Estimation in multisensor networked systems with scarce measurements and time varying delays

I. Peñarrocha[a,*], R. Sanchis[a], P. Albertos[b]

[a]Departament d'Enginyeria de Sistemes Industrials i Disseny, Universitat Jaume I de Castelló, Campus de Riu Sec, 12071 Castelló, Spain.
[b]Department of Systems Engineering and Control, Universidad Politécnica de Valencia, E-46071, Valencia, Spain

Abstract

In this paper, the problem of estimating signals from a dynamic system at regular periods from scarce, delayed and possibly time disordered measurements acquired through a network is addressed. A model based predictor that takes into account the delayed and irregularly gathered measurements from different devices is used. Robustness of the predictor to the time-delays and scarce data availability as well as disturbance and noise attenuation is dealt with via $\mathcal{H}_\infty$ performance optimization. The result is a time variant estimator gain that depends on the measurement characteristics, but belonging to an offline precalculated finite set, and hence, the online needed computer resources are low. An alternative to reduce the number of gains to be stored has been proposed, based on defining the gain as a function of the sampling parameters. The idea allows reaching a compromise between online computer cost and performance.

Keywords: Sensor Fusion; Scarce Measurements; Unconventional Sampling; Time-varying delay; Packet dropout; Communication constraint; Networked control systems

1. Introduction

In the last years, many processes in industry are controlled or monitored using sensors, controllers and actuators connected to a network. The multi-sensor fusion for this processes has been widely studied in the literature. In this sense, several approaches have been proposed in order to estimate the state or outputs of a system when the sensors information is acquired through a network with packet dropout, network-induced delays, or accessibility constraints due to, for instance, its shared nature. This problem can be addressed as a multi-sensor filtering problem with scarce, delayed and out-of-sequence measurements. Most of the approaches can be

*Corresponding author. Tel: +34 964728177, Fax: +34 964728170
Email addresses: ipenarro@esid.uji.es (I. Peñarrocha), rsanchis@esid.uji.es (R. Sanchis), pedro@aii.upv.es (P. Albertos)

classified into two main streams: Kalman filter based approaches (computed on line), and gain scheduling approaches (computed offline).

The first class of approaches lead to high computational cost algorithms to be executed in real-time that try to minimize the estimation error, updating the covariance matrix and the gain of the filter, in each step. In works like [12] and [22] an internal model of the system has been used to estimate the states and the missing outputs of irregular measured systems with the use of a Kalman filter. In [6] several possibilities have been analyzed to incorporate delayed measurements to the Kalman filter. In several recent works, as [18], [14], [8], [10] the use of a Kalman filter to estimate the state in networked control systems with varying delays and data dropout has been analyzed. These approaches can be applied when the data availability and the delays are time varying, but the resulting algorithms require a high computational on line effort. Moreover, the Kalman filter does not guarantee an optimal performance when either there is uncertainty on the model or the disturbances are not gaussian noises with known statistics. Furthermore, these approaches do not give any a priori information on the achievable performance before its use, and hence, it is not very useful to take decisions at the design stage about sensor bandwidth or sampling rate assignment to achieve some desired performance.

The second type of approaches try to solve a complex optimization problem offline, but lead to a time invariant observer that can be implemented in a low computing cost online estimation algorithm. The optimization problem is based on maximizing disturbances attenuation, and is solved using Linear Matrix Inequalities (LMI) techniques. This strategy gives as a result some a priori information on the achievable performance at the design stage, and hence, these techniques can be used to make a priori bandwidth or sampling rate assignment to achieve a desired attenuation of the disturbances and noise measurements on the system.

In several works, like [2, 9, 16, 17, 3, 4, 19, 20], the robust performance on the filtering problem on discrete-time systems has been studied using LMI techniques, but without considering the case of scarce irregular measurements. Some of the results of those works have been adapted in this paper to solve the scarce and delayed measurements prediction problem.

More recently, several authors have dealt with the multi-sensor fusion in networked control systems with data dropout and varying delays via LMI approaches, as for example, [11], [21], [5] and [15]. All of them propose a time invariant filter/observer with a constant gain, that is designed assuming a given stochastic behavior of the data availability. Furthermore, some of those works substitute the dropped measurements by the old ones and, thus, an additional error is introduced. Also, most of those works do not take into account the problem of estimating unknown inputs with packet dropout or time-varying delays.

The output prediction under scarce measurements was addressed by the authors using input-output models in previous works, as [1], where the nominal stability was only guaranteed for the periodic data availability case. In [13], the output prediction in systems with scarce measurements and time varying delays was solved, with a guarantee on stability and maximizing measurement noise and disturbances attenuation, using an input-output model, but the approach was only applicable to SISO systems.

This work deals with the estimation of signals of a system (inputs or outputs, not necessarily directly measured) at a regular period, using the measurements of several sensors that are acquired irregularly and scarcely in time through a network, and with different time-varying delays, but assuming that the measurements are time-tagged and reception acknowledgement is available. As a difference with other approaches in the literature, the approach presented here exploits all the information of the system, measurement noises, packet dropout and induced delays to design a time variant estimator whose gain varies as a function of the measurement characteristics, but belonging to a set of offline precalculated gains, such that the disturbance and noise attenuation is optimized. The problem of designing the set of gains is solved via LMI techniques. The online computer implementation cost is much lower than the Kalman filter based approaches, at the expense of a lower (suboptimal) performance, but the achieved performance is still much better than the constant gain approaches. In this sense, the proposed approach represents a compromise between performance and computing cost that lies in between the Kalman filter based approaches and the constant gain approaches. Furthermore, the set of precalculated gains can be selected to be larger or shorter as a compromise between available online implementation resources and performance. The approach can also be used to determine the minimum sensors data availability that is necessary to assure a given desired bound on the estimation error.

The layout of this paper is as follows: In section 2 the plant and the sampling scenario are described. In section 3 the proposed prediction model and the predictor algorithm are defined. Predictor design is addressed through $\mathcal{H}_\infty$ performance on the disturbances attenuation in section 4. Some numerical examples are analyzed on section 5, showing the validity of the proposed design strategy. Finally, on section 6 the main conclusions are summarized.

2. Problem statement

2.1. The plant

Consider a continuous linear time invariant process with different inputs and outputs and consider that the value of some of the variables is desired to be known periodically each T seconds. Let us assume that a discrete-time equivalent model of the process at period T is available, that is described by equations

$$\boldsymbol{x}[t+1] = \boldsymbol{A}_x \boldsymbol{x}[t] + \boldsymbol{B}_u \boldsymbol{u}[t] + \boldsymbol{w}[t], \tag{1a}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state and $\boldsymbol{u} \in \mathbb{R}^{n_u}$ are the recognizable inputs of the model, that can be manipulated inputs provided by any local controller connected to the plant or measurable disturbances whose effect on the state evolution can be modeled, and $\boldsymbol{w} \in \mathbb{R}^{n_x}$ are bounded disturbances. It is assumed that $rank(\boldsymbol{B}_u) = n_u$. The vector of signals that is desired to be known at each period is assumed to be

$$\boldsymbol{y}[t] = \boldsymbol{C}_{xy} \boldsymbol{x}[t] + \boldsymbol{D}_{uy} \boldsymbol{u}[t], \tag{1b}$$

where $\boldsymbol{y}[t] \in \mathbb{R}^{n_y}$ can be any linear combination of the states and recognizable inputs.

In this work not only the measured outputs but also the inputs of the process are assumed to be transmitted through the network and, hence, are subject to delays and data drop outs. In order to account for this, the recognizable inputs are assumed to vary slowly with time, and a simple way of modeling this fact is to define the inputs as

$$\boldsymbol{u}[t] = \boldsymbol{u}[t-1] + \Delta\boldsymbol{u}[t], \tag{2}$$

where $\Delta\boldsymbol{u} \in \mathbb{R}^{n_u}$ are assumed to be bounded signals. The bound in $\Delta\boldsymbol{u}$ in fact imposes a bound in the rate of change of $\boldsymbol{u}$. Using this model for the inputs implies that their estimate will remain constant when there is no measurement available.

2.2. Networked sampling scenario

Let us also assume that there are several noisy sensors that measure some of the process variables (states, outputs, manipulable inputs or other recognizable disturbances) and different remote but accessible controllers that can transmit the value of some of the manipulable inputs. Those signals are assumed to be acquired through a network that is not always available and that can introduce some time-varying delay. Furthermore, the measurements can be scheduled to be taken scarcely in time in order to reduce the network load, or due to the characteristics of the measurement process (slow analysis or image processing). As a consequence, the measurements can arrive to the estimation unit only at scarce periods of time and possibly disordered in time.

The values of some of the variables are assumed to be received at different sampling instants $t = t_k$, $k \in \mathbb{N}$ (at least one measurement is supposed to be available at instant $t = t_k$), that represents the instant in which the t-th input update occurs and the k-th sample (formed with the values of the received messages) is given. The acquired measurements can be expressed as

$$m_{i,k} = \boldsymbol{c}_{xi}\boldsymbol{x}[t_k - d_{i,k}] + \boldsymbol{c}_{ui}\boldsymbol{u}[t_k - d_{i,k}] + v_{i,k}, \qquad i = 1, \ldots, n_m \tag{3}$$

where n_m is the number of measured signals, $m_{i,k}$ is the available measured value from sensor i at time $t = t_k$, $d_{i,k}$ is the delay (measured in number of control periods) from the instant the measurement was taken (or applied by a remote controller) until it was received, and $v_{i,k}$ is the measurement noise. In this work, time-tagged messages through the network are assumed, and therefore, $d_{i,k}$ is assumed to be a known value for every measured signal value.

Note that if the i-th measured signal is an output of the system, then $\boldsymbol{c}_{ui}$ will be a null vector. If the i-th acquired signal refers to the j-th input $u_j[t]$, then $\boldsymbol{c}_{xi}$ will be a null vector, and $\boldsymbol{c}_{ui}$ will be a null vector containing only an entry of 1 on the j-th position. If that input is provided by the controller that generates it, then a null noise $v_{i,k}$ will also be assumed.

The number of discrete periods between available measurements (from t_{k-1} to t_k) is denoted with $N_k = t_k - t_{k-1}$ and, therefore, $t_k = \sum_{i=1}^{k} N_i$. The value N_k is assumed to be time variant, but belonging to a known finite set

$$N_k \in \mathcal{N} = \{\nu_1, \nu_2, \ldots, \nu_{n_\mathcal{N}}\}, \tag{4}$$

while the delay of the different measurements, $(d_{i,k})$ is also assumed to vary in a known finite set of possible values

$$d_{i,k} \in \mathcal{D}_i = \{\delta_{i1}, \ldots, \delta_{ip_i}\}. \tag{5}$$

Let us define D_k as the vector gathering the delays of the measurements available at instant t_k. According to the previous assumption, D_k can take values in a known finite set. If the process has an additional constant delay in any of its outputs, that delay can be assigned to all the sensors that measure that output, adding that value in the corresponding $d_{i,k}$.

In order to denote which of the n_m sensors are available at each sampling instant, the availability matrix $\boldsymbol{\alpha}_k$ at sampling instant k is defined as the diagonal matrix with non-null entries defined as

$$\boldsymbol{\alpha}_{k\,[ii]} = \begin{cases} 1, & \text{if } m_{i,k} \text{ is available,} \\ 0, & \text{if } m_{i,k} \text{ is not available,} \end{cases}$$

If in a given instant $t = t_k$ all measurements are available, then $\boldsymbol{\alpha}_k = \boldsymbol{I}$. Depending on the measurement pattern, the matrix $\boldsymbol{\alpha}_k$ can take different values. It is assumed that those possible values belong to a known finite set

$$\boldsymbol{\alpha}_k \in \Xi = \{\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_p\}. \tag{6}$$

In the most general case, any combination of available measurements is possible leading to $p = 2^{n_m} - 1$ possible values.

Now, let us define the sampling scenario parameter s_k as the combination of sensors and inputs availability, time between samples and sensors delay $(\boldsymbol{\alpha}_k, N_k, D_k)$ that defines a sample. This parameter enumerates all the possible sampling situations as

$$s_k \in \mathcal{S} = \{1, 2, \ldots, n_{\mathcal{S}}\}, \tag{7}$$

where $n_{\mathcal{S}}$ is the number of possible combinations. All the variables that define the sampling scenario can be expressed as a function of this parameter, i.e., $N_k = N(s_k)$, $D_k = D(s_k)$, $\boldsymbol{\alpha}_k = \boldsymbol{\alpha}(s_k)$.

In order the estimation problem to be solvable, the system (1) must be detectable under the assumed sampling scenario. This detectability can be guaranteed if the system is detectable from each of the measurable outputs, and if at every measurement instant, the number of measured outputs is larger or equal than the number of unavailable inputs.

3. Proposed approach

3.1. Prediction model

In order to obtain an unbiased estimation of the desired signals $\boldsymbol{y}[t]$ at period T, from the scarce, delayed and out-of-sequence measurements $m_{i,k}$ acquired through the network, an extended order model is proposed to be used. The system dynamics, including the desired outputs, can be rewritten as

$$\begin{bmatrix} \boldsymbol{x}[t+1] \\ \boldsymbol{u}[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}_x & \boldsymbol{B}_u \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{x}[t] \\ \boldsymbol{u}[t] \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{B}_w & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} \boldsymbol{w}[t] \\ \Delta\boldsymbol{u}[t] \end{bmatrix} \tag{8}$$

$$\boldsymbol{y}[t] = \underbrace{\begin{bmatrix} \boldsymbol{C}_{xy} & \boldsymbol{D}_{uy} \end{bmatrix}}_{\boldsymbol{C}_y} \begin{bmatrix} \boldsymbol{x}[t] \\ \boldsymbol{u}[t] \end{bmatrix} \tag{9}$$

Defining a new extended state vector $\boldsymbol{z}[t] \in \mathbb{R}^n$ $(n = n_x + n_u)$ as

$$\boldsymbol{z}[t] = \begin{bmatrix} \boldsymbol{x}[t]^T & \boldsymbol{u}[t]^T \end{bmatrix}^T, \tag{10}$$

the previous system dynamics can be expressed as

$$\boldsymbol{z}[t+1] = \boldsymbol{A}\boldsymbol{z}[t] + \boldsymbol{B}\bar{\boldsymbol{w}}[t], \tag{11}$$

$$\boldsymbol{y}[t] = \boldsymbol{C}_y \boldsymbol{z}[t], \tag{12}$$

where

$$\bar{\boldsymbol{w}}[t] = \begin{bmatrix} \boldsymbol{w}[t]^T & \Delta\boldsymbol{u}[t]^T \end{bmatrix}^T$$

is the new disturbance vector.

The measurement equation can be expressed as $(i = 1, \dots, n_m)$

$$m_{i,k} = \underbrace{\begin{bmatrix} \boldsymbol{c}_{xi} \; \boldsymbol{c}_{ui} \end{bmatrix}}_{\boldsymbol{c}_i} \boldsymbol{z}[t_k - d_{i,k}] + v_{i,k} = \boldsymbol{c}_i \, \boldsymbol{z}[t_k - d_{i,k}] + v_{i,k}. \tag{13}$$

3.2. Estimation algorithm

The previous extended order model is used to estimate the desired system signals as follows. The extended state (including states and inputs) is initially observed running the model in open loop and assuming null values on vector $\bar{\boldsymbol{w}}[t]$, leading to

$$\hat{\boldsymbol{z}}[t|t-1] = \boldsymbol{A}\,\hat{\boldsymbol{z}}[t-1]. \tag{14a}$$

Depending on the availability of a new measurement at $t = t_k$ (i.e. some $m_{i,k}$ is available), the estimated state is updated by

$$\hat{\boldsymbol{z}}[t_k] = \hat{\boldsymbol{z}}[t_k|t_k-1] + \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k} \left(m_{i,k} - \boldsymbol{c}_i \, \hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1] \right) \boldsymbol{\alpha}_{k[ii]}, \tag{14b}$$

where $\boldsymbol{\ell}_{i,k}$ is the gain vector used to update the estimated state with the measurement $m_{i,k}$. Finally, the vector of desired signals is estimated as

$$\hat{\boldsymbol{y}}[t] = \boldsymbol{C}_y \, \hat{\boldsymbol{z}}[t]. \tag{14c}$$

The state estimation of the delayed state at the time of output measurement ($\hat{\boldsymbol{z}}[t_k - d_i|t_k - 1]$) used in (14b), is defined as the delayed state estimation vector that fulfills

$$\hat{\boldsymbol{z}}[t_k|t_k-1] = \boldsymbol{A}^{d_{i,k}} \, \hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1]$$

and, therefore, it can be calculated as

$$\hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1] = \boldsymbol{A}^{-d_{i,k}} \, \hat{\boldsymbol{z}}[t_k|t_k - 1], \tag{15}$$

i.e., the vector $\hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1]$ is the estimation of the state at instant $t - d_{i,k}$ with the available information until $t_k - 1$ calculated running back the model and assuming a null vector $\bar{\boldsymbol{w}}[t]$. Note that the values of the inputs $\boldsymbol{u}[t]$ are assumed to be constant during the $d_{i,k}$ periods, due to the block structure of matrix $\boldsymbol{A}$, with an identity matrix in the last block (the signals $\boldsymbol{u}[t]$ remain unchanged when matrix $\boldsymbol{A}$ is powered to any value). As the inputs are assumed to have bounded derivative, the introduced error is assumed to be small, and furthermore, with the proposed approach, that error is taken into account to bound the state estimation error.

4. Predictor design

The dynamics of the estimator depends on the matrix gain

$$\boldsymbol{L}_k = \begin{bmatrix} \boldsymbol{\ell}_{1,k} & \boldsymbol{\ell}_{2,k} & \cdots & \boldsymbol{\ell}_{n_m,k} \end{bmatrix} \tag{16}$$

defined at measuring instants ($t = t_k$), that must be designed to assure: the predictor stability, robustness to the irregular data availability and time varying delay, and a proper attenuation of the disturbances and measurement noises. The predictor gain is assumed to be time varying in general, but belonging to a finite set of possible gains. The possibility of defining a single constant matrix gain is a particular case that will also be analyzed.

In order to design the predictor gain (16) with these properties, the prediction error dynamic equation must be obtained.

Lemma 1 (Prediction error dynamics). *The prediction error dynamics of the algorithm* (14) *applied to system* (1) *when there is no modeling error and there is at least one measurement available every N_k input periods (with N_k time variant), is described by the linear time-variant system*

$$\tilde{\boldsymbol{z}}_k = \boldsymbol{\mathcal{A}}_k\, \tilde{\boldsymbol{z}}_{k-1} + \boldsymbol{\mathcal{B}}_k \mathbb{W}_k \tag{17a}$$
$$\tilde{\boldsymbol{y}}_k = \boldsymbol{C}_y\, \tilde{\boldsymbol{z}}_k, \tag{17b}$$

where the extended state estimation error vector is defined when a measurement is available ($t = t_k$) as

$$\tilde{\boldsymbol{z}}_k \equiv \tilde{\boldsymbol{z}}[t_k] = \boldsymbol{z}[t_k] - \hat{\boldsymbol{z}}[t_k],$$

while the output prediction error is defined as $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}[t_k] - \hat{\boldsymbol{y}}[t_k]$, and where $\mathbb{W}_k$ is a vector gathering the disturbances between measurements defined as

$$\mathbb{W}_k = \begin{bmatrix} \boldsymbol{v}_k^T & \bar{\boldsymbol{w}}[t_k - 1]^T & \bar{\boldsymbol{w}}[t_k - 2]^T & \cdots & \bar{\boldsymbol{w}}[t_k - \beta]^T \end{bmatrix}^T,$$

being $\beta = \max\{\mathcal{D}_i, \mathcal{N}\}$, $i = 1, \ldots, n_m$ the maximum integer from the set defined by all the possible delays $d_{i,k}$ and number of inter-sampling periods N_k. Matrices $\boldsymbol{\mathcal{A}}_k$ and $\boldsymbol{\mathcal{B}}_k$ are defined as follows:

$$\boldsymbol{\mathcal{A}}_k = (\boldsymbol{I} - \boldsymbol{L}_k\, \boldsymbol{\alpha}_k\, \boldsymbol{C}_{d,k})\, \boldsymbol{A}^{N_k}, \tag{18}$$

$$\boldsymbol{C}_{d,k} = \begin{bmatrix} \boldsymbol{c}_1\, \boldsymbol{A}^{-d_{1,k}} \\ \cdots \\ \boldsymbol{c}_{n_m}\, \boldsymbol{A}^{-d_{n_m,k}} \end{bmatrix}_{n_m \times n}, \tag{19}$$

$$\boldsymbol{\mathcal{B}}_k = \begin{bmatrix} -\boldsymbol{L}_k\, \boldsymbol{\alpha}_k & \quad \boldsymbol{\Lambda}(N_k) - \boldsymbol{L}_k\, \boldsymbol{\alpha}_k\, \boldsymbol{\mathcal{C}}_{d,k} \end{bmatrix}_{n \times (n_m + \beta n)} \tag{20}$$

being $\boldsymbol{\alpha}_k \equiv \boldsymbol{\alpha}[t_k]$ *the availability matrix, and*

$$\boldsymbol{\mathcal{C}}_{d,k} = \begin{bmatrix} \boldsymbol{c}_1\, \boldsymbol{A}^{-d_{1,k}}\, (\boldsymbol{\Lambda}(N_k) - \boldsymbol{\Lambda}(d_{1,k})) \\ \cdots \\ \boldsymbol{c}_{n_m}\, \boldsymbol{A}^{-d_{n_m,k}}\, (\boldsymbol{\Lambda}(N_k) - \boldsymbol{\Lambda}(d_{n_m,k})) \end{bmatrix}_{n_m \times \beta n},$$

with $\boldsymbol{\Lambda}(N)$ *the matrix defined as*

$$\boldsymbol{\Lambda}(N) = [\underbrace{\overbrace{\boldsymbol{B}\ \ \boldsymbol{A}\boldsymbol{B}\ \ \boldsymbol{A}^2\boldsymbol{B}\ \ \cdots\ \ \boldsymbol{A}^{N-1}\boldsymbol{B}}^{N}\ \ \boldsymbol{0}\ \ \cdots\ \ \boldsymbol{0}}_{\beta}]_{n \times \beta n}. \tag{21}$$

Proof. At the measuring instant, $t = t_k$, expression (13) holds and, therefore, the extended state prediction (14b) can be expressed as

$$\hat{\boldsymbol{z}}[t_k] = \hat{\boldsymbol{z}}[t_k|t_k - 1] + \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, (\boldsymbol{c}_i\, (\boldsymbol{z}[t_k - d_{i,k}] - \hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1]) + v_{i,k})\, \boldsymbol{\alpha}_{k\,[ii]}. \tag{22}$$

Vectors $\boldsymbol{z}[t_k - d_{i,k}]$ and $\hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1]$ can be expressed as a function of $\boldsymbol{z}[t_k - N_k]$ and $\hat{\boldsymbol{z}}[t_k - N_k]$ (the instant when the previous measurement was received and the last prediction update was made) if expressions (1a) and (14a) are applied recursively, leading to

$$\begin{aligned} \boldsymbol{z}[t_k - d_{i,k}] - \hat{\boldsymbol{z}}[t_k - d_{i,k}|t_k - 1] = \boldsymbol{A}^{N_k - d_{i,k}}\, (\boldsymbol{z}[t_k - N_k] - \hat{\boldsymbol{z}}[t_k - N_k]) \\ + \sum_{j=1}^{N_k} \boldsymbol{A}^{j-1-d_{i,k}} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - j] - \sum_{j=1}^{d_{i,k}} \boldsymbol{A}^{j-1-d_{i,k}} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - j]. \end{aligned} \tag{23}$$

The vector $\hat{\boldsymbol{z}}[t_k|t_k - 1]$ can also be expressed by means of $\hat{\boldsymbol{z}}[t_k - N_k]$ applying (14a) recursively, leading to

$$\hat{\boldsymbol{z}}[t_k|t_k - 1] = \boldsymbol{A}^{N_k} \hat{\boldsymbol{z}}[t_k - N_k]. \tag{24}$$

The state $\boldsymbol{z}[t_k]$ can also be expressed (by means of (1a)) as a function of $\boldsymbol{z}[t_k - N_k]$:

$$\boldsymbol{z}[t_k] = \boldsymbol{A}^{N_k} \boldsymbol{z}[t_k - N_k] + \sum_{i=1}^{N_k} \boldsymbol{A}^{i-1} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - i]. \tag{25}$$

Introducing expressions (23) and (24) in (22) and subtracting the resulting expression from (25), it yields

$$\begin{aligned} \underbrace{\boldsymbol{z}[t_k] - \hat{\boldsymbol{z}}[t_k]}_{=\tilde{\boldsymbol{z}}_k} &= \boldsymbol{A}^{N_k}\, \underbrace{(\boldsymbol{z}[t_k - N_k] - \hat{\boldsymbol{z}}[t_k - N_k])}_{=\tilde{\boldsymbol{z}}_{k-1}} + \sum_{j=1}^{N_k} \boldsymbol{A}^{j-1} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - j] \\ &- \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, \boldsymbol{c}_i\, \boldsymbol{A}^{-d_{i,k}} \boldsymbol{A}^{N_k}\, \underbrace{(\boldsymbol{z}[t_k - N_k] - \hat{\boldsymbol{z}}[t_k - N_k])}_{=\tilde{\boldsymbol{z}}_{k-1}} \boldsymbol{\alpha}_{k\,[ii]} - \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, v_{i,k}\, \boldsymbol{\alpha}_{k\,[ii]} \\ &- \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, \boldsymbol{c}_i\, \boldsymbol{A}^{-d_{i,k}} \left(\sum_{j=1}^{N_k} \boldsymbol{A}^{j-1} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - j] - \sum_{j=1}^{d_{i,k}} \boldsymbol{A}^{j-1} \boldsymbol{B}\bar{\boldsymbol{w}}[t_k - j] \right) \boldsymbol{\alpha}_{k\,[ii]} \end{aligned}$$

that, taking into account that at the measuring instant $t_k - N_k = t_{k-1}$, leads to

$$\begin{aligned}
\tilde{\boldsymbol{z}}_k =& \left(\boldsymbol{I} - \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, \boldsymbol{c}_i\, \boldsymbol{A}^{-d_{i,k}}\, \boldsymbol{\alpha}_{k[ii]}\right) \left(\boldsymbol{A}^{N_k}\, \tilde{\boldsymbol{z}}_{k-1} + \sum_{j=1}^{N_k} \boldsymbol{A}^{j-1} \boldsymbol{B} \bar{\boldsymbol{w}}[t_k - j]\right) \\
&+ \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, \boldsymbol{c}_i\, \boldsymbol{A}^{-d_{i,k}}\, \boldsymbol{\alpha}_{k[ii]} \sum_{j=1}^{d_{i,k}} \boldsymbol{A}^{j-1} \boldsymbol{B} \bar{\boldsymbol{w}}[t_k - j] - \sum_{i=1}^{n_m} \boldsymbol{\ell}_{i,k}\, v_{i,k}\, \boldsymbol{\alpha}_{k[ii]} \qquad (26a) \\
\tilde{\boldsymbol{y}}_k =& \boldsymbol{C}_y\, \tilde{\boldsymbol{z}}_k. \qquad (26b)
\end{aligned}$$

Expression (17a) is finally derived using matricial notation. The instantaneous output prediction error $\tilde{\boldsymbol{y}}_k$ is obtained by means of the vector $\boldsymbol{C}_y$

$$\tilde{\boldsymbol{y}}_k = \boldsymbol{y}[t_k] - \hat{\boldsymbol{y}}[t_k] = \boldsymbol{C}_y\, (\boldsymbol{z}[t_k] - \hat{\boldsymbol{z}}[t_k]) = \boldsymbol{C}_y \tilde{\boldsymbol{z}}_k$$

that is updated every time a measurement is available. The extended state estimation error vector is defined when a measurement is available ($t = t_k$) as

$$\tilde{\boldsymbol{z}}_k \equiv \tilde{\boldsymbol{z}}[t_k] = \boldsymbol{z}[t_k] - \hat{\boldsymbol{z}}[t_k],$$

while the output prediction error is defined as $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}[t_k] - \hat{\boldsymbol{y}}[t_k]$.

Expression (17) represents an internal realization of the prediction error dynamics. The vector of disturbances $\mathbb{W}_k$, can be considered as inputs, the estimation error is the state, and the prediction error is the output. The goal of the present work is to find a procedure to design the matrix $\boldsymbol{L}_k$ (time variant in a finite set, or constant) such that the system (17) attains prescribed stability and disturbance attenuation conditions.

In this work the design of gain $\boldsymbol{L}_k$ is addressed defining a new different matrix for each set $(N_k, D_k, \boldsymbol{\alpha}_k)$, i.e., for each possible value of the sampling parameter s_k. The calculation of matrices $\boldsymbol{L}_k = \boldsymbol{L}(s_k)$ is done off-line only once and gives as a result a finite set of gains

$$\boldsymbol{L}_k = \boldsymbol{L}(s_k) \in \boldsymbol{\mathcal{L}} = \{\boldsymbol{L}(1), \boldsymbol{L}(2), \ldots, \boldsymbol{L}(n_{\mathcal{S}})\}. \qquad (27)$$

where $n_{\mathcal{S}}$ is the number of possible sampling scenarios. Every time a new measurement arrives (with values of one or more sensors), a different gain $\boldsymbol{L}_k$ is applied, depending on the value s_k, to update the state estimation (equation (14b))

Note the difference with Kalman filter, where the gain varies arbitrarily with time because the time-varying gain $\boldsymbol{L}_k$ is computed (on line) with every new measurement. The Kalman filter obtains the optimum gain (the one that minimizes the ℓ_2 norm of the output error) with every sample under the assumption of white noises of zero mean with known variance. With the Kalman filter it is necessary to run the equations of the state estimation and covariance estimation, and their respective update equations (with matrix inversions involved), and can have a very high computational cost when dealing with out-of-sequence measurements, especially if it is compared with the low online computational cost of the proposed methodology.

Remark 1. Defining the predictor gain matrix as in (27), the predictor error dynamics (17) can be written parametrically as

$$\tilde{\boldsymbol{z}}_k = \boldsymbol{\mathcal{A}}(s_k)\,\tilde{\boldsymbol{z}}_{k-1} + \boldsymbol{\mathcal{B}}(s_k)\mathbb{W}_k \tag{28a}$$
$$\tilde{\boldsymbol{y}}_k = \boldsymbol{C}_y\,\tilde{\boldsymbol{z}}_k, \tag{28b}$$

where

$$\boldsymbol{\mathcal{A}}(s_k) = (\boldsymbol{I} - \boldsymbol{L}(s_k)\,\boldsymbol{\alpha}(s_k)\,\boldsymbol{C}_d(s_k))\,\boldsymbol{A}^{N(s_k)}, \qquad \boldsymbol{C}_d(s_k) = \begin{bmatrix} \boldsymbol{c}_1\,\boldsymbol{A}^{-d_1(s_k)} \\ \vdots \\ \boldsymbol{c}_{n_m}\,\boldsymbol{A}^{-d_{n_m}(s_k)} \end{bmatrix}_{n_m\times n},$$

$$\boldsymbol{\mathcal{B}}(s_k) = \begin{bmatrix} -\boldsymbol{L}(s_k)\,\boldsymbol{\alpha}_k & \boldsymbol{\Lambda}(N(s_k)) - \boldsymbol{L}(s_k)\,\boldsymbol{\alpha}(s_k)\,\boldsymbol{\mathcal{C}}_d(s_k) \end{bmatrix}$$

and

$$\boldsymbol{\mathcal{C}}_d(s_k) = \begin{bmatrix} \boldsymbol{c}_1\,\boldsymbol{A}^{-d_1(s_k)}\,(\boldsymbol{\Lambda}(N(s_k)) - \boldsymbol{\Lambda}(d_1(s_k))) \\ \vdots \\ \boldsymbol{c}_{n_m}\,\boldsymbol{A}^{-d_{n_m}(s_k)}\,(\boldsymbol{\Lambda}(N(s_k)) - \boldsymbol{\Lambda}(d_{n_m}(s_k))) \end{bmatrix}_{n_m\times\beta n}.$$

With the introduction of parameter s_k, the error dynamics is represented as a linear time parametric varying system. As the parameter belongs to a finite set of previously known values (the possible sampling scenarios), the error dynamics behaves as a jump linear system and some known LMI techniques can be applied to obtain the set of gains $\boldsymbol{L}(s_k)$.

5. $\mathcal{H}_\infty$ disturbance attenuation based predictor design

Theorem 2 (Robust $\mathcal{H}_\infty$ performance). *Consider the predictor algorithm* (14) *applied to system* (1) *and assume that there is at least one available measurement every $N_k \in \mathcal{N}$ periods. Assume that the sensors availability in each sampling time is given by the matrix $\boldsymbol{\alpha}_k \in \Xi$ and that the associated delay to each sensor is on the set $d_{i,k} \in \mathcal{D}_i$, being $n_{\mathcal{S}}$ the number of possible $s_k \in \mathcal{S}$ sampling scenarios. For given $\gamma_{v_1}, \ldots, \gamma_{v_{n_m}}, \gamma_{u_1}, \ldots, \gamma_{u_{n_u}}, \gamma_w \in \mathbb{R}^+$, assume that there exist real matrices $\boldsymbol{P}(s_k) = \boldsymbol{P}(s_k)^T \succ \boldsymbol{0}$, $\boldsymbol{X}(s_k) \in \mathbb{R}^{n\times n_m}$ such that, for any trajectory $\{s_k\}$ of the sampling scenario*

$$\begin{bmatrix} \boldsymbol{P}(s_k) & \boldsymbol{M}_A(s_k) & \boldsymbol{M}_B(s_k) \\ \boldsymbol{M}_A(s_k)^T & \boldsymbol{P}(s_{k-1}) - \boldsymbol{C}_y^T\boldsymbol{C}_y & \boldsymbol{0} \\ \boldsymbol{M}_B(s_k)^T & \boldsymbol{0} & \boldsymbol{\Gamma} \end{bmatrix} \succeq 0, \qquad s_k, s_{k-1} \in \mathcal{S}\times\mathcal{S} \tag{29}$$

with

$$\boldsymbol{M}_A(s_k) = (\boldsymbol{P}(s_k) - \boldsymbol{X}(s_k)\boldsymbol{\alpha}(s_k)\,\boldsymbol{C}_d(s_k))\;\boldsymbol{A}^{N(s_k)}, \tag{30a}$$
$$\boldsymbol{M}_B(s_k) = \begin{bmatrix} -\boldsymbol{X}(s_k)\boldsymbol{\alpha}(s_k) & \boldsymbol{P}(s_k)\boldsymbol{\Lambda}(N(s_k)) - \boldsymbol{X}(s_k)\boldsymbol{\alpha}(s_k)\,\boldsymbol{\mathcal{C}}_d(s_k) \end{bmatrix} \tag{30b}$$

being

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_v \oplus (\gamma_w\boldsymbol{I}) \oplus \boldsymbol{\Gamma}_u \bigoplus_{i=1}^{(\beta-1)(n+n_u)} 0,$$
$$\boldsymbol{\Gamma}_v = \mathrm{diag}\{\gamma_{v_1}, \ldots, \gamma_{v_{n_m}}\}, \quad \boldsymbol{\Gamma}_u = \mathrm{diag}\{\gamma_{u_1}, \ldots, \gamma_{u_{n_u}}\}.$$

and matrix $\mathbf{\Lambda}(N(s_k))$ the one defined by (21).Then, defining the predictor gain as $\boldsymbol{L}(s_k) = \boldsymbol{P}(s_k)^{-1}\boldsymbol{X}(s_k)$, the prediction error of the algorithm defined by (14) converges asymptotically to zero in the absence of disturbances and, under zero initial condition, the output prediction error at the measuring instants is bounded by

$$\|\tilde{\boldsymbol{y}}_k\|^2_{RMS} < \sum_{i=1}^{n_m} \gamma_{v_i} \|v_{i,k}\|^2_{RMS} + \gamma_w \|\boldsymbol{w}_k\|^2_{RMS} + \sum_{i=1}^{n_u} \gamma_{u_i} \|\Delta u_{i,k}\|^2_{RMS} \tag{31}$$

where $\| \bullet \|_{RMS}$ stands for root mean square norm of a signal.

Proof. Introducing $\boldsymbol{X}(s_k) = \boldsymbol{P}(s_k)\boldsymbol{L}(s_k)$ in (29) and applying Schur complements one obtains

$$\begin{bmatrix} \begin{pmatrix} \boldsymbol{\mathcal{A}}(s_k)^T\boldsymbol{P}(s_k)\boldsymbol{\mathcal{A}}(s_k) \\ -\boldsymbol{P}(s_{k-1}) + \boldsymbol{C}_y^T\boldsymbol{C}_y \end{pmatrix} & \boldsymbol{\mathcal{A}}(s_k)^T\boldsymbol{P}(s_k)\boldsymbol{\mathcal{B}}(s_k) \\ \boldsymbol{\mathcal{B}}(s_k)^T\boldsymbol{P}(s_k)\boldsymbol{\mathcal{A}}(s_k) & \left(\boldsymbol{\mathcal{B}}(s_k)^T\boldsymbol{P}(s_k)\boldsymbol{\mathcal{B}}(s_k) - \mathbf{\Gamma}\right) \end{bmatrix} \prec \mathbf{0} \tag{32}$$

As $\boldsymbol{C}_y^T\boldsymbol{C}_y \succeq 0$, inequality (32) implies

$$\tilde{\boldsymbol{z}}_{k-1}^T \left(\boldsymbol{\mathcal{A}}(s_k)^T\boldsymbol{P}(s_k)\boldsymbol{\mathcal{A}}(s_k) - \boldsymbol{P}(s_{k-1})\right)\tilde{\boldsymbol{z}}_{k-1} < 0.$$

Assuming that there are no disturbances or measurement noise, using (28), the above expression leads to

$$\tilde{\boldsymbol{z}}_k^T\boldsymbol{P}(s_k)\tilde{\boldsymbol{z}}_k^T - \tilde{\boldsymbol{z}}_{k-1}^T\boldsymbol{P}(s_{k-1})\tilde{\boldsymbol{z}}_{k-1} \prec \mathbf{0},$$

which assures asymptotical convergence of the prediction error if the Lyapunov function $\mathcal{V}_k = \tilde{\boldsymbol{z}}_k^T\boldsymbol{P}(s_k)\tilde{\boldsymbol{z}}_k$ is defined.

Now, multiplying inequality (32) by $[\tilde{\boldsymbol{z}}_{k-1}\ \mathbb{W}_k]^T$ on the left, and by its transpose on the right, it leads

$$\tilde{\boldsymbol{z}}_k^T\boldsymbol{P}(s_k)\tilde{\boldsymbol{z}}_k - \tilde{\boldsymbol{z}}_{k-1}^T\boldsymbol{P}(s_{k-1})\tilde{\boldsymbol{z}}_{k-1} + \tilde{\boldsymbol{y}}_{k-1}^T\tilde{\boldsymbol{y}}_{k-1} - \mathbb{W}_k^T\mathbf{\Gamma}\mathbb{W}_k < 0,$$

where the predictor dynamic error (28) has been taken into account. Assuming a null initial prediction error ($\tilde{\boldsymbol{z}}_0 = \mathbf{0}$) and adding from $k=1$ to $k=K$ it leads to

$$\tilde{\boldsymbol{z}}_K^T\boldsymbol{P}(s_K)\tilde{\boldsymbol{z}}_K + \sum_{k=1}^{K}\left(\tilde{\boldsymbol{y}}_{k-1}^T\tilde{\boldsymbol{y}}_{k-1} - \mathbb{W}_k^T\mathbf{\Gamma}\mathbb{W}_k\right) < 0. \tag{33}$$

As $\boldsymbol{P}(s_k) \succ 0$, then $\tilde{\boldsymbol{x}}_K^T\boldsymbol{P}(s_K)\tilde{\boldsymbol{x}}_K > \mathbf{0}$, leading to

$$\sum_{k=1}^{K}\left(\boldsymbol{y}_{k-1}^T\tilde{\boldsymbol{y}}_{k-1} - \mathbb{W}_k^T\mathbf{\Gamma}\mathbb{W}_k\right) < 0. \tag{34}$$

Introducing the definitions of $\mathbf{\Gamma}$ and $\mathbb{W}_k$, and taking into account that the elements of the disturbance vector are not correlated, it can be written that

$$\sum_{k=1}^{K}\left(\tilde{\boldsymbol{y}}_{k-1}^T\tilde{\boldsymbol{y}}_{k-1} - \sum_{i=1}^{m_m}\gamma_{v_i}v_{i,k}^2 - \gamma_w\boldsymbol{w}_k^T\boldsymbol{w}_k + \sum_{i=1}^{n_u}\gamma_{u_i}\Delta u_{i,k}^2\right) < 0.$$

Dividing by K and taking limit when K tends to ∞, the RMS norm of the signals is obtained, and then (31) follows.

Remark 2. If the previous LMI problem has a feasible solution, then the system is detectable for the considered sampling scenarios. On the other hand, if it is unfeasible, then the system may be non detectable for some of the sampling scenarios of the considered set. This will happen for those scenarios where the number of measured outputs is lower than the unavailable inputs. Those cases, as far as they do not contain enough information to improve the estimation of the extended state, must be removed from the set of possible scenarios. However the acquired information in that sampling instant should not be discarded, but instead, it should be merged with the next available sampling data. The fact of removing those scenarios may have an impact on the maximum bound on N_k and $d_{i,k}$ to be taken into account in the estimator design.

Remark 3. If those non detectable scenarios are removed, then, the previous linear matrix inequalities are a sufficient condition to achieve a given RMS gain and to assure asymptotical stability. As the sampling modeling presented in this work leads to a discrete-time switched linear system, the condition presented here is not necessary for asymptotical stability (as shown in [7]), but is a necessary condition for polyquadratic stability, and, therefore, is less conservative than establishing quadratic stability with a unique matrix P for the Lyapunov stability assessment.

Remark 4. If the RMS norms of disturbances, RMS norm of the derivative of each input, and noise measurement of each sensor are assumed to be known, it is possible to minimize the upper bound on $\|\tilde{\boldsymbol{y}}_k\|_{RMS}$ minimizing the sum

$$\sum_{i=1}^{n_m} \gamma_{v_i} \|v_{i,k}\|_{RMS}^2 + \gamma_w \|\boldsymbol{w}_k\|_{RMS}^2 + \sum_{i=1}^{n_u} \gamma_{u_i} \|\Delta u_{i,k}\|_{RMS}^2$$

along all variables γ_{v_i} $(i = 1, \ldots, n_m)$, γ_w, γ_{u_i} $(i = 1, \ldots, n_u)$ $\boldsymbol{P}(s_k)$ and $\boldsymbol{X}(s_k)$ $(s_k = 1, \ldots, n_{\mathcal{S}})$ that satisfy the LMI (29).

Remark 5. In the previous theorem, different matrices $\boldsymbol{X}(s_k)$ and $P(s_k)$ have been assumed for every s_k. This would lead to a set of $n_{\mathcal{S}}$ different predictor gains that should be stored in order to apply the correct one depending on the values of s_k. The theorem, however, is also valid if some restrictions are imposed on the matrices $\boldsymbol{X}(s_k)$ and $\boldsymbol{P}(s_k)$ in order to reduce the number of gains to be stored. The purpose is to make the gain robust against the variation of N_k, $\boldsymbol{\alpha}_k$ and D_k.

The most general case can be defined as follows. Let us divide the total set of possible delays, set of available sensors and inter measuring periods, $\mathcal{S}$, into r disjoint subsets, $\mathcal{S}_i$, $i = 1, \ldots, r$ and define r different matrices $\boldsymbol{P}(i)$ and $\boldsymbol{X}(i)$, $i = 1, \ldots, r$ such that $\boldsymbol{P}(s_k) = \boldsymbol{P}(i)$ and $\boldsymbol{X}(s_k) = \boldsymbol{X}(i)$ if $s_k \in \mathcal{S}_i$. As a result, a reduced set of r gain matrices $\boldsymbol{L}(i) = \boldsymbol{P}(i)^{-1}\boldsymbol{X}(i)$, $i = 1, \ldots, r$ must be stored. Which gain will be used in a given measuring instant will depend on the subset $\mathcal{S}_i$, that the set $(N_k, \boldsymbol{\alpha}_k, D_k)$ belongs to. The drawback of imposing this restriction is that the achieved performance will decrease as the number of subsets is reduced. The number of gains to be stored can be used to find a compromise between implementation resources and performance.

An interesting particular case (that leads to the simplest predictor algorithm) is defined when there is only one subset, i.e. two constant matrices $\boldsymbol{P}(i) = \boldsymbol{P}$

and $\boldsymbol{X}(i) = \boldsymbol{X},\ i = 1, \ldots, n_{\mathcal{S}}$ are used in the LMI set, leading to a constant gain $\boldsymbol{L} = \boldsymbol{P}^{-1}\boldsymbol{X}$.

Remark 6. One approach to reduce the number of gain matrices to be stored, but maintaining a performance that approaches the optimum, consists of defining the gain $\boldsymbol{L}(s_k)$ as a function of some parameters related to the sampling scenario, in such a way that the function can be included in the LMIs to be solved. The idea is to use a function that somehow approximates the values of the gains $\boldsymbol{L}(s_k)$ that are obtained solving the full optimization problem (with $n_{\mathcal{S}}$ different gains). For that purpose, the following general function is proposed:

$$\boldsymbol{L}(s_k) = \sum_{i=0}^{q} \boldsymbol{L}_i f_i(N_k, D_k), \tag{35}$$

where $\boldsymbol{L}_i \in \mathbb{R}^{n,n_m},\ \ i = 0, \ldots, q$ are the gains that must be stored. The inclusion of the function (35) in the LMIs to be solved is straightforward, as it simply consists of using $\boldsymbol{P}(s_k) = \boldsymbol{P}$ and $\boldsymbol{X}(s_k) = \boldsymbol{X}\boldsymbol{M}(N_k, D_k)$ with

$$\boldsymbol{M}(N_k, D_k) = [f_0(N_k, D_k), f_1(N_k, D_k), \cdots, f_q(N_k, D_k)]^T \tag{36}$$

leading to the gains defined by equation (35) with $[\boldsymbol{L}_0, \boldsymbol{L}_1, \cdots, \boldsymbol{L}_q] = \boldsymbol{P}^{-1}\boldsymbol{X}$. The higher the value of q, the larger implementation memory required and the better achieved performance. A simple type of functions that can be used are polynomial type. For example, the following functions can be used

$$\begin{aligned} f_0(N_k, D_k) &= \boldsymbol{I}, & f_1(N_k, D_k) &= N_k\boldsymbol{I}, \\ f_2(N_k, D_k) &= diag\{d_{1,k}, \cdots, d_{n_m,k}\}, & f_3(N_k, D_k) &= N_k^2\boldsymbol{I}, \\ f_4(N_k, D_k) &= diag\{d_{1,k}^2, \cdots, d_{n_m,k}^2\}, & f_5(N_k, D_k) &= N_k diag\{d_{1,k}, \cdots, d_{n_m,k}\}. \end{aligned} \tag{37}$$

For most of the plants that have been tested, taking $q = 5$ with the previous functions results in a reasonable fit of the optimal gains, leading to a performance that is similar to the optimum one, but with a much lower computer memory requirement. Taking $q = 2$ results in a lower computer cost algorithm, but a worse performance. An example illustrates this compromise between cost and performance in the next section.

Remark 7. The proposed approach can also be used to solve the inverse problem, that is, given a maximum allowable bound of the estimation error norm, $\tilde{y}^2_{\max}$, determine the worst sampling scenario for which it is possible to assure that $\|\tilde{\boldsymbol{y}}_k\|^2_{RMS} < \tilde{y}^2_{\max}$. One possibility is to assume that the set of possible measurement delays is fixed, and to calculate the widest set of possible values of N_k, $\nu_{\mathcal{N},max}$, for which the above condition holds.

The problem could be solved with the following algorithm:

1. Start with $\nu_{\mathcal{N}} = 1$, i.e. $\mathcal{N} = \{1\}$
2. Construct the different sampling scenarios s_k for the given set $\mathcal{N}$
3. Solve the LMI problem with the additional constraint

$$\sum_{i=1}^{n_m} \gamma_{v_i} \|v_{i,k}\|^2_{RMS} + \gamma_w \|\boldsymbol{w}_k\|^2_{RMS} + \sum_{i=1}^{n_u} \gamma_{u_i} \|\Delta u_{i,k}\|^2_{RMS} < \tilde{y}^2_{\max}$$

4. If it is feasible, $\nu_{\mathcal{N}} = \nu_{\mathcal{N}} + 1$, $\mathcal{N} = \mathcal{N} \cup \{\nu_{\mathcal{N}}\}$ and go to step 2.
5. If it is not feasible, $\mathcal{N}$ is the widest sampling scenario for which the estimator assures $\|\tilde{\boldsymbol{y}}_k\|^2_{RMS} < \tilde{y}^2_{\max}$.

In the next section, an example illustrates the idea.

6. Examples

6.1. Example 1

Let us consider a discrete time SISO system defined by model (1) with matrices

$$\boldsymbol{A}_x = \begin{bmatrix} 0 & 1 \\ -0.5478 & 1.4981 \end{bmatrix}, \quad \boldsymbol{B}_u = \begin{bmatrix} -0.2796 \\ -0.2571 \end{bmatrix}, \quad \boldsymbol{C}_{xy} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{uy} = \begin{bmatrix} 0 \end{bmatrix}.$$

A state disturbance with a RMS norm $\|w_i[t]\|_{RMS} = 0.01$ $(i = 1, 2)$is assumed. The input of the plant is assumed to vary slowly on time, in such a way that its variations between input updates are bounded by $\|\Delta u[t]\|_{RMS} \leq 1$. The measurements of the output of the system are corrupted by a norm bounded noise with $\|v[t]\|_{RMS} = 0.1$. It is assumed that both the sensor measurements and applied inputs are received by the estimator through a network, resulting in a delay that can vary from 1 to 5 periods. It is also assumed that some of that values are lost due to packet dropout. As a result, the number of inter-sampling periods between consecutive data reception is assumed to belong to the set $N_k \in \{2, \ldots, 8\}$. Four strategies in the estimator gain definition are analyzed for comparison purposes:

A1 A single constant gain.

A2 A different gain for every value of the sampling scenario parameter, leading to $n_{\mathcal{S}}$ matrix gains to be stored.

A3 A gain that depends linearly on the inter-sampling period N_k, and on the delays $d_{1,k}$ and $d_{2,k}$ (i.e., expression (37) with $q = 2$, leading to 3 matrix gains to be stored).

A4 A gain that depends quadratically on N_k and $d_{i,k}$ (i.e., expression (37) with $q = 5$, leading to 6 gains to be stored).

The bound on the state estimation error achieved with each strategy (by means of the optimization problem presented in the previous section) is summarized on table 1, where the number of matrices to be stored, as well as the obtained γ_v, γ_w, γ_u during the optimization is also shown. It has been also added the resulting root mean square state estimation errors during a simulation of 10^6 samples for both the sampling instants ($\|\tilde{\boldsymbol{x}}_k\|_{RMS}$) and the control periods ($\|\tilde{\boldsymbol{x}}[t]\|_{RMS}$), including the results that are obtained if a Kalman filter is implemented with an extended state approach to include the delayed measurements [10]. The best compromise between performance and resources is in this case the strategy $A4$, because a very similar performance to that of strategy $A2$ is achieved, while the number of gains to be stored is much lower (6 instead of 35). The Euclidean norm of the matrix gain to be applied is shown in Figure 1 as a function of the value of the sampling scenario, for the four strategies. Finally, the initial transient for the state estimation is shown in Figure 2 when using approaches $A1$ and $A2$. The $A2$ approach results in a better performance.

Strategy	A1	A2	A3	A4	Kalman
Number of matrices	1	35	3	6	-
$\Vert\tilde{\boldsymbol{y}}_k\Vert_{RMS}$	1.8253	1.5733	1.6696	1.5855	-
γ_v	10.0855	10.8730	11.6070	10.9791	-
$10^{-3}\cdot\gamma_w$	3.6692	3.9066	4.1850	3.7830	-
γ_u	2.8638	1.9759	2.2529	2.0257	-
$\Vert\tilde{\boldsymbol{x}}_k\Vert_{RMS}$	1.3629	1.1045	1.1484	1.1234	1.0867
$\Vert\tilde{\boldsymbol{x}}[t]\Vert_{RMS}$	2.7813	2.5707	2.6360	2.6163	2.5627

Table 1: Performance, resources and simulation results comparison for Example 1.

6.2. Example 2

Let us consider a MIMO unstable plant described by matrices

$$\boldsymbol{A}_x=\begin{bmatrix}0.7 & 0 & 0.5\\ 0 & 1.1 & 0.8\\ 0 & 0 & 1.5\end{bmatrix},\qquad \boldsymbol{B}_u=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix},\quad \boldsymbol{C}_{xy}=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 0\end{bmatrix},\quad \boldsymbol{D}_{uy}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

where a state disturbance with norm $\Vert\boldsymbol{w}[t]\Vert_{RMS}=0.01$ is assumed. In this case, the inputs of the plant are assumed to be known without delays (the estimator is assumed to be implemented in the controller-actuator node). However, the two measured outputs are assumed to be acquired through a network with varying delay and packet dropout, and corrupted by noise measurements with norms $\Vert v_1[t]\Vert_{RMS}=0.1$ and $\Vert v_2[t]\Vert_{RMS}=0.01$. The delays are assumed to vary between 1 and 3, i.e., $d_{1,k}\in\{1,2,3\}$ and $d_{2,k}\in\{1,2,3\}$. In this example, the analysis is focused on finding the maximum allowable packet dropout for which a stable state estimator can be found. The set of possible inter-sampling periods is denoted as $N_k=\{1,\ldots,\bar{N}\}$. The idea is then to look for the maximum $\bar{N}$ for which a finite bound on $\Vert\tilde{\boldsymbol{x}}[t]\Vert_{RMS}$ can be found for each one of the four strategies considered in Example 1.

The algorithm described in Remark 7 is run for the four strategies using a maximum admisible bound of $\Vert\tilde{\boldsymbol{x}}_k\Vert^2\leq 10$. The constant gain approach A1 only assures the bound for $\bar{N}=1$ (standard sampling), and results in a stabilizing estimator only until $\bar{N}=2$, although for that value, a very poor performance is achieved ($\Vert\tilde{\boldsymbol{x}}_k\Vert^2_{RMS}=75.7$). For the third strategy, A3, a stabilizing estimator is found for a maximum value $\bar{N}=7$, while the desired performance is only assured for a value $\bar{N}=6$. A maximum of $\bar{N}=10$ has been found for strategy A4, with the quadratic approximation. With the strategy A2, the estimator assures the desired bound for values larger than $\bar{N}=12$. The different strategies are compared in Table 2, that shows the estimation error bound for the stable cases as a function of the size of the inter-sampling periods set, $\bar{N}$. The worst performance is achieved for strategy A1. The approach A2 leads to the lower estimation error, followed by A4 and A3. The drawback of strategy A2 is that the number of gains to be stored is $n_{\mathcal{S}}=9\cdot\bar{N}$, that can be a high value for large $\bar{N}$.

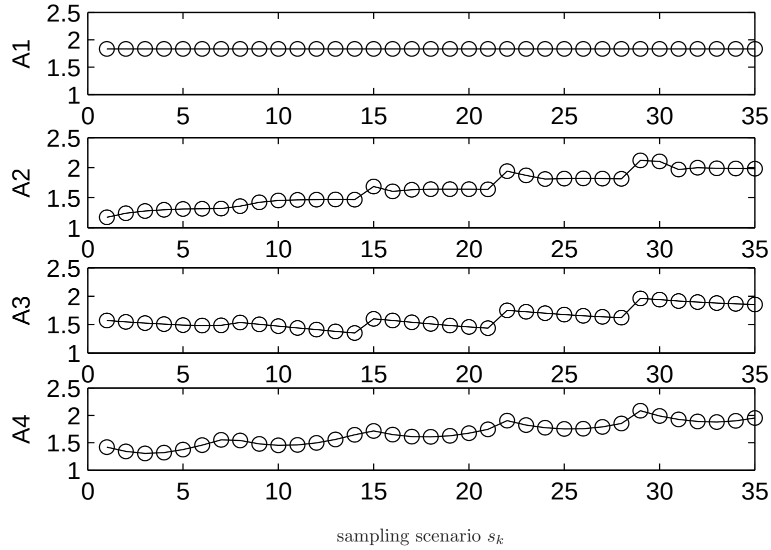

Figure 1: Values of the Euclidean norm of the gain to be applied as a function of the sampling scenario in Example 1

7. Conclusions

In this work, the design of a signal estimation algorithm for multisensor systems under scarce, delayed and irregularly time-spaced out-of-sequence networked measurements has been addressed. The estimation algorithm uses the internal model of the system, the scarce and delayed sensor measurements and its availability to predict the desired signals at a desired constant period. The number of periods between consecutive measurements (N_k), the delay associated to each sensor measurement ($d_{i,k}$) and the possible combinations of sensors availabilities ($\mathbf{\Delta}_k$) are assumed to be time-varying values that belong to known finite sets and that are known at the instant when the measurements are received.

The main contribution is a procedure for designing the gains of a low computing cost algorithm, that guarantees stability and performance against disturbances, despite the non periodic data availability pattern, the measurement delays, the partial availability of sensors and the time disordered data reception.

The attenuation of disturbances and measurement noises has been taken into account by $\mathcal{H}_\infty$ performance. A predictor design procedure has been proposed based on the available information about the disturbances in order to minimize the norm of the prediction error signals. The result is a finite set of predictor gain matrices that are applied depending on the characteristics of the last measurement ($\mathbf{\Delta}_k$, D_k and N_k) (as a difference with other approaches in the literature, where a constant gain is proposed).

The online computer implementation cost is much lower than the Kalman filter based approaches, at the expense of a lower (suboptimal) performance, but the

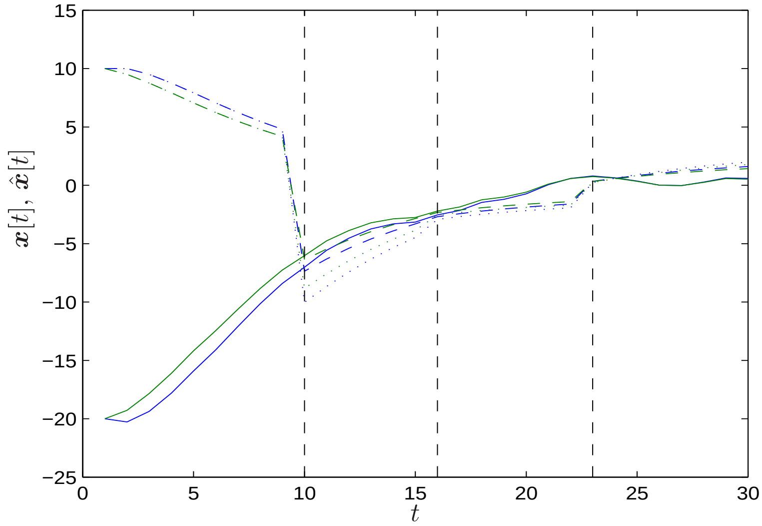

Figure 2: Transient detail for the state and state estimation with strategy A1 (dotted), and A2 (dashed). The instants of time in which measurements are acquired are marked by vertical lines

achieved performance is still much better than the constant gain approaches. In this sense, the proposed approach represents a compromise between performance and computing cost that lies in between the Kalman filter based approaches and the constant gain approaches. In order to reduce the number of precalculated gains that must be stored, a polynomial type function has also been proposed as an alternative to define the gains as a function of the sampling parameters. This idea allows to reach a compromise between online implementation resources and performance. The approach can also be used to solve the inverse problem, i.e. to determine the worst sampling scenario for which the estimation error can be assured to be below a predefined value.

Acknowledgements

This work has been granted by the CICYT project number DPI2008-06731-C02-02/DPI.

[1] P. Albertos, R. Sanchis, and A. Sala. Output prediction under scarce data operation: Control applications. *Automatica*, 35:1671–1681, 1999.

[2] J.C. Geromel, J. Bernussou, G. Garcia, and M.C de Oliveira. H_2 and H_∞ robust filtering for discrete-time linear systems. *SIAM J. Control Optim.*, 38(5):1353–1368, 2000.

[3] B. Halder, B. Hassibi, and T. Kailath. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ estimation: Preliminary

N	A1	A2	A3	A4
1	5.903	1.203	1.483	1.366
2	75.707	1.316	1.677	1.475
3	-	1.474	2.005	1.676
4	-	1.563	2.413	1.774
5	-	1.632	3.155	1.876
6	-	1.684	4.957	1.998
7	-	1.723	15.887	2.164
8	-	1.752	-	2.385
9	-	1.773	-	2.723
10	-	1.789	-	4.796
11	-	1.800	-	-
12	-	1.811	-	-

Table 2: Achievable bound on estimation error as a function of the maximum inter-sampling periods, for the different gain scheduling strategies

analytic characterization and a numerical solution. *Proceedings of the 1996 IFAC World Congress*, J:37–42, 1996.

[4] B. Hassibi and T. Kailath. On nonlinear filters for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ estimation. *Proceedings of the American Control Conference*, 1997.

[5] X. He, Z. Wang, and D. Zhou. Robust $\mathcal{H}_\infty$ filtering for networked systems with multiple state delays. *International Journal of Control*, 80(8):1217–1232, 2007.

[6] T. D. Larsen, N. A. Andersen, O. Ravn, and N. K. Poulsen. Incorporation of time delayed measurements in a discrete-time kalman filter. *In Proceeding ofIEEE 37th Conference of Decision and Control*, pages 3972–3977, December 1998.

[7] H. Lin and P.J Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2), 2009.

[8] M. Moayedi, Y. K. Foo, and Y. C. Soh. Filtering for networked control systems with single/multiple measurement packets subject to multiple-step measurement delays and multiple packet dropouts. *International Journal of Systems Science*, 42(3):335–348, 2011.

[9] R.M. Palhares and P.L.D. Peres. Robust filtering with guaranteed energy-to-peak performance – an LMI approach. *Automatica*, 36:851–858, 2000.

[10] I. Peñarrocha, R. Sanchis, and J.A. Romero. State estimator for multisensor systems with irregular sampling and time-varying delays. *International Journal of Systems Science*, (iFirst):1–13, 2011.

[11] M. Sahebsara, T. Chen, and S. L. Shah. Optimal $\mathcal{H}_2$ filtering in networked control systems with multiple packet dropout. *IEEE Transactions on Automatic Control*, 52(8):1508–1513, 2007.

[12] R. Sanchis and P. Albertos. Virtual sensors: model based output prediction. *First EurAsia Conference on Advances in Information and Communication Technology*, 2002.

[13] R. Sanchis, I. Peñarrocha, and P. Albertos. Design of robust output predictors under scarce measurements with time-varying delays. *Automatica*, 43:281–289, 2007.

[14] L. Schenato. Optimal estimation in networked control systems subject to random delay and packet drop. *IEEE Transactions on Automatic Control*, 53(5):1311–1317, 2008.

[15] H. Song, L. Yu, and W. . Zhang. Networked $\mathcal{H}_\infty$ filtering for linear discrete-time systems. *Information Sciences*, 181(3):686–696, 2011.

[16] J. Wang, C. Wang, and H. Gao. Robust $\mathcal{H}_\infty$ filtering for LPV discrete-time state-delayed systems. *Nature and Science*, 2(2):36–45, 2004.

[17] Z. Wang, B. Huang, and H. Unbehauen. Robust $\mathcal{H}_\infty$ observer design of linear state delayed systems with parametric uncertainty: the discrete-time case. *Automatica*, 35:1161–1167, 1999.

[18] Y. Xia, J. Shang, J. Chen, and G. . Liu. Networked data fusion with packet losses and variable delays. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 39(5):1107–1120, 2009.

[19] L. Xie, L. Lu, D. Zhang, and H. Zhang. Improved robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering for uncertain discrete-time systems. *Automatica*, 40:873–880, 2004.

[20] S. Xu and T. Chen. Robust $\mathcal{H}_\infty$ filtering for uncertain impulsive stochastic systems under sampled measurements. *Automatica*, 39:509–516, 2003.

[21] F. Yang, Z. Wang, Y. S. Hung, and M. Gani. $\mathcal{H}_\infty$ control for networked systems with random communication delays. *IEEE Transactions on Automatic Control*, 51(3):511–518, 2006.

[22] H. Zhang, M.V. Basin, and M. Skliar. Optimal state estimation with continuous, multirate and randomly sampled measurements. *Proceedings of the American Control Conference*, 2004.