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Orthogonal ℓ_1-sets and extreme non-Arens regularity of preduals of von Neumann algebras ☆

M. Filali [a], J. Galindo [b,*]

[a] *Department of Mathematical Sciences, University of Oulu, Oulu, Finland*
[b] *Instituto Universitario de Matemáticas y Aplicaciones (IMAC), Universidad Jaume I, E-12071, Castellón, Spain*

A R T I C L E I N F O

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A B S T R A C T

A Banach algebra $\mathcal{A}$ is Arens-regular when all its continuous functionals are weakly almost periodic, in symbols when $\mathcal{A}^* = \mathcal{WAP}(\mathcal{A})$. To identify the opposite behaviour, Granirer called a Banach algebra extremely non-Arens regular (enAr, for short) when the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ contains a closed subspace that has $\mathcal{A}^*$ as a quotient. In this paper we propose a simplification and a quantification of this concept. We say that a Banach algebra $\mathcal{A}$ is r-enAr, with $r \geq 1$, when there is an isomorphism with distortion r of $\mathcal{A}^*$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. When $r = 1$, we obtain an isometric isomorphism and we say that $\mathcal{A}$ is isometrically enAr. We then identify sufficient conditions for the predual $\mathfrak{V}_*$ of a von Neumann algebra $\mathfrak{V}$ to be r-enAr or isometrically enAr. With the aid of these conditions, the following algebras are shown to be r-enAr:

(i) the weighted semigroup algebra of any weakly cancellative discrete semigroup, when the weight is diagonally bounded with diagonal bound $c \geq r$. When the weight is multiplicative, i.e., when $c = 1$, the algebra is isometrically enAr,
(ii) the weighted group algebra of any non-discrete locally compact infinite group and for any weight,
(iii) the weighted measure algebra of any locally compact infinite group, when the weight is diagonally bounded with diagonal bound $c \geq r$. When the weight is multiplicative, i.e., when $c = 1$, the algebra is isometrically enAr.

The Fourier algebra $A(G)$ of a locally compact infinite group G is shown to be isometrically enAr provided that (1) the local weight of G is greater or equal than its compact covering number, or (2) G is countable and contains an infinite amenable subgroup.

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* Corresponding author.
E-mail addresses: mfilali@cc.oulu.fi (M. Filali), jgalindo@uji.es (J. Galindo).

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1. Introduction

In [1], Richard Arens showed how to extend the product of a Banach algebra $\mathcal{A}$ to its second dual $\mathcal{A}^{**}$. He in fact observed that there are two symmetric and identically natural ways of performing this extension. Each of these paths however leads to a *different* multiplication on $\mathcal{A}^{**}$. With one of these multiplications, left translations are weak*-weak*-continuous but right multiplications may fail to be so. With the other multiplication, the situation is reversed.

There is a subspace of $\mathcal{A}^*$ on which both multiplications coincide, the space $\mathcal{WAP}(\mathcal{A})$ of weakly almost periodic functionals elements. This follows from an important property of weakly almost periodic functionals: Grothendieck's double limit criterion. According to this criterion, $f \in \mathcal{WAP}(\mathcal{A})$ if and only if for every pair of bounded nets $(a_\alpha)_{\alpha\in\Lambda_1}$ and $(b_\beta)_{\beta\in\Lambda_2}$,

$$\lim_\alpha \lim_\beta f(a_\alpha b_\beta) = \lim_\beta \lim_\alpha f(a_\alpha b_\beta),$$

whenever both limits exist.

So, when $\mathcal{A}^* = \mathcal{WAP}(\mathcal{A})$, i.e., when the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ is trivial, there is only one Arens product, which is separately weak*-weak*-continuous. In such a situation, the algebra $\mathcal{A}$ is said to be *Arens regular*. Otherwise, the algebra $\mathcal{A}$ is said to be *non-Arens regular* or *Arens irregular*. C*-algebras constitute the paradigmatic example of Arens regular Banach algebras. If $\mathcal{A}$ is a C^*-algebra, its universal representation identifies $\mathcal{A}$ with a norm-closed algebra of operators on a Hilbert space. By the Sherman-Takeda theorem, $\mathcal{A}^{**}$ may be identified with the closure of the universal representation of $\mathcal{A}$ in the weak operator topology, and both Arens products coincide with the multiplication of operators, see [4].

Most of the algebras of functions arising in harmonic analysis turned out to be non-Arens regular, even dramatically so. Not only $\mathcal{WAP}(\mathcal{A})$ is often different from $\mathcal{A}^*$ but the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ tends to be as large as $\mathcal{A}^*$. A name for this situation was coined by Granirer in [18] when he called a Banach algebra $\mathcal{A}$ *extremely non-Arens regular* (enAr, for short) if $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ contains a closed subspace that has $\mathcal{A}^*$ as a quotient. The group algebra $L^1(G)$ of an infinite locally compact group G is an important example of a Banach algebra that is enAr, see [12]. Extreme non-Arens regularity of the Fourier algebra $A(G)$ is more subtle. First, the question of whether $A(G)$ is non-Arens regular is still not completely settled. It is known that $A(G)$ is not Arens regular if G contains an infinite amenable subgroup or if G is not discrete. The first assertion was obtained by Forrest in [17, Proposition 3.7], improving upon results of Lau and Wong [27, Proposition 5.3], and the second assertion was proved by Forrest [17, Corollary 3.2]. If G is far enough from being discrete, then $A(G)$ is even enAr. For this to happen, it is enough that the minimal cardinal of an open base at the identity, $\chi(G)$, is larger than $\kappa(G)$, the minimal number of compact sets required to cover G. This was proved by Granirer [18] when $\chi(G) = \omega$ and by Hu [21] in the general case.

We refer to our recent paper [13], or the expository papers [14] and [15], for a wider background, and an extended list of references on Arens irregularity. In our paper [13] we devised a general method for proving when a Banach algebra is enAr. This method showed how the two main properties that trigger non-Arens regularity in the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ -non-compactness and bounded approximate identity in $L^1(G)$, non-discreteness and amenability in $A(G)$- are actually particular cases of a single one, the existence of ℓ^1-bases with a certain multiplicative triangle-like structure in a bounded subset of the algebra.

In the present paper we consider preduals of von Neumann algebras. In this special case, one may take advantage of the additional structure available and require the above ℓ^1-bases to be orthogonal. With that requirement, it is possible to construct bounded linear isomorphisms of $\mathcal{A}^*$ into the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ in such a way that their distortion is controlled. When the above mentioned triangles lie in the unit sphere of $\mathcal{A}$, and this can be achieved in many cases, these isomorphisms become isometries.

The availability of these constructions has led us to propose a new version of extreme non-Arens regularity by requiring an isomorphic (or an isometric) copy of $\mathcal{A}^*$ into the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. This definition seems to be more natural and still holds in most of the known classes of enAr algebras.

We remark that, to the best of our knowledge, isometries of $\mathcal{A}^*$ into the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ have only been obtained before in the particular case of $A = L^1(G)$, [12], with quite a different approach.

In particular, isomorphisms or isometries of $\mathcal{A}^*$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ could not be obtained from our previous paper [13], as it is not possible to deduce (to the best of our ability) Theorem 3.12 below from Theorem 3.9 of [13]. The strategy using lower and upper triangles (see Section 3) is indeed used in both theorems, as it is the natural method with this problem and has been used in the past by many authors, as mentioned in [13]. But the proofs of these two theorems are in essence different. In the general case treated in [13], $\ell^1(\eta)$-bases are used to find a closed subspace of $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ having $\mathcal{A}^*$ as a quotient via a Hahh-Banach argument, but we have no control on how to embed $\mathcal{A}^*$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$, while in Theorem 3.12 the isomorphisms, or isometries, are defined concretely using the orthogonal projections in the von Neumann algebras.

We remark in closing that there is no reason for Banach algebras that happen to be the predual of von Neumann algebras to be enAr. They can even be Arens regular as is the case with the semigroup algebra ℓ^1 with pointwise product as already observed by Arens in [1], or with many weighted semigroup algebras $\ell^1(S, w)$, see Remark 4.8.

2. Outline of the paper

As stated above, Granirer defined a Banach algebra $\mathcal{A}$ to be enAr when the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ contains a closed linear subspace which has $\mathcal{A}^*$ as a continuous linear image. We start by modifying this definition.

Definition 2.1. We say that a Banach algebra $\mathcal{A}$ is *r-enAr*, where $r \geq 1$, when there is a linear isomorphism of $\mathcal{A}^*$ in the quotient $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ with distortion r, i.e., when there is a linear isomorphism $\mathcal{E}\colon \mathcal{A}^* \to \mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ with

$$\|\mathcal{E}\|\|\mathcal{E}^{-1}\| = r.$$

When $r = 1$, the map $\|\mathcal{E}^{-1}\|\mathcal{E}$ is a linear isometry and we say that $\mathcal{A}$ *is isometrically enAr*.

In Section 3, a combination of the strategy used in the general theorem of [13] with the concept of orthogonal family sets, natural conditions under which the predual of a von Neumann algebra is r-enAr for some $r \geq 1$. This is Theorem 3.12. The following two sections apply Theorem 3.12 to the Banach algebras in harmonic analysis which are non-Arens regular. We extend the main theorem in [12] to the weighted group algebra and prove that the weighted group algebra $L^1(G, w)$ is isometrically enAr for any non-discrete locally compact group G and for any weight function on G. If the weight function w is diagonally bounded with bound c, then the weighted semigroup algebra $\ell^1(S, w)$ for any infinite, discrete, weakly cancellative semigroup S is r-enAr with $r \leq c$. When the weight function w is diagonally bounded on G, the same is also true for the weighted measure algebra $M(G, w)$ for any infinite locally compact group G. Both weighted algebras $\ell^1(S, w)$ and $M(G, w)$ are isometrically enAr when the weight is multiplicative.

In our last section we show that the conditions of Theorem 3.12 are met when $A(G)$ contains either a TI-net (a net converging to a topologically invariant mean in $A(G)^{**}$) or a bai-sequence (a sequence converging to a right identity in $A(G)^{**}$, see section 3.2 for the definitions). This implies that $A(G)$ is isometrically enAr when G satisfies the condition $\chi(G) \geq \kappa(G)$ or when G is a second countable group containing a non-compact amenable open subgroup. This strengthens the corresponding results in [13], [18] and [21].

3. Triangles and weakly almost periodic functionals

In this section we put forward the main tools developed in this paper. We begin with a number of definitions designed to set the ground for the somewhat subtle combinatorial arguments needed in our subsequent results. These definitions were introduced in Section 3 of [13], we reproduce them here for the benefit of the reader.

By a directed set, it is always meant a set Λ together with a preorder $\preceq$ with the additional property that every pair of elements has an upper bound. We will use a single letter, Λ usually, to denote a directed set, the existence of $\preceq$ is implicitly assumed.

Definition 3.1. Let $(\Lambda, \preceq)$ be a directed set and let Λ_1, Λ_2 be two cofinal subsets of Λ. If $\mathcal{U}$ is a subset of $\Lambda_1 \times \Lambda_2$, we say that

(i) $\mathcal{U}$ is *vertically cofinal*, when for every $\alpha \in \Lambda_1$, there exists $\beta(\alpha) \in \Lambda_2$ such that $(\alpha, \beta) \in \mathcal{U}$ for every $\beta \in \Lambda_2$, $\beta \succeq \beta(\alpha)$.
(ii) $\mathcal{U}$ is *horizontally cofinal*, when for every $\beta \in \Lambda_2$, there exists $\alpha(\beta) \in \Lambda_1$ such that $(\alpha, \beta) \in \mathcal{U}$ for every $\alpha \in \Lambda_1$, $\alpha \succeq \alpha(\beta)$.

Definition 3.2. Let $\mathcal{U}$ and X be two sets. We say that

(i) X is *indexed* by $\mathcal{U}$, when there exists a surjective map $x : \mathcal{U} \to X$. When $\mathcal{U} \subset \Lambda \times \Lambda$, for some other set Λ, we say that X is *double-indexed* by $\mathcal{U}$ and write $X = \{x_{\alpha\beta} : (\alpha, \beta) \in \mathcal{U}\}$, where $x_{\alpha\beta} = x(\alpha, \beta)$.
(ii) If X is double-indexed by $\mathcal{U}$, we say it is *vertically injective* if $x_{\alpha\beta} = x_{\alpha'\beta'}$ implies $\beta = \beta'$ for every $(\alpha, \beta) \in \mathcal{U}$. If $x_{\alpha\beta} = x_{\alpha'\beta'}$ implies $\alpha = \alpha'$ for every $(\alpha, \beta) \in \mathcal{U}$, we say that X is *horizontally injective*.

Definition 3.3. Let $\mathcal{A}$ be a Banach algebra, $(\Lambda, \preceq)$ be a directed set and Λ_1, Λ_2 be two cofinal subsets of Λ. Consider two subsets, A and B, of $\mathcal{A}$ indexed, respectively, by Λ_1 and Λ_2, i.e.,

$$A = \{a_\alpha \colon \alpha \in \Lambda_1\} \quad \text{and} \quad B = \{b_\alpha \colon \alpha \in \Lambda_2\}.$$

(i) The sets

$$T^u_{AB} = \{a_\alpha b_\beta \colon (\alpha, \beta) \in \Lambda_1 \times \Lambda_2,\ \alpha \prec \beta\} \quad \text{and}$$
$$T^l_{AB} = \{a_\alpha b_\beta \colon (\alpha, \beta) \in \Lambda_1 \times \Lambda_2,\ \beta \prec \alpha\}$$

are called, respectively, the *upper* and *lower triangles defined by A and B*.
(ii) A set $X \subseteq \mathcal{A}$ is said to *approximate segments in* T^u_{AB}, if there exists a vertically cofinal set $\mathcal{U}$ in $\Lambda_1 \times \Lambda_2$ so that X is double-indexed as $X = \{x_{\alpha\beta} \colon (\alpha, \beta) \in \mathcal{U}\}$, and for each $\alpha \in \Lambda_1$,

$$\lim_{\beta \succeq \beta(\alpha)} \|x_{\alpha\beta} - a_\alpha b_\beta\| = 0.$$

Note that, by considering an appropriate subset of X we can assume that $(\alpha, \beta) \in \mathcal{U}$ implies $\beta \succ \alpha$.
(iii) A set $X \subseteq \mathcal{A}$ is said to *approximate segments in* T^l_{AB}, if there exists a horizontally cofinal set $\mathcal{U}$ in $\Lambda_1 \times \Lambda_2$ so that X is double-indexed as $X = \{x_{\alpha\beta} \colon (\alpha, \beta) \in \mathcal{U}\}$, and for each $\beta \in \Lambda_2$,

$$\lim_{\alpha \succeq \alpha(\beta)} \|x_{\alpha\beta} - a_\alpha b_\beta\| = 0.$$

As before, we can assume here that $(\alpha, \beta) \in \mathcal{U}$ implies $\alpha \succ \beta$.

Definition 3.4. Let $\mathfrak{V}$ be a von Neumann algebra with predual $\mathfrak{V}_*$, and let $a \in \mathfrak{V}_*$ be a positive normal functional. The *support projection* of a is the smallest projection $S(a) \in \mathfrak{V}$ such that $\langle a, S(a)\rangle = \langle a, I\rangle = \|a\|$.

Definition 3.5. Let $\mathfrak{V}$ be a von Neumann algebra and let $A \subseteq \mathfrak{V}_*^+$. We say that A is an *orthogonal* $\ell^1(\eta)$-set with bound M and constant $K > 0$ if

(i) $|A| = \eta$,
(ii) $S(a)\, S(a') = 0$ whenever $a, a' \in A$, $a \neq a'$ and
(iii) $K \leq \|a\| \leq M$ for every $a \in A$.

Remarks 3.6.

(i) *If one considers the action of $\mathfrak{V}$ on $\mathfrak{V}_*$ given by $\langle Fa, H\rangle = \langle a, FH\rangle$ for every $F, H \in \mathfrak{V}$ and every $a \in \mathfrak{V}_*$, then for every $a \in \mathfrak{V}_*^+$, $S(a)a = a$.*
This follows from applying the Cauchy Schwarz inequality (see e.g. [24, 4.3.1]), to $a \in \mathfrak{V}_*^+$, $S(a) - I$ and an arbitrary $Q \in \mathfrak{V}$:

$$|\langle a, (S(a) - I)Q\rangle|^2 \leq \langle a, (S(a) - I)^2\rangle \cdot \langle a, QQ^*\rangle.$$

Since $|\langle S(a)a - a, Q\rangle|^2 = |\langle a, (S(a) - I)Q\rangle|^2$, and, by definition of $S(a)$, $\langle a, (S(a) - I)^2\rangle = 0$, the equality $S(a)a = a$ follows.

(ii) *If $S(a)$ and $S(b)$ are orthogonal, then $\langle a, S(b)\rangle = 0$.* This is a consequence of the preceding item

$$\langle a, S(b)\rangle = \langle S(a)a, S(b)\rangle = \langle a, S(a)S(b)\rangle = 0.$$

(iii) *We have chosen the term* orthogonal $\ell^1(\eta)$-set *because these sets are equivalent to the unit ℓ^1-basis, i.e., the closed vector space they span is isomorphic to $\ell^1(\eta)$.* To see this let $a_1, \ldots, a_k \in A$ and $z_1, \ldots, z_k \in \mathbb{C}$. Taking into account that $\left\|\sum_{k=1}^n \frac{\overline{z_k}}{|z_k|} S(a_k)\right\|_{\mathfrak{V}} \leq 1$, we have that

$$\begin{aligned}\left\|\sum_{k=1}^n z_k a_k\right\| &\geq \left|\left\langle \sum_{k=1}^n z_k a_k, \sum_{k=1}^n \frac{\overline{z_k}}{|z_k|} S(a_k)\right\rangle\right| \\ &= \left|\sum_{k=1}^n \sum_{j=1}^n z_k \frac{\overline{z_j}}{|z_j|} \langle S(a_j), a_k\rangle\right| \\ &= \left|\sum_{k=1}^n |z_k| \langle S(a_k), a_k\rangle\right| = \sum_{k=1}^n |z_k| \|a_k\| \geq K \sum_{k=1}^n |z_k|,\end{aligned}$$

and this shows that A is equivalent to the unit ℓ^1-basis.

3.1. Auxiliary lemmas

The following definition and its consequence, recorded in [12], will prove convenient to exploit Definition 3.5.

Definition 3.7. Let E_1 and E_2 be Banach spaces, $\mathcal{T}\colon E_1 \to E_2$ be a bounded linear map, F be a closed subspace of E_2, and let $c > 0$. We say that $\mathcal{T}$ is c-preserved by F when the following property holds

$$\|\mathcal{T}\xi - \phi\| \geq c\|\xi\|, \quad \text{for all } \phi \in F \text{ and } \xi \in E_1. \qquad (*)$$

The proof of next lemma is similar to that of [12, Lemma 2.2].

Lemma 3.8. *Let $\mathcal{T}\colon E_1 \to E_2$ be a bounded linear isomorphism of the Banach spaces E_1 into E_2 and let D, F be closed linear subspaces of E_2 with $D \subseteq F$. Denote by $Q\colon E_2 \to E_2/D$ the quotient map. If, for some $c > 0$, T is c-preserved by F, then the map $Q \circ \mathcal{T}\colon E_1 \to E_2/D$ is a linear isomorphism with distortion at most $\frac{\|\mathcal{T}\|}{c}$.*

3.2. Weak almost periodicity

We materialize here the facts about weak almost periodicity mentioned in the introduction. If $\mathcal{A}$ is a Banach algebra, the left and right actions of $a \in \mathcal{A}$ on $f \in \mathcal{A}^*$ are defined, respectively, by:

$$\langle a \cdot f, b\rangle = \langle f, ba\rangle, \quad \langle fa, b\rangle = \langle f, ab\rangle, \qquad \text{for every } b \in \mathcal{A}.$$

In the following definitions we use the symbol $\mathcal{A}_1$ to denote the unit ball of the Banach space $\mathcal{A}$.

Definition 3.9. Let $\mathcal{A}$ be a Banach algebra. A functional $f \in \mathcal{A}^*$ is said to be *weakly almost periodic* if the left orbit $\mathcal{A}_1 \cdot f = \{a \cdot f\colon a \in \mathcal{A}_1\}$ is relatively weakly compact.

Theorem 3.10 *([28]). Let $\mathcal{A}$ be a Banach algebra and $f \in \mathcal{A}^*$. The following are equivalent:*

(i) *f is weakly almost periodic.*
(ii) *The right orbit $f\mathcal{A}_1 = \{fa\colon a \in \mathcal{A}_1\}$ is relatively weakly compact.*
(iii) *(Grothendieck's double limit criterion)*

$$\lim_\alpha \lim_\beta f(a_\alpha b_\beta) = \lim_\beta \lim_\alpha f(a_\alpha b_\beta)$$

for every pair of bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\mathcal{A}$ for which both limits exist.

One more lemma is needed before we state our main theorem. It is an easy corollary of the strong convergence of the sum of orthogonal projections on a Hilbert space, and might be well-known. We include the proof for completeness.

Lemma 3.11. *Let $\{P_\xi\}_{\xi\prec\eta}$ be a family of orthogonal projections on a Hilbert space $\mathbb{H}$ and $\mathbf{v} = (z_\xi)_{\xi\prec\eta} \in \ell^\infty(\eta)$. Then $\sum_{\xi\prec\eta} z_\xi P_\xi$ converges strongly to a bounded operator $P_{\mathbf{v}}$ on $\mathbb{H}$. Moreover, $\|P_{\mathbf{v}}\| = \|\mathbf{v}\|_\infty$.*

Proof. Note first that, for every finite subset F of η, $P_F = \sum_{\xi\in F} P_\xi$ is a projection and, thus, $\|P_F\| = 1$.

Then, for any finite subset F of η and any vector w in the Hilbert space $\mathbb{H}$, we have

$$\begin{aligned}\|\sum_{\xi\in F} z_\xi P_\xi w\|^2 &= \langle \sum_{\xi\in F} z_\xi P_\xi w, \sum_{\xi\in F} z_\xi P_\xi w\rangle = \sum_{\xi\in F} |z_\xi|^2 \langle P_\xi w, P_\xi w\rangle \\ &< \|\mathbf{v}\|_\infty^2 \sum_{\xi\in F} \langle P_\xi w, P_\xi w\rangle \le \|\mathbf{v}\|_\infty^2 \|P_F\| \|w\|^2 = \|\mathbf{v}\|_\infty^2 \|w\|^2.\end{aligned} \tag{3.1}$$

Let now $\eta^{<\omega}$ denote the set of all finite subsets of η, direct it by set inclusion, and consider for each $w \in \mathbb{H}$ the net in $\mathbb{H}$ given by $\left(\sum_{\xi\in F} z_\xi P_\xi w\right)_{F\in\eta^{<\omega}}$.

Since the net of projections $(P_F)_{F\in\eta^{<\omega}}$ is convergent (see for instance [24, Proposition 2.5.6]), there exists for each $\epsilon > 0$, $F_0 \in \eta^{<\omega}$ such that $\|P_{F_1} w - P_{F_2} w\| < \epsilon$ for every $F_1, F_2 \in \eta^{<\omega}$, with $F_1, F_2 \supseteq F_0$. We have then that

$$\left\|\sum_{\xi\in F_1} z_\xi P_\xi w - \sum_{\xi\in F_2} z_\xi P_\xi w\right\|^2 = \sum_{\xi\in F_1\Delta F_2} |z_\xi|^2 \|P_\xi w\|^2 \leq \|\mathbf{v}\|_\infty^2 \sum_{\xi\in F_1\Delta F_2} \|P_\xi w\|^2$$

$$= \left\|\sum_{\xi\in F_1} P_\xi w - \sum_{\xi\in F_2} P_\xi w\right\|^2 = \|P_{F_1} w - P_{F_2} w\| < \epsilon$$

whenever $F_1, F_2 \in \eta^{<\omega}$, $F_1 \supseteq F_0, F_2 \supseteq F_0$.

The net $\left(\sum_{\xi\in F} z_\xi P_\xi w\right)_{F\in\eta^{<\omega}}$ is accordingly a Cauchy net in $\mathbb{H}$. We denote by $P_{\mathbf{v}}w$ its limit in $\mathbb{H}$. Then it is clear from (3.1) that $w \mapsto P_{\mathbf{v}}w$ defines a bounded operator $P_{\mathbf{v}}$ on $\mathbb{H}$ with $\|P_{\mathbf{v}}\| = \|\mathbf{v}\|_\infty$. □

In our previous paper [13], we dealt with Banach algebras that contain $\ell^1(\eta)$-bases that approximate segments in triangles. For such an $\ell^1(\eta)$-base X, contained in the Banach algebra $\mathcal{A}$, we constructed an isomorphism $\ell^\infty(\eta) \to \langle X\rangle^*/\mathcal{WAP}(\mathcal{A})\big|_{\langle X\rangle}$. This led to the extreme non-Arens regularity of $\mathcal{A}$, in the sense of Granirer, as soon as the density character of $\mathcal{A}$ is not larger than η.

In our next Theorem we assume that $\mathcal{A}$ is a subalgebra of the predual of a von Neumann algebra. Under this condition, we are able to find an isomorphism $\ell^\infty(\eta) \to \mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. This will lead us to show that, in such a situation, there is an isomorphic, and in many cases a linear isometry, copy of $\mathcal{A}^*$ in $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. The term $\mathrm{tr}(\Lambda)$ that appears in its statement makes reference to the *true cardinality* of the directed set Λ, $\mathrm{tr}\,(\Lambda) = \min_{\xi\in\Lambda} |\{\alpha \in \Lambda \colon \xi \prec \alpha\}|$, see [10]. As explained in [13, Page 1845], this concept is needed to delimitate pathological situations that will not be pertinent to our applications.

Theorem 3.12. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\mathcal{A}$ is a subalgebra of the predual $\mathfrak{V}_*$ of a von Neumann algebra $\mathfrak{V}$. Let η be an infinite cardinal number and suppose that $\mathcal{A}$ contains two bounded subsets A and B indexed by a directed set $(\Lambda, \preceq)$ with $\mathrm{tr}\,(\Lambda) = \eta$. Suppose also that $\mathcal{A}$ contains two other disjoint sets X_1 and X_2 with the following properties*

(i) *X_1 and X_2 approximate segments in T^u_{AB} and T^l_{AB}, respectively.*
(ii) *$X_1 \cup X_2$ is an orthogonal $\ell^1(\eta)$-set (as a subset of $\mathfrak{V}_*$) with constant K and bound M, i.e., with $K \leq \|x\| \leq M$ for every $x \in X_1 \cup X_2$.*
(iii) *X_1 is vertically injective and X_2 is horizontally injective.*

Then there is a linear isomorphism $\mathcal{E} \colon \ell^\infty(\eta) \to \mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$ with distortion at most $\frac{M}{K}$.

In particular, $\|\mathcal{E}^{-1}\|\mathcal{E}$ is a linear isometry when $K = M$.

Proof. Put $A = \{a_\alpha \colon \alpha \in \Lambda\}$ and $B = \{b_\beta \colon \beta \in \Lambda\}$. Let

$$X_1 = \{x_{\alpha\beta} \colon (\alpha, \beta) \in \mathcal{U}_1\} \quad \text{and } X_2 = \{x_{\alpha\beta} \colon (\alpha, \beta) \in \mathcal{U}_2\}$$

be the sets which approximate T^u_{AB} and T^l_{AB}, respectively, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are vertically and horizontally cofinal in $\Lambda \times \Lambda$, respectively.

Introduce on $\Lambda \times \Lambda$ an equivalence relation by the rule $(\alpha, \beta) \sim (\alpha', \beta')$ if and only if $x_{\alpha\beta} = x_{\alpha'\beta'}$. Note that, by vertical/horizontal injectivity, $(\alpha, \beta) \sim (\alpha', \beta')$ implies $\beta = \beta'$ when $\beta \succ \alpha$ and implies $\alpha = \alpha'$ when $\alpha \succ \beta$. We shall denote the equivalence class of $(\alpha, \beta) \in \Lambda \times \Lambda$ by $[\alpha, \beta]$. Restrict this equivalence relation to $\mathcal{U}_1$ and $\mathcal{U}_2$. Denote the quotient spaces

$$\mathcal{U}_1/\!\sim \quad \text{and} \quad \mathcal{U}_2/\!\sim,$$

respectively, by R_1 and R_2. By [10, Lemma, p. 61], we can partition the set Λ into η-many cofinal subsets Λ_ξ, with the cardinality of each Λ_ξ equals η. Then, for an element $\mathbf{v} = \left(z_\xi\right)_{\xi \prec \eta} \in \ell^\infty(\eta)$, we consider the sum

$$\sum_{\xi \prec \eta} z_\xi \left(\sum_{\substack{[\alpha,\beta] \in R_1 \\ \beta \in \Lambda_\xi}} S(x_{\alpha\beta}) - \sum_{\substack{[\alpha,\beta] \in R_2 \\ \alpha \in \Lambda_\xi}} S(x_{\alpha\beta}) \right).$$

Since $X_1 \cup X_2$ is an orthogonal set, and $x_{\alpha\beta} \neq x_{\alpha'\beta'}$ when $[\alpha, \beta] \neq [\alpha', \beta']$, all the projections appearing in this sum are pairwise orthogonal. So by Lemma 3.11, the above sum converges strongly to an element of $\mathfrak{V}$. We label this element as $P_{\mathbf{v}}$.

We now define $\mathcal{T} \colon \ell^\infty(\eta) \to \mathcal{A}^*$ by

$$\mathcal{T}(\mathbf{v}) = P_{\mathbf{v}}\big|_{\mathcal{A}}, \quad \text{for each} \quad \mathbf{v} \in \ell^\infty(\eta).$$

Recalling Lemma 3.1 it is obvious, that for every $\mathbf{v} \in \ell^\infty(\eta)$,

$$\|\mathcal{T}(\mathbf{v})\|_{\mathcal{A}^*} \leq \|P_{\mathbf{v}}\|_{\mathfrak{V}} = \|\mathbf{v}\|_\infty. \tag{3.2}$$

It is easy to check in fact that $\mathcal{T}$ is an isomorphism. (When $\mathcal{A} = \mathfrak{V}_*$, $\mathcal{T}$ is even an isometry.)

Next we prove that $\mathcal{T}$ is $\frac{K}{M}$-preserved by $\mathcal{WAP}(\mathcal{A})$. We consider $\mathbf{v} = (z_\xi)_{\xi \prec \eta} \in \ell^\infty(\eta)$ and $\phi \in \mathcal{WAP}(\mathcal{A})$ and let $\varepsilon > 0$ be fixed as well. Let now $\xi \prec \eta$ be fixed. Since A and B are bounded and $\phi \in \mathcal{WAP}(\mathcal{A})$, we can assume, after taking suitable subnets on the cofinal set Λ_ξ, that the following equality holds

$$\lim_\alpha \lim_\beta \langle \phi, a_\alpha b_\beta \rangle = \lim_\beta \lim_\alpha \langle \phi, a_\alpha b_\beta \rangle .$$

Mark these iterated limits by L, put $L_\alpha = \lim_\beta \langle \phi, a_\alpha b_\beta \rangle$ and $M_\beta = \lim_\alpha \langle \phi, a_\alpha b_\beta \rangle$. Then $\lim_\alpha L_\alpha = \lim_\beta M_\beta = L$, and so for a fixed $\varepsilon > 0$, we may choose α_0 and β_0 in Λ_ξ such that

$$|L - L_{\alpha_0}| < \varepsilon/4 \quad \text{and} \quad |L - M_{\beta_0}| < \varepsilon/4. \tag{3.3}$$

For these fixed α_0 and β_0, there are β_1 and α_1 in Λ_ξ such that

$$|\phi(a_{\alpha_0} b_\beta) - L_{\alpha_0}| < \varepsilon/4 \quad \text{for all } \beta \succ \beta_1, \tag{3.4}$$

$$|\phi(a_\alpha b_{\beta_0}) - M_{\beta_0}| < \varepsilon/4 \quad \text{for all } \alpha \succ \alpha_1. \tag{3.5}$$

Putting together (3.3)–(3.5), we obtain $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \Lambda_\xi$, such that

$$|\langle \phi, a_{\alpha_0} b_\beta \rangle - \langle \phi, a_\alpha b_{\beta_0} \rangle| < \varepsilon \quad \text{for all } \beta \succ \beta_1 \text{ and } \alpha \succ \alpha_1. \tag{3.6}$$

Since

$$\lim_\beta \|a_{\alpha_0} b_\beta - x_{\alpha_0 \beta}\|_{\mathcal{A}} = \lim_\alpha \|a_\alpha b_{\beta_0} - x_{\alpha \beta_0}\|_{\mathcal{A}} = 0$$

and Λ_ξ is cofinal, we can find $\alpha_2, \beta_2 \in \Lambda_\xi$ with $\alpha_0, \beta_1 \prec \beta_2$ and $\beta_0, \alpha_1 \prec \alpha_2$ such that

$$\|a_{\alpha_0} b_{\beta_2} - x_{\alpha_0 \beta_2}\|_{\mathcal{A}} \leq \varepsilon \text{ and } \|a_{\alpha_2} b_{\beta_0} - x_{\alpha_2 \beta_0}\|_{\mathcal{A}} \leq \varepsilon, \tag{3.7}$$

and so

$$\|a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\|_{\mathcal{A}} \leq 2M + 2\varepsilon. \tag{3.8}$$

Recalling that (see (ii) of Remarks 3.6)

$$\langle S(x_{\alpha\beta}), x_{\alpha'\beta'}\rangle = 0$$

whenever $[\alpha,\beta] \neq [\alpha',\beta']$ and taking into account that

$$\langle S(x_{\alpha\beta}), x_{\alpha\beta}\rangle = \|x_{\alpha\beta}\| \geq K \quad \text{for every} \quad \alpha,\ \beta \in \Lambda,$$

we see that

$$|\langle \mathcal{T}(\mathbf{v}), x_{\alpha_0\beta_2} - x_{\alpha_2\beta_0}\rangle| = |z_\xi| \left(\langle S(x_{\alpha_0\beta_2}), x_{\alpha_0\beta_2}\rangle + \langle S(x_{\alpha_2\beta_0}), x_{\alpha_2\beta_0}\rangle\right) \geq 2K|z_\xi|. \tag{3.9}$$

Using (3.2) and (3.9), it follows that

$$\begin{aligned}
\left|\langle \mathcal{T}(\mathbf{v}) - \phi, a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\rangle\right| &= \left|\langle \mathcal{T}(\mathbf{v}), a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\rangle - \langle \phi, a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\rangle\right| \\
&= \Big|\langle \mathcal{T}(\mathbf{v}), (a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}) - (x_{\alpha_0\beta_2} - x_{\alpha_2\beta_0})\rangle \\
&\quad + \langle \mathcal{T}(\mathbf{v}), x_{\alpha_0\beta_2} - x_{\alpha_2\beta_0}\rangle - \langle \phi, a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\rangle\Big| \\
&\geq \left|\langle \mathcal{T}(\mathbf{v}), x_{\alpha_0\beta_2} - x_{\alpha_2\beta_0}\rangle\right| - \left|\langle \mathcal{T}(\mathbf{v}), a_{\alpha_0}b_{\beta_2} - x_{\alpha_0\beta_2}\rangle\right| \\
&\quad - \left|\langle \mathcal{T}(\mathbf{v}), a_{\alpha_2}b_{\beta_0} - x_{\alpha_2\beta_0}\rangle\right| - \left|\langle \phi, a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\rangle\right| \\
&\geq 2K\,|z_\xi| - 2\varepsilon\,\|\mathcal{T}(\mathbf{v})\|_{\mathcal{A}^*} - \varepsilon \\
&\geq 2K\,|z_\xi| - 2\varepsilon\,\|\mathbf{v}\|_\infty - \varepsilon.
\end{aligned}$$

Next, we use (3.8) to obtain

$$\begin{aligned}
\|\mathcal{T}(\mathbf{v}) - \phi\|_{\mathcal{A}^*} &\geq \frac{1}{\|a_{\alpha_0}b_{\beta_2} - a_{\alpha_2}b_{\beta_0}\|_{\mathcal{A}}}\left(2K\,|z_\xi| - 2\varepsilon\,\|\mathbf{v}\|_\infty - \varepsilon\right) \\
&\geq \frac{1}{2M + 2\varepsilon}\left(2K\,|z_\xi| - 2\varepsilon\,\|\mathbf{v}\|_\infty - \varepsilon\right).
\end{aligned} \tag{3.10}$$

Since $\xi \prec \eta$ and $\varepsilon > 0$ were chosen arbitrarily, we conclude that

$$\frac{K}{M}\,\|\mathbf{v}\|_\infty \leq \|\mathcal{T}(\mathbf{v}) - \phi\|_{\mathcal{A}^*} \quad \text{for every} \quad \mathbf{v} \in \ell^\infty(\eta),\ \phi \in \mathcal{WAP}(\mathcal{A}).$$

The map $\mathcal{T}$ is therefore a linear isomorphism that is $\frac{K}{M}$-preserved by $\mathcal{WAP}(\mathcal{A})$.

If Q is the quotient map of $\mathcal{A}^*$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$, then Lemma 3.8 shows that $\mathcal{E} = Q \circ \mathcal{T}$ is the sought after isomorphism with distortion at most $\frac{M}{K}$.

If $K = M$, then $\|\mathcal{E}^{-1}\|\mathcal{E}$ becomes clearly an isometry. □

The conditions of Theorem 3.12 imply clearly that $\mathcal{A}$ is non-Arens regular. If, in addition $d(\mathcal{A}) = \eta$, then $\mathcal{A}$ is even r-enAr with $r \leq M/K$. This follows from the following well-known fact (see, e.g. [21]). Recall that the density character of a normed space $\mathcal{A}$, denoted by $d(\mathcal{A})$, is the cardinality of the smallest norm-dense subset of $\mathcal{A}$.

Lemma 3.13. *If $\mathcal{A}$ is a normed space with density character $d(\mathcal{A}) = \eta$, then there is a linear isometry of $\mathcal{A}^*$ into $\ell^\infty(\eta)$.*

Proof. If $\{x_\alpha : \alpha < \eta\}$ is a norm-dense subset in the unit ball of $\mathcal{A}$, the required isometry $\mathcal{I} : \mathcal{A}^* \to \ell^\infty(\eta)$ is defined by:

$$\mathcal{I}(\psi) = v_\psi,$$

where v_ψ in $\ell^\infty(\eta)$ is given by $v_\psi(\alpha) = \langle \psi, x_\alpha \rangle$. □

Corollary 3.14. *Let $\mathcal{A}$ be as in Theorem 3.12 and suppose that $d(\mathcal{A}) = \eta$. Then $\mathcal{A}$ is r-enAr with $r \leq \frac{M}{K}$. In particular, $\mathcal{A}$ is isometrically enAr when $K = M$.*

The presence of a bai or of a TI-net is usually the reason behind the non-Arens regularity of a given Banach algebra. A weaker form of these nets is in fact enough to deduce non-Arens regularity. Here are the necessary definitions.

Definition 3.15. In a Banach algebra $\mathcal{A}$, a net $\{a_\alpha \colon \alpha \in \Lambda\}$, with $\|a_\alpha\| = 1$ for every $\alpha \in \Lambda$, is a *weak bounded approximate identity* (weak bai for short) if

$$\lim_\alpha \|a_\alpha a_\beta - a_\beta\| = \lim_\alpha \|a_\beta a_\alpha - a_\beta\| = 0 \quad \text{for each} \quad \beta \in \Lambda.$$

Definition 3.16. Let $\mathfrak{V}$ be a von Neuman algebra. A net $\{a_\alpha\}_{\alpha\in\Lambda}$ of normal states of $\mathfrak{V}$ is a *weak TI-net* if

$$\lim_\alpha \|a_\alpha a_\beta - a_\alpha\| = \lim_\alpha \|a_\beta a_\alpha - a_\alpha\| = 0 \quad \text{for each} \quad \beta \in \Lambda.$$

If we require that $\lim_\alpha \|a_\alpha a - a_\alpha\| = \lim_\alpha \|a a_\alpha - a_\alpha\| = 0$ for every normal state a of $\mathfrak{V}$, and not only for members of the net itself, then we obtain the familiar concept of a *TI-net*. Here TI stands for topological invariance, the term was introduced by Chou [6]. They also appeared in the work by Lau, see for example [26].

A separable Banach algebra satisfying the conditions of the following theorem was proved to be enAr in the sense of Granirer in [13, Theorems 4.2 and 4.4]. When $\mathcal{A}$ is in addition a subalgebra of the predual of a von Neumann algebra, Theorem 3.12 implies as we see next that $\mathcal{A}$ is isometrically enAr.

Theorem 3.17. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\mathcal{A}$ is a subalgebra of the predual $\mathfrak{V}_*$ of a von Neumann algebra $\mathfrak{V}$. Let η be an infinite cardinal number and suppose that $\mathcal{A}$ contains either a weak bai or a weak TI-net $\{a_\alpha\}_{\alpha\in\Lambda}$ of true cardinality η such that $\{a_\alpha \colon \alpha \in \Lambda\}$ is an orthogonal $\ell^1(\eta)$-set. Then*

(i) *there is an isometry $\mathcal{E} \colon \ell^\infty(\eta) \to \mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$,*
(ii) *in particular, $\mathcal{A}$ is non-Arens regular,*
(iii) *$\mathcal{A}$ is isometrically enAr, if in addition $d(\mathcal{A}) \leq \eta$.*

Proof. Take $\Lambda_1,\ \Lambda_2 \subset \Lambda$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$ in such a way that both Λ_1 and Λ_2 are cofinal for $\preceq$. Put then

$$A = \{a_\alpha \colon \alpha \in \Lambda_1\} \quad \text{and} \quad B = \{a_\alpha \colon \alpha \in \Lambda_2\}.$$

If $\{a_\alpha\}$ is a weak bai, define the elements $x_{\alpha\beta}$ by

$$x_{\alpha\beta} = \begin{cases} a_\alpha, & \text{if} \quad (\alpha,\beta) \in \Lambda \times \Lambda_1,\ \beta \succ \alpha \\ a_\beta, & \text{if} \quad (\alpha,\beta) \in \Lambda_2 \times \Lambda,\ \alpha \succ \beta. \end{cases}$$

If $\{a_\alpha\}$ is a weak TI-net, define the elements $x_{\alpha\beta}$ by

$$x_{\alpha\beta} = \begin{cases} a_\beta, & \text{if} \quad (\alpha,\beta) \in \Lambda \times \Lambda_1,\ \beta \succ \alpha \\ a_\alpha, & \text{if} \quad (\alpha,\beta) \in \Lambda_2 \times \Lambda,\ \alpha \succ \beta. \end{cases}$$

In each case, let

$$X_1 = \{x_{\alpha\beta} \colon (\alpha,\beta) \in \Lambda \times \Lambda_1,\ \beta \succ \alpha\} \text{ and } X_2 = \{x_{\alpha\beta} \colon (\alpha,\beta) \in \Lambda_2 \times \Lambda,\ \alpha \succ \beta\}.$$

So here $X_1 \subseteq A$ and $X_2 \subseteq B$ are double-indexed by

$$\mathcal{U}_1 = \{(\alpha,\beta) \in \Lambda \times \Lambda_1 : \beta \succ \alpha\} \quad \text{and} \quad \mathcal{U}_2 = \{(\alpha,\beta) \in \Lambda_2 \times \Lambda : \alpha \succ \beta\},$$

respectively. The sets $\mathcal{U}_1$ and $\mathcal{U}_2$ are clearly vertically cofinal and horizontally cofinal, respectively.

In the first situation, for every $\alpha \in \Lambda$, the approximate identity property yields

$$\lim_{\substack{\beta\in\Lambda_1\\ \beta\succ\alpha}} \|x_{\alpha\beta} - a_\alpha a_\beta\| = \lim_{\substack{\beta\in\Lambda_1\\ \beta\succ\alpha}} \|a_\alpha - a_\alpha a_\beta\| = \lim_\beta \|a_\alpha - a_\alpha a_\beta\| = 0.$$

Similarly, for each $\beta \in \Lambda$,

$$\lim_{\substack{\alpha\in\Lambda_2\\ \alpha\succ\beta}} \|x_{\alpha\beta} - a_\alpha a_\beta\| = \lim_{\substack{\alpha\in\Lambda_2\\ \alpha\succ\beta}} \|a_\beta - a_\alpha a_\beta\| = \lim_\alpha \|a_\beta - a_\alpha a_\beta\| = 0$$

In the second situation, the weak TI-net property yields

$$\lim_{\substack{\beta\in\Lambda_1\\ \beta\succ\alpha}} \|x_{\alpha\beta} - a_\alpha a_\beta\| = \lim_{\substack{\beta\in\Lambda_1\\ \beta\succ\alpha}} \|a_\beta - a_\alpha a_\beta\| = \lim_\beta \|a_\beta - a_\alpha a_\beta\| = 0 \quad \text{for every} \quad \alpha \in \Lambda,$$

$$\lim_{\substack{\alpha\in\Lambda_2\\ \alpha\succ\beta}} \|x_{\alpha\beta} - a_\alpha a_\beta\| = \lim_{\substack{\alpha\in\Lambda_2\\ \alpha\succ\beta}} \|a_\alpha - a_\alpha a_\beta\| = \lim_\alpha \|a_\alpha - a_\alpha a_\beta\| = 0 \quad \text{for every} \quad \beta \in \Lambda.$$

Hence, in each case, X_1 approximates segments of T^u_{AB} and X_2 approximates segments of T^l_{AB}.

It is clear that X_1 is vertically injective and X_2 is horizontally injective. Since by assumption, the norm of each $x_{\alpha\beta}$ is one and $X_1 \cup X_2$ is an orthogonal $\ell^1(\eta)$-set, Theorem 3.12 provides a linear isometry of $\ell^\infty(\eta)$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. In particular, $\mathcal{A}$ is non-Arens regular.

This isometry together with the condition $\eta \geq d(\mathcal{A})$ implies, by Lemma 3.13, that there is an isometry of $\mathcal{A}^*$ into $\mathcal{A}^*/\mathcal{WAP}(\mathcal{A})$. $\mathcal{A}$ is therefore isometrically enAr. $\square$

4. Extreme non-Arens regularity of the weighted convolution algebras

We apply in this section Theorem 3.12 to the weighted semigroup algebra of an infinite discrete weakly cancellative semigroup, the weighted group algebra and the weighted measure algebra of an infinite locally compact group. In each case, the algebra is r-enAr, where r is at most equal to the diagonal bound of the weight. When the weight is multiplicative, these algebras are all isometrically enAr. When G is non-discrete, the weighted group algebra is isometrically enAr for any weight.

Some Definitions 4.1. We first introduce some terminology concerning weighted convolution algebras, for a more detailed discussion we refer the reader to [8].

Let S be a semigroup with a topology.

(i) A *weight* on S is a continuous function $w : S \to (0, \infty)$ which is submultiplicative, that is, with

$$w(st) \le w(s)w(t) \quad \text{for every} \quad s, t \in S.$$

If S is a group with identity e, we shall assume in addition that $w(e) = 1$.

(ii) Following [3] and [7], we let Ω be the continuous function on $S \times S$ given by

$$\Omega(s, t) = \frac{w(st)}{w(s)w(t)}.$$

Note that $0 < \Omega(s, t) \le 1$ for every s, $t \in S$.

(iii) The weight function is called *diagonally bounded* on S if there exists $c > 0$ such that

$$w(s)w(t) \le c\, w(st) \quad \text{whenever} \quad s, t \in S.$$

In other words, the weight function is diagonally bounded if $\Omega(s, t) \ge \frac{1}{c}$ for every s, $t \in S$. When S is a group, it is usual to define the weight w as diagonally bounded by $c > 0$ when

$$\sup_{s \in S} w(s)w(s^{-1}) \le c.$$

It is easy to check that the two definitions are the same in this case.

(iv) Let G be a locally compact group. For a function space $\mathcal{F}(G)$ contained in $L^\infty(G)$, the corresponding weighted space is defined, following [19] and [8] as

$$\mathcal{F}(G, w^{-1}) = \{f : S \to \mathbb{C} : w^{-1} f \in \mathcal{F}(G), \}$$

with the norm given by $\|f\|_w = \|w^{-1} f\|_\infty$, for any $f \in \mathcal{F}(G, w^{-1})$.
If $\mathcal{F}(G)$ is contained in $L^1(G)$, then the weighted space for w is defined as

$$\mathcal{F}(G, w) = \{f : S \to \mathbb{C} : w\, f \in \mathcal{F}(G)\}$$

and the norm given by $\|f\|_w = \|wf\|_1$, for any $f \in \mathcal{F}(G, w)$.
The space $L^\infty(G, w^{-1})$ can then be identified with the Banach dual space of $L^1(G, w)$ via the pairing

$$< f, \phi >= \int_G f(x)\overline{\phi(x)}dx$$

for each $f \in L^1(G, w)$ and $\phi \in L^\infty(G, w^{-1})$. When $w \ge 1$, $L^1(G, w)$ is called *Beurling algebra* and is studied for instance in [8].
In the same vein, $M(G, w)$ will denote the space of all complex-valued measures regular Borel measures on G such that

$$\|\mu\|_w = \int_G w(s) d|\mu|(s)$$

is finite. $(M(G, w), \|\mu\|_w)$ can be identified with the Banach dual space of $C_0(G, w^{-1})$ as defined above.

(v) When S is a discrete semigroup, we shall consider the semigroup algebra $\ell^1(S)$, its Banach dual space $\ell^\infty(S)$ of all bounded functions on S, and their corresponding weighted spaces $\ell^1(S, w)$ and $\ell^\infty(S, w^{-1})$ which are defined exactly as done above in the group case.

Remark 4.2. Let $\mathcal{WAP}(G)$ be the space of weakly almost periodic functions on a locally compact group (or on a discrete semigroup) G. The literature contains a number of different definitions for the weighted space $\mathcal{WAP}(G, w^{-1})$ of $\mathcal{WAP}(G)$. The latest is in [8], where the space $\mathcal{WAP}(G, w^{-1})$ is defined as a subspace of $L^\infty(G, w^{-1})$, as in 4.1 (iv) above. With this definition, using the fact that $L^\infty(G)/\mathcal{WAP}(G)$ contains an isometric copy of $L^\infty(G)$ (proved in [12, Theorem B] for the non-discrete case, and [5, Theorem 4.3] or [16, Theorem 3.3] for the discrete case), it is very quick to show that the quotient $L^\infty(G, w^{-1})/\mathcal{WAP}(G, w^{-1})$ contains an isometric copy of $L^\infty(G, w^{-1})$ for any infinite locally compact group G. However, this does not show that $L^1(G, w)$ is enAr, for $\mathcal{WAP}(G, w^{-1})$ can be different from $\mathcal{WAP}(L^1(G, w))$. They can actually be very different. If w is the weight on $\mathbb{Z}$ given by $w(n) = (1 + |n|)^\alpha$, with $\alpha > 0$, it is shown in [8, Example 9.1] that $\ell^1(\mathbb{Z}, w)$ is Arens regular, i.e., $\mathcal{WAP}(\ell^1(\mathbb{Z}, w)) = \ell^\infty(\mathbb{Z}, w^{-1})$. Since, as mentioned above, the quotient $\ell^\infty(\mathbb{Z}, w^{-1})/\mathcal{WAP}(\mathbb{Z}, w^{-1})$ is at least as large as $\ell^\infty(\mathbb{Z}, w^{-1})$, we conclude that $\mathcal{WAP}(\mathbb{Z}, w^{-1})$ is much smaller that $\mathcal{WAP}(\ell^1(\mathbb{Z}, w))$ in this case.

A more suitable definition for the weighted almost periodic functions in the context of Arens regularity was given by Baker and Rejali in [3]. We will not need this definition of the space of functions $\mathcal{WAP}(G, w^{-1})$ in this paper. Theorem 3.12 deals directly with the space of functionals $\mathcal{WAP}(L^1(G, w))$. We will not go any further with this matter at the moment, but we hope to return to it in forthcoming work.

4.1. Weighted semigroup algebras

We start with the weighted semigroup algebra of an infinite, discrete, weakly cancellative semigroup. We see here that, whenever the weight is diagonally bounded by some $c > 0$, Theorem 3.12 applies and shows that $\ell^1(S, w)$ is r-enAr, where $r \leq c$. When w is multiplicative, $\ell^1(S, w)$ is therefore isometrically enAr. This latter fact was proved also in [16], in [5, Theorem 4.5] and [12, Theorem 6.4] when $w = 1$.

Recall first that a semigroup S is called *weakly cancellative* if the sets

$$s^{-1}t = \{u \in S : su = t\} \quad \text{and} \quad ts^{-1} = \{u \in S : us = t\}$$

are finite for every $s, t \in S$. We shall use the notations

$$s^{-1}B = \{t \in S : st \in B\} \quad \text{and} \quad A^{-1}B = \bigcup_{s \in A} s^{-1}B,$$

where $s \in S$ and $A, B \subseteq S$. The sets Bs^{-1} and BA^{-1} are defined similarly.

Theorem 4.3. *Let S be a weakly cancellative, infinite, discrete semigroup, and let w be a weight on S that is diagonally bounded with bound c. Then the weighted semigroup algebra $\ell^1(S, w)$ is r-enAr, where $r \leq c$.*

In particular, $\ell^1(S, w)$ is isometrically enAr when $c = 1$.

Proof. Let the weight w on S be diagonally bounded by $c > 0$ so that

$$\frac{1}{c} \leq \Omega(s, t) = \frac{w(st)}{w(s)w(t)} \leq 1$$

for every s, $t \in S$.

Let Λ denote the initial ordinal with cardinal $\eta := |S|$ and note that $\mathrm{tr}(\Lambda) = \eta$. Let as well $\{S_\alpha\}_{\alpha<\eta}$ be an increasing cover of S made of subsets with $|S_\alpha| \leq \alpha$ for every $\alpha < \eta$, and collect by induction a faithfully indexed set $X = \{s_\alpha : \alpha < \eta\}$ such that

$$(S_\alpha s_\alpha) \cap (S_\beta s_\beta) = (s_\alpha S_\alpha) \cap (s_\beta S_\beta) = \varnothing \quad \text{for every } \alpha < \beta < \eta. \tag{4.1}$$

This is possible since S is weakly cancellative, and so, for every $\alpha, \beta < \eta$,

$$|(S_\beta^{-1} S_\alpha s_\alpha) \cup (s_\alpha S_\alpha S_\beta^{-1})| \leq \max\{\alpha, \beta\} < \eta.$$

Note that for each $\alpha < \eta$, there exists $\beta(\alpha) < \eta$, $\beta(\alpha) > \alpha$ such that

$$s_\alpha s_\beta \in S_\beta s_\beta \quad \text{for every} \quad \beta \geq \beta(\alpha). \tag{4.2}$$

Similarly, for each $\beta < \eta$, there exists $\alpha(\beta) < \eta$, $\alpha(\beta) > \beta$ such that

$$s_\alpha s_\beta \in s_\alpha S_\alpha \quad \text{for every} \quad \alpha \geq \alpha(\beta). \tag{4.3}$$

Split Λ into two cofinal subsets Λ_1 and Λ_2, let

$$A = \left\{ \frac{\delta_{s_\alpha}}{w(s_\alpha)} : \alpha \in \Lambda_1 \right\} \quad \text{and} \quad B = \left\{ \frac{\delta_{s_\alpha}}{w(s_\alpha)} : \alpha \in \Lambda_2 \right\}$$

and note that

$$\frac{\delta_{s_\alpha}}{w(s_\alpha)} * \frac{\delta_{s_\beta}}{w(s_\beta)} = \frac{\delta_{s_\alpha s_\beta}}{w(s_\alpha) w(s_\beta)}.$$

Now, for each $\alpha < \eta$, define

$$x_{\alpha\beta} = \frac{\delta_{s_\alpha s_\beta}}{w(s_\alpha) w(s_\beta)}, \text{ if } \beta \in \Lambda_1, \ \beta \geq \alpha(\beta),$$

and for each $\beta < \eta$, define

$$x_{\alpha\beta} = \frac{\delta_{s_\alpha s_\beta}}{w(s_\alpha) w(s_\beta)}, \text{ if } \alpha \in \Lambda_2, \ \alpha \geq \beta(\alpha).$$

Then put

$$X_1 = \{x_{\alpha\beta} \colon \alpha < \eta, \, \beta \in \Lambda_1, \beta \geq \alpha(\beta)\} \quad \text{and}$$
$$X_2 = \{x_{\alpha\beta} \colon \alpha \in \Lambda_1, \, \beta \in \Lambda_2, \alpha \geq \beta(\alpha)\},$$

that is, X_1 and X_2 are double-indexed by the vertically and horizontally cofinal sets

$$\{(\alpha, \beta) \in \Lambda \times \Lambda_1 : \beta \geq \alpha(\beta)\} \quad \text{and} \quad \{(\alpha, \beta) \in \Lambda_2 \times \Lambda : \alpha \geq \beta(\alpha)\},$$

respectively. Using properties (4.1), (4.2) and (4.3), we see that X_1 and X_2 are disjoint and are, respectively, vertically and horizontally injective (and hence have cardinality η). Since $S(\delta_s/w(s)) = w(s)\mathbb{1}_{\{s\}} \in \ell^\infty(S, w^{-1})$ for each $s \in S$, and $\frac{1}{c} \leq \|x_{\alpha\beta}\|_w \leq 1$ for every $x_{\alpha\beta} \in X_1 \cup X_2$, we see that $X_1 \cup X_2$ is an orthogonal $\ell^1(\eta)$-set with bound 1 and constant $K = 1/c$ in the sense of Definition 3.5. Since X_1 approximates segments in T^u_{AB} and X_2 approximates segments in T^l_{AB}, we see that all conditions of Theorem 3.12 are met so that the desired result that $\ell^1(S, w)$ is r-enAr with $r \leq c$.

When $c = 1$, $\ell^1(S, w)$ is clearly isometrically enAr. □

4.2. Weighted group algebras

We start with the weighted analogue of [30, Theorem 2]. With the same proof, this result is valid for any weight function w on G.

Lemma 4.4. *Let G be a locally compact group. Then for any weight w on G, we have $\mathcal{WAP}(L^1(G,w)) \subseteq \mathcal{UC}(G,w^{-1})$.*

Theorem 4.5. *Let G be an infinite, non-discrete, locally compact group and $\mathcal{F}(w)$ be any closed subspace of $\mathcal{CB}(G,w^{-1})$. Then there exists a linear isometric copy of $L^\infty(G,w^{-1})$ in the quotient space $L^\infty(G,w^{-1})/\mathcal{F}(w)$.*

Proof. By [12, Theorem 6.3], there is a linear isometry of $L^\infty(G)$ into $L^\infty(G)$ which is 1-preserved by $\mathcal{CB}(G)$. Since $L^\infty(G)$ and $\mathcal{CB}(G)$ are linearly isometric to their weighted analogues, this gives a linear isometry of $L^\infty(G,w^{-1})$ into $L^\infty(G,w^{-1})$ which is 1-preserved by $\mathcal{CB}(G,w^{-1})$. Lemma 3.8 provides then the desired linear isometry $L^\infty(G,w^{-1})$ into $L^\infty(G,w^{-1})/\mathcal{F}(w)$. □

This leads immediately to the isometric enArity of $L^1(G,w)$ for any weight w on G when G is infinite and non-discrete.

Corollary 4.6. *Let G be an infinite, non-discrete, locally compact group and w be any weight on G. Then the weighted group algebra $L^1(G,w)$ is isometrically enAr for any weight function w on G.*

Proof. Since Lemma 4.4 shows that

$$\mathcal{WAP}(L^1(G,w)) \subseteq \mathcal{UC}(G,w^{-1}) \subseteq \mathcal{CB}(G,w^{-1}),$$

Theorem 4.5 provides the required isometry

$$L^\infty(G,w^{-1}) \to L^\infty(G,w^{-1})/\mathcal{WAP}(L^1(G,w)). \quad \square$$

Here are our corollaries.

Corollary 4.7. *Let G be an infinite locally compact group.*

(i) *If G is not discrete, then the weighted group algebra $L^1(G,w)$ is isometrically enAr for any weight w on G.*
(ii) *If G is an infinite discrete group, then the weighted group algebra $\ell^1(G,w)$ is r-enAr, where $r \leq c$, for any weight w on G that is diagonally bounded with bound c.*

Remark 4.8. As mentioned earlier, with the weight given on $\mathbb{Z}$ by $w(n) = (1+|n|)^\alpha$, $\alpha > 0$, the weighted group algebra $\ell^1(\mathbb{Z},w)$ is even Arens regular. So when w is not diagonally bounded, Corollary 4.7 fails badly.

4.3. Weighted measure algebras

Let G be an infinite locally compact group. Next theorem deals with $M(G,w)$, where w is diagonally bounded. As in Remark 4.8, the theorem fails when w is not diagonally bounded.

Theorem 4.9. *Let G be an infinite locally compact group G and w be a diagonally bounded weight on G with bound $c > 0$. Then the weighted measure algebra $M(G,w)$ is r-enAr with $r \leq c$. In particular, $M(G,w)$ is isometrically enAr when w is multiplicative.*

Proof. For $\eta = |G|$, let A, B, X_1 and X_2 be as in the proof of Theorem 4.3. Regarding these as subsets of $M(G,w)$ and regarding the elements $S(\delta_s/w(s)) = w(s)\mathbb{1}_{\{s\}}$ as projections in $M(G,w)^*$, all the conditions for Theorem 3.12 to apply are satisfied. Therefore, we have a linear isomorphism of $\ell^\infty(\eta)$ in the quotient $M(G,w)^*/\mathcal{WAP}(M(G,w))$ with distortion at most c.

Since by [23, Theorem 5.5], $|G|$ is the density of $M(G)$, Corollary 4.7 yields the theorem. □

5. The Fourier algebra

We summarize first the basic facts on the Fourier algebra that will be needed in the remainder of this section. For more details, the reader is directed to [11] or Chapter 2 of [25].

The Fourier algebra is the collection of all functions h on G of the form $h = f * \widetilde{g}$ with $f, g \in L^2(G)$ and $\widetilde{g}(s) = \overline{g(s^{-1})}$. The norm of $A(G)$ is given by

$$\|h\| = \inf\{\|f\|_2\|g\|_2 : h = f * \widetilde{g},\ f,\, g \in L^2(G)\}.$$

We may remark that, when G is abelian, $A(G)$ identifies with $L^1(\widehat{G})$ via the Fourier transform. The results stated in Section 4 for the group algebra show therefore that the Fourier algebra $A(G)$ is isometrically enAr when G is Abelian. Non-Arens regularity of $A(G)$ has however turned out to be more resistant and a complete solution to the regularity problem for the Fourier algebra is not known yet.

The Banach dual of $A(G)$ is isometrically isomorphic to the group von Neumann algebra $VN(G)$, which is the closure in the weak operator topology of the linear span of $\{\lambda(x) : x \in G\}$ in $\mathcal{B}(L^2(G))$, where λ is the left regular representation of G on $L^2(G)$. This linear isometry identifies each $T \in VN(G)$ with an element $\varphi_T \in A(G)^*$ such that

$$\varphi_T\left(\overline{f} * \check{g}\right) = \langle Tg, f\rangle,$$

where $\check{g}(s) = g(s^{-1})$ and the bracket refers to the $L^2(G)$ inner product.

Under this identification, normal sates of $VN(G)$ correspond to the set

$$\mathcal{P}_1(G) = \{\varphi \in A(G)\colon \varphi \text{ is positive definite and } \|\varphi\| = \varphi(e) = 1\}.$$

So here, a TI-net is a net $\{\varphi_\alpha\colon \alpha \in \Lambda\}$ in $\mathcal{P}_1(G)$ with the property

$$\lim_\alpha \|\varphi_\alpha\varphi - \varphi_\alpha\| = \lim_\alpha \|\varphi\varphi_\alpha - \varphi_\alpha\| = 0 \quad \text{for every} \quad \varphi \in \mathcal{P}_1(G).$$

We apply, in this section, Theorem 3.12 to show that

(1) for every locally compact group G, there is a linear isometry from $\ell^\infty(\chi(G))$ into $VN(G)/\mathcal{WAP}(A(G))$, and
(2) the existence in G of an open, non-compact, amenable subgroup implies that there is a linear isometry from ℓ^∞ into $VN(G)/\mathcal{WAP}(A(G))$.

Since the density character of $A(G)$ is $\max\{\kappa(G), \chi(G)\}$, (1) shows automatically that the Fourier algebra $A(G)$ is isometrically enAr for those groups with $\chi(G) \geq \kappa(G)$. This implies Hu's theorem [21] to the effect that in this situation, $A(G)$ is enAr in the sense of Granirer. See also [22] for further results on quotients of $VN(G)$. The interesting application of (2) is when G is discrete (otherwise, (2) is an easy consequence of (1) even without assuming that G has an amenable subgroup). So when G is a countable discrete group with an infinite amenable subgroup (such as the free group $\mathbb{F}_r$ with r generators, where $r \geq 2$), statement (2) implies that $A(G)$ is isometrically enAr.

Our approach is based on the following two lemmas:

Lemma 5.1 *(Theorem 2.4 of [6]). Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of normal states of a von Neumann algebra $\mathfrak{V}$ such that $\lim_n \|a_n - a\| = 2$ for each normal state $a \in \mathfrak{V}_*$. Then there exist positive integers $n_1 < n_2 < \cdots$ and a sequence of normal states $\{b_j\}_{j\in\mathbb{N}}$ such that*

(i) $\lim_j \|a_{n_j} - b_j\| = 0$ *and*
(ii) *The sequence $\{b_j\}_{j\in\mathbb{N}}$ is an orthogonal ℓ^1-set.*

Lemma 5.2. *Let G be a locally compact group and let Q be a projection in $VN(G)$. Take $h \in L^2(G)$, with $Qh \neq 0$ and define $\phi = \frac{1}{\|Qh\|_2} Qh \in L^2(G)$ and $\psi = \phi * \widetilde{\phi}$. Then $\psi \in A(G) \cap \mathcal{P}_1(G)$ and $S(\psi) \leq Q$.*

Proof. It is clear that $\psi \in A(G) \cap \mathcal{P}_1(G)$. To prove that $S(\psi) \leq Q$, we only have to recall that $S(\psi)$ is the smallest projection in $VN(G)$ with $\langle \psi, S(\psi)\rangle = \|\psi\|$ and observe that:

$$\begin{aligned}\langle \psi, Q\rangle_{\langle A(G),VN(G)\rangle} &= \langle Q\phi, \phi\rangle_{\langle L^2(G),L^2(G)\rangle} \\ &= \frac{1}{\|Qh\|_2^2} \langle QQh, Qh\rangle_{\langle L^2(G),L^2(G)\rangle} \\ &= 1 = \psi(e) = \|\psi\|. \quad \square\end{aligned}$$

Next we proceed to check that the Fourier algebra of a non-discrete locally compact group contains TI-nets which are orthogonal $\ell^1(\chi(G))$-sets.

Theorem 5.3. *If G is a non-discrete locally compact group, then $A(G)$ contains a TI-net of true cardinality $\chi(G)$ that is an orthogonal $\ell^1(\chi(G))$-set.*

Proof. If G is metrizable, this was shown by Chou in [6]. The proof there consists in observing that every TI-sequence in $\mathcal{P}_1(G)$ satisfies the conditions of Lemma 5.1 (this is [6, Lemma 3.2]) and, hence, has a subsequence that can be approximated by another TI-sequence which is also an orthogonal ℓ^1-set. TI-sequences that are orthogonal ℓ^1-sets can therefore be found as long as TI-sequences are available, which is the case in non-discrete groups, see [29, Proposition 3] or [13, Lemma 5.2].

We now assume that G is not metrizable. Let $\eta = \chi(G)$ and $\{U_\alpha : \alpha < \eta\}$ be a base of symmetric neighbourhoods of the identity e. For each $\alpha < \eta$, let V_α be a neighbourhood of e with $V_\alpha^4 \subset U_\alpha$ and put $\mathcal{B} = \{V_\alpha : \alpha < \eta\}$.

We consider the family of compact subgroups $\{N_\alpha : \alpha < \eta\}$ given by the Kakutani-Kodaira theorem for the base $\mathcal{B}$ (see [20, Proposition 4.3 and its proof]). Recall, in particular, that $N_{\alpha+1} \subset N_\alpha \cap V_\alpha$ for every $\alpha < \eta$. Consider then the projections

$$P_\alpha : L^2(G) \to L^2(G/N_{\alpha+1})$$

given by $P_\alpha(f) = \lambda_{N_\alpha+1} * f$, where $L^2(G/N_{\alpha+1})$ stands for the functions of $L^2(G)$ that are constant on the cosets of $N_{\alpha+1}$ and $\lambda_{N_\alpha+1}$ stands for the regular representation of $N_{\alpha+1}$. This is an increasing net of projections. Putting $Q_\alpha = P_{\alpha+1} - P_\alpha$, we obtain an orthogonal net of projections $\{Q_\alpha : \alpha < \eta\}$.

Since $N_{\alpha+1} \setminus N_{\alpha+2} \neq \varnothing$ and N_α's are closed, we may pick for each $\alpha < \eta$, a symmetric compact neighbourhood $W_\alpha \subset V_\alpha$ of e such that $N_{\alpha+1} \setminus N_{\alpha+2}W_\alpha \neq \varnothing$, and so $N_{\alpha+1}W_\alpha \setminus N_{\alpha+2}W_\alpha$ has non-empty interior in G. Define $h_\alpha := 1_{N_{\alpha+2}W_\alpha}$. Put

$$\phi_\alpha = \frac{1}{\|Q_\alpha h_\alpha\|_2} Q_\alpha h_\alpha \quad \text{and} \quad \psi_\alpha = \phi_\alpha * \widetilde{\phi_\alpha}.$$

Note that $P_{\alpha+1}h_\alpha = h_\alpha$, and so $Q_\alpha h_\alpha = h_\alpha - P_\alpha h_\alpha$ for each $\alpha < \eta$. We claim that $h_\alpha \neq P_\alpha h_\alpha$ so that $Q_\alpha h_\alpha$ is not zero for each $\alpha < \eta$. Let $x = pw$ be any point in $N_{\alpha+1}W_\alpha \setminus N_{\alpha+2}W_\alpha$ with $p \in N_{\alpha+1}$ and $w \in W_\alpha$. Then $h_\alpha(x) = 0$, while

$$\begin{aligned} P_\alpha h_\alpha(x) &= P_\alpha h_\alpha(w) \\ &= \int\limits_{N_{\alpha+1}} h_\alpha(t^{-1}w) d\lambda_{N_{\alpha+1}}(t) = \lambda_{N_\alpha+1}\left(wW_\alpha N_{\alpha+2} \cap N_{\alpha+1}\right) \neq 0, \end{aligned}$$

where the latter value is non-zero because the interior in $N_{\alpha+1}$ of the set $wW_\alpha N_{\alpha+2} \cap N_{\alpha+1}$ is non-empty since it contains $N_{\alpha+2}$.

Hence $P_\alpha h_\alpha$ and h_α differ on the set $N_{\alpha+1}W_\alpha \setminus N_{\alpha+2}W_\alpha$, which is of positive measure in G (having non-empty interior). We conclude that $P_\alpha h_\alpha \neq h_\alpha$ for each $\alpha < \eta$, as wanted.

Lemma 5.2 then implies that $S(\psi_\alpha) \leq Q_\alpha$, showing that $\{\psi_\alpha : \alpha < \eta\}$ is an orthogonal $\ell^1(\eta)$-set.

Since the support of $Q_\alpha h_\alpha = h_\alpha - \lambda_{N_{\alpha+1}} * h_\alpha$ is clearly contained in $N_{\alpha+1}W_\alpha$, which is in turn contained in V_α^2, we see that

$$\operatorname{supp}(\psi_\alpha) = \operatorname{supp}\left(\phi_\alpha * \widetilde{\phi_\alpha}\right) \subseteq V_\alpha^4 \subseteq U_\alpha \quad \text{for each} \quad \alpha < \eta,$$

and [29, Proposition 3] proves that $\{\psi_\alpha : \alpha < \eta\}$ is a TI-net as well. Since this net is directed with the natural order of the ordinal η, its true cardinality is η. □

An easy argument (see Theorem 4 of [29]) shows that accumulation points in $VN(G)^*$ of TI-net in $A(G)$ are topologically invariant means on $VN(G)$. Theorem 5.3 therefore yields an easier and shorter proof to Hu's theorem on the number of topologically invariant means on the von Neumann algebra $VN(G)$ when G is not metrizable.

Corollary 5.4 *(Theorem 3.3 of [6] for metrizable G and Theorem 5.9 of [20] for nonmetrizable G). Let G be a nondiscrete locally compact group. Then the number of topologically invariant means on the von Neumann algebra $VN(G)$ is $2^{2^{\chi(G)}}$.*

Proof. Let $\{\psi_\alpha : \alpha < \eta\}$ be the orthogonal TI-net constructed in Theorem 5.3, where $\eta = \chi(G)$. Let η have the discrete topology and consider the map $I : \eta \to B$, where B is the unit ball in $VN(G)^*$, given by $I(\alpha) = \psi_\alpha$ for $\alpha < \eta$. Extend this map to $\tilde{I} : \beta\eta \to B$, where $\beta\eta$ is the Stone-Čech compactification of η. The TI-net being orthogonal implies immediately that the map $\tilde{I}$ is injective. For if x and y are distinct in $\beta\eta$, pick two disjoint subsets X and Y in η with $x \in \overline{X}$ and $y \in \overline{Y}$ (the closure is in $\beta\eta$). If $S = \sum_{\alpha \in X} S(\psi_\alpha)$, then $\langle \tilde{I}(x), S\rangle = 1$ while $\langle \tilde{I}(y), S\rangle = 0$. The map $\tilde{I}$ being injective yields the claim since the cardinality of $\beta\eta$ is 2^{2^η}. □

Theorem 5.3 will be applied in Corollary 5.6 to prove that $A(G)$ is isometrically enAr when $\chi(G) \geq \kappa(G)$. If G is discrete, and so $\chi(G) < \kappa(G)$, this approach cannot be followed. But when G is in addition amenable, TI-nets can be replaced, to the same effect, by weak bounded approximate identities.

Following [11], let $P(G)$ be the space of continuous positive definite functions on G and $B(G)$ be its the linear span. The space $B(G)$ is a Banach algebra, called the Fourier-Stieltjes algebra, and if $C^*(G)$ is the group C^*-algebra of G, then $B(G)$ is its Banach dual.

Theorem 5.5. *If G is a locally compact group that contains a σ-compact, non-compact open amenable subgroup H, then $A(G)$ has an orthogonal weak bai.*

Proof. It is a well-known theorem of Leptin that $A(H)$ contains a sequential bai $\{v_n\}_{n\in\mathbb{N}}$, see, e.g., [25, Theorem 2.7.2]. It is then clear that, in the $\sigma(B(H), C^*(H))$-topology, $\lim_n v_n = \mathbf{1}$ where $\mathbf{1}$ denotes the constant 1-function.

Since H is not compact, the regular representation of H is disjoint from the trivial one-dimensional representation. It then follows from [2, Corollaire 3.13] (or [25, Proposition 2.8.9]) that, for any $u \in A(H)$,

$$\|\mathbf{1} - u\|_{B(H)} = 1 + \|u\|_{A(H)}.$$

Let now $u \in A(H)$ be an arbitrary positive definite function with $\|u\|_{A(H)} = 1$ (i.e., an arbitrary normal state u of $VN(H) = A(H)^*$). Given $\varepsilon > 0$ there is then $T_\varepsilon \in C^*(H)$ with $\|T_\varepsilon\| \leq 1$ such that

$$|\langle \mathbf{1} - u, T_\varepsilon\rangle| > 2 - \varepsilon.$$

As a consequence, there is $n_\varepsilon \in \mathbb{N}$ such that, for $n \geq n_\varepsilon$,

$$|\langle v_n - u, T_\varepsilon\rangle| \geq 2 - \varepsilon.$$

It follows that

$$\lim_n \|v_n - u\|_{A(H)} = 2. \tag{5.1}$$

Lemma 5.1 now provides an orthogonal ℓ^1-sequence $\{u_j\}_{j\in\mathbb{N}}$ and a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}$ of $\{v_n\}_{n\in\mathbb{N}}$ such that

$$\lim_j \|v_{n_j} - u_j\|_{A(H)} = 0. \tag{5.2}$$

We next consider the restriction and extension maps, $R\colon A(G) \to A(H)$ and $\Phi\colon A(H) \to A(G)$, the latter one defined by $\Phi(u)(s) = u(s)$ if $s \in H$ and $\Phi(u)(s) = 0$ if $s \notin H$. The adjoint $R^*\colon VN(H) \to VN(G)$ of R is then a multiplicative linear isometry (see [9, Proposition 7.3.5], this is considerably easier when H is open) and $\|\Phi(u)\|_{A(G)} = \|u\|_{A(H)}$, see [25, Proposition 2.4.1].

The sought after orthogonal weak bai will be the sequence $(\Phi(u_j))_{j\in\mathbb{N}}$ as we check next.

Let $\{S(u_j)\}_{j\in\mathbb{N}}$ be the family of orthogonal projections corresponding to sequence $\{u_j\}_{j\in\mathbb{N}}$, and consider, for each $j \in \mathbb{N}$, the operator in $VN(G)$ given by $R^*(S(u_j))$.

Since normal states of $VN(G)$ correspond precisely to positive definite functions of $A(G)$ and these are clearly preserved by R, R^* must preserve self-adjointness. This, together with the multiplicative character of R^*, implies that the operators $R^*(S(u_j))$ are projections. Finally, since

$$\langle \Phi(u_j), R^*(S(u_j))\rangle = \langle u_j, S(u_j)\rangle = 1 = u_j(e) = \Phi(u_j)(e)$$

for each $j \in \mathbb{N}$, we deduce according to Definition 3.4, that $S(\Phi(u_j)) \leq R^*(S(u_j))$ for each $j \in \mathbb{N}$. Thus $(\Phi(u_j))_{j\in\mathbb{N}}$ is orthogonal in the sense of Definition 3.5.

That $(\Phi(u_j))_{j\in\mathbb{N}}$ is a weak bai follows from the bai property of the sequence $(v_{n_j})_{j\in\mathbb{N}}$, the approximation property (5.2) and the following inequality, valid for every $j,k\in\mathbb{N}$:

$$\begin{aligned}\|\Phi(u_j)\Phi(u_k)-\Phi(u_k)\|_{A(G)} &= \|u_ju_k-u_k\|_{A(H)}\\ &\le \|u_ju_k-v_{n_j}u_k\|_{A(H)}+\|v_{n_j}u_k-u_k\|_{A(H)}. \quad \square\end{aligned}$$

Corollary 5.6. *Let G be a locally compact group. $A(G)$ is isometrically enAr if G satisfies any of the following conditions:*

(i) $\chi(G)\ge\kappa(G)$, *or*
(ii) *G is second countable and contains a non-compact open amenable subgroup.*

Proof. (i) To obtain the linear isometry of $VN(G)$ into $VN(G)/\mathcal{WAP}(A(G))$, simply combine Theorems 3.17 and 5.3 and the fact that, under the hypothesis of (1), $d(A(G))=\chi(G)$.

(ii) We only have to put together Theorems 5.5 and 3.17. $\square$

Final remarks. We believe the techniques of the present paper can be applied to many other instances. Obvious candidates are Fourier-Stieltjes algebras and weighted Fourier algebras.

It is natural to wonder how much the concept of extreme Arens regularity introduced in Definition 2.1 strengthens the original one by Granirer. We have been unable to produce an example of a Banach algebra that is extremely non-Arens regular in the sense of Granirer's and is not in our sense. Another question we could not answer is whether the weighted group algebra $\ell^1(G,w)$ is isometrically enAr for weights diagonally bounded by $c>1$, even if G is the additive group $\mathbb{Z}$ of integers. The memoir [8], which provides a rich list of examples of weighted (semi)group algebras, might be a good starting point to deal with these questions.

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References

[1] R. Arens, The adjoint of a bilinear operation, Proc. Am. Math. Soc. 2 (1951) 839–848.
[2] G. Arsac, Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire, Pub. Dép. Math. (Lyon) 13 (2) (1976) 1–101.
[3] J.W. Baker, A. Rejali, On the Arens regularity of weighted convolution algebras, J. Lond. Math. Soc. (2) 40 (3) (1989) 535–546.
[4] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Springer-Verlag, Berlin, 1973.
[5] A. Bouziad, M. Filali, On the size of quotients of function spaces on a topological group, Stud. Math. 202 (2011) 243–259.
[6] C. Chou, Topological invariant means on the von Neumann algebra $VN(G)$, Trans. Am. Math. Soc. 273 (1) (1982) 207–229.
[7] I.G. Craw, N.J. Young, Regularity of multiplication in weighted group algebras and semigroup algebras, Q. J. Math. Oxf. Ser. (2) 25 (1974) 351–358.
[8] H.G. Dales, A.T.-M. Lau, The second duals of Beurling algebras, Mem. Am. Math. Soc. 177 (2005).
[9] Antoine Derighetti, Convolution Operators on Groups, Lecture Notes of the Unione Matematica Italiana, vol. 11, Springer, Heidelberg, 2011, UMI, Bologna.
[10] E.K. van Douwen, The number of cofinal ultrafilters, Topol. Appl. 39 (1991) 61–63.
[11] P. Eymard, L' algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. Fr. 92 (1964) 181–236.
[12] M. Filali, J. Galindo, Extreme non-Arens regularity of the group algebra, Forum Math. 30 (5) (2018) 1193–1208.
[13] M. Filali, J. Galindo, On the extreme non-Arens regularity of Banach algebras, J. Lond. Math. Soc. (2) 104 (4) (2021) 1840–1860.
[14] M. Filali, J. Galindo, ℓ^1-bases in Banach algebras and Arens irregularities in harmonic analysis, in: Mahmoud Filali (Ed.), Banach Algebras and Applications, De Gruyter, 2020, pp. 95–132.
[15] M. Filali, A.I. Singh, Recent developments on Arens regularity and ideal structure of the second dual of a group algebra and some related topological algebras, in: General Topological Algebras, Tartu, 1999, in: Math. Stud. (Tartu), vol. 1, Est. Math. Soc., 2001, pp. 95–124.

[16] M. Filali, T. Vedenjuoksu, Extreme non-Arens regularity of semigroup algebras, Topol. Proc. 33 (2009) 185–196.

[17] B. Forrest, Arens regularity and discrete groups, Pac. J. Math. 151 (1991) 217–227.

[18] E.E. Granirer, Day points for quotients of the Fourier algebra $A(G)$, extreme nonergodicity of their duals and extreme non-Arens regularity, Ill. J. Math. 40 (3) (1996) 402–419.

[19] N. Grønbæk, Amenability of weighted convolution algebras on locally compact groups, Trans. Am. Math. Soc. 319 (1990) 765–775.

[20] Z. Hu, On the set of topologically invariant means on the von Neumann algebra VN(G), Ill. J. Math. 39 (1995) 463–490.

[21] Z. Hu, Extreme non-Arens regularity of quotients of the Fourier algebra $A(G)$, Colloq. Math. 72 (2) (1997) 237–249.

[22] Z. Hu, The von Neumann algebra VN(G) of a locally compact group and quotients of its subspaces, Can. J. Math. 49 (6) (1997) 1117–1138.

[23] Z. Hu, M. Neufang, Decomposability of von Neumann algebras and the Mazur property of higher level, Can. J. Math. 58 (2006) 768–795.

[24] R.V. Kadison, J.R. Ringrose, Fundamentals of the theory of operator algebras, vol. I, in: Elementary Theory, in: Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997, Reprint of the 1983 original.

[25] E. Kaniuth, A.T.-M. Lau, Fourier and Fourier-Stieltjes Algebras on Locally Compact Groups, Mathematical Surveys and Monographs, American Mathematical Society, 2018.

[26] A.T.-M. Lau, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups, Fundam. Math. 118 (3) (1983) 161–175.

[27] A.T.-M. Lau, J.C.S. Wong, Weakly almost periodic elements in $L^\infty(G)$ of a locally compact group, Proc. Am. Math. Soc. 107 (1989) 1031–1036.

[28] J.S. Pym, The convolution of functionals on spaces of bounded functions, Proc. Lond. Math. Soc. (3) 15 (1965) 84–104.

[29] P.F. Renaud, Invariant means on a class of von Neumann algebras, Trans. Am. Math. Soc. 170 (1972) 285–291.

[30] A. Ülger, Continuity of weakly almost periodic functionals on $L^1(G)$, Q. J. Math. Oxf. Ser. (2) 37 (148) (1986) 495–497.