BILINEAR ISOMETRIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. Let $X, Y, Z$ be compact Hausdorff spaces and let $E_1, E_2, E_3$ be Banach spaces. If $T: C(X, E_1) \times C(Y, E_2) \to C(Z, E_3)$ is a bilinear isometry which is stable on constants and $E_3$ is strictly convex, then there exists a nonempty subset $Z_0$ of $Z$, a surjective continuous mapping $h: Z_0 \to X \times Y$ and a continuous function $\omega: Z_0 \to \text{Bil}(E_1 \times E_2, E_3)$ such that

$$T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z)))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$. This result generalizes the main theorems in [2] and [6].

1. Introduction.

Let $X$ be a compact Hausdorff space and $E$ a Banach space. Let $C(X)$ (resp. $C(X, E)$) denote the Banach spaces of all continuous scalar-valued (resp. vector-valued) functions on $X$ endowed with the supremum norm, $\|\cdot\|_\infty$. A bilinear mapping $T: C(X) \times C(Y) \to C(Z)$ which satisfies

$$\|T(f, g)\|_\infty = \|f\|_\infty\|g\|_\infty$$

for every $(f, g) \in C(X) \times C(Y)$ is called a bilinear isometry.

In [6], Moreno and Rodriguez proved the following bilinear version of the well-known Holsztyński’s Theorem on non-surjective linear isometries of $C(X)$-spaces ([5] and, also, [1]):

Let $T: C(X) \times C(Y) \to C(Z)$ be a bilinear isometry. Then there exist a closed subset $Z_0$ of $Z$, a surjective continuous mapping $h: Z_0 \to X \times Y$ and a norm-one continuous function $a \in C(Z)$ such that $T(f, g)(z) = a(z)f(\pi_X(h(z)))g(\pi_Y(h(z)))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$. The proof of this result rests heavily on the powerful Stone-Weierstrass Theorem. In [3], the authors extend...
these results to certain subspaces of continuous scalar-valued functions, where Stone-Weierstrass Theorem is not applicable.

The concept of bilinear isometry can be naturally extended to the context of spaces of vector-valued continuous functions. Examples of bilinear isometries defined on these spaces can be found, for instance, in [7, Proposition 5.2], where the author provide certain compact spaces $X$ and Banach spaces $E$ for which there exists a bilinear isometry $T : C(X, E) \times C(X, E) \to C(Y, E)$.

In this paper we study the conditions under which we can obtain a representation of such bilinear isometries on this vector-valued setting. Thus, given three Banach spaces $E_1$, $E_2$ and $E_3$, we prove that if $T : C(X, E_1) \times C(Y, E_2) \to C(Z, E_3)$ is a bilinear isometry which is stable on constants (see Definition 3) and $E_3$ is strictly convex, then there exists a nonempty subset $Z_0$ of $Z$, a surjective continuous mapping $h : Z_0 \to X \times Y$ and a continuous function $\omega : Z_0 \to \text{Bil}(E_1 \times E_2, E_3)$ such that

$$T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z)))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$.

It can be easily checked that this result contains the main theorems in [6] and in [2] (see the concluding remarks at the end of the paper).

2. Notation and previous lemmas.

Let $E$ be a Banach space and let $S_E$ denote the unit sphere of $E$.

For any $e \in E$, we denote by $\tilde{e}$ the element of $C(X, E)$ which is constantly equal to $e$. For any $x \in X$ and $e \in S_E$, let

$$C_{x,e} := \{ f \in C(X, E) : 1 = \|f\|_\infty \text{ and } f(x) = e \}.$$  

We shall write $\text{Bil}(E_1 \times E_2, E_3)$ to denote the space of jointly continuous bilinear mappings between $E_1 \times E_2$ and $E_3$ endowed with the strong operator topology.

In the sequel we shall assume that $T : C(X, E_1) \times C(Y, E_2) \to C(Z, E_3)$ is a bilinear mapping which satisfies

$$\|T(f, g)\|_\infty = \|f\|_\infty \|g\|_\infty$$

for every $(f, g) \in C(X, E_1) \times C(Y, E_2)$, which is to say that $T$ is bilinear isometry.

**Lemma 1.** Assume $(x, y) \in X \times Y$ and $(e, e') \in S_{E_1} \times S_{E_2}$. The set

$$I_{x,y,e,e'} := \{ z \in Z : 1 = \|T(f, g)\|_\infty = \|(T(f, g)(z)), (f, g) \in C_{x,e} \times C_{y,e'} \}$$

is nonempty.
Proof. For any $f \in C(X, E_1)$ and $g \in C(Y, E_2)$, let us define the following compact subset of $Z$: $M_{f,g} := \{ z \in Z : \|T(f, g)(z)\| \geq \frac{1}{2} \}$. It is apparent that $I_{x,y,e,e'}$ is a closed subset of $M_{f,g}$. Hence, in order to prove that $I_{x,y,e,e'}$ is nonempty, it suffices to check that if $f_1, ..., f_n$ belong to $C_{x,e}$ and $g_1, ..., g_n$ belong to $C_{y,e'}$, then

$$\bigcap_{i,j}\{z \in Z : 1 = \|T(f_i, g_j)(z)\| = \|(T(f_i, g_j)(z))\| \neq \emptyset.$$ 

Let $f_0 \in C(X, E_1)$ and $g_0 \in C(Y, E_2)$ defined as follows:

$$f_0 := \sum_{i=1}^{n} f_i \text{ and } g_0 := \sum_{j=1}^{n} g_j.$$ 

It is clear that $\|f_0(x)\| = n = \|f_0\|_{\infty}$ and $\|g_0(y)\| = n = \|g_0\|_{\infty}$.

Hence, $\|T(f_0, g_0)\|_{\infty} = \|f_0\|_{\infty} \cdot \|g_0\|_{\infty} = n^2$ since $T$ is a bilinear isometry and, consequently, there exists $z_0 \in Z$ such that

$$n^2 = \|T(f_0, g_0)(z_0)\| = \left\| \sum_{i,j} T(f_i, g_j)(z_0) \right\| \leq \sum_{i,j} \|T(f_i, g_j)(z_0)\| \leq n^2.$$ 

This fact yields $\|T(f_i, g_j)(z_0)\| = 1$ for all $i, j$, which is to say that

$$z_0 \in \bigcap_{i,j}\{z \in Z : 1 = \|T(f_i, g_j)(z)\| = \|(T(f_i, g_j)(z))\| \}. \quad \square$$

Lemma 2. Assume $E_3$ is strictly convex and fix $(x_0, y_0) \in X \times Y$ and $(e, e') \in S_{E_1} \times S_{E_2}$.

1. If $f(x_0) = 0$ for some $f \in C(X, E_1)$ and $g' \in C_{y_0,e'}$, then $T(f, g')(z) = 0$ for all $z \in I_{x_0,y_0,e,e'}$.  
2. If $g(y_0) = 0$ for some $g \in C(Y, E_2)$ and $f' \in C_{x_0,e}$, then $T(f', g)(z) = 0$ for all $z \in I_{x_0,y_0,e,e'}$.

Proof. 1) Let us choose $z_0 \in I_{x_0,y_0,e,e'}$. Define a linear isometry $T' : C(X, E_1) \to C(Z, E_3)$ as $T'(f) := T(f, g')$.

We shall first check that if $f \in C(X, E_1)$ vanishes on an open neighborhood, $U$, of $x_0$, then $(T'f)(z_0) = 0$. With no loss of generality, we shall assume that $\|f\|_{\infty} = 1$.

Let us take $\xi \in C(X)$ such that $1 = |\xi(x_0)| = \|\xi\|_{\infty}$ and such that its support is included in $U$. We can now define two functions in $C(X, E_1)$ as follows:

$$g := f + \xi e$$
and
\[ h := \frac{1}{2}(g + \xi e). \]
It is clear that \( g(x_0) = h(x_0) = \xi(x_0)e \) and that \( \|\xi e\|_\infty = \|g\|_\infty = \|h\|_\infty = 1 \). Therefore, since \( z_0 \in I_{x_0,y_0,e,e'} \), then
\[ \|T'(\xi e)(z_0)\| = \|T'(g)(z_0)\| = \|T'(h)(z_0)\| = 1. \]
Now, as \( T'(h)(z_0) \) is on the segment which joins \( T'(\xi e)(z_0) \) and \( T'(g)(z_0) \), the strict convexity of \( E \) yields \( T'(\xi e)(z_0) = T'(g)(z_0) \), which is to say that \( T'(f)(z_0) = 0 \).

Let us now define two linear functionals on \( C(X, E_1) \) as follows:
\[ \hat{T}'z_0(f) := T'(f)(z_0) \]
and \( \hat{x}_0(f) := f(x_0) \). It is not hard to check that the functions in \( C(X, E_1) \) which vanish on a neighborhood of \( x_0 \) are dense in the kernel of \( \hat{x}_0 \), \( \ker(\hat{x}_0) \), which is closed due to the continuity of this functional. Consequently, the above paragraph yields the inclusion \( \ker(\hat{x}_0) \subseteq \ker(\hat{T}'z_0) \); that is, if \( f(x_0) = 0 \), then \( T'(f)(z_0) = 0 \), as was to be proved.

(2) The proof of (2) is similar to (1). \( \square \)

**Definition 2.** For any pair \( (x, y) \in X \times Y \), we define the set
\[ I_{x,y} := \bigcup_{(e,e') \in S_{E_1} \times S_{E_2}} I_{x,y,e,e'}. \]

**Lemma 3.** Assume \( E_3 \) is strictly convex. Let \( (x_0, y_0) \in X \times Y \) and suppose that there exist \((\hat{f}, \hat{g}) \in C(X, E_1) \times C(Y, E_2)\) which vanish on \( x_0 \) and \( y_0 \) respectively. Then \( T(\hat{f}, \hat{g})(z) = 0 \) for all \( z \in I_{x_0,y_0} \).

**Proof.** Assume first that there exist \((f, g) \in C(X, E_1) \times C(Y, E_2)\) which vanish on certain neighborhoods, \( U \) and \( V \), of \( x_0 \) and \( y_0 \) respectively. Then we claim that \( T(f, g)(z) = 0 \) for all \( z \in I_{x_0,y_0} \).

To this end, fix \( z_0 \in I_{x_0,y_0} \). Then \( z_0 \in I_{x_0,y_0,e,e'} \) for some \((e, e') \in S_{E_1} \times S_{E_2} \). Assume, with no loss of generality, \( \|f\|_\infty \leq 1 \) and \( \|g\|_\infty \leq 1 \).

Let us consider \((f_1, g_1) \in C(X) \times C(Y)\) such that \( \text{supp}(f_1) \subseteq U \) and \( \text{supp}(g_1) \subseteq V \), and \( 1 = \|f_1\|_\infty = f_1(x_0) \) and \( 1 = \|g_1\|_\infty = g_1(y_0) \).

It is then clear that \( \|f + f_1e\|_\infty = \|f(x_0) + f_1(x_0)e\| = \|e\| = 1 \) and \( \|g + g_1e'\|_\infty = \|g(y_0) + g_1(y_0)e'\| = \|e'\| = 1 \). Consequently, since \( z_0 \in I_{x_0,y_0,e,e'} \),
\[ \|T(f + f_1e, g + g_1e')(z_0)\| = 1, \]
\[ \|T(f_1e, g_1e')(z_0)\| = 1 \]
and
\[ \left\| T\left( \frac{f}{2} + f_1e, g + g_1e' \right) (z_0) \right\| = 1. \]
On the other hand, by Lemma 2, we know that $T(f, g_1 e')(z_0) = T(f_1 e, g)(z_0) = 0$. Therefore

$$\frac{T(f + f_1 e, g + g_1 e')(z_0) + T(f_1 e, g_1 e')(z_0)}{2} = \frac{T(f, g)(z_0)}{2} + T(f_1 e, g_1 e')(z_0) = T\left(\frac{f}{2} + f_1 e, g + g_1 e'\right)(z_0).$$

This means that $T\left(\frac{f}{2} + f_1 e, g + g_1 e'\right)(z_0)$ is on the segment which joins $T(f + f_1 e, g + g_1 e')(z_0)$ and $T(f_1 e, g_1 e')(z_0)$. Hence, since $E_3$ is strictly convex, $T(f + f_1 e, g + g_1 e')(z_0)$ and $T(f_1 e, g_1 e')(z_0)$ coincide, which is to say, again by Lemma 2, that $T(f, g)(z_0) = 0$.

Let us now take a sequence $(f_n) \in C(X, E_1)$ convergent to $\tilde{f}$ and such that $f_n \equiv 0$ on a certain neighborhood $U_n$ of $x_0$. Similarly, take a sequence $(g_n) \in C(Y, E_2)$ convergent to $\tilde{g}$ and such that $g_n \equiv 0$ on a certain neighborhood $V_n$ of $y_0$. Fix $z_0 \in I_{x_0, y_0}$. Then we can define a linear functional on $C(X, E_1) \times C(Y, E_2)$ as follows: $T_{z_0}(f, g) := T(f, g)(z_0)$. It is apparent, from the above paragraph, that $T_{z_0}(f_n, g_n) = 0$ for all $n \in N$. On the other hand, by the Uniform Boundedness Theorem (see, e.g., [4, 11.15 Theorem]), we deduce that $(T_{z_0}(f_n, g_n))$ converges to $T_{z_0}(\tilde{f}, \tilde{g}) = T(\tilde{f}, \tilde{g})(z_0)$. This fact yields $T(\tilde{f}, \tilde{g})(z_0) = 0$.

\[\square\]

**Definition 4.** We say that $T$ is stable on constants if, given $(f, g) \in C(X, E_1) \times C(Y, E_2)$ and $z \in Z$, then

$$\|T(f, \tilde{e}_2)(z)\| = \|T(f, \tilde{e}'_2)(z)\|$$

for every pair $e_2, e'_2 \in S_{E_2}$ and

$$\|T(\tilde{e}_1, g)(z)\| = \|T(\tilde{e}'_1, g)(z)\|$$

for every pair $e_1, e'_1 \in S_{E_1}$.

**Lemma 4.** Assume $E_3$ is strictly convex. Fix $(x_0, y_0) \in X \times Y$ and assume that $T$ is stable on constants.

1. If $f(x_0) = 0$ for some $f \in C(X, E_1)$ (resp. $g(y_0) = 0$ for some $g \in C(Y, E_2)$), then $T(f, g)(z) = 0$ for all $z \in I_{x_0, y_0}$ and all $g \in C(Y, E_2)$ (resp. all $f \in C(X, E_1)$).

2. Furthermore, $T(f, g)(z) = T(\tilde{f}(x_0), g(y_0))(z)$ for all $z \in I_{x_0, y_0}$ and all $(f, g) \in C(X, E_1) \times C(Y, E_2)$.

**Proof.** (1) Let us take $(f, g) \in C(X, E_1) \times C(Y, E_2)$ such that $f(x_0) = 0$ and assume, with no loss of generality, that $\|g(y_0)\| = 1$. 


Fix \( z_0 \in I_{x,0,0} \). Then \( z_0 \in I_{x,0,0,e,e'} \) for some \((e,e') \in S_{E_1} \times S_{E_2}\). By Lemma 2, we know that \( T(f, e')(z_0) = 0 \).

By Lemma 3, \( T(f, g - g(y_0))(z_0) = 0 \), which yields \( T(f, g)(z_0) = T(f, g)(z_0) \).

Therefore, since \( T \) is stable on constants, we have

\[
0 = T(f, e')(z_0) = T(f, g(y_0))(z_0) = T(f, g)(z_0).
\]

(2) Take now a pair \((f, g) \in C(X, E_1) \times C(Y, E_2)\) and define the function \( f' := f - \widehat{f(x_0)} \). Since \( f'(x_0) = 0 \), then, by (a), \( T(f - \widehat{f(x_0)}, g)(z) = 0 \) for all \( z \in I_{x,0,0} \), which is to say, by the bilinearity of \( T \), that \( T(f, g)(z) = T(f(x_0), g)(z) \) for all \( z \in I_{x,0,0} \).

Next, define the function \( g' := g - g(y_0) \). Since \( g'(y_0) = 0 \), then, again by (a), \( T(f(x_0), g - g(y_0))(z) = 0 \) for all \( z \in I_{x,0,0} \), which yields \( T(f, g)(z) = T(f(x_0), g)(z) = T(f(x_0), g(y_0))(z) \).

3. The main result.

**Theorem 1.** Let \( T : C(X, E_1) \times C(Y, E_2) \rightarrow C(Z, E_3) \) be a bilinear isometry which is stable on constants and assume that \( E_3 \) is strictly convex. Then there exists a nonempty subset \( Z_0 \) of \( Z \), a surjective continuous mapping \( h : Z_0 \rightarrow X \times Y \) and a continuous function \( \omega : Z_0 \rightarrow \text{Bil}(E_1 \times E_2, E_3) \) such that \( T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z))) \) for all \( z \in Z_0 \) and every pair \((f, g) \in C(X, E_1) \times C(Y, E_2)\).

**Proof.** Let us suppose that \((x, y)\) and \((x', y')\) belong to \( X \times Y \) and are distinct. Then we claim that \( I_{x,y} \cap I_{x',y'} = \emptyset \). Assume, contrary to what we claim, that there exists \( z \in I_{x,y} \cap I_{x',y'} \). Let us suppose, with no loss of generality, that \( x \neq x'\).

- If \( y \neq y' \), then we can choose \( f \in C_{x,e} \) and \( g \in C_{y,e'} \) for some \( e, e' \in S_E \) with \( f(x') = g(y') = 0 \). Consequently, \( \|T(f, g)(z)\| = 1 \), but, by Lemma 3, \( T(f, g)(z) = 0 \), which is a contradiction.
- If \( y = y' \), then we can choose \( f \in C_{x,e} \) and \( g \in C_{y,e'} \) for some \( e, e' \in S_E \) with \( f(x') = 0 \). Consequently, \( \|T(f, g)(z)\| = 1 \), but, by Lemma 4, \( T(f, g)(z) = 0 \), which is a contradiction.

Let us next define a subset \( Z_0 \) of \( Z \) as follows:

\[
Z_0 := \bigcup_{(x,y) \in X \times Y} I_{x,y}
\]
Now we can define a linear map $\omega$ from $Z_0$ to $Bil(E_1 \times E_2, E_3)$ as

$\omega(z)(e, e') := T(\widetilde{e}, \widetilde{e}')(z)$ where $(e, e') \in E_1 \times E_2$. Hence, by Lemma 4,

$$T(f, g)(z) = T(\widetilde{f(x_0)}, \widetilde{g(y_0)})(z) = \omega(z)(f(x_0), g(y_0))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$.

To prove the continuity of $\omega$, let $(z_\alpha)$ be a net convergent to $z_0 \in Z_0$. Fix $(e, e') \in E_1 \times E_2$. Then $\|\omega(z_\alpha)(e, e') - \omega(z_0)(e, e')\| = \|T(\widetilde{e}, \widetilde{e}')(z_\alpha) - T(\widetilde{e}, \widetilde{e}')(z_0)\|$. Since $(T(\widetilde{e}, \widetilde{e}')(z_\alpha))$ converges to $T(\widetilde{e}, \widetilde{e}')(z_0)$, the continuity of $\omega$ is then verified.

Let us next define a mapping $h : Z_0 \longrightarrow X \times Y$ as $h(z) := (x, y)$ where $z \in J_{x,y}$. We claim that $h$ is continuous. To this end, fix $z_0 \in Z_0$ and let $h(z_0) = (x_0, y_0)$. Let $U$ be a neighborhood of $x_0$ and choose $f \in C(X, E_1)$ such that $1 = \|f\|_\infty = \|f(x_0)\|$ and $\|f\|_\infty < 1$ off $U$. Let $s(x_0) = \sup_{x \in X \setminus U} \|f(x)\|$. It is apparent that $s(x_0) < 1$. In like manner, let $V$ be a neighborhood of $y_0$ and choose $g \in C(Y, E_2)$ such that $1 = \|g\|_\infty = \|g(y_0)\|$ and $\|g\|_\infty < 1$ off $V$. Let $s(y_0) = \sup_{y \in Y \setminus U} \|g(y)\|$. As above, $s(y_0) < 1$.

Since $h(z_0) = (x_0, y_0)$, then $\|T(f, g)(z_0)\| = \|T(f, g)\|_\infty = 1$. Let $s := \max\{s(x_0), s(y_0)\}$ and define the following open neighborhood of $z_0$:

$$W := \{z \in Z_0 : \|T(f, g)(z)\| > s\}.$$

Fix $z_1 \in W$ and suppose that $h(z_1) := (x_1, y_1)$. Then, by the above representation of $T$,

$$s < \|T(f, g)(z_1)\| = \|\omega(z_1)(f(x_1), g(y_1))\| = \|T(f(x_1), g(y_1))(z_1)\| \leq \|T(f(x_1), g(y_1))\|_\infty = \|f(x_1)\|_\infty \cdot \|g(y_1)\|_\infty = \|f(x_1)\| \cdot \|g(y_1)\|$$

and, consequently, $\|f(x_1)\| > s \geq s(x_0)$ and $\|g(y_1)\| > s \geq s(y_0)$. This yields $x_1 \in U$ and $y_1 \in V$, which is to say that $h(W) \subseteq U \times V$ and the proof is done.

Finally, it is clear that $T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z))))$.

**Concluding remarks.**

(1) To be stable on constants can be regarded as a necessary condition in the following sense: Let $T : C(X, E_1) \times C(Y, E_2) \longrightarrow$
Let $C(Z, E_3)$ be a bilinear isometry which can be written as
\[ T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z))) \]
for all $z \in Z$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$, where $h$ is a surjective continuous mapping from $Z$ onto $X \times Y$ and $\omega(z) \in \text{Bil}(E_1 \times E_2, E_3)$. Then
\[ \|T(f, \tilde{e})(z)\| = \|\omega(z)(f(\pi_X(h(z))), e)\| = \|f(\pi_X(h(z)))\| \]
for all $e \in E_2$ and all $z \in Z$; that is, $T$ is stable on constants.

(2) It is clear that if we assume $E_1$, $E_2$ and $E_3$ to be the field of real or complex numbers, then $T$ is stable on constants. Hence, Theorem 1 is an extension, indeed a vector-valued version, of the main result in [6].

(3) In like manner, Theorem 1 contains the main theorem in [2], by assuming $Y$ to be a singleton and $E_2$ to be the field of real or complex numbers. Indeed, it is a routine matter to verify that, in this context, Lemma 4 and Theorem 1 remain true even if we do not assume $T$ to be stable on constants.

(4) Typical examples of bilinear isometries can be defined as follows: assume that there exists a continuous surjection $h : X \rightarrow X \times X$ and let $E$ be a Banach algebra. Then we can define a mapping $T(f, g)(z) := f(\pi_1(h(z)))g(\pi_2(h(z)))$ for all $z \in X$ and every pair $(f, g) \in C(X, E) \times C(X, E)$. It is apparent that $T$ is a bilinear isometry which is stable on constants.

References

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