarity of the group G is that it contains a nonempty open subsemigroup C such that C^{-1} is closed and discrete, i.e., the inversion in G is extremely discontinuous.

We also prove that for every continuous real-valued function g on a feebly compact paratopological group G , one can find a continuous homomorphism φ of G onto a second countable Hausdorff *topological group* H and a continuous real-valued function h on H such that $g = h \circ \varphi$. In particular, every feebly compact paratopological group is \mathbb{R}_3 -factorizable. This generalizes a theorem of Comfort and Ross established in 1966 for real-valued functions on pseudocompact topological groups.

Keywords Paratopological group · Feebly compact · 2-pseudocompact · precompact · Regularization · \mathbb{R}_3 -factorizable

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1 Introduction

One of the first results on the automatic continuity of multiplication and inversion in semitopological groups was obtained by Montgomery in [20]. He proved that every completely metrizable *semitopological* group G (a group with separately continuous multiplication) is in fact a *paratopological* group, i.e., multiplication in G is jointly continuous. In addition, if G is separable, then the inversion in G is also continuous and G is a topological group. In 1960, Żelazko [36] eliminated the assumption about the separability of G in the latter fact by showing that every completely metrizable semitopological group is a topological group.

Compactness has an even stronger impact on automatic continuity of operations in Topological Algebra than complete metrizability. By the celebrated theorem of Numakura in [21], every compact Hausdorff topological semigroup with two-sided cancellation is a topological group. In particular, compact Hausdorff paratopological groups are topological groups. The latter fact has been considerably generalized by Ellis, Grant, Brand, Bouziad, Bokalo and Guran, Romaguera and Sanchis, Kenderov–Kortezov–Moors, and some others (see [11, 15, 9, 8, 7, 30, 18]).

The next step in the study of automatic continuity in paratopological groups was done by Reznichenko in [28] where he proved that every pseudocompact paratopological group G is a topological group, i.e., the inversion in G is continuous. In [2], Arhangel'skii and Reznichenko extended this result to *regular* pseudocompact paratopological groups. Since the term ‘pseudocompact’ is usually reserved for Tychonoff spaces, we will replace it with ‘feebly compact’ when referring to regular or Hausdorff spaces. Thus, a space is called *feebly compact* if every locally finite family of open sets in the space is finite. It is clear that feeble compactness and pseudocompactness coincide in the class of completely regular spaces.

It was recently shown by Ravsky in [23, Example 3] that there exists a feebly compact Hausdorff paratopological group which fails to be a topological group. In other words, Arhangel'skii–Reznichenko's theorem cannot be extended to Hausdorff paratopological groups. Furthermore, under Martin's Axiom, there exists a countably compact Hausdorff paratopological group with discontinuous inversion [25, Example 2].

In this article we continue the study of pseudocompact and 2-pseudocompact paratopological groups originated in [1, 2, 25, 27, 28, 30]. The authors of [1] and [25] presented several conditions which, combined with feeble compactness or 2-pseudocompactness, imply that a paratopological group is in fact a topological group. For example, every feebly compact periodic Hausdorff paratopological group is a topological group according to [25, Proposition 4], while [25, Proposition 5] implies that every 2-pseudocompact paratopological group of countable pseudocharacter is also a topological group.

Rephrasing Question A from [1], Ravsky asked in [25] if every 2-pseudocompact Hausdorff paratopological group was a topological group. Since countable compactness implies 2-pseudocompactness in paratopological groups, it follows from [25, Example 2] that, consistently, the answer to Ravsky's question is “no”. We answer this question in Theorem 1 in the negative by constructing in *ZFC* an example of a 2-pseudocompact, feebly compact Hausdorff paratopological group G that fails to be

a topological group. Our group G has the Baire property but it is not precompact, so Theorem 1 provides an answer (again in the negative) to the third item of Problem 1 in [25]. It is worth mentioning that the group G is Fréchet–Urysohn and the product $G \times X$ is feebly compact for every feebly compact space X (see Remark 1).

It is well known that every infinite precompact topological group G contains a countable non-closed subset—it suffices to take a countably infinite subgroup C of G and put $A = C \setminus \{e\}$, where e is the identity of G . Then A cannot be closed in G since otherwise C would be a discrete subgroup of G and both C and G would not be precompact. The same conclusion is valid for pseudocompact topological groups since they are precompact by [10, Theorem 1.1]. In Theorem 2 we present an example of an infinite feebly compact Hausdorff paratopological group G all countable subsets of which are closed. The peculiarity of the group G is that it contains a nonempty open subsemigroup C such that C^{-1} is a closed discrete subset of G , i.e., the inversion in G is “extremely discontinuous”. It is clear that any paratopological group with this combination of properties cannot be 2-pseudocompact. Nevertheless, in Theorem 3 we construct a feebly compact, 2-pseudocompact Hausdorff paratopological group H such that all countable subsets of H are closed.

In Section 3 we consider continuous real-valued functions on feebly compact paratopological groups. No separation axioms are imposed on the groups there. We prove in Theorem 5 that for every continuous real-valued function g on a feebly compact paratopological group G one can find a continuous homomorphism φ of G onto a second countable Hausdorff topological group H and a continuous real-valued function h on H such that $g = h \circ \varphi$. In particular we see that every feebly compact paratopological group is \mathbb{R}_3 -factorizable. This result extends a similar property of continuous real-valued functions on pseudocompact topological groups (see [10, Theorem 1.5]).

By a theorem in [10], every pseudocompact topological group is precompact, while Theorem 3.8 from [34] implies that all precompact topological groups are \mathbb{R} -factorizable. Somewhat surprisingly, Theorem 5 is not valid for precompact paratopological groups, even if one considers only Tychonoff paratopological groups. Once again, this result shows the big difference in the properties of the two classes of objects of Topological Algebra.

1.1 Notation and terminology

A space X is said to be *feebly compact* if every locally finite family of open sets in X is finite. In the class of Tychonoff spaces, feeble compactness is equivalent to pseudocompactness.

A sequence $\{U_n : n \in \omega\}$ of subsets of a space X is *decreasing* if $U_n \supset U_{n+1}$ for each $n \in \omega$. A paratopological group G is *2-pseudocompact* if $\bigcap_{n=0}^{\infty} cl_G U_n^{-1} \neq \emptyset$ for each decreasing sequence $\{U_n : n \in \omega\}$ of nonempty open subsets of G . Clearly, every countably compact paratopological group is 2-pseudocompact.

A paratopological group G is *precompact* if for every neighborhood U of the identity in G one can find a finite set F in G such that $FU = G = UF$. A standard example of a precompact paratopological group is the unit circle \mathbb{T}_{Sor} endowed with

the *Sorgenfrey topology* which is obtained as the quotient group \mathbb{S}/\mathbb{Z} and where \mathbb{S} is the usual Sorgenfrey line. Notice that the paratopological group \mathbb{T}_{Sor} is Hausdorff and zero-dimensional (hence completely regular), but it obviously fails to be a topological group.

Similarly, a paratopological group G is τ -*narrow*, for a given cardinal $\tau \geq \omega$, if for every neighborhood U of the identity in G one can find a subset F of G such that $FU = G = UF$ and $|F| \leq \tau$. The class of τ -narrow paratopological groups is productive, but it is not closed with respect to taking subgroups. In fact, a subgroup of a precompact paratopological group can even fail to be ω -narrow (this follows from [4, Corollary 5]).

Given a Hausdorff space X with topology τ , one defines the *semiregularization* of τ , denoted by τ_r , as a topology on X whose base consists of regular open subsets of (X, τ) (see [33, 17]). The topology τ_r is coarser than τ and the space $X_r = (X, \tau_r)$, the *semiregularization* of X , is Hausdorff and *semiregular*, i.e., the family of regular open sets in X_r constitutes a base for X_r . If G is a Hausdorff paratopological group, then the semiregularization G_r of G is a *regular* paratopological group (see [23, Example 1.9]). This fact permits us to call G_r the *regularization* of G .

Let $X = \prod_{i \in I} X_i$ be a product of spaces with the usual Tychonoff product topology. For every $i \in I$, denote by π_i the projection of X onto the factor X_i . If U is a canonical open set in X (i.e., an element of the *canonical base* for X , see [12, Section 2.3]), we put

$$c(U) = \{i \in I : \pi_i(U) \neq X_i\}.$$

Clearly $c(U)$ is a finite subset of I .

A nonempty subset S of an Abelian group G is called *independent* if the equality $x_1^{n_1} \cdots x_k^{n_k} = 1_G$ with $n_1, \dots, n_k \in \mathbb{Z}$ and distinct $x_1, \dots, x_k \in S$ implies $x_1^{n_1} = \cdots = x_k^{n_k} = 1_G$. It is clear that a set $S \subseteq G$ consisting of elements of infinite order is independent if and only if the former equality implies that $n_1 = \cdots = n_k = 0$. A set $C \subseteq G$ is called a *subsemigroup* of G if $x \cdot y \in C$ for all $x, y \in C$.

2 Three examples of feebly compact paratopological groups

In [1] Sanchis and Alas posed the question whether every 2-pseudocompact Hausdorff topological group G was precompact (or, equivalently, whether G was a topological group). In [25, Problem 1] O. Ravsky asked whether every feebly compact Hausdorff paratopological group with the Baire property was a topological group. He also raised a question equivalent to the one in [1]: Is every 2-pseudocompact Hausdorff paratopological group a topological group? Notice that all 2-pseudocompact paratopological groups have the Baire property by [1, Theorem 2.2]. A consistent counterexample to both questions was presented in [25, Example 2].

In Theorem 1 below we answer all these questions in the negative by constructing in *ZFC* a 2-pseudocompact, feebly compact Hausdorff paratopological group G that fails to be precompact and, therefore, cannot be a topological group. Then, in Remark 1, we comment on several additional properties of the group G from Theorem 1.

We start with two simple lemmas.

Lemma 1 *Let \mathbb{T} be circle group endowed with the usual topology. One can find a family $\mathcal{S} = \{S_\alpha : \alpha \in \omega_1\}$ of infinite pairwise disjoint subsets of \mathbb{T} satisfying the following conditions:*

- (1) $S_\alpha = \{t_{\alpha,n} : n \in \omega\}$ is dense in \mathbb{T} and consists of elements of infinite order, for each $\alpha \in \omega_1$;
- (2) $S = \bigcup_{\alpha \in \omega_1} S_\alpha$ is an independent subset of \mathbb{T} .

Proof Let $\{U_n : n \in \omega\}$ be a countable neighborhood base for the space \mathbb{T} . Suppose that for some $\alpha \in \omega_1$, we have defined a family $\{S_\nu : \nu \in \alpha\}$ of infinite disjoint subsets of \mathbb{T} such that $S_\nu = \{t_{\nu,n} : n \in \omega\}$ is dense in \mathbb{T} for each $\nu \in \alpha$ and the set $T_\alpha = \bigcup_{\nu \in \alpha} S_\nu$ is independent. We define S_α as follows. Since U_0 is uncountable, there exists an element $t_{\alpha,0} \in U_0$ of infinite order in \mathbb{T} such that $\langle t_{\alpha,0} \rangle \cap \langle T_\alpha \rangle = \{1\}$. Suppose that for some $n \in \omega$, we have chosen elements $t_{\alpha,0} \in U_0, \dots, t_{\alpha,n} \in U_n$ of infinite order such that the set $T_\alpha \cup S_{\alpha,n}$ is independent in \mathbb{T} , where $S_{\alpha,n} = \{t_{\alpha,i} : i \leq n\}$. Again, since U_{n+1} is uncountable, we can find an element $t_{\alpha,n+1} \in U_{n+1}$ of infinite order such that $\langle t_{\alpha,n+1} \rangle \cap \langle T_\alpha \cup S_{\alpha,n} \rangle = \{1\}$. Clearly, the set $T_\alpha \cup \{t_{\alpha,i} : i \leq n+1\}$ is independent.

Continuing this way, we obtain a set $S_\alpha = \{t_{\alpha,n} : n \in \omega\}$ which intersects every nonempty open set in \mathbb{T} . It also follows from our choice of the elements $t_{\alpha,n}$ that the sets T_α and S_α are disjoint and $T_\alpha \cup S_\alpha$ is independent in \mathbb{T} . This finishes our construction of the family \mathcal{S} satisfying conditions (1) and (2) of the lemma. \square

The following result remains valid in the non-Abelian case if one adds an additional assumption about the family \mathcal{C} :

Lemma 2 *Let $\Sigma = (\Sigma, \tau)$ be a paratopological Abelian group with identity e and \mathcal{C} a family of subsemigroups of Σ closed under finite intersections such that $e \in C$ for each $C \in \mathcal{C}$. Let also $\sigma = \langle \tau, \mathcal{C} \rangle$ be the topology on Σ with the base $\{xC \cap U : x \in \Sigma, C \in \mathcal{C}, x \in U \in \tau\}$. Then $G = (\Sigma, \sigma)$ is again a paratopological group. In addition, if the space Σ is regular and the family \mathcal{C} satisfies*

- (α) $C \cap xC$ is dense in Σ for all $x \in \Sigma$ and $C \in \mathcal{C}$,

then

- (1) $cl_\Sigma V = cl_G(C \cap V)$, for all $V \in \tau$ and $C \in \mathcal{C}$;
- (2) $\sigma_r = \tau$, i.e., the regularization of G coincides with Σ ;
- (3) G is Hausdorff and if $C \notin \tau$ for some $C \in \mathcal{C}$, then G is not regular.

Proof Let \mathcal{N}_e be the family of open neighborhoods of e in (Σ, τ) . It is easy to see that the family

$$\mathcal{B}_e = \{U \cap C : U \in \mathcal{N}_e, C \in \mathcal{C}\}$$

satisfies Pontryagin's four conditions (see [31, Proposition 2.1], hence $\sigma = \langle \tau, \mathcal{C} \rangle$ is a paratopological group topology on Σ with \mathcal{B}_e being a local base for $G = (\Sigma, \sigma)$ at the identity e .

Suppose now that Σ is regular and \mathcal{C} satisfies condition (α) of the lemma.

- (1) Let $V \in \tau$ and $C \in \mathcal{C}$. Then $cl_G(C \cap V) \subseteq cl_G V \subseteq cl_\Sigma V$ since σ is finer than τ . Conversely, take any element $x \in cl_\Sigma V$ and let U be a neighborhood of x in G . Then there exists an open neighborhood O of x in Σ and an element $D \in \mathcal{C}$ such that

$xD \cap O \subseteq U$. To show that $x \in cl_G(C \cap V)$ we have to verify that $(xD \cap O) \cap (C \cap V) \neq \emptyset$. Clearly, $E = C \cap D$ is in \mathcal{C} . Since $x \in cl_\Sigma V$ and $V \in \tau$, the set $O \cap V$ is open in Σ and nonempty. The density of $xE \cap E$ in Σ implies that $\emptyset \neq (xE \cap E) \cap (O \cap V) \subseteq (xD \cap O) \cap (C \cap V)$. This shows that $cl_\Sigma V \subseteq cl_G(C \cap V)$ and proves (1) of the lemma.

(2) Since $\tau \subseteq \sigma$, the regularity of τ implies that $\tau \subseteq \sigma_r$. To show that $\sigma_r \subseteq \tau$ it suffices to note that, by (1), $V \subseteq cl_\Sigma V = cl_G(V \cap C)$ for all $V \in \tau$ and $C \in \mathcal{C}$. We conclude, therefore, that $\sigma_r = \tau$.

(3) It is clear that G is Hausdorff since the topology σ of G finer than the regular (hence Hausdorff) topology τ . If $C \notin \tau$ for some $C \in \mathcal{C}$, then σ is strictly finer than τ . Since $\sigma_r = \tau$ by (2), the topology σ cannot be regular. \square

Theorem 1 *There exists a feebly compact, 2-pseudocompact Hausdorff paratopological group G with the Fréchet-Urysohn property which fails to be a topological group. Furthermore, G has the Baire property, is countably cellular, but it is neither regular nor ω -narrow (nor precompact).*

Proof Let Π be the compact group \mathbb{T}^{ω_1} with the usual pointwise multiplication and Tychonoff product topology. We will use multiplicative notation for the group operation \mathbb{T} , so 1 is the identity element of \mathbb{T} .

For every element $x \in \Pi$, let

$$\text{supp}(x) = \{\alpha < \omega_1 : x(\alpha) \neq 1\}.$$

Then $\Sigma = \{x \in \Pi : |\text{supp}(x)| \leq \omega\}$ is a dense subgroup of Π . In addition, the group Σ has the Fréchet-Urysohn property [3, Corollary 1.6.33].

Let $S = \{t_n : n \in \omega\}$ be a dense independent subset of \mathbb{T} whose elements have infinite order (see Lemma 1). Denote by K the subgroup of \mathbb{T} generated by S . Every element $t \in K$ distinct from 1 has a unique representation in the form $t = t_{n_1}^{p_1} \cdots t_{n_k}^{p_k}$, where k is a positive integer, $0 \leq n_1 < \cdots < n_k < \omega$, and p_1, \dots, p_k are nonzero integers. For such an element $t \in K$, we put

$$p(t) = \sum_{1 \leq i \leq k} p_i.$$

We can now define a subset C_0 of Σ by

$$C_0 = \{x \in \Sigma : (\exists \alpha < \omega_1) [x(\alpha) \in K \ \& \ p(x(\alpha)) > 0 \ \& \ (\forall \beta > \alpha) [x(\beta) = 1]]\}.$$

It is clear from the definition that $C = C_0 \cup \{e\}$ is a subsemigroup of Σ , i.e., $a \cdot b \in C$ for all $a, b \in C$; here e stands for the identity element of Σ .

Let us refine the topology τ of Σ inherited from Π by declaring the set C open. In other words, a base of the new topology of Σ , say σ , consists of the sets $xC \cap U$, where $x \in \Sigma$ and $x \in U \in \tau$. Since C is a subsemigroup of Σ and σ is finer than the original topology τ of Σ , it follows from Lemma 2 that $G = (\Sigma, \sigma)$ is a Hausdorff paratopological group. It is immediate from the definition of C that both C and C^{-1} are dense in (Σ, τ) . In fact, we claim that C satisfies condition (α) of Lemma 2, i.e., the set $xC \cap C$ is dense in (Σ, τ) for each $x \in \Sigma$.

Indeed, for every $\alpha \in \omega_1$, let D_α be the set of all $x \in C_0$ such that $\max \text{supp}(x) \geq \alpha$. Clearly, D_α is a dense subsemigroup of (Σ, τ) and $D_\alpha \subseteq C_0 \subseteq C$. Take an arbitrary

element $x \in \Sigma$ and choose $\alpha \in \omega_1$ such that $\text{supp}(x) \subseteq \alpha$. It is clear that $xd \in D_\alpha \subseteq C$ for each $d \in D_\alpha$ and, hence, $xD_\alpha \subseteq xC \cap C$. Since the sets D_α and xD_α are dense in (Σ, τ) , so is $xC \cap C$. This proves our claim.

For the further argument we formulate a simple property of the subsemigroups D_α 's which is immediate from the definition of the subsemigroups (and which has been implicitly used in the previous paragraph):

Claim 1. *If $\alpha < \omega_1$ and an element $x \in \Sigma$ satisfies $\text{supp}(x) \subseteq \alpha$, then $xD_\alpha = D_\alpha$.*

(A) It is clear that C is not open in (Σ, τ) , so the space G is not regular by (3) of Lemma 2. Since every Hausdorff topological group is a regular space, we see that G fails to be a topological group.

(B) Our first aim is to verify that G has *countable tightness* (see [12, 1.7.13]). Suppose that $x \in cl_G P$, for some $x \in G$ and $P \subseteq G$. We claim that there exists a countable set $Q \subseteq P$ such that $x \in cl_G Q$. Our argument involves a standard inductive construction.

Take an arbitrary nonempty countable set $A_0 \subseteq \omega_1$ such that $\text{supp}(x) \subseteq A_0$. Let us identify every point $y \in \mathbb{T}^{A_0}$ with the corresponding point \bar{y} of Σ which coincides with y on A_0 and $\bar{y}(\alpha) = 1$ for each $\alpha \in \omega_1 \setminus A_0$. Since the subspace \mathbb{T}^{A_0} of (Σ, τ) is first countable, there exists a countable set $Q_0 \subseteq P$ with the property that $xC \cap U \cap Q_0 \neq \emptyset$ for every canonical open neighborhood U of x in Π such that $c(U) \subseteq A_0$.

Suppose that for some $n \in \omega$, we have defined countable sets $A_n \subseteq \omega_1$ and $Q_n \subseteq P$ with the property that $xC \cap U \cap Q_n \neq \emptyset$ for every canonical open neighborhood U of x in Π such that $c(U) \subseteq A_n$. Then we put $A_{n+1} = A_n \cup \bigcup_{y \in Q_n} \text{supp}(y)$. Since the set A_{n+1} is countable and the space $\mathbb{T}^{A_{n+1}}$ is first countable, there exists a countable set $Q_{n+1} \subseteq P$ with the property that $xC \cap U \cap Q_{n+1} \neq \emptyset$ for every canonical open neighborhood U of x in Π such that $c(U) \subseteq A_{n+1}$. This finishes our construction of the sequences $\{A_n : n \in \omega\}$ and $\{Q_n : n \in \omega\}$.

Let $A = \bigcup_{n \in \omega} A_n$ and $Q = \bigcup_{n \in \omega} Q_n$. Clearly, the sets $A \subseteq \omega_1$ and $Q \subseteq P$ are countable. It follows from our construction that $\text{supp}(x) \subseteq A_0 \subseteq A$ and $\text{supp}(y) \subseteq A$ for each $y \in Q$. It remains to show that $x \in cl_G Q$. Take an arbitrary open neighborhood U of x in G . There exists a canonical open neighborhood V of x in Π such that $xC \cap V \subseteq U$. Then $B = c(V) \cap A$ is a finite subset of A . Since the sequence $\{A_n : n \in \omega\}$ is increasing, we can find $n \in \omega$ such that $B \subseteq A_n$. Let $W = \pi_A^{-1} \pi_A(V)$, where $\pi_A : \Pi \rightarrow \mathbb{T}^A$ is the projection. Then W is a canonical open neighborhood of x in Π and $c(W) = B \subseteq A_n$. Hence, by our choice of Q_n , there exists a point $y \in xC \cap W \cap Q_n$. Since $Q_n \subseteq Q$, we see that $\text{supp}(y) \subseteq A$. In particular, $y(\alpha) = 1$ for each $\alpha \in \omega_1 \setminus A$. This implies that $y \in xC \cap V$ since the canonical open set V contains the point x and, therefore, $1 \in \pi_\alpha(V)$ for each $\alpha \in \omega_1 \setminus A$ (notice that $\pi_\alpha(V) = \pi_\alpha(W)$ for each $\alpha \in A$). Therefore, $xC \cap V \cap Q \neq \emptyset$ and we conclude that $x \in cl_G Q$. This proves that G has countable tightness.

The stronger fact that G has the Fréchet-Urysohn property is now almost immediate. First, every countable subspace of (Σ, τ) is first countable (see [3, Proposition 1.6.31]). Hence, by the definition of the topology σ on Σ , the same holds for countable subspaces of the space $G = (\Sigma, \sigma)$. If $x \in cl_G P$ for some $x \in G$ and $P \subseteq G$, we already know that there exists a countable set $Q \subseteq P$ such that $x \in cl_G Q$. Since the subspace $\{x\} \cup Q$ of G is first countable, it has a countable local base at the point

x . Therefore, there exists a sequence from Q converging to x , i.e., G has the Fréchet-Urysohn property.

(C) To show that G is 2-pseudocompact we need to establish the following:

Claim 2. *For every nonempty open set V in $\Sigma = (\Sigma, \tau)$, there exists an ordinal $\alpha_V \in \omega_1$ such that*

$$D_\alpha^{-1} \cap cl_\Sigma V \subseteq cl_G(D_\alpha^{-1} \cap V) \quad (*)$$

whenever $\alpha_V \leq \alpha < \omega_1$.

Proof of Claim 2. Let V be a nonempty open set in (Σ, τ) . Take an open set \tilde{V} in Π such that $\tilde{V} \cap \Sigma = V$. Since Π has countable cellularity, there exists a countable family γ of canonical open sets in Π such that $\bigcup \gamma$ is a dense subset of \tilde{V} . Clearly we can assume that each $U \in \gamma$ is a proper subset of \mathbb{T}^{ω_1} , so $c(U) \neq \emptyset$ for each $U \in \gamma$. Then $A = \bigcup_{U \in \gamma} c(U)$ is a countable subset of ω_1 and we can find an ordinal $\alpha_V \in \omega_1$ such that $A \subseteq \alpha_V$. Let us show that α_V is as required.

Take an ordinal α with $\alpha_V \leq \alpha < \omega_1$ and an arbitrary point $x \in D_\alpha^{-1} \cap cl_\Sigma V$ and let $O \cap xC$ be a basic open neighborhood of x in G , where O is a canonical open neighborhood of x in Π . We have to verify that $(O \cap xC) \cap (D_\alpha^{-1} \cap V) \neq \emptyset$ or, equivalently, that $(O \cap xC) \cap (D_\alpha^{-1} \cap U) \neq \emptyset$ for some $U \in \gamma$.

Since $x \in cl_\Sigma V$ and $x \in O \cap \Sigma \in \tau$, it follows from our choice of the family γ that $O \cap U \neq \emptyset$ for some $U \in \gamma$. Let $O = \prod_{\beta \in \omega_1} O_\beta$ and $U = \prod_{\beta \in \omega_1} U_\beta$. Then $U_\beta = \mathbb{T}$ for each $\beta \geq \alpha$ since $c(U) \subseteq \alpha_V \subseteq \alpha$. Put $\delta = \max c(U)$ and note that $\delta < \alpha$. We define elements $c, d \in C$ as follows. First, $O_\delta \cap U_\delta$ is a nonempty open subset of \mathbb{T} , so we use the density of $S = \{t_n : n \in \omega\}$ in \mathbb{T} to take $n \in \omega$ such that $t_n \in (O_\delta \cap U_\delta) \cdot x(\delta)^{-1}$. Let $c(\delta) = t_n$ and $c(\beta) = 1$ for each $\beta > \delta$. Hence, independently of the choice of the values $c(\beta)$ for $\beta < \delta$, we will have that $c \in D_\delta \subseteq C$. Let also $d(\delta) = c(\delta)^{-1} \cdot x(\delta)^{-1}$. Then $d(\delta)^{-1} = c(\delta) \cdot x(\delta) \in U_\delta \cap O_\delta$. We put $d(\beta) = x(\beta)^{-1}$ for each $\beta > \delta$. Then $d(\beta)^{-1} = x(\beta) \in O_\beta$. Finally, given an ordinal $\beta < \delta$, pick an element $d(\beta) \in O_\beta^{-1} \cap U_\beta^{-1}$ and put $c(\beta) = d(\beta)^{-1} \cdot x(\beta)^{-1}$. It is immediate from our definitions that $xc = d^{-1}$. Since $x^{-1} \in D_\alpha$ and $d(\beta) = x(\beta)^{-1}$ for each $\beta \geq \alpha > \delta$, we see that $d \in D_\alpha$. Our definition of the element d and the facts that U and O are canonical open sets in Π and $\max c(U) = \delta$ imply that $d^{-1} \in O \cap U$. Therefore,

$$xc = d^{-1} \in (O \cap U) \cap (xC \cap D_\alpha^{-1}) = (O \cap xC) \cap (D_\alpha^{-1} \cap U) \neq \emptyset.$$

This implies that $x \in cl_G(D_\alpha^{-1} \cap V)$ and finishes the proof of Claim 2.

Let us turn back to the proof of the 2-pseudocompactness of G . Suppose that $\{U_n : n \in \omega\}$ is a decreasing sequence of nonempty open sets in G . For every $n \in \omega$, choose $x_n \in U_n$ and an open neighborhood V_n of x_n in Π such that $x_n C \cap V_n \subseteq U_n$. Put $O_n = \bigcup_{k \geq n} (x_k C \cap V_k)$. It is clear that O_n is open in G and $O_n \subseteq U_n$ for each $n \in \omega$. Hence it suffices to verify that the set $\bigcap_{n \in \omega} cl_G O_n^{-1}$ is nonempty.

For every $n \in \omega$, put $W_n = \bigcup_{k \geq n} V_k$. Then $\{W_n : n \in \omega\}$ is a decreasing sequence of nonempty open sets in Π . Applying Claim 2 we find, for every $n \in \omega$, an ordinal $\alpha_n \in \omega_1$ such that $D_\alpha^{-1} \cap cl_\Sigma W_n^{-1} \subseteq cl_G(D_\alpha^{-1} \cap W_n^{-1})$ for each α with $\alpha_n \leq \alpha < \omega_1$.

Take a countable ordinal β such that $\alpha_n \leq \beta$ and $\text{supp}(x_n) \subseteq \beta$ for each $n \in \omega$. By Claims 1 and 2, we have that

$$\begin{aligned} \bigcap_{n \in \omega} cl_G O_n^{-1} &= \bigcap_{n \in \omega} cl_G \left(\bigcup_{k \geq n} x_k^{-1} C^{-1} \cap V_k^{-1} \right) \supseteq \bigcap_{n \in \omega} cl_G \left(\bigcup_{k \geq n} x_k^{-1} D_\beta^{-1} \cap V_k^{-1} \right) \\ &= \bigcap_{n \in \omega} cl_G \left(\bigcup_{k \geq n} D_\beta^{-1} \cap V_k^{-1} \right) = \bigcap_{n \in \omega} cl_G (D_\beta^{-1} \cap W_n^{-1}) \\ &\supseteq \bigcap_{n \in \omega} (D_\beta^{-1} \cap cl_\Sigma W_n^{-1}) = D_\beta^{-1} \cap \bigcap_{n \in \omega} cl_\Sigma W_n^{-1}. \end{aligned}$$

Since the space Π is compact, the intersection $\bigcap_{n \in \omega} cl_\Pi W_n^{-1}$ is nonempty. The sets W_n 's being open in Π , each $cl_\Pi W_n^{-1}$ is a G_δ -set in Π according to [3, Corollary 1.6.19]. Hence $\bigcap_{n \in \omega} cl_\Pi W_n^{-1}$ is also a G_δ -set in Π . Since the sets D_β and D_β^{-1} meet every nonempty G_δ -set in Π , we conclude that $D_\beta^{-1} \cap \bigcap_{n \in \omega} cl_\Sigma W_n^{-1} \neq \emptyset$. Therefore, the intersection $\bigcap_{n \in \omega} cl_G O_n^{-1}$ is nonempty and G is 2-pseudocompact.

(D) We already know that the regularization of $G = (\Sigma, \sigma)$ coincides with the countably compact (hence pseudocompact) topological group (Σ, τ) . Since a paratopological group is feebly compact iff so is its regularization (see [25, Lemma 13]), we conclude that G is feebly compact.

Notice that every 2-pseudocompact paratopological group has the Baire property (see [1, Theorem 2.2] or [25, Lemma 4]) and the cellularity of every feebly compact paratopological group is countable [27].

Finally, we show that G cannot be covered by countably many translates of the open set C , which implies that G is neither ω -narrow nor precompact. Take any countable set $F \subseteq G$. Since F is a subset of Σ , there exists $\alpha^* \in \omega_1$ such that $x(\alpha) = 1$ for all $x \in F$ and $\alpha \geq \alpha^*$. Let y be an element of G such that $y(\alpha^*) \notin K$ and $y(\alpha) = 1$ for each $\alpha > \alpha^*$. It is clear that $y \in G \setminus FC$, i.e., G is not ω -narrow. \square

Remark 1 a) The topology τ of the group Σ inherited from Π is the finest topological group topology on Σ coarser than the topology σ on Σ defined in Theorem 1. In other words, using the terminology of [4], one can say that the group reflection of the paratopological group G is topologically isomorphic to the subgroup (Σ, τ) of Π .

Indeed, since the paratopological group $G = (\Sigma, \sigma)$ is commutative, a base of open neighborhoods of the group reflection of G at the identity e consists of the sets $W \cdot W^{-1}$, where W runs over the family of open neighborhoods of e in G (this follows from [5, Theorem 1]). Every neighborhood of e in G contains a set W of the form $V \cap C$, where V is a canonical open neighborhood of e in Π . A simple calculation shows that $V \cap \Sigma \subseteq WW^{-1}$, whence it follows that the group reflection of G is topologically isomorphic to (Σ, τ) .

b) The group G is not precompact, but it is topologically isomorphic to a subgroup of a precompact Hausdorff paratopological group. Indeed, the identity mapping of $G = (\Sigma, \sigma)$ onto (Σ, τ) is a continuous isomorphism of G onto a precompact topological group. Therefore, according to [4, Corollary 5], G admits a topological embedding as a subgroup into a precompact Hausdorff paratopological group.

c) A subset B of a space Y is said to be *(feebly) bounded* in X if every locally finite family of open sets in X meeting B is finite. According to [16, Theorem 1.2], if A is a bounded subset of a topological group X and B is a (feebly) bounded subset of a space Y , then the product $A \times B$ is (feebly) bounded in $X \times Y$. We claim that the paratopological group G has a slightly weaker property: *For every space Y , $G \times B$ is (feebly) bounded in $X \times Y$, for every (feebly) bounded subset B of Y .* In particular, $G \times Y$ is feebly compact, for every feebly compact space Y . In other words, the space G belongs to the natural analog of Frolík's class \mathcal{P} obtained by omitting any separation restrictions on the factors (see [13]). According to [35, Theorem 1.2] and [6, Proposition 1], both facts will follow if we show that every infinite disjoint family γ of nonempty open sets in G contains a disjoint subfamily $\{U_n : n \in \omega\}$ such that the set

$$Q = \bigcap_{F \in \mathcal{F}} cl_G \left(\bigcup_{n \in F} U_n \right) \quad (1)$$

is nonempty for every filter \mathcal{F} of infinite subsets of ω . In fact, we will show that the set Q in (1) is nonempty for *every* sequence $\{U_n : n \in \omega\}$ of nonempty open sets in G and every filter \mathcal{F} on ω . We can assume without loss of generality that each U_n has the form $x_n C \cap V_n$, where V_n is a canonical open set in Π with $x_n \in V_n$.

First we note that if X is a space and X_r is semiregularization of X , then $cl_X U = cl_{X_r} U$ for each open subset U of X . This follows immediately from the fact that $V = X \setminus cl_X U$ is a regular open set in X and, hence, V is open in X_r . Therefore,

$$cl_G \left(\bigcup_{n \in F} U_n \right) = cl_\Sigma \left(\bigcup_{n \in F} U_n \right) = cl_\Sigma \left(\bigcup_{n \in F} (\Sigma \cap V_n) \right) = cl_\Sigma \left(\Sigma \cap \bigcup_{n \in F} V_n \right) \quad (2)$$

for each $F \in \mathcal{F}$, since $U_n = x_n C \cap V_n$ is dense in the open subspace $\Sigma \cap V_n$ of Σ for each $n \in \omega$.

Since every V_n depends on finitely many coordinates, there exists a countable set $A \subset \omega_1$ such that $c(V_n) \subseteq A$ and, hence, $V_n = \pi_A^{-1} \pi_A(V_n)$ for each $n \in \omega$. Put

$$R = \bigcap_{F \in \mathcal{F}} cl_\Pi \left(\bigcup_{n \in F} V_n \right).$$

It follows from the compactness of Π that the set R is not empty. In addition, since $c(V_n) \subseteq A$ for each $n \in \omega$ and the projection $\pi_A : \mathbb{T}^{\omega_1} \rightarrow \mathbb{T}^A$ is open, we conclude that $R = \pi_A^{-1} \pi_A(R)$. Take any point $x \in R \cap \Sigma$. It follows from (2) that $x \in cl_G(\bigcup_{n \in F} U_n)$ for each $F \in \mathcal{F}$, i.e., the set Q in (1) is not empty. This finishes the proof that the product $G \times Y$ is feebly compact for every feebly compact space Y .

d) It is a well-known fact that every pseudocompact topological group G is precompact in its finest uniformity (which coincides with any of the natural uniformities on G , see [10] or [34, Corollary 2.13]). The situation is quite different for paratopological groups. Indeed, since precompact 2-pseudocompact paratopological groups are topological groups, Theorem 1 provides an example of a feebly compact, 2-pseudocompact Hausdorff paratopological group whose finest quasi-uniformity fails to be precompact (compare with [30, Lemma 3.3]).

It is easy to verify that every infinite pseudocompact topological group contains a countable non-closed subset (see [3, Corollary 1.4.24]). Since pseudocompact topological groups are precompact by [10, Theorem 1.1], the former fact is an immediate corollary to Protasov's theorem in [22]: Every infinite precompact topological group contains a countable discrete subset that accumulates at the identity of the group. In the next theorem we show that feebly compact Hausdorff paratopological groups can fail to have this property.

Theorem 2 *There exists an infinite feebly compact Hausdorff paratopological group G with the Baire property such that all countable subsets of G are closed. In addition, G contains a nonempty open subsemigroup C such that the set C^{-1} is closed and discrete in G , so G is not 2-pseudocompact.*

Proof As in the proof of Theorem 1, let Σ be the Σ -product of ω_1 copies of the circle group \mathbb{T} endowed with the topology τ inherited from the product group $\Pi = \mathbb{T}^{\omega_1}$. Then Σ is a countably compact (hence pseudocompact) topological group. Let us refine the topology of Σ by declaring a family $\{C_\alpha : \alpha \in \omega_1\}$ of dense subsemigroups of Σ to be open in the new topology. In what follows we use notation introduced in the proof of Theorem 1. In particular, we take a dense independent subset $S = \{t_n : n \in \omega\}$ of \mathbb{T} consisting of elements of infinite order and denote by K the subgroup of \mathbb{T} generated by S .

For a countable ordinal α , let

$$D_\alpha = \{x \in \Sigma : (\exists \beta \geq \alpha) [x(\beta) \in K \ \& \ p(x(\beta)) > 0 \ \& \ (\forall \gamma > \beta) [x(\gamma) = 1]]\}$$

and

$$C_\alpha = D_\alpha \cup \{e\},$$

where e is the neutral element of Σ . It is clear that each C_α is a dense subsemigroup of Σ and that $C_\nu \subseteq C_\mu$ whenever $\mu < \nu < \omega_1$.

Since the family $\mathcal{C} = \{C_\alpha : \alpha \in \omega_1\}$ is closed under finite intersections, Lemma 2 implies that $\sigma = \langle \tau, \mathcal{C} \rangle$ is a paratopological group topology on Σ . Let $G = (\Sigma, \sigma)$. Since $\tau \subseteq \sigma$, the paratopological group $G = (\Sigma, \sigma)$ is Hausdorff.

Let us verify that the family \mathcal{C} satisfies condition (α) of Lemma 2. Take arbitrary $x \in \Sigma$ and $\alpha \in \omega_1$. Choose $\beta \in \omega_1$ such that $\alpha \leq \beta$ and $\text{supp}(x) \subseteq \beta$ and take any element $d \in D_\beta$. It is clear that $xd \in C_\alpha$ and, hence, $xD_\beta \subseteq xC_\beta \cap C_\alpha \subseteq xC_\alpha \cap C_\alpha$. Since the sets C_β and $D_\beta = C_\beta \setminus \{e\}$ are dense in (Σ, τ) , so are xD_β and $xC_\alpha \cap C_\alpha$. This implies (α) . Hence, according to Lemma 2, $\sigma_r = \tau$.

(A) By [25, Lemma 13], the space $G = (\Sigma, \sigma)$ is feebly compact if and only if so is $G_r = (\Sigma, \tau)$. Since the space G_r is countably compact (see [3, Corollary 1.6.34]), we deduce that G is feebly compact.

(B) To show that G has the Baire property, we first note that for arbitrary sequences $\{\alpha_n : n \in \omega\} \subseteq \omega_1$ and $\{x_n : n \in \omega\} \subseteq G$, there exists an ordinal $\alpha \in \omega_1$ such that $D_\alpha \subseteq \bigcap_{n \in \omega} x_n C_{\alpha_n}$. Indeed, take $\alpha \in \omega_1$ such that $\alpha_n \leq \alpha$ and $\text{supp}(x_n) \subseteq \alpha$ for each $n \in \omega$. It follows from our choice of α and the definition of D_α that $x_n D_\alpha = D_\alpha$ for each $n \in \omega$. Therefore, $D_\alpha = \bigcap_{n \in \omega} x_n D_\alpha \subseteq \bigcap_{n \in \omega} x_n C_{\alpha_n}$.

Suppose for a contradiction that G fails to have the Baire property. Then we can find a sequence $\{W_n : n \in \omega\}$ of open dense sets in G and a nonempty open set O in G

such that $O \cap \bigcap_{n \in \omega} W_n = \emptyset$. Choose a point $x_0 \in O \cap W_0$ and take an ordinal $\alpha_0 \in \omega_1$ and an open set U_0 in (Σ, τ) containing x_0 such that $V_0 = x_0 C_{\alpha_0} \cap U_0 \subseteq O \cap W_0$. Suppose that we have defined a basic open set $V_n = x_n C_{\alpha_n} \cap U_n$ in G such that $x_n \in V_n \subseteq W_n$. Then we choose a point $x_{n+1} \in W_{n+1} \cap U_n$, a countable ordinal α_{n+1} and an open neighborhood U_{n+1} of x_{n+1} in (Σ, τ) such that $V_{n+1} = x_{n+1} C_{\alpha_{n+1}} \cap U_{n+1} \subseteq W_{n+1}$ and $cl_{\Sigma} U_{n+1} \subseteq U_n$. This finishes our construction of the sequences $\{\alpha_n : n \in \omega\} \subseteq \omega_1$, $\{x_n : n \in \omega\} \subseteq G$, and $\{U_n : n \in \omega\} \subseteq \tau$. We have just proved that there exists an ordinal $\alpha \in \omega_1$ such that $D_\alpha \subseteq \bigcap_{n \in \omega} x_n C_{\alpha_n}$. Since the space $\{\Sigma, \tau\}$ is countably compact, the intersection $P = \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} cl_{\Sigma} U_n$ is a nonempty G_δ -set in (Σ, τ) . Therefore, $D_\alpha \cap P \neq \emptyset$ since D_α meets all nonempty G_δ -sets in (Σ, τ) . This implies that the intersection $\bigcap_{n \in \omega} V_n$ is nonempty. However, $\bigcap_{n \in \omega} V_n \subseteq O \cap \bigcap_{n \in \omega} W_n = \emptyset$, which is a contradiction. Thus G has the Baire property.

(C) Let us show that all countable subsets of G are closed. Take an arbitrary countable set $A \subseteq G \setminus \{e\}$. It suffices to show that $e \notin cl_G(A)$. Since $|A| \leq \omega$, we can find an ordinal $\alpha \in \omega_1$ such that $\text{supp}(x) \subseteq \alpha$ for each $x \in A$. It follows from the definition of C_α that $C_\alpha \cap A = \emptyset$ and since $e \in C_\alpha \in \sigma$, we conclude that $e \notin cl_G(A)$.

(D) Here we verify that C^{-1} is closed and discrete in G , where $C = C_0$. If $x \in G \setminus C^{-1}$, then $x C \cap C^{-1} = \emptyset$ —otherwise $x \in C^{-1} \cdot C^{-1} = C^{-1}$. So C^{-1} is closed in G . Further, take any $x \in C^{-1}$ and choose $\alpha \in \omega_1$ such that $\text{supp}(x) \subseteq \alpha$. Then $x C_\alpha \cap C^{-1} = \{x\}$. Indeed, it follows from $e \in C_\alpha$ that $x \in x C_\alpha \cap C^{-1}$. It is also clear from the choice of α that $x d \in D_\alpha$ for each $d \in D_\alpha$, while the sets D_α and C^{-1} are disjoint. Hence $x D_\alpha \cap C^{-1} = \emptyset$ and $x C_\alpha \cap C^{-1} = \{x\}$. Therefore C^{-1} is discrete.

(E) It remains to show that G is not 2-pseudocompact. Take any decreasing sequence $\{U_n : n \in \omega\}$ of nonempty open sets in (Σ, τ) with empty intersection. Since $C = C_0$ is dense in (Σ, τ) , each set $V_n = U_n \cap C$ is nonempty and open in G . However, the intersection

$$\bigcap_{n \in \omega} cl_G V_n^{-1} = \bigcap_{n \in \omega} cl_G (C^{-1} \cap U_n^{-1}) = \bigcap_{n \in \omega} (C^{-1} \cap U_n^{-1}) = C^{-1} \cap \left(\bigcap_{n \in \omega} U_n \right)^{-1}$$

is empty and the group G fails to be 2-pseudocompact. This finishes the proof of the theorem. \square

The group G in Theorem 2 is not 2-pseudocompact since it contains a nonempty open subsemigroup C such that C^{-1} is closed and discrete in G . Modifying the construction in Theorem 1, we present below an example of a feebly compact, 2-pseudocompact Hausdorff paratopological group H such that all countable subsets of H are closed.

Theorem 3 *There exists an infinite feebly compact, 2-pseudocompact Hausdorff paratopological group H in which all countable subsets are closed.*

Proof We keep notation of Theorem 1 and denote by Σ the Σ -product of ω_1 copies of the circle group \mathbb{T} endowed with topology τ inherited from \mathbb{T}^{ω_1} .

Let $\mathcal{S} = \{S_\alpha : \alpha \in \omega_1\}$ be a family of infinite disjoint subset of \mathbb{T} satisfying conditions (1) and (2) of Lemma 1. Then $S = \bigcup_{\alpha \in \omega_1} S_\alpha$ is an independent subset of \mathbb{T} consisting of elements of infinite order. Every element $x \in \langle S \rangle$ distinct from the identity

has a unique representation in the form $x = t_1^{p_1} \cdots t_n^{p_n}$, where t_1, \dots, t_n are pairwise distinct elements of S and p_1, \dots, p_n are nonzero integers. We put $p(x) = \sum_{i=1}^n p_i$. For every $\alpha \in \omega_1$, denote by K_α the subgroup of \mathbb{T} generated by the set $\bigcup \{S_\beta : \alpha \leq \beta < \omega_1\}$. Notice that $K_\beta \subseteq K_\alpha$ if $\alpha < \beta < \omega_1$. We define subsemigroups E_α and F_α of Σ , for each $\alpha \in \omega_1$, by letting

$$E_\alpha = \{x \in \Sigma : (\exists \beta \in \omega_1)[x(\beta) \in K_\alpha \ \& \ p(x(\beta)) > 0 \ \& \ (\forall \gamma > \beta)[x(\gamma) = 1]]\}$$

and

$$F_\alpha = E_\alpha \cup \{e\},$$

where e is the identity of Σ . Evidently, $F_\beta \subseteq F_\alpha$ whenever $\alpha < \beta < \omega_1$, so the family $\mathcal{F} = \{F_\alpha : \alpha \in \omega_1\}$ is closed under finite intersections.

Let $\sigma = \langle \tau, \mathcal{F} \rangle$ be the topology on Σ with the base

$$\{xF_\alpha \cap U : x \in \Sigma, \alpha \in \omega_1, x \in U \in \tau\}.$$

It follows from Lemma 2 that $H = (\Sigma, \sigma)$ is a Hausdorff paratopological group. We leave to the reader a routine verification of the fact that \mathcal{F} satisfies condition (α) of Lemma 2, i.e., $F_\alpha \cap xF_\alpha$ is dense in Σ for all $x \in \Sigma$ and $\alpha \in \omega_1$. Therefore, the regularization of H coincides with $\Sigma = (\Sigma, \tau)$ and the space H is feebly compact since Σ is countably compact.

To show that H is 2-pseudocompact, it suffices to note that each $S_\alpha \subseteq K_\alpha$ is a dense independent subset of \mathbb{T} and apply the argument in the part (C) of the proof of Theorem 1. In particular, one can show without substantial changes that for every nonempty open subset V of (Σ, τ) , there exists a countable ordinal α_V such that the inclusion $L_\beta^{-1} \cap cl_\Sigma V \subseteq cl_H(L_\beta^{-1} \cap V)$ is valid for every ordinal β satisfying $\alpha_V \leq \beta < \omega_1$, where $L_\beta = \{x \in E_\beta : \max \text{supp}(x) \geq \beta\}$.

Finally, we verify that all countable subsets of H are closed. To this end, it suffices to show that the identity e of H is not in the closure of any countable set $A \subseteq H \setminus \{e\}$. Let $B = F_0 \cap A$. For every $x \in B$, put $\beta_x = \max \text{supp}(x)$ and $t_x = x(\beta_x)$. Since the sets B and $T = \{t_x : x \in B\}$ are countable, there exists an ordinal $\alpha \in \omega_1$ such that T is contained in the subgroup of \mathbb{T} generated by the set $\{t_v : v < \alpha\}$. Hence $T \cap K_\alpha = \emptyset$ and, consequently, $F_\alpha \cap B = \emptyset$. It now follows from $F_\alpha \subseteq F_0$ that $F_\alpha \cap A = F_\alpha \cap (F_0 \cap A) = F_\alpha \cap B = \emptyset$. Since F_α is an open neighborhood of e in H , we conclude that $e \notin cl_H A$. This completes the proof of the theorem. \square

Remark 2 The paratopological groups G and H constructed in Theorems 2 and 3, respectively, are connected and locally connected. Let us verify the (local) connectedness of H , the argument in the case of G is the same. Clearly, the semiregularization X_r of an arbitrary space X is connected if and only if X is connected. Since the subspace $(\Sigma, \tau) = H_r$ of \mathbb{T}^{ω_1} is connected, so is H . As for the local connectedness of H , one applies a similar argument along with the fact that the semiregularization O_r of an open subspace of a space X is a subspace of the semiregularization X_r of X .

The groups constructed in Theorems 2 and 3 are not precompact nor ω -narrow (notice that feebly compact precompact paratopological groups are topological groups according to [25, Proposition 2]). This is why we raise the following question:

Problem 1 Does there exist an infinite precompact Hausdorff paratopological group all countable subsets of which are closed?

We also repeat the last item of Problem 2 from [25]:

Problem 2 Is every 2-pseudocompact regular paratopological group a topological group?

It is known that the answer to Problem 2 is “yes” if 2-pseudocompactness is strengthened to *C-compactness* (see [30, Proposition 3.4]). Below we explain the terminology and extend the latter fact to *locally C-compact* paratopological groups.

A space X which carries two (arbitrary) topologies τ and γ is called a *bitopological space* and denoted by (X, τ, γ) . Notice that a paratopological group (G, τ) can be regarded as a bitopological space by considering the topologies τ and τ^{-1} on G . Given a bitopological space (X, τ, γ) , a function $f: (X, \tau, \gamma) \rightarrow (\mathbb{R}, u, l)$ is said to be *bicontinuous* if both $f: (X, \tau) \rightarrow (\mathbb{R}, u)$ and $f: (X, \gamma) \rightarrow (\mathbb{R}, l)$ are continuous functions (here u and l stand for the upper and lower topology on \mathbb{R} , respectively).

The notion of 2-pseudocompactness is related to the study of the concept of boundedness in the realm of bitopological spaces. Indeed, 2-pseudocompact T_0 paratopological groups (G, τ) can be characterized by the property that every bicontinuous function on (G, τ, τ^{-1}) is bounded (see [14, Corollary 1.5] for a more general result). The situation becomes rather clearer if we move on to another close notion of boundedness, the so called *C-compactness*. A bitopological space is called *C-compact* if $(f \vee g)(X)$ is a compact subset of $\mathbb{R} = (\mathbb{R}, u \vee l)$ whenever $f: (X, \tau, \gamma) \rightarrow (\mathbb{R}, u, l)$ and $g: (X, \gamma, \tau) \rightarrow (\mathbb{R}, u, l)$ are bicontinuous functions. Here $f \vee g$ stands for $\max\{f, g\}$.

By [30, Proposition 3.4], every *C-compact* paratopological group is a pseudocompact topological group. The following result, related to Problem 2, is a local version of this fact; its proof makes use of the theory of quasi-uniformizable topological spaces. We recall that a quasi-uniform space (X, \mathcal{U}) is called *precompact* if for every $U \in \mathcal{U}$ there is a finite set F such that $X = \bigcup_{x \in F} U(x)$, where $U(x) = \{y \in X : (x, y) \in U\}$. The quasi-uniformity \mathcal{U} is said to be *hereditarily precompact* if the restriction $\mathcal{U}|_Y$ ($Y \times Y$) is a precompact quasi-uniformity for every subset $Y \subset X$. A group is called *locally C-compact* if its identity has a *C-compact* neighborhood.

In the theorem below ‘regular’ does not include either T_0 or T_1 separation axiom. The reader whose preference in interpreting regularity follows Engelking’s book [12] can skip the major part of the following argument where we pass from a given paratopological group G to a quotient group G/K .

Theorem 4 *Every locally C-compact regular paratopological group G is a locally pseudocompact topological group.*

Proof Let e be the neutral element of G and C the closure of $\{e\}$ in G . Notice that $C \subseteq V^{-1}$ for every neighborhood V of the neutral element e of G . It is easy to see that $K = C \cap C^{-1}$ is a normal (perhaps nonclosed) subgroup of G . Indeed, both C and C^{-1} are subsemigroups of G since so is $\{e\}$ (see [3, Proposition 1.4.10]). It is also clear that $K^{-1} = K$, so K is a subgroup of G . It follows from the definition of C and

the continuity of inner automorphisms in G that $x^{-1}Cx \subseteq C$ for each $x \in G$, i.e., the subsemigroup C is invariant in G . Therefore C^{-1} is also invariant in G and so is the subgroup $K = C \cap C^{-1}$ of G .

We claim that $UK = U$ for every open set $U \subseteq G$. If not, we can find elements $u \in U$ and $k \in K$ such that $uk \notin U$. Then $V = u^{-1}U$ is an open neighborhood of the neutral element e in G which does not contain k , thus contradicting the fact that $k \in K \subseteq V$.

Let $\pi: G \rightarrow G/K$ be the quotient homomorphism of paratopological groups. Clearly π is an open mapping. The above observation implies that $U = \pi^{-1}\pi(U)$ for every open set $U \subseteq G$. Hence the quotient paratopological group G/K is a regular space.

We claim that G/K is a T_0 -space. Indeed, by the homogeneity argument, it suffices to verify that for every $\bar{x} \in G/K$ distinct from the neutral element \bar{e} of G/K , there exists an open set $O \subseteq G/K$ which contains exactly one of the points \bar{x}, \bar{e} . Take $x \in G$ such that $\pi(x) = \bar{x}$. Then $x \notin K$, so there exists an open neighborhood U of e in G such that $x \notin U \cap U^{-1}$ (otherwise $x \in K$). If $x \notin U$, then $O = \pi(U)$ is an open neighborhood of \bar{e} in G/K which does not contain \bar{x} . If $x \in U^{-1}$, then $e \notin xU$ and, hence, $O = \pi(xU)$ is an open neighborhood of \bar{x} which does not contain \bar{e} .

Let V be a C -compact neighborhood of the identity in G . Then $W = \pi(V)$ is a C -compact neighborhood of \bar{e} in G/K , so the group G/K is also locally C -compact. Since G/K is a T_0 -space, [29, Lemma 1 and Theorem 1] imply that the restriction of the left quasi-uniformity of G/K to $W \times W$ is hereditarily precompact. Hence, by [19, Theorem 2], G/K is a topological group. Clearly, K inherits the antidiscrete topology from G , so K is also a topological group. Therefore, by [26, Lemma 4], G is a topological group as well.

Finally, the local pseudocompactness of G follows from the definition of (local) C -compactness. □

3 \mathbb{R} -factorizability of feebly compact paratopological groups

Following [32] we say that a paratopological group G is \mathbb{R}_i -factorizable for $i = 0, 1, 2, 3$ if for every continuous real-valued function g on G one can find a continuous homomorphism $\pi: G \rightarrow H$ onto a second countable paratopological group H satisfying the T_i separation axiom and a continuous real-valued function h on H such that $g = h \circ \pi$.

$$\begin{array}{ccc}
 G & \xrightarrow{g} & \mathbb{R} \\
 & \searrow \pi & \nearrow h \\
 & H &
 \end{array}$$

In the next theorem we do not require that the group G satisfy any separation axiom.

Theorem 5 *Every feebly compact paratopological group G is \mathbb{R}_3 -factorizable. Furthermore, for every continuous real-valued function g on G , there exist a continuous*

homomorphism φ of G onto a second countable Hausdorff topological group H and a continuous function h on H such that $g = h \circ \varphi$.

Proof Let τ be the topology of a feebly compact paratopological group G . Denote by τ_r the regularization of the topology τ and let $G_r = (G, \tau_r)$. It is clear that τ_r is coarser than τ . By [23, Example 1.9], G_r with the same multiplication is a paratopological group. Therefore, according to [23, Proposition 1.5], the paratopological group G_r satisfies the T_3 separation axiom (however, G_r can fail to be even a T_0 -space). Since the topology τ_r is coarser than τ , the group G_r is feebly compact. Applying [25, Lemma 6], we conclude that G_r is a topological group.

Let g be a continuous real-valued function on G . Then g is continuous on G_r since the space \mathbb{R} is regular. Indeed, take an arbitrary point $x \in G_r$ and an open neighborhood U of $g(x)$ in \mathbb{R} . There exists an open neighborhood V of $g(x)$ in \mathbb{R} such that $\overline{V} \subseteq U$. Since g is continuous, we can find an open neighborhood O of x in G such that $g(O) \subseteq V$. Then $W = \text{Int}_G \overline{O}$ (the closure is taken in G) is an open neighborhood of x in G_r and

$$g(W) \subseteq g(\overline{O}) \subseteq \overline{g(O)} \subseteq \overline{V} \subseteq U.$$

This proves the continuity of g on G_r . The same function g considered on the space G_r will be denoted by g_r .

Denote by K the closure of the set $\{e\}$ in G_r , where e is the identity element in G_r . We claim that the function g_r is constant on each coset of K in G_r . Indeed, take an arbitrary element $x \in G_r$. Since G_r is a topological group, the closure of the set $\{x\}$ in G_r coincides with xK . The continuity of the function $g_r: G_r \rightarrow \mathbb{R}$ implies that $g_r(xK) \subseteq \overline{\{g_r(x)\}} = \{g_r(x)\}$, i.e., g_r is constant on xK .

Let $p: G_r \rightarrow G_r/K$ be the quotient homomorphism. Then the quotient group $L = G_r/K$ satisfies the T_1 separation axiom and therefore L is a completely regular topological group. Clearly L is feebly compact as a continuous homomorphic image of G_r . Hence L is pseudocompact. Since g_r is constant on the cosets of K in G_r , there exists a real-valued function f on the group L satisfying $g_r = f \circ p$. The function f is continuous because the homomorphism p is continuous and open.

Since every pseudocompact topological group is precompact and precompact groups are \mathbb{R} -factorizable (see [3, Corollary 8.1.17]), we see that the group L is \mathbb{R} -factorizable. Hence we can find a continuous homomorphism $\pi: L \rightarrow H$ onto a second countable topological group H and a continuous real-valued function h on H such that $f = h \circ \pi$. Let $i: G \rightarrow G_r$ be the identity mapping.

$$\begin{array}{ccccc} G & \xrightarrow{g} & \mathbb{R} & & \\ \downarrow i & \nearrow g_r & \uparrow f & \nwarrow h & \\ G_r & \xrightarrow{p} & G_r/K & \xrightarrow{\pi} & H \end{array}$$

Then the continuous homomorphism $\varphi = \pi \circ p \circ i$ of G onto the second countable topological group H and the function h satisfy the equality $g = h \circ \varphi$. Since H is regular, the group G is \mathbb{R}_3 -factorizable. \square

It is worth noting that unlike in topological groups, Theorem 5 is not valid for precompact paratopological groups. Indeed, the circle group \mathbb{T}_{Sor} endowed with the Sorgenfrey topology is precompact but it is not even \mathbb{R}_1 -factorizable, see [32, Example 3.3]. Another relevant peculiarity of paratopological groups is that feebly compact Hausdorff paratopological groups need not be precompact (see Theorem 1), while every feebly compact topological group satisfying the T_0 separation axiom is completely regular, pseudocompact, and hence precompact by [10, Theorem 1.1]. In fact, a feebly compact (not necessarily Hausdorff or even T_0) paratopological group G is a topological group iff G is precompact iff G is regular. Indeed, by [2, Theorem 1.7], every regular feebly compact paratopological group is a topological group. Further, by [24, Proposition 3.1], every precompact paratopological group is saturated, while saturated feebly compact paratopological groups are topological groups according to [25, Proposition 2].

Added in proof. a) Recently O. Ravsky succeeded in proving that every 2-pseudocompact paratopological group is feebly compact. Therefore, the feeble compactness of the paratopological groups G and H in Theorems 1 and 3, respectively, can alternatively be deduced from this fact.

b) The second listed author has constructed a precompact completely regular paratopological group K of cardinality \mathfrak{c}^+ such that all subsets of the group K of cardinality less than or equal to \mathfrak{c} are closed. This solves Problem 1 in the affirmative. O. Ravsky proved that all regular 2-pseudocompact paratopological groups are topological groups, thus solving Problem 2.

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