PSEUDOCOMPACT GROUP TOPOLOGIES WITH NO INFINITE COMPACT SUBSETS

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Abstract. We show that every Abelian group satisfying a mild cardinal inequality admits a pseudocompact group topology from which all countable subgroups inherit the maximal totally bounded topology (we say that such a topology satisfies property $\sharp$).

Every pseudocompact Abelian group G with cardinality $|G| \leq 2^{2^{\mathfrak{c}}}$ satisfies this inequality and therefore admits a pseudocompact group topology with property $\sharp$. Under the Singular Cardinal Hypothesis (SCH) this criterion can be combined with an analysis of the algebraic structure of pseudocompact groups to prove that every pseudocompact Abelian group admits a pseudocompact group topology with property $\sharp$.

We also observe that pseudocompact Abelian groups with property $\sharp$ contain no infinite compact subsets and are examples of Pontryagin reflexive precompact groups that are not compact.

1. Introduction

A topological space X is pseudocompact if every real-valued continuous function on X is bounded. Pseudocompactness is greatly enhanced by the addition of algebraic structure. This fact was discovered in 1966 by Comfort and Ross [9] who proved that pseudocompact topological groups are totally bounded or, what is the same, that they always appear as *subgroups* of compact groups. They went even further and precisely identified pseudocompact groups among subgroups of topological groups: a subgroup of a compact group is pseudocompact if, and only if, it is G_δ-dense in its closure (i.e., meets every nonempty G_δ-subset of its closure).

A powerful tool to study totally bounded topologies on Abelian groups is Pontryagin duality. This is because a totally bounded group topology is

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always induced by a group of characters [8] and Pontryagin duality is based on relating a topological group with its group of continuous characters. We recall here that a character of a group G is nothing but a homomorphism of G into the multiplicative group $\mathbb{T}$ of complex numbers of modulus one.

If G is an Abelian topological group, the topology of uniform convergence on compact subsets of G makes the group of continuous characters of G, denoted $G^\wedge$, into a topological group. Evaluations then define a homomorphism $\alpha_G\colon G \to G^{\wedge\wedge}$ between G and the group of all continuous characters on the dual group, the so-called *bidual* group $G^{\wedge\wedge}$. When α_G is a topological isomorphism we say that G is Pontryagin reflexive. It will be necessary for the development of this paper to keep in mind that character groups of discrete groups are compact groups. Even if it is not relevant for our purposes we cannot resist here to add that character groups of compact groups are again discrete, and that the Pontryagin van-Kampen theorem proves that all locally compact Abelian groups (discrete and compact ones are thus comprised) are reflexive.

In the present paper Pontryagin duality will appear both as a tool for constructing pseudocompact group topologies and as an objective itself. To be precise, this paper is motivated by the following two questions

Question 1.1 ([3])**.** *Is every Pontryagin reflexive totally bounded Abelian group a compact group?*

Question 1.2 ([12], Question 25 of [13])**.** *Does every pseudocompact Abelian group admit a pseudocompact group topology with no infinite compact subsets?*

In this paper we obtain a negative answer to Question 1.1 and a positive answer, valid under the Singular Cardinal Hypothesis (SCH), to Question 1.2. The focus of the paper will be on Question 1.2 with the analysis of Question 1.1 and its relation with Question 1.2 deferred to Section 6.

It should be noted, in a direction opposite to Question 1.2, that every pseudocompact group admits pseudocompact group topology with nontrivial convergent sequences, see [19].

Our approach to Question 1.2 consists in combining techniques that can be traced back at least to [25] with the ideas of [18]. Our construction actually produces pseudocompact Abelian groups with all countable subgroups h-embedded. This is stronger (see Section 2) that finding pseudocompact group topologies with no infinite compact subsets. With the aid of results from [23] this construction will yield a wide range of negative answers to Question 1.1. As pointed to us by M. G. Tkachenko, Question 1.1 has been answered independently in [1].

On notation and terminology. All groups considered in this paper will be Abelian. So, the specification *Abelian group* to be found at some points will respond only to a matter of emphasis. To further avoid the cumbersome use of the word "Abelian", free Abelian groups will simply be termed as *free groups*.

The symbol $\mathbb{P}$ will denote the set of all prime numbers. *Faute de mieux*, we will use the unusual symbol $\mathbb{P}^{\uparrow}$ to denote the set of all prime powers, i.e., an integer $k \in \mathbb{P}^{\uparrow}$ if, and only if, $k = p^n$ for some $p \in \mathbb{P}$ and some positive integer n.

For a set X and a cardinal number α, $[X]^\alpha$ stands for the collection of all subsets of X with cardinality α.

Following Tkachenko [25], we say that a subgroup H of a topological group G is h-embedded if every homomorphism of H to the unit circle $\mathbb{T}$ can be extended to a *continuous* homomorphism of G to $\mathbb{T}$. If G is totally bounded and H is h-embedded in G, then the topology of H must equal the maximal totally bounded topology of H (or, using van Douwen's terminology, $H = H^\sharp$).

The cardinal function $m(\alpha)$ will be often used. The cardinal $m(\alpha)$ is defined for every infinite cardinal α as the least cardinal number of a G_δ-dense subset of a compact group K_α of weight α. It is proved in [7] that this definition does not depend on the choice K_α and therefore makes sense. The same reference contains proofs of the following basic essential features of $m(\alpha)$:

$$\log(\alpha) \leq m(\alpha) \leq (\log(\alpha))^\omega \quad \text{and} \quad \mathrm{cf}(m(\alpha)) > \omega, \quad \text{for every } \alpha \geq \omega.$$

These inequalities have a much simpler form if *Singular Cardinal Hypothesis* (SCH) is assumed. SCH is a condition consistent with ZFC that follows from (but is much weaker than) the *Generalized Continuum Hypothesis* (GCH). Under SCH every infinite cardinal α satisfies

$$m(\alpha) = (\log(\alpha))^\omega.$$

It is well known that every compact group has cardinality 2^κ for some cardinal κ. The question on which cardinals can appear as the cardinal of a pseudocompact group is not so readily answered. We will say that a cardinal

κ is *admissible* provided there is a pseudocompact group of cardinal κ. The first obstructions to admissibility were found by van Douwen [15], the main one being that the cardinality $|G|$ of a pseudocompact group cannot be a strong limit cardinal of countable cofinality; see [11, Chapter 3] for more information on admissible cardinals.

Most of our results concern constructing pseudocompact group topologies on a given Abelian group G. As indicated in the introduction, every pseudocompact group topology is totally bounded and a totally bounded group topology $\mathcal{T}$ on an Abelian group G is always induced by a unique group of characters $H \subset Hom(G, \mathbb{T})$, [8, 9]. To stress this latter fact we will usually refer to $\mathcal{T}$ as $\mathcal{T}_H$. Recall that the topology $\mathcal{T}_H$ is Hausdorff if, and only if, the subgroup H separates points of G.

We have also introduced above the symbol $G^\wedge$ to denote the group of all continuous characters of a topological Abelian group equipped with the compact-open topology. We will use in this context the subscript $_d$ to indicate that G carries the discrete topology. Thus $(G_d)^\wedge$ equals the set $Hom(G, \mathbb{T})$ of all homomorphisms into $\mathbb{T}$. Being a closed subgroup of $\mathbb{T}^G$, $(G_d)^\wedge$ is always a compact group.

Several purely algebraic notions from the theory of infinite Abelian groups will be necessary, as for instance the notion of basic subgroup and the related one of pure subgroup. We refer to [17] for the meaning and significance of these properties. As usual, the symbol $t(G)$ stands for the torsion subgroup of the group G and $r_0(G)$ denotes the torsion-free rank of G.

2. The dual property to pseudocompactness

The following theorem is at the heart of the relationship between questions 1.2 and 1.1.

Theorem 2.1 ([23])**.** *Let $(G, \mathcal{T}_H)$, $H \subset \mathrm{Hom}(G, \mathbb{T})$, be a Hausdorff Abelian totally bounded group. $(G, \mathcal{T}_H)$ is pseudocompact if, and only if, every countable subgroup of $(H, \mathcal{T}_G)$ is h-embedded in $(G_d)^\wedge$.*

Definition 2.2. We say that a topological group G has property $\sharp$ if every countable subgroup of G is h-embedded in G.

Thus property $\sharp$ is, in the terminology of [23], the dual property of pseudocompactness.

The relation between property $\sharp$ and Question 1.2 is clear from the following Lemma. Although a combination of Propositions 3.4 and 4.4 of [23] would provide an indirect proof, we offer a direct proof for the reader's convenience.

Lemma 2.3. *Let $(G, \mathcal{T}_H)$ denote a totally bounded group with property $\sharp$. Then $(G, \mathcal{T}_H)$ has no infinite compact subsets.*

Proof. We first see that all countable subgroups of G are $\mathcal{T}_H$-closed. Suppose otherwise that $x \in \mathrm{cl}_{(G,\mathcal{T}_H)} N \setminus N$ with N a countable subgroup of G. The subgroup $\widetilde{N} = \langle N \cup \{x\} \rangle$ is also countable and, by hypothesis, inherits its maximal totally bounded group topology from $(G, \mathcal{T}_H)$. Since subgroups are necessarily closed in that topology, it follows that N is closed in $\widetilde{N}$, which goes against $x \in \widetilde{N} \setminus N$.

Now suppose K is an infinite compact subset of G and let $S \subset K$ be a countable subset of K. Define $\widetilde{G} = \langle S \rangle$ and denote by $\widetilde{G}$ and $\overline{(\widetilde{G}, \mathcal{T}_H)}$ the completions of $\widetilde{G}^\sharp$ and $(\widetilde{G}, \mathcal{T}_H)$ respectively. Since $\langle S \rangle$ is h-embedded the identity function $j \colon \widetilde{G}^\sharp \to (\widetilde{G}, \mathcal{T}_H)$, extends to a topological isomorphism $\bar{\jmath} \colon b\widetilde{G} \to \overline{(\widetilde{G}, \mathcal{T}_H)}$. Then $\bar{\jmath}(\mathrm{cl}_{b\widetilde{G}} S) = \mathrm{cl}_{\overline{(\widetilde{G},\mathcal{T}_H)}} j(S) \subset K$, therefore $\mathrm{cl}_{\overline{(\widetilde{G},\mathcal{T}_H)}} j(S) = \mathrm{cl}_{(\widetilde{G},\mathcal{T}_H)} S$ and, it follows from the preceding paragraph that $\mathrm{cl}_{b\widetilde{G}} S = \bar{\jmath}(\mathrm{cl}_{b\widetilde{G}} S) \subset \langle S \rangle$.

But a well known theorem of van Douwen [16] (see also [20] and [2, Theorem 9.9.51] for different proofs and [21] for extensions of that result) states that $|\,\mathrm{cl}_{b(\widetilde{G})} S| = 2^{\mathfrak{c}}$ and therefore it is impossible that $\bar{\jmath}(\mathrm{cl}_{b\widetilde{G}} S)S \subset \langle S \rangle$. □

We establish next some easily deduced permanence properties.

Proposition 2.4. *The class of groups having property $\sharp$ is closed for finite products.*

Proof. Let G_1 and G_2 be two topological Abelian groups with property $\sharp$ and let N be a countable subgroup of $G_1 \times G_2$. Let h be a homomorphism from N to $\mathbb{T}$. By considering an arbitrary extension of h to $G_1 \times G_2$ we may assume that h is actually defined on $G_1 \times G_2$. Since both $\pi_1(N)$ and $\pi_2(N)$ are countable there will be continuous homomorphisms $h_i \colon G_i \to \mathbb{T}$, $i = 1, 2$, with $h_1(x) = h(x, 0)$ and $h_2(y) = h(0, y)$ for all $x \in \pi_1(N)$ and $y \in \pi_2(N)$. The homomorphism $\bar{h} \colon G_1 \times G_2 \to \mathbb{T}$ given by $\bar{h}(x, y) = h_1(x) \cdot h_2(y)$ is then a continuous extension of h. □

Lemma 2.5. *Let $\pi : K \to L$ be a continuous surjection between two compact Abelian groups K and L and suppose that N is a subgroup of L that, as subspace of L, carries the maximal totally bounded topology. If M is a* *subgroup of K such that $\pi_{\restriction M}$ is a group isomorphism between M and N, then M also inherits from K the maximal totally bounded topology.*

Proof. Denote by $\mathcal{T}_K$ and $\mathcal{T}_L$ the topologies that M inherit from K and L respectively (the latter obtained through $\pi_{\restriction M}$). Since π is continuous, the topology $\mathcal{T}_K$ is finer than $\mathcal{T}_L$, but $\mathcal{T}_K$ is the maximal totally bounded topology, therefore $\mathcal{T}_K = \mathcal{T}_L$. □

3. Property ♯ on torsion-free and bounded groups

We will make a heavy use of powers of groups in the sequel. If σ is a cardinal number, K^σ stands for such powers. We use calligraphical letters, to denote sets of coordinates, that is, subsets of σ. If $\mathcal{D} \subset \sigma$, we will denote by $\pi^K_{\mathcal{D}}$ the projection from K^σ to $K^{\mathcal{D}}$, if no confusion is possible we will simply use $\pi_{\mathcal{D}}$.

Lemma 3.1. *Let G be a metrizable group and let $\sigma \geq \mathfrak{c}$ and α be cardinal numbers with $m(\sigma) \leq \alpha$, and $\alpha^\omega \leq \sigma$.*

Then there exists an independent G_δ-dense subset $D \subseteq G^\sigma$ with cardinality $m(\sigma)$, $D = \{d_\eta \colon \eta < m(\sigma)\}$, and two families of sets of coordinates $\{\mathcal{S}_\theta \colon \theta \in [\alpha]^\omega\}, \{\mathcal{N}_\eta \colon \eta < \alpha\} \subset \sigma$ such that:

1. $|\mathcal{S}_\theta| = \sigma$.
2. $\mathcal{S}_\theta \cap \mathcal{S}_{\theta'} = \emptyset$, *if* $\theta \neq \theta'$.
3. $\left| \mathcal{S}_\theta \setminus \bigcup_{\eta\in\theta} \mathcal{N}_\eta \right| = \sigma$ *for every* $\theta \in [\alpha]^\omega$.
4. *Every subset $\{g_\eta \colon \eta < \alpha\}$ of G^σ with $\pi_{\mathcal{N}_\eta}(g_\eta) = \pi_{\mathcal{N}_\eta}(d_\eta)$, for all $\eta < \alpha$ is G_δ-dense.*

Proof. Let $\mathcal{A}_\beta = \{a_\gamma \colon \gamma < \sigma\}$ be a set with $|\mathcal{A}_\beta| = \sigma$ and consider the disjoint union $\mathcal{A} = \bigcup_{\beta<\mathfrak{c}} \mathcal{A}_\beta$. We identify G^σ with $G^{\mathcal{A}}$ and α with $[\mathfrak{c}]^\omega \times \alpha$. Since $\alpha^\omega \leq \sigma$, we can as well decompose each $\mathcal{A}_\beta$ as a disjoint union $\mathcal{A}_\beta = \bigcup_{\widetilde{\theta}\in[[\mathfrak{c}]^\omega\times\alpha]^\omega} \mathcal{A}_{\beta,\widetilde{\theta}}$ of sets of cardinality $|\mathcal{A}_{\beta,\widetilde{\theta}}| = \sigma$.

For each $N \in [\mathfrak{c}]^\omega$, let next $F_N = \{f_{(N,\eta)} \colon \eta < \alpha\}$ be an independent G_δ-dense subset of the product $G^{\cup_{\gamma\in N}\mathcal{A}_\gamma}$ (note that $m(\sigma) \leq \alpha$ and that G is metrizable). Assume that each $f_{(N,\eta)}$ actually belongs to $G^{\mathcal{A}}$ by putting $\pi_{\mathcal{A}_\gamma}(f_{(N,\eta)}) = 0$ if $\gamma \notin N$.

We now order $\alpha = [\mathfrak{c}]^\omega \times \alpha$ lexicographically and define the sets $N_{\widetilde{\eta}}$, $\widetilde{\eta} \in [\mathfrak{c}]^\omega \times \alpha$ and $\mathcal{S}_{\widetilde{\theta}}$, $\widetilde{\theta} \in [[\mathfrak{c}]^\omega \times \alpha]^\omega$. For $\widetilde{\eta} = (N, \eta) \in [\mathfrak{c}]^\omega \times \alpha$ define $\mathcal{N}_{(N,\eta)} = \bigcup_{\gamma\in N} \mathcal{A}_{\gamma,\widetilde{\eta}}$ and given $\widetilde{\theta} = \{(N_k, \eta_k) \colon k < \omega,\ (N_k, \eta_k) \in [\mathfrak{c}]^\omega \times \alpha\}$, we define $\mathcal{S}_{\widetilde{\theta}} = \mathcal{A}_{\beta_0,\widetilde{\theta}}$ where β_0 is such that $\beta \in N_k$ for some k, implies $\beta < \beta_0$ (recall that $\mathfrak{c}$ has uncountable cofinality). By construction of the sets $\mathcal{A}_{\beta,\widetilde{\theta}}$, we have $\mathcal{S}_{\widetilde{\theta}} \cap \mathcal{S}_{\widetilde{\theta}'} = \emptyset$, when $\widetilde{\theta} \neq \widetilde{\theta}'$. Condition (3) obviously holds, since $S_{\widetilde{\theta}}$ and $\bigcup_{\widetilde{\eta}\in\widetilde{\theta}} \mathcal{N}_{\widetilde{\eta}}$ are even disjoint.

Define finally $D = \{f_{\widetilde{\eta}} \colon \widetilde{\eta} \in [\mathfrak{c}]^\omega \times \alpha\} = \cup_{N\in[\mathfrak{c}]^\omega} F_N$.

Suppose $\widetilde{D} = \{g_{\widetilde{\eta}} \colon \widetilde{\eta} \in [\mathfrak{c}]^\omega \times \alpha\}$ is such that $\pi_{\mathcal{N}_{\widetilde{\eta}}}(g_{\widetilde{\eta}}) = \pi_{\mathcal{N}_{\widetilde{\eta}}}(f_{\widetilde{\eta}})$, for all $\widetilde{\eta} \in [\mathfrak{c}]^\omega \times \alpha$.

To check that $\widetilde{D}$ is indeed G_δ-dense we choose a G_δ-subset U of $G^{\mathcal{A}}$. There will be then $N = \{\alpha_n \colon n < \omega\} \in [\mathfrak{c}]^\omega$ and a G_δ-set $V \subset G^{\cup \mathcal{A}_{\alpha_n}}$ such that $\{\bar{x} \in G^{\mathcal{A}} \colon \pi_{\cup_n \mathcal{A}_{\alpha_n}}(\bar{x}) \in V$ for each $n < \omega\} \subset U$. Since F_N is G_δ-dense in $G^{\bigcup_{\gamma\in N}\mathcal{A}_\gamma} = G^{\bigcup_n \mathcal{A}_{\alpha_n}}$, there will be an element $f_{(N,\eta)} \in F_N$ with $\pi_{\cup_n \mathcal{A}_{\alpha_n}}(f_{(N,\eta)}) \in V$ for every $\alpha_n \in N$.

As $g_{(N,\eta)}$ and $f_{(N,\eta)}$ have the same $\cup_{\gamma\in N}\mathcal{A}_\gamma$-coordinates, we conclude that $g_{(N,\eta)} \in U \cap \widetilde{D}$. □

If χ is a homomorphism between two groups G_1 and G_2 and σ is a cardinal number, we denote by χ^σ the product homomorphism $\chi^\sigma \colon G_1^\sigma \to G_2^\sigma$ defined by $\chi^\sigma((g_\eta)_{\eta<\sigma}) = (\chi(g_\eta))_{\eta<\sigma}$. It is easily verified that, for any $\mathcal{D} \subseteq \sigma$, the projections $\pi_{\mathcal{D}}^{G_i} : G_i^\sigma \to G_i^{\mathcal{D}}$, $i = 1, 2$ satisfy

$$\pi_{\mathcal{D}}^{G_2} \circ \chi^\sigma = \chi^{\mathcal{D}} \circ \pi_{\mathcal{D}}^{G_1}$$

Corollary 3.2. *Let $\chi \colon G_1 \to G_2$ be a surjective homomorphism between two metrizable groups G_1 and G_2. If σ and α are cardinal numbers with $m(\sigma) \leq \alpha$ and $\alpha^\omega \leq \sigma$, then it is possible to find an independent G_δ-dense subset D of G_1^σ satisfying the properties of Proposition 3.1 such that in addition $\chi^\sigma(D)$ is an independent subset of G_2^σ.*

Proof. It suffices to repeat the proof of Lemma 3.1 taking care to choose the sets F_N in such a way that $\chi^{\cup_{\gamma\in N}\mathcal{A}_\gamma}(F_N)$ is also independent. □

Proposition 3.3. *Let $\chi \colon G \to \mathbb{T}$ be a surjective character of a compact metrizable group G. If σ and α are cardinal numbers with $m(\sigma) \leq \alpha$, and $\alpha^\omega \leq \sigma$, then the topological group G^σ contains an independent G_δ-dense subset F of cardinality α such that F and $\chi^\sigma(F)$ generate isomorphic groups with property $\sharp$.*

Proof. We begin with a G_δ-dense subset of G^σ, $D = \{d_\eta \colon \eta < \alpha\}$, with the properties of Lemma 3.1 and Corollary 3.2. We have thus two families of sets $\{\mathcal{S}_\theta, \,:\, \theta \in [\alpha]^\omega\}$, $\{\mathcal{N}_\eta, \,:\, \eta < \alpha\} \subset \sigma$ with the properties (1) through (4) of that Lemma.

Next, for every $\theta \in [\alpha]^\omega$, we choose and fix a set of coordinates $\mathcal{D}_\theta \subseteq \sigma$ of cardinality $|\mathcal{D}_\theta| = \sigma$ in such a way that

$$\mathcal{D}_\theta \subseteq \mathcal{S}_\theta \setminus \bigcup_{\eta\in\theta} \mathcal{N}_\eta$$

(recall that by Lemma 3.1, $\left|\mathcal{S}_\theta \setminus \bigcup_{\eta\in\theta} \mathcal{N}_\eta\right| = \sigma$)

Given each $\theta \in [\alpha]^\omega$, we consider the free subgroup $\langle \chi^\sigma(d_\eta) \colon \eta \in \theta\rangle$ and equip it with its maximal totally bounded topology. Denoting the resulting topological group as $\langle \chi^\sigma(d_\eta) \colon \eta \in \theta\rangle^\sharp$, and taking into account that it has weight $\mathfrak{c}$, we can find an embedding

$$j_\theta \colon \langle \chi^\sigma(d_\eta) \colon \eta \in \theta\rangle^\sharp \hookrightarrow \mathbb{T}^{\mathcal{D}_\theta}. \tag{3.1}$$

For each $\theta \in [\alpha]^\omega$ and each $\eta \in \theta$, let $g_{\eta,\theta}$ denote an element of $G^{\mathcal{D}_\theta}$ with $\chi^{\mathcal{D}_\theta}(g_{\eta,\theta}) = j_\theta(\chi^\sigma(d_\eta))$. Observe that the set $\{g_{\eta,\theta} \colon \eta \in \theta\}$ is independent.

We finally define the elements f_η, $\eta < \alpha$, by the rules:

$$\pi^G_{\mathcal{D}_\theta}(f_\eta) = g_{\eta,\theta}, \text{ if } \theta \in [\alpha]^\omega \text{ is such that } \eta \in \theta, \qquad \text{and}$$
$$\pi^G_\gamma(f_\eta) = \pi^G_\gamma(d_\eta) \text{ if } \gamma \notin \mathcal{D}_\theta \text{ for any } \theta \in [\alpha]^\omega \text{ with } \eta \in \theta.$$

Let us see that $F = \{f_\eta \colon \eta < \alpha\}$ satisfies the desired properties:

(1) *F and $\chi^\sigma(F)$ are independent.* Suppose that $\sum_{k=1}^m n_k f_{\eta_k} = 0$ with $n_k \in \mathbb{Z}$. Choose then $\theta \in [\alpha]^\omega$ with $\eta_1, \ldots, \eta_m, \in \theta$. Since $\pi^G_{\mathcal{D}_\theta}(f_{\eta_k}) = g_{\eta_k,\theta}$ and the set $\{g_{\eta,\theta} \colon \eta \in \theta\}$ is independent, the independence of F follows. Since $\pi_{\mathcal{D}_\theta}(\chi^\sigma(f_\eta)) = \chi^{\mathcal{D}_\theta}(g_{\eta,\theta})$, $\chi^\sigma(F)$ is also independent. It is easy to see, now, that $\langle F\rangle$ and $\langle \chi^\sigma(F)\rangle$ are isomorphic.

(2) *The subgroup $\langle \chi^\sigma(F)\rangle$ has property $\sharp$.* Let N be a countable subgroup of $\langle \chi^\sigma(F)\rangle$. Let $\theta \in [\alpha]^\omega$ be such that $N \subseteq \langle \chi^\sigma(f_\eta) \colon \eta \in \theta\rangle$ and define $N_\theta := \langle f_\eta \colon \eta \in \theta\rangle$.

Observe finally that $\pi^{\mathbb{T}}_{\mathcal{D}_\theta}(N) = \chi^{\mathcal{D}_\theta}(\pi^G_{\mathcal{D}_\theta}(N_\theta))$. This last subgroup is just $j_\theta(\langle \chi^\sigma(d_\eta) \colon \eta \in \theta\rangle)$ and the latter carries by construction its maximal totally bounded topology, since the restriction of $\pi^{\mathbb{T}}_{\mathcal{D}_\theta} \colon \mathbb{T}^\sigma \to \mathbb{T}^{\mathcal{D}_\theta}$ to N is a group isomorphism onto $\pi^{\mathbb{T}}_{\mathcal{D}_\theta}(N) = \chi^{\mathcal{D}_\theta}(\pi^G_{\mathcal{D}_\theta}(N_\theta))$, Lemma 2.5 applies.

(3) *$\langle F\rangle$ has property $\sharp$.* Take $\pi = \chi^\sigma$, $K = G^\sigma$ and $L = \mathbb{T}^\sigma$. Bearing in mind that the restriction to $\langle F\rangle$ is an isomorphism because F and $\chi^\sigma(F)$ are independent sets, Lemma 2.5 applies again.

(4) *F is a G_δ-dense subset of G^σ.* Observe that, for every $\eta < \alpha$, f_η coincides with d_η on the set of coordinates $\mathcal{N}_\eta$, for $\mathcal{D}_\theta \subseteq \mathcal{S}_\theta \setminus \bigcup_{\eta\in\theta} \mathcal{N}_\eta$.

Since D has the properties of Lemma 3.1, we conclude that F is G_δ-dense.

□

Proposition 3.4. *Let σ and α be cardinal numbers with $m(\sigma) \leq \alpha$, and $\alpha^\omega \leq \sigma$. The topological group $\mathbb{Z}(p)^\sigma$ contains an independent G_δ-dense subset H with property $\sharp$.*

Proof. Proceed exactly as in Proposition 3.3 and construct an embedding into $\mathbb{Z}(p)^\sigma$. To obtain the $\sharp$-property we identify countable subgroups with Bohr groups of the form $(\oplus_\omega \mathbb{Z}(p))^\sharp$. □

4. The algebraic structure of pseudocompact Abelian groups

We obtain here some results on the algebraic structure of pseudocompact that will be useful in the next section. The first of them is inspired (and shares a part of its proof) from the first part of the proof of Lemma 3.2 of [18]. We sketch here the proof for the reader's convenience. We thank Dikran Dikranjan for pointing a misguiding sentence in a previous version of this proof.

Lemma 4.1. *Every Abelian group admits a decomposition*

$$G = \left(\bigoplus_{p^k \in \mathbb{P}_0^\uparrow} \bigoplus_{\gamma(p^k)} \mathbb{Z}(p^k) \right) \bigoplus H$$

where $\mathbb{P}_0^\uparrow$ is a finite subset of $\mathbb{P}^\uparrow$ and H is a subgroup of G with

$$|nH| = |H|, \text{ for all } n \in \mathbb{N}.$$

Proof. Decompose $t(G) = \bigoplus_p G_p$ as a direct sum of p-groups G_p and let B_p denote a basic subgroup of G_p for each p. This in particular means that B_p is a direct sum of cyclic p-groups,

$$B_p = \bigoplus_{n<\omega} B_{p,n} \text{ with } B_{p,n} \cong \bigoplus_{\beta_{p^n}} \mathbf{Z}(p^n)$$

and that G_p/B_p is divisible. Define $\mathcal{D} = \{|B_{p,n}|\colon p^n \in \mathbb{P}^\uparrow\}$. If $\mathcal{D}$ has no maximum or $\beta_0 = \max \mathcal{D}$ is attained at an infinite number of $|B_{p,n}|$'s we stop here. If, otherwise, $\beta_0 = \max \mathcal{D} = |B_{p_1,n_1}| = \ldots = |B_{p_r,n_r}|$ and $|B_{p_j,n_j}| < \beta_0$ for all the remaining $p_j^{n_j} \in \mathbb{P}^\uparrow$ we repeat the process with the set $\mathcal{D} \setminus |B_{p_1,n_1}|$. After a finite number of steps we obtain in this manner a finite collection of cardinals $F \subset \mathcal{D}$ such that either:

(1) *Case 1:* the supremum $\beta := \sup(\mathcal{D} \setminus F)$ is not attained, or
(2) *Case 2:* the supremum $\beta := \sup(\mathcal{D} \setminus F)$ is attained infinitely often, i.e., there is an infinite subset $I \subset \mathbb{P}^{\uparrow}$ with $|B_{p,n}| = \beta$ for all $p^n \in I$.

Define $\mathbb{P}_0^{\uparrow} = \{p^n \in \mathbb{P}^{\uparrow} \colon |B_{p,n}| \in F\}$ (observe that $\mathbb{P}_0^{\uparrow}$ is necessarily finite), and set $\gamma(p_k^{n_k}) = |B_{p_k,n_k}|$ if $p_k^{n_k} \in \mathbb{P}_0^{\uparrow}$. Since the subgroups B_{p_k,n_k} are bounded pure subgroups, there will be [17, Theorem 27.5] a subgroup H of G such that

$$G = \left(\bigoplus_{p_k^{n_k} \in \mathbb{P}_0^{\uparrow}} \bigoplus_{\gamma(p_k^{n_k})} B_{p_k,n_k} \right) \bigoplus H,$$

For each prime p, consider a p-basic subgroup $B_{p,H} = \oplus_n B_{p,n,H}$ of H_p, the p-part of $t(H)$, it is immediately checked that either $B_{p,H}$ itself (if $p \notin \mathbb{P}_0^{\uparrow}$) or $B_{p,H} \bigoplus \left(\bigoplus_{\substack{p_k^{n_k} \in \mathbb{P}_0^{\uparrow} \\ p_k = p}} \bigoplus_{\gamma(p_k^{n_k})} B_{p_k,n_k} \right)$ (if $p \in \mathbb{P}_0^{\uparrow}$) is also p-basic in G.

Since different basic subgroups are necessarily isomorphic [17, Theorem 35], we have that $B_{p,H}$ or $B_{p,H} \bigoplus \left(\bigoplus_{\substack{p_k^{n_k} \in \mathbb{P}_0^{\uparrow} \\ p_k = p}} \bigoplus_{\gamma(p_k^{n_k})} B_{p_k,n_k} \right)$ is isomorphic to B_p. We have therefore that, for each p, either $\sup |B_{p,n,H}|$ is not attained (case 1 above) or attained at infinitely many p^n's (case 2).

Let now n be any natural number. Then $|nB_{p_k,n_k,H}| = |B_{p_k,n_k,H}|$ unless $p_k^{n_k}$ divides n. Since this will only happen for finitely $p_k^{n_k}$'s, we conclude, in both cases 1 and 2 that $|nB_{p,H}| = |B_{p,H}|$.

Using that $B_{p,H}$ is pure in H_p and that $H_p/B_{p,H}$ is divisible we have that,

$$\begin{aligned} |nH_p| &= \left| \frac{nH_p}{nB_{p,H}} \right| + |nB_{p,H}| \\ &= \left| n \left(\frac{H_p}{B_{p,H}} \right) \right| + |B_{p,H}| \\ &= \left| \frac{H_p}{B_{p,H}} \right| + |B_{p,H}| = |H_p|. \end{aligned}$$

Since $|H| = \sum_p H_p + r_0(H)|$ for every infinite group H and $r_0(nH) = r_0(H)$ we have finally that $|H| = |nH|$, for every $n \in \mathbb{Z}$. $\square$

The terminology introduced in the next definition is motivated, in the present context, by Theorem 4.4 below.

Definition 4.2. If G is an Abelian group, the set $\mathbb{P}_0^{\uparrow}$ of Lemma 4.1 can be partitioned as $\mathbb{P}_0^{\uparrow} = \mathbb{P}_1^{\uparrow} \cup \mathbb{P}_2^{\uparrow}$ with $p_i^{n_i} \in \mathbb{P}_1^{\uparrow}$ if, and only if, $\gamma(p_i^{n_i}) > r_0(G)$.

The cardinal numbers $\gamma(p_i^{n_i})$ with $p_i^{n_i} \in \mathbb{P}_1^{\uparrow}$ will be called the *dominant ranks* of G.

Lemma 4.3. *If G is a nontorsion pseudocompact group, then there is a positive integer such that:*

$$m(w(nG)) \leq r_0(nG) \leq 2^{w(nG)}. \tag{4.1}$$

Proof. If nG is metrizable for some $n \in \mathbb{N}$, then nG is a compact metrizable group. Therefore $r_0(nG) = \mathfrak{c}$ and the inequalities in (4.1) hold for this n.

If nG is not metrizable for any $n \in \mathbb{N}$, then G is, in the terminology of [10], *nonsingular*. Combining Lemma 3.3 and Theorem 1.15 of [10], there must be $n \in N$ such that $r_0(nG)$ is the cardinal of a pseudocompact group of weight $w(nG)$. Therefore

$$m(w(nG)) \leq r_0(nG) \leq 2^{w(nG)}.$$

□

Theorem 4.4. *Let G be an Abelian group. If G admits a pseudocompact group topology, then G can be decomposed as*

$$G = \left(\bigoplus_{p^k \in \mathbb{P}_1^{\uparrow}} \bigoplus_{\gamma(p^k)} \mathbb{Z}(p^k) \right) \oplus G_0$$

where $\gamma(p_i^{k_i})$, $p_i^{k_i} \in \mathbb{P}_1^{\uparrow}$, are the dominant ranks of G and there is a cardinal $\omega_d(G)$ such that

$$m(\omega_d(G)) \leq r_0(G) \leq |G_0| \leq 2^{\omega_d(G)}. \tag{4.2}$$

Proof. Since every pseudocompact torsion group must be of bounded order, the theorem is trivial (and vacuous) for such groups, we may assume that G is nontorsion.

Decompose G as in Lemma 4.1:

$$\left(\bigoplus_{p^k \in \mathbb{P}_0^{\uparrow}} \bigoplus_{\gamma(p^k)} \mathbb{Z}(p^k) \right) \bigoplus H$$

with $\mathbb{P}_0^{\uparrow}$ a finite subset of $\mathbb{P}^{\uparrow}$ and

$$|nH| = |H| \text{ for all } n \in \mathbb{N}.$$

Split $\mathbb{P}_0^{\uparrow} = \mathbb{P}_1^{\uparrow} \cup \mathbb{P}_2^{\uparrow}$ as in Definition 4.2 and define

$$G_0 = \bigoplus_{p_i^{k_i} \in \mathbb{P}_2^{\uparrow}} \bigoplus_{\gamma(p_i^{k_i})} \mathbb{Z}(p_i^{k_i}) \bigoplus H.$$

We will prove that the inequalities 4.2 hold for $w_d(G) = w(nG_0)$.

Lemma 4.3 proves that there is some $n \in \mathbb{N}$ with

$$m(w(nG_0)) \le r_0(G_0) \le 2^{w(nG_0)}. \tag{4.3}$$

If $|G_0| = \gamma(p_i^{k_i})$ for some $p_i^{k_i} \in \mathbb{P}_2^{\uparrow}$, it follows from the definition of $P_2^{\uparrow}$ that $|G_0| = r_0(G)$ and (4.2) is deduced from (4.3). If, otherwise, $|G_0| = |H|$, then $|nG_0| \ge |nH| = |H| = |G_0|$ and we deduce that $|G_0| = |nG_0|$ and thus that $|G_0| \le 2^{w(nG_0)}$. This together with (4.3) gives again (4.2) with $w_d(G) = w(nG_0)$. □

Remark 4.5. The cardinal $w_d(G)$ used in Theorem 4.4 is precisely the *divisible weight* of G that was introduced and studied by Dikranjan and Giordano-Bruno [10]. We refer the reader to that paper to get an idea of the important role played by the divisible weight in the structure of pseudocompact groups. One of its applications (Theorem 1.19 loc. cit.) is to prove that $r_0(G)$ is an admissible cardinal for every pseudocompact group G, a fact first proved by Dikranjan and Shakhmatov in [14].

5. Pseudocompact groups with property ♯

The results of the previous sections will be used here to obtain sufficient conditions for the existence of pseudocompact group topologies with property ♯.

Lemma 5.1. *Let $\pi\colon G_1 \to G_2$ be a quotient homomorphism between two Abelian topological groups G_1 and G_2 and let L be a compact Abelian group. Assume that the following conditions hold:*

1. *G_1 contains a free G_δ-dense subgroup H_1 such that H_1 and $\pi(H_1)$ are isomorphic and have property ♯.*
2. *G_1 contains another free subgroup H_2 such that $H_1 \cap H_2 = \{0\}$, $H_1 + H_2$ and $\pi(H_1 + H_2)$ are isomorphic and have property ♯.*
3. *$m(w(L)) \le |H_2|$.*

Under these conditions the product $G_1 \times L$ contains a G_δ-dense subgroup $\widetilde{H}$ such that both $\widetilde{H}$ and $\pi\left(p_1(\widetilde{H})\right)$ have property ♯, where $p_1\colon G_1 \times L \to G_1$ denotes the first projection.

Proof. We first enumerate the elements of H_1 and H_2 as $H_1 = \{f_\beta\colon \kappa < \beta\}$ and $H_2 = \{g_\eta\colon \eta < \alpha\}$. Since $m(w(L)) \le \alpha = |H_2|$, we can also enumerate a G_δ-dense subgroup D of L (allowing repetitions if necessary) as $D = \{d_\eta\colon \eta < \alpha\}$. We now define the subgroup $\widetilde{H}$ of $G_1 \times L$ as

$$\widetilde{H} = \langle\, (f_\kappa + g_\eta, d_\eta)\colon \eta < \alpha, \kappa < \beta \,\rangle\,.$$

It is easy to check that $\widetilde{H}$ is a G_δ-dense subgroup of $G_1 \times L$ with $\widetilde{H} \cap \{0\} \times L = \{(0,0)\}$.

Since the homomorphism p_1 is continuous and establishes a group isomorphism between $\widetilde{H}$ and $H_1 + H_2$, Lemma 2.5 shows that $\widetilde{H}$ has property $\sharp$. The same argument applies to the group $\pi\left(p_1(\widetilde{H})\right) = \pi(H_1 + H_2)$. $\square$

Definition 5.2. Let $\alpha \ge \omega$ be a cardinal. We say that α *satisfies property* $(*)$ *if:*

$$\text{there is a cardinal } \kappa \text{ with } \kappa^\omega \le \alpha \le 2^\kappa \tag{*}$$

Every cardinal α with $\alpha^\omega = \alpha$ satisfies property (*). This condition is equivalent to the condition $(m(\alpha))^\omega \le \alpha$.

To apply Lemma 5.1 we need the following result:

Theorem 5.3 (Theorem 4.5 of [4])**.** *Let $G = (G, \mathcal{T}_1)$ be a pseudocompact Abelian group with $w(G) = \alpha > \omega$, and set*

$$\sigma = \min\{r_0(N) : N \textit{ is a closed } G_\delta\textit{-subgroup of } G\}.$$

If $\alpha^\omega \le \sigma$ and if $\lambda \ge \omega$ satisfies $m(\lambda) \le \sigma$, then G admits a pseudocompact group topology $\mathcal{T}_2$ such that $w(G, \mathcal{T}_2) = \alpha + \lambda$ and $\mathcal{T}_1 \bigvee \mathcal{T}_2$ is pseudocompact. Moreover, every closed G_δ-subgroup of $(G, \mathcal{T}_1)$ is G_δ-dense $(G, \mathcal{T}_2)$.

Corollary 5.4. *Let σ, α and λ be cardinals with $\alpha^\omega \le \sigma$ and $m(\lambda) \le \sigma$. If H is a free, dense subgroup of $\mathbb{T}^\sigma$ with property $\sharp$ and cardinality α, then $\mathbb{T}^\sigma$ contains another subgroup H_2 with $H \cap H_2 = \{0\}$, $|H_2| = \lambda + \alpha$ and such that $H + H_2$ has property $\sharp$.*

Proof. Let $F(\sigma)$ denote the free Abelian group of rank σ. We apply Theorem 5.3 to the pseudocompact group $(F(\sigma), \mathcal{T}_H)$ defined by H. We obtain thus a pseudocompact topology $\mathcal{T}_{H_2}$ on $F(\sigma)$ induced by a subgroup H_2 of $\mathbb{T}^\sigma$ of cardinality $|H_2| = \alpha + \lambda$ such that $\mathcal{T}_H \bigvee \mathcal{T}_{H_2} = \mathcal{T}_{H+H_2}$ is pseudocompact. By Theorem 2.1 the subgroup $H + H_2$ has property $\sharp$ and, since closed G_δ-subgroups of $\mathcal{T}_H$ are G_δ-dense in $\mathcal{T}_{H_2}$, we also have that $H \cap H_2 = \{0\}$. $\square$

Theorem 5.5. *Let G be a pseudocompact Abelian group with dominant ranks $\gamma(p_1^{n_1}),\dots,\gamma(p_k^{n_k})$ and suppose that $\gamma(p_i^{n_i})$, $1\le i\le k$, satisfy property (*). If $r_0(G)$ also satisfies property (*) for some κ with $m(|G_0|)\le 2^\kappa$, then G admits a pseudocompact topology with property $\sharp$.*

Proof. Decompose, following Theorem 4.4, G as a direct sum

$$G=\left(\bigoplus_{\gamma(p_1^{n_1})}\mathbb{Z}(p_1^{n_1})\bigoplus\cdots\bigoplus_{\gamma(p_k^{n_k})}\mathbb{Z}(p_k^{n_k})\right)\bigoplus G_0$$

Let F denote a free Abelian group of cardinality $r_0(G)$ contained in G_0 and denote by $D(F)$ and $D(t(G_0))$ divisible hulls of F and $t(G_0)$, respectively. There is then a chain of group embeddings (here we use [17, Lemmas 16.2 and 24.3])

$$F\overset{j_1}{\to}G_0\overset{j_2}{\to}D(F)\oplus D(t(G_0)) \tag{5.1}$$

Denote by χ the quotient homomorphism obtained as the dual map of the canonical embedding $\mathbb{Z}\to\mathbb{Q}$. Observe that identifying F with $\oplus_{r_0(G)}\mathbb{Z}$ and $D(F)$ with $\oplus_{r_0(G)}\mathbb{Q}$, the dual map of $j_2\circ j_1$ is exactly $\chi^{r_0(G)}$.

Taking $\sigma=r_0(G)$, $G=\mathbb{Q}_d^\wedge$ and $\alpha=\kappa^\omega$, we can apply Proposition 3.3 to get a G_δ-dense subgroup H_1 of $(D(F)_d)^\wedge=\left(\mathbb{Q}_d^\wedge\right)^{r_0(G)}$ with $|H_1|=\kappa^\omega$ and such that H_1 and $\chi^{r_0(G)}(H_1)$ are isomorphic and have property $\sharp$ (notice that κ^ω and $r_0(G)$ satisfy the hypothesis of that Proposition).

We now apply Corollary 5.4 to $\chi^{r_0(G)}(H_1)$ to obtain another free subgroup H_2' of $\mathbb{T}^{r_0(G)}$ with $\chi^{r_0(G)}(H_1)\cap H_2'=\{0\}$, $|H_2'|=2^\kappa$ and such that $\chi^{r_0(G)}(H_1)+H_2'$ has property $\sharp$. By lifting (through $\chi^{r_0(G)}$) the free generators of H_2' to $(D(F)_d)^\wedge$, we obtain a free subgroup H_2 of $(D(F)_d)^\wedge$ such that $H_1\cap H_2=\{0\}$ and $|H_2|=2^\kappa$. Clearly H_1+H_2 is isomorphic to $\chi^{r_0(G)}(H_1)+H_2'$ and therefore H_1+H_2 has property $\sharp$ by Lemma 2.5.

We finally apply Lemma 5.1. The role of $G_1\times L$ is played by $(D(F)_d)^\wedge\times\left(D(t(G_0))_d\right)^\wedge$; G_2 is here identified with $\mathbb{T}^{r_0(G)}$ and π is $\chi^{r_0(G)}$. Lemma 5.1 then provides a G_δ-dense subgroup $\widetilde{H}$ of $\left(D(F)_d\right)^\wedge\times\left(D(t(G_0))_d\right)^\wedge$ such that both $\widetilde{H}$ and $\chi^{r_0(G)}(p_1(\widetilde{H}))$ have property $\sharp$. This subgroup generates a pseudocompact topology $\mathcal{T}_{\widetilde{H}}$ on $D(F)\oplus D(t(G_0))$ with property $\sharp$ that makes F pseudocompact (the induced topology on F is just the topology inherited from $\chi^{r_0(G)}(p_1(\widetilde{H})))$. Since G_0 sits between F and $D(F) \oplus D(t(G_0))$, it follows that the restriction of $\mathcal{T}_{\widetilde{H}}$ to G_0 is pseudocompact and has property $\sharp$.

By Proposition 3.4 the bounded group $\bigoplus_{\alpha(p_1^{n_1})} \mathbb{Z}(p_1^{n_1}) \bigoplus \cdots \bigoplus_{\alpha(p_k^{n_k})} \mathbb{Z}(p_k^{n_k})$ also admits a pseudocompact group topology with property $\sharp$ and the theorem follows. $\square$

Dikranjan and Shakmatov [12] prove under a set-theoretic axiom called ∇_κ (that implies $\mathfrak{c} = \omega_1$ and $2^{\mathfrak{c}} = \kappa$ with κ being any cardinal $\kappa \geq \omega_2$) that every pseudocompact group of cardinality at most $2^{\mathfrak{c}}$ has a pseudocompact group topology with no infinite compact subsets. It follows from Theorem 5.5 that the result is true in ZFC, even for larger cardinalities.

Corollary 5.6. *Let G be a pseudocompact Abelian group of cardinality $|G| \leq 2^{2^{\mathfrak{c}}}$. Then G admits a pseudocompact topology with property $\sharp$ (and thus a pseudocompact topology with no infinite compact subsets).*

Proof. Since a pseudocompact group with $r_0(G) < \mathfrak{c}$ is a bounded group it will suffice to check that every cardinal α with $\alpha \leq 2^{2^{\mathfrak{c}}}$ satisfies property (*). Theorem 5.5 will then be applied. We consider the following two cases:

Case 1: $\mathfrak{c} \leq \alpha \leq 2^{\mathfrak{c}}$. In this case we put $\kappa = \mathfrak{c}$.

Case 2: $\alpha > 2^{\mathfrak{c}}$. Choose $\kappa = 2^{\mathfrak{c}}$ for this case.

Observe that in both cases $|m(|G|)| \leq 2^\kappa$ and hence that all hypothesis of Theorem 5.5 are fulfilled. $\square$

By van Douwen's theorem [15], a strong limit admissible cardinal must have uncountable cofinality. Under mild set-theoretic assumptions this implies that admissible cardinals must have property (*). It suffices, for instance, to assume the *Singular Cardinal Hypothesis* SCH.

Theorem 5.7 (Theorem 3.5 of [6] and Lemma 3.4 of [11])**.** *If SCH is assumed, then every admissible cardinal has property* (*)*.*

Combining Theorem 4.4 and Theorem 5.5, it turns out that, under SCH, every pseudocompact group admits a pseudocompact group topology with property $\sharp$.

Theorem 5.8 (SCH)**.** *Every pseudocompact Abelian group G admits a pseudocompact group topology with property $\sharp$.*

Proof. Let $\gamma(p_1^{n_1}) \geq \cdots \geq \gamma(p_k^{n_k})$ be the dominant ranks of G. Then $|G| = \gamma(p_1^{n_1})$ and, $\gamma(p_1^{n_1})$ is admissible. Since we can assume that $n_i < n_j$ when

$j > i$ and $p_i = p_j$, p_1G will be a pseudocompact group of cardinality $|p_1G| = \gamma(p_2^{n_2})$. Proceeding in the same way we obtain that the dominant ranks are admissible cardinals. By Theorem 5.7 all these cardinals must satisfy property (*). Theorem 4.4 shows, on the other hand, that the cardinal $r_0(G)$ is also admissible and, actually:

$$m(w_d(G_0)) \leq r_0(G_0) = r_0(G) \leq |G_0| \leq 2^{w_d(G_0)}$$

In order to apply Theorem 5.5 and finish the proof, we must show that $r_0(G)$ also satisfies property (*) for some cardinal κ with $m(|G_0|) \leq 2^\kappa$.

We have two possibilities:

Case 1: $m(w_d(G_0)) \leq r_0(G) \leq (w_d(G_0))^\omega$. In this case, we put $\kappa = \log(w_d(G_0))$. Then, bearing in mind that, under SCH, we have $m(\alpha) = (\log(\alpha))^\omega$ for every infinite cardinal α, we get:

$$\kappa^\omega = \Big(\log\big(w_d(G_0)\big)\Big)^\omega = m\big(w_d(G_0)\big) \leq r_0(G)$$

and

$$r_0(G) \leq \big(w_d(G_0)\big)^\omega \leq \Big(2^{\log\left(w_d(G_0)\right)}\Big)^\omega = (2^\kappa)^\omega = 2^\kappa.$$

So property (*) is checked. On the other hand,

$$m(|G_0|) \leq m\big(2^{w_d(G_0)}\big) = \Big(\log\big(2^{w_d(G_0)}\big)\Big)^\omega \leq \big(w_d(G_0)\big)^\omega \leq 2^\kappa$$

Case 2: $\big(w_d(G_0)\big)^\omega \leq r_0(G) \leq 2^{w_d(G_0)}$. In this case, property (*) and condition $m(|G_0|) \leq 2^\kappa$ are obviously fulfilled with $\kappa = w_d(G_0)$. □

Theorem 5.8 relies quite strongly on SCH. It uses the construction of Theorem 5.5 made applicable to all admissible cardinals by Theorem 5.7. We do not know whether SCH is essential for Theorem 5.8, i.e., whether the theorem is true for pseudocompact groups whose cardinal does not satisfy property (*).

Indeed, admissible cardinals not satisfying property (*) are hard to find in the literature. The following (consistent) example, suggested to us by W.W. Comfort and based on a construction due to Gitik and Shelah, produces one such cardinal. We refer to Remark 3.14 of the forthcoming paper [5] for additional remarks concerning the Gitik-Shelah models. This same paper contains related results concerning the cardinals $m(\alpha)$ and, more generally, the density character of powers of discrete groups in the κ-box topology.

Example 5.9. A pseudocompact group G whose cardinality does not satisfy property (*).

Proof. Gitik and Shelah, [22], construct a model where $m(\aleph_\omega) = \aleph_{\omega+1}$ while $2^{\aleph_\omega} = (\aleph_\omega)^\omega = \aleph_{\omega+2}$. This means that the compact group $\{1,-1\}^{\aleph_\omega}$ has a G_δ-dense subgroup G of cardinality $|G| = \aleph_{\omega+1}$. Let us denote for simplicity $\alpha = \aleph_{\omega+1}$.

Suppose that α satisfies property (*). There is then a cardinal κ with

$$\kappa^\omega \leq \alpha \leq 2^\kappa. \tag{5.2}$$

Since $\alpha^\omega \geq (\aleph_\omega)^\omega = \aleph_{\omega+2} > \alpha$, we see that $\kappa^\omega \neq \alpha$. It follows then from (5.2) that $\kappa^\omega \leq \aleph_\omega \leq 2^\kappa$. But then $m(\aleph_\omega) \leq m(2^\kappa) \leq \kappa^\omega \leq \aleph_\omega$, whereas, by construction, $m(\aleph_\omega) = \aleph_{\omega+1}$. This contradiction shows that α does not satisfy property (*). □

6. Property ♯ and the duality of totally bounded Abelian groups

Pontryagin duality was designed to work in locally compact Abelian groups and usually works better for complete groups. This behaviour raised the question (actually our first motivating Question 1.1) as to whether all totally bounded reflexive group should be compact, [3]. We see next that this is not the case.

Theorem 6.1. *If a pseudocompact Abelian group contains no infinite compact subsets, then it is Pontryagin reflexive.*

Proof. Let $G = (G, \mathcal{T}_H)$ be a pseudocompact group with no infinite compact subsets. The group of continuous characters of G is then precisely H and since G has no infinite compact subsets, the topology of this dual group will equal the topology of pointwise convergence on G, therefore $G^\wedge = (H, \mathcal{T}_G)$ (see in this connection [24]). By Theorem 2.1, $(H, \mathcal{T}_G)$ must be again a totally bounded group with property ♯ and hence with no infinite compact subsets, the same argument as above then shows that $G^{\wedge\wedge} = (H, \mathcal{T}_G)^\wedge = (G, \mathcal{T}_H)$ and therefore that G is reflexive. □

This last theorem combined with Lemma 2.3 and the results of Section 5 provides a wide range of examples that answer negatively Question 1.1. This question has also been answered independently in [1] where another collection of examples has been obtained.

Corollary 6.2 (SCH)**.** *Every infinite pseudocompact Abelian group G supports a noncompact, pseudocompact group topology $\mathcal{T}_H$ such that $(G, \mathcal{T}_H)$ is reflexive.*

Corollary 6.3. *Every infinite pseudocompact Abelian group G with $|G| \leq 2^{2^{\mathfrak{c}}}$ supports a noncompact, pseudocompact group topology $\mathcal{T}_H$ such that $(G, \mathcal{T}_H)$ is reflexive.*

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References

[1] S. Ardanza-Trevijano, M.J. Chasco, X. Domínguez, and M. G. Tkachenko. Precompact noncompact reflexive abelian groups. *Forum Mathematicum*. To appear.

[2] A. Arhangel′skii and M. G. Tkachenko. *Topological groups and related structures*, volume 1 of *Atlantis Studies in Mathematics*. Atlantis Press, Paris, 2008.

[3] M. J. Chasco and E. Martín-Peinador. An approach to duality on abelian precompact groups. *Journal of Group Theory*, 11:5:635–643, 2008.

[4] W. W. Comfort and J. Galindo. Pseudocompact topological group refinements of maximal weight. *Proc. Amer. Math. Soc.*, 131(4):1311–1320 (electronic), 2003.

[5] W. W. Comfort and I. Gotchev. Cardinal invariants for κ-box products: weight, density, character and Souslin number. Preprint., 2010.

[6] W. W. Comfort and D. Remus. Imposing pseudocompact group topologies on abelian groups. *Fund. Math.*, 142(3):221–240, 1993.

[7] W. W. Comfort and L. C. Robertson. Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups. *Pacific J. Math.*, 119(2):265–285, 1985.

[8] W. W. Comfort and K. A. Ross. Topologies induced by groups of characters. *Fund. Math.*, 55:283–291, 1964.

[9] W. W. Comfort and Kenneth A. Ross. Pseudocompactness and uniform continuity in topological groups. *Pacific J. Math.*, 16:483–496, 1966.

[10] D. Dikranjan and A. Giordano Bruno. w-divisible groups. *Topology Appl.*, 155(4):252–272, 2008.

[11] D. Dikranjan and D. Shakhmatov. Algebraic structure of pseudocompact groups. *Mem. Amer. Math. Soc.*, 133(633):x+83, 1998.

[12] D. Dikranjan and D. Shakhmatov. Forcing hereditarily separable compact-like group topologies on abelian groups. *Topology Appl.*, 151(1-3):2–54, 2005.

[13] D. Dikranjan and D. Shakhmatov. Selected topics from the structure theory of topological groups. In Elliott Pearl, editor, *Open problems in topology*, pages 389–406. Elsevier, 2007.

[14] D. Dikranjan and D. Shakhmatov. Algebraic structure of pseudocompact abelian groups. Preprint, 2008.

[15] E. K. van Douwen. The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality. *Proc. Amer. Math. Soc.*, 80(4):678–682, 1980.

[16] E. K. van Douwen. The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G. *Topology Appl.*, 34(1):69–91, 1990.

[17] L. Fuchs. *Infinite abelian groups. Vol. I.* Academic Press, New York, 1970.

[18] J. Galindo and S. García-Ferreira. Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences. *Topology Appl.*, 154(2):476–490, 2007.

[19] J. Galindo, S. García-Ferreira, and A. H. Tomita. Pseudocompact group topologies with prescribed topological subspaces. *Sci. Math. Jpn.*, 70(3):269–279, 2009.

[20] J. Galindo and S. Hernández. On a theorem of van Douwen. *Extracta Math.*, 13(1):115–123, 1998.

[21] J. Galindo and S. Hernández. The concept of boundedness and the Bohr compactification of a MAP abelian group. *Fund. Math.*, 159(3):195–218, 1999.

[22] M. Gitik and S. Shelah. On densities of box products. *Topology Appl.*, 88:219–237, 1998.

[23] S. Hernández and S. Macario. Dual properties in totally bounded abelian groups. *Arch. Math. (Basel)*, 80(3):271–283, 2003.

[24] S. U. Raczkowski and F. Javier Trigos-Arrieta. Duality of totally bounded abelian groups. *Bol. Soc. Mat. Mexicana (3)*, 7(1):1–12, 2001.

[25] M. G. Tkačenko. Compactness type properties in topological groups. *Czechoslovak Math. J.*, 38(113)(2):324–341, 1988.

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