Dynamics of weighted composition operators on weighted Banach spaces of entire functions

María J. Beltrán-Meneu *

Abstract

We study the dynamics of the weighted composition operator $C_{w,\varphi}$ on the weighted Banach spaces of entire functions $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$. We characterize the continuity and compactness of the operator and, in the case of affine symbols $\varphi(z) = az+b, a, b \in \mathbb{C}$, and exponential weights, we analyze when the operator is power bounded, (uniformly) mean ergodic and hypercyclic. Continuous weighted composition operators when $|a| = 1$ are just multiples of composition operators λC_φ $\lambda \in \mathbb{C}$. When $|a| < 1$, we consider as a multiplier w the product of a polynomial by an exponential function. For multiples of composition operators, we get a complete characterization of power boundedness and mean ergodicity and we study the hypercyclicity in terms of λ. An example of a power bounded but not mean ergodic operator on $H_v^0(\mathbb{C})$ is provided. For the case of composition operators, we obtain the spectrum and a complete characterization of the dynamics.

Keywords: Weighted composition operator; Weighted Banach spaces of entire functions; Power bounded operator; Mean ergodic operator; Hypercyclic operator.

2020 Mathematics Subject Classification: 46E15, 47A16, 47A35, 47B33.

1 Introduction and outline of the paper

The purpose of this paper is to study the dynamics of the weighted composition operator $C_{w,\varphi} : f \to w(f \circ \varphi)$ on the weighted Banach spaces of entire functions $H_{v_m}(\mathbb{C})$ and $H_{v_m}^0(\mathbb{C})$, where $v_m(z) = e^{-m|z|}$, $z \in \mathbb{C}$, $m > 0$, is an exponential weight. More precisely, we study when $C_{w,\varphi}$ is power bounded, (uniformly) mean ergodic and hypercyclic. We refer to the next section for the precise notation and definitions.

Weighted Banach spaces of holomorphic functions have been widely studied. They appear in a natural way in the study of the growth of analytic functions. See for example [9] and [26] and the references therein.

*Departament d'Educació i Didàctiques Específiques, Universitat Jaume I, Av. Vicent Sos Baynat, s/n, E-12071. Castelló de la Plana, Spain.

Composition operators between weighted spaces of holomorphic functions have been studied by several authors. Boundedness and compactness on weighted Banach spaces of holomorphic functions on the disc $\mathbb{D}$ were characterized by Bonet, Domański, Lindström and Taskinen in [12], and on weighted inductive limits of Banach spaces of holomorphic functions defined on arbitrary open subsets of $\mathbb{C}$ by Bonet, Friz and Jordá in [13]. There is a vast literature about composition operators on Banach spaces of holomorphic functions. We refer the reader to the books by Cowen and MacCluer [18] and Shapiro [32].

The study of $C_{w,\varphi}$ on different spaces of functions has been a very active area of research. For example, it is well-known that all isometries of the Hardy spaces $H^p(\mathbb{D})$ for $1 \leq p < \infty$, $p \neq 2$, are weighted composition operators [18]. The boundedness and compactness of $C_{w,\varphi}$ on weighted Banach spaces of holomorphic functions on the disc $\mathbb{D}$ have been characterized by Contreras and Hernández-Díaz in [17] and by Montes-Rodríguez in [29].

The dynamics of composition and weighted composition operators on spaces of holomorphic functions has recently attracted the attention of many researchers. See for instance [5], [6], [7], [11], [16], [21] [22], [27], [31], and the references therein.

Concerning weighted Banach spaces of holomorphic functions on $\mathbb{D}$, Bonet and Ricker [15] studied the mean ergodicity of multiplication operators, Miralles and Wolf [28] the hypercyclicity of the composition operator C_φ, and Liang and Zhou [23] the hypercyclicity of its multiples λC_φ, $\lambda \in \mathbb{C}$. The power boundedness and mean ergodicity of weighted composition operators on $H_v(\mathbb{D})$ have been studied by Wolf in [33].

To our knowledge, this is the first attempt to study weighted composition operators on weighted Banach spaces of entire functions and their dynamics. In Section 2 we fix the notation and review some basic results. In Section 3 we characterize the continuity and compactness of $C_{w,\varphi}$ on general weighted Banach spaces of entire functions, obtaining analogous results to those in [17] and [13] for the corresponding spaces on the disc and for unweighted composition operators, respectively. It is known (see [2, Corollary 30] and [14, Proposition 3.1]) that if the composition operator C_φ on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ is continuous, then the symbol φ must be affine, that is, $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$. In Proposition 6 we give a condition under which the continuity of the weighted composition operator also implies the affinity of the symbol. In Theorem 8 we characterize the continuity and compactness of $C_{w,\varphi}$ on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ for affine symbols:

Theorem A
Given $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, the weighted composition operator $C_{w,\varphi}$ is continuous on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ if and only if $\|C_{w,\varphi}\| = \sup_{z\in\mathbb{C}} |w(z)|e^{m(|az+b|-|z|)} < \infty$, and compact if and only if $\lim_{|z|\to\infty} |w(z)|e^{m(|az+b|-|z|)} = 0$. As a consequence we get:

(i) If $|a| > 1$, $C_{w,\varphi}$ can never be continuous on $H_{v_m}(\mathbb{C})$, neither on $H^0_{v_m}(\mathbb{C})$.

(ii) If $|a| = 1$, $C_{w,\varphi}$ is continuous if and only if $w \equiv \lambda$, $\lambda \in \mathbb{C}$, and it is never compact.

(iii) If $|a| < 1$, the continuity and compactness depends on the multiplier. For instance:

a) $C_{w,\varphi}$ is continuous and compact if w is a polynomial.

b) If $w(z) = p_N(z)e^{q_M(z)}$, with p_N and q_M polynomials of degrees N and $M \neq 0$:

- $C_{w,\varphi}$ is continuous if and only if $w(z) = p_N(z)e^{b_1 z+b_0}$ and $|b_1| < m(1-|a|)$ or $w(z) = \lambda e^{b_1 z+b_0}$, $\lambda \in \mathbb{C}$, and $|b_1| = m(1-|a|)$.
- $C_{w,\varphi}$ is compact if and only if $w(z) = p_N(z)e^{b_1 z+b_0}$ and $|b_1| < m(1-|a|)$.

In Section 4 we study the spectrum of the composition operator and the power boundedness, (uniform) mean ergodicity and hypercyclicity of $C_{w,\varphi}$ on the spaces $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$ when $\varphi(z) = az + b$. We compile the main results of Proposition 15 and Theorems 16, 18, 19 and 20 in the next Theorem. The case $a = 1$ refers to multiples of the translation operator. It should be noted that it provides an example of a uniformly mean ergodic operator whose iterates converge to 0 in the strong operator topology but not in the operator norm. It is also worth mentioning that in the case of $|a| < 1$ we provide an example of a power bounded operator on $H^0_v(\mathbb{C})$ which is not mean ergodic.

Theorem B
Given $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, the operator $C_{w,\varphi}$ on $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$ is never weakly supercyclic if $a \neq 1$. Moreover it satisfies:

(i) When $|a| < 1$ and $w(z) = p_N(z)e^{b_1 z+b_0}$, $b_0, b_1 \in \mathbb{C}$, p_N a polynomial of degree N :

a) If $|b_1| < m(1-|a|)$, then $C_{w,\varphi}$ is uniformly mean ergodic whenever it is power bounded. This is satisfied, for instance, if w is a polynomial.

- If $w(z) = \lambda e^{b_1 z+b_0}$, $\lambda \in \mathbb{C}$, power boundedness, mean ergodicity and uniformly mean ergodicity are equivalent to $|\lambda| \leq |e^{\frac{b_1 b}{a-1}-b_0}|$. Moreover, $\|C^k_{w,\varphi}\| \to 0$ if $|\lambda| < |e^{\frac{b_1 b}{a-1}-b_0}|$.
- If $w(z) = \lambda(z - \frac{b}{1-a})^N e^{b_1 z+b_0}$, $N \neq 0$, $\|C^k_{w,\varphi}\| \to 0$, so it is always power bounded and uniformly mean ergodic.

b) If $w(z) = \lambda e^{b_1 z+b_0}$, $\lambda \in \mathbb{C}$, and $|b_1| = m(1-|a|)$, $b_1 \neq 0$, then $C_{w,\varphi}$ is power bounded if and only if $|\lambda| \leq |e^{\frac{b_1 b}{a-1}-b_0}|$.

- If $|\lambda| < \left|e^{\frac{b_1 b}{a-1}-b_0}\right|$, then $\|C^k_{w,\varphi}\| \to 0$, so $C_{w,\varphi}$ is power bounded and uniformly mean ergodic.
- If $|\lambda| > \left|e^{\frac{b_1 b}{a-1}-b_0}\right|$, then $C_{w,\varphi}$ is neither mean ergodic nor power bounded.
- If $\lambda = e^{\frac{b_1 b}{a-1}-b_0}$ and $a \in \mathbb{R}$, $a > 0$, then $C_{w,\varphi}$ is power bounded but not mean ergodic on $H^0_{v_m}(\mathbb{C})$.

(ii) When $|a| = 1, a \neq 1$, $C_{w,\varphi}$, which is necessarily of the form λC_φ, $\lambda \in \mathbb{C}$, satisfies:

a) If $|\lambda| < 1$, $\|C^k_{w,\varphi}\| \to 0$, thus it is power bounded and uniformly mean ergodic.

b) If $|\lambda| > 1$, it is neither power bounded nor mean ergodic.

c) If $|\lambda| = 1$, it is power bounded and mean ergodic on $H^0_{v_m}(\mathbb{C})$. It is uniformly mean ergodic if and only if $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$. If $a^n \neq 1$ for every $n \in \mathbb{N}$, it is not mean ergodic on $H_{v_m}(\mathbb{C})$.

(iii) When $a = 1$, $b \neq 0$, $C_{w,\varphi}$, which is necessarily of the form λC_φ, $\lambda \in \mathbb{C}$, satisfies:

a) If $|\lambda| < e^{-m|b|}$, $\|C^k_{w,\varphi}\| \to 0$, then it is power bounded, uniformly mean ergodic and not hypercyclic on $H^0_{v_m}(\mathbb{C})$.

b) If $|\lambda| > e^{-m|b|}$, it is neither power bounded nor mean ergodic. In this case, it is hypercyclic if $|\lambda| < e^{m|b|}$ and not hypercyclic if $|\lambda| > e^{m|b|}$.

c) If $|\lambda| = e^{-m|b|}$, it is power bounded, hence not hypercyclic, and $C^k_{w,\varphi} \to 0$ in $H^0_{v_m}(\mathbb{C})$ in the strong operator topology but not in the operator norm. It is uniformly mean ergodic if and only if $\lambda \neq e^{-m|b|}$.

In the last section of the paper we compile the results for the relevant case $w \equiv 1$, that is, for composition operators.

2 Notation and preliminaries

Our notation is standard. We denote by $H(\mathbb{C})$ the space of entire functions endowed with the compact open topology τ_{co} of uniform convergence on the compact subsets of $\mathbb{C}$, and by $\mathbb{D}$ the open unit disc centered at zero. Given two entire functions w and φ, the *weighted composition operator* $C_{w,\varphi}$ on $H(\mathbb{C})$ is defined by

$$C_{w,\varphi}(f) = w(f \circ \varphi),\ \ f \in H(\mathbb{C}).$$

The function φ is called *symbol* and w is called *multiplier*. $C_{w,\varphi}$ combines the composition operator $C_\varphi : f \mapsto f \circ \varphi$ with the pointwise multiplication operator $M_w : f \mapsto w \cdot f$.

We say that $v : \mathbb{C} \to]0, \infty[$ is a *weight* if it is continuous, decreasing and radial, that is, $v(z) = v(|z|)$ for every $z \in \mathbb{C}$. It is *rapidly decreasing* if $\lim_{r\to\infty} r^k v(r) = 0$ for all $k \in \mathbb{N}$.

For an arbitrary weight v on $\mathbb{C}$ we consider the *weighted Banach spaces of entire functions* with O- and o-growth conditions

$$H_v(\mathbb{C}) = \{f \in H(\mathbb{C}) : \|f\|_v := \sup_{z\in\mathbb{C}} v(z)|f(z)| < \infty\},$$

$$H^0_v(\mathbb{C}) = \{f \in H(\mathbb{C}) : \lim_{|z|\to\infty} v(z)|f(z)| = 0\}.$$

$(H_v(\mathbb{C}), \| \|_v)$ and $(H^0_v(\mathbb{C}), \| \|_v)$ are Banach spaces, and $(H^0_v(\mathbb{C}), \| \|_v) \hookrightarrow (H_v(\mathbb{C}), \| \|_v) \hookrightarrow (H(\mathbb{C}), \tau_{co})$ with continuous inclusions. If we assume v is rapidly decreasing, then $H^0_v(\mathbb{C})$ and $H_v(\mathbb{C})$ contain the polynomials. We denote by B_v and B^0_v the closed unit balls of $H_v(\mathbb{C})$ and $H^0_v(\mathbb{C})$, respectively. B_v is compact with respect to τ_{co}.

Given a weight v, its *associated weight* $\widetilde{v}$ is defined as

$$\widetilde{v}(z) := \frac{1}{\sup\{|f(z)| : f \in H_v(\mathbb{C}), \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v(\mathbb{C})'}},$$

where $\delta_z : H_v(\mathbb{C}) \to \mathbb{C}$, $f \mapsto f(z)$ is the continuous evaluation at z.

It is known (see [9, Proposition 1.2]) that $v \leq \widetilde{v}$, $H_v(\mathbb{C}) = H_{\widetilde{v}}(\mathbb{C})$ isometrically and $H^0_{\widetilde{v}}(\mathbb{C})$ is a closed subspace of $H^0_v(\mathbb{C})$. Although the last two spaces do not coincide in general, it follows from results in [8] and [12] that for rapidly decreasing weights, $H^0_v(\mathbb{C}) = H^0_{\widetilde{v}}(\mathbb{C})$. A weight v is said to be *essential* if there exists a constant $C > 0$ such that $v(z) \leq \widetilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{C}$. As mentioned in [9], many results on weighted spaces of analytic functions and on weighted composition operators defined on them have to be formulated in terms of the associated weights and not directly on the given weights, since they satisfy nice additional properties. The spaces under consideration $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$, are associated to the essential weights $v_m(z) = e^{-m|z|}, m > 0$.

Let X be a Banach space and $T : X \to X$ a continuous and linear operator on X. We say that $x_0 \in X$ is a *fixed point* if $T(x_0) = x_0$, and that it is *periodic* if there exists $n \in \mathbb{N}$ such that $T^n(x_0) = x_0$, where $T^n := T \circ \overset{n)}{\cdots} \circ T$. The operator T is said to be *power bounded* if $\sup_n \|T^n\| < \infty$ and it is called *mean ergodic* if the Cesàro means $(T_{[n]})_n$,

$$T_{[n]} := \frac{1}{n}\sum_{j=1}^{n} T^j, \quad n \in \mathbb{N},$$

converge to some P in the strong operator topology, i.e., if for every $x \in X$ the limit $\lim_{n\to\infty} T_{[n]}(x)$ exists in X. If $(T_{[n]})_n$ converges in $L(X)$ then T is called *uniformly mean ergodic*.

A power bounded operator T is mean ergodic precisely when $X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}$. Moreover, $\mathrm{Im}P = \mathrm{Ker}(I - T)$ and $\mathrm{Ker}P = \overline{\mathrm{Im}(I - T)}$. Clearly, if T is mean ergodic, then $\lim_{n\to\infty} ||T^n x||/n = 0$ for each $x \in X$, and if it is uniformly mean ergodic, $\lim_{n\to\infty} ||T^n||/n = 0$. If this condition is satisfied, Lin proved in [24] that the operator T is uniformly mean ergodic if and only if $\mathrm{Im}(I - T)$ is closed. For a Grothendieck Dunford-Pettis space X, Lotz proved that and operator $T \in L(X)$ satisfying $\|T^n/n\| \to 0$ is mean ergodic if and only if it is uniformly mean ergodic [25]. $H_{v_m}(\mathbb{C})$ is a Grothendieck Dunford-Pettis space (see [26]).

An abstract result of Yosida and Kakutani ([34, Theorem 4 and Corollary on pages 204-205] implies that every compact power bounded operator on a Banach space is uniformly mean ergodic. Yosida (see [30, Theorem 1.3]) also proved that in power bounded operators on Banach spaces, the convergence of the Cesàro means in the strong operator topology is equivalent to the convergence in the weak operator topology. Troughout the paper, we will use the following well-known fact: if T is power bounded and the sequence $(T_{[n]})_n$ converges to a continuous operator T on some dense set $D \subseteq X$, then T is mean ergodic.

An operator $T : X \to X$ is called *topologically transitive* if, for any pair U, V of non-empty open subsets of X, there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, and T

is called *topologically mixing* if there exists some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \in \mathbb{N}$, $n \geq N$. T is said to be *hypercyclic* if it has a dense orbit, that is, if there is some $x \in X$ such $\mathrm{Orb}(T, x) = \{T^n x : n = 0, 1, \dots\}$ is dense in X. Any such vector is called a *hypercyclic vector*. By the *Birkhoff's transitivity criterion*, T is hypercyclic if and only if it is topologically transitive. If $\mathrm{Orb}(T, \mathrm{span}\{x\}) = \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, \dots\}$ is dense in X, we say that T is *supercyclic* and x is a *supercyclic vector* for T, and if $\mathrm{span}\{\mathrm{Orb}(T, x)\} = \mathrm{span}\{T^n x : \ n = 0, 1, \dots\}$ is dense, it is said to be *cyclic* and x is called a *cyclic vector*. In the case we consider the density in the weak topology, the operator is said to be weakly hypercyclic, weakly supercyclic or weakly cyclic, respectively. An operator $T : X \to X$ is called *chaotic* if it is hypercyclic and it has a dense set of periodic points.

For a good exposition of ergodic theory we refer the reader to the monograph by [30], and for the subject of linear dynamics, to the monographs by Bayart and Matheron [1] and by Grosse-Erdmann and Peris [20].

3 Boundedness and compactness of weighted composition operators

We begin this section characterizing the continuity and compactness of weighted composition operators on general weighted Banach spaces of entire functions. The first two lemmata follow by an adaptation of the proofs of [17, Proposition 3.1 and Proposition 3.2], stated for weighted Banach spaces of holomorphic functions on the unit disc (see also [12] and [13, Proposition 5]).

Lemma 1 *Given two weights u_1 and u_2 on $\mathbb{C}$, the following are equivalent:*

(i) $C_{w,\varphi} : H_{u_1}(\mathbb{C}) \to H_{u_2}(\mathbb{C})$ *is continuous.*

(ii) $C_{w,\varphi}(H_{u_1}(\mathbb{C})) \subseteq H_{u_2}(\mathbb{C})$.

(iii) $\|C_{w,\varphi}\| := \sup_{z \in \mathbb{C}} \frac{|w(z)| u_2(z)}{\widetilde{u_1}(\varphi(z))} < \infty$.

If u_1 is essential, $C_{w,\varphi}$ is continuous if and only if $\sup_{z \in \mathbb{C}} \frac{|w(z)| u_2(z)}{u_1(\varphi(z))} < \infty$.

Lemma 2 *Given two rapidly decreasing weights u_1 and u_2 on $\mathbb{C}$, the following are equivalent:*

(i) $C_{w,\varphi} : H^0_{u_1}(\mathbb{C}) \to H^0_{u_2}(\mathbb{C})$ *is continuous.*

(ii) $C_{w,\varphi}(H^0_{u_1}(\mathbb{C})) \subseteq H^0_{u_2}(\mathbb{C})$.

(iii) $w \in H^0_{u_2}(\mathbb{C})$ *and* $\|C_{w,\varphi}\| := \sup_{z \in \mathbb{C}} \frac{|w(z)| u_2(z)}{\widetilde{u_1}(\varphi(z))} < \infty$.

If u_1 is essential, $C_{w,\varphi}$ is continuous if and only if $w \in H^0_{u_2}(\mathbb{C})$ and $\sup_{z \in \mathbb{C}} \frac{|w(z)| u_2(z)}{u_1(\varphi(z))} < \infty$.

The next lemma follows proceeding as in the proof of [13, Theorem 8], where the result is stated for unweighted composition operators.

Lemma 3 *Given two weights u_1 and u_2, consider the following assertions:*

(i) $C_{w,\varphi} : H_{u_1}(\mathbb{C}) \to H^0_{u_2}(\mathbb{C})$ *is compact.*

(ii) $C_{w,\varphi} : H_{u_1}(\mathbb{C}) \to H_{u_2}(\mathbb{C})$ *is compact and* $C_{w,\varphi}(H_{u_1}(\mathbb{C})) \subseteq H^0_{u_2}(\mathbb{C})$.

(iii) $C_{w,\varphi} : H^0_{u_1}(\mathbb{C}) \to H^0_{u_2}(\mathbb{C})$ *is compact.*

(iv) $\lim_{|z|\to\infty} \frac{|w(z)|u_2(z)}{\widetilde{u_1}(\varphi(z))} = 0$.

Then (i) ⇒ (ii), (ii) ⇒ (iii) and (iv) ⇒ (i). If we assume $\overline{B^0_v}^{\tau_{co}} = B_v$ *then (iii) ⇒ (iv) and all the conditions are equivalent.*

Now we look at the symbol of the operator. If $|\varphi(z)| = O(|z|)$, it is trivial that φ must be affine. In the next results we give some conditions under which only affine symbols can induce continuous weighted composition operators.

Remark 4 For an essential weight u satisfying $u(\varphi(z)) = O(u(z))$, if the operator $C_{w,\varphi}$ is continuous on $H_u(\mathbb{C})$, by Lemma 1 there exists $C > 0$ such that $\sup_{z\in\mathbb{C}} |w(z)| \leq C \sup_{z\in\mathbb{C}} \frac{u(\varphi(z))}{u(z)} < \infty$, and so, $w \equiv \lambda$ for some $\lambda \in \mathbb{C}$. Then, if $C_{w,\varphi} = \lambda C_\varphi$, the symbol must be affine by [2, Corollary 30] (see also [14, Proposition 3.1]). In particular:

- If $C_{w,\varphi} : H_u(\mathbb{C}) \to H_u(\mathbb{C})$ is continuous and there exists $R > 0$ such that $|z| \leq |\varphi(z)|$ for every $|z| \geq R$, then φ must be affine. Thus, if $\varphi(z) = p_N(z)$, $N \geq 2$, that is, a polynomial of degree greater than or equal to 2, the operator $C_{w,\varphi}$ can never be continuous.
- If $C_{w,\varphi} : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$ is continuous and there exists $R > 0, M \geq 0$ such that $|z| \leq M + |\varphi(z)|$ for all $|z| \geq R$, then φ must be affine.

The proof of the next result is analogous to the one in [14, Proposition 3.1], stated for unweighted composition operators.

Proposition 5 *Consider two weights u_1 and u_2 such that, for some $\alpha > 1$,*

$$\lim_{|z|\to\infty} \frac{|w(z)|u_2(z)}{\widetilde{u_1}(\alpha z)} = \infty.$$

If the operator $C_{w,\varphi} : H_{u_1}(\mathbb{C}) \to H_{u_2}(\mathbb{C})$ *is continuous, then φ is affine, that is, there exist* $a, b \in \mathbb{C}$ *such that* $\varphi(z) = az + b$.

Proof. It is enough to see that $\sup_{z\in\mathbb{C},|z|\geq 1}\frac{|\varphi(z)|}{|z|}<\infty$. If we assume the contrary, then there exists $(z_k)_k$ such that $|z_k|\to\infty$ and $|\varphi(z_k)|\geq k|z_k|$ for every $k\in\mathbb{N}$. This fact and the continuity condition in Lemma 1 imply

$$\sup_{k\in\mathbb{N}}\frac{|w(z_k)|u_2(z_k)}{\widetilde{u_1}(\alpha z_k)}\leq\sup_{k\in\mathbb{N}}\frac{|w(z_k)|u_2(z_k)}{\widetilde{u_1}(kz_k)}\leq\sup_{k\in\mathbb{N}}\frac{|w(z_k)|u_2(z_k)}{\widetilde{u_1}(\varphi(z_k))}<\infty,$$

a contradiction. $\square$

In the rest of the section we study the continuity and compactness of $C_{w,\varphi}$ on the spaces $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$, $v_m(z)=e^{-m|z|}$, $m>0$. As the weight v_m is essential, Lemma 1 and Proposition 5 yield a condition under which continuity implies the symbol must be affine:

Proposition 6 *Assume w is a multiplier such that there exists $\alpha>1$ with*

$$\lim_{|z|\to\infty}|w(z)|e^{|z|m(\alpha-1)}=\infty. \tag{3.1}$$

If $C_{w,\varphi}: H_{v_m}(\mathbb{C})\to H_{v_m}(\mathbb{C})$ is continuous, that is, if $\sup_{z\in\mathbb{C}}|w(z)|e^{m(|\varphi(z)|-|z|)}<\infty$, then φ must be affine. As a consequence, if C_φ is continuous, then φ must be affine.

Example 7 The following multipliers satisfy (3.1), and so, only can induce continuous weighted composition operators on $H_{v_m}(\mathbb{C})$ with affine symbols:

- $w\in H^0_{v_m}(\mathbb{C})$ such that $|w(z)|\geq\delta$ for every $|z|\geq R$, for some $R,\delta>0$. For instance, if w is a polynomial.
- $w(z)=p_N(z)e^{q_M(z)}$, with p_N and q_M polynomials of degrees $N\geq 0$ and $M\leq 1$, respectively.

However, for $w(z)=p_N(z)e^{q_M(z)}$, $M>1$, (3.1) is not satisfied.

In the rest of the paper, we focus on affine symbols $\varphi(z)=az+b$, $a,b\in\mathbb{C}$. As the weights v_m, $m>0$, are rapidly decreasing, we get the following:

Theorem 8 *$C_{w,\varphi}$ is continuous on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ if and only if*

$$\|C_{w,\varphi}\|=\sup_{z\in\mathbb{C}}|w(z)|e^{m(|az+b|-|z|)}<\infty, \tag{3.2}$$

and $C_{w,\varphi}$ is compact on both spaces if and only if

$$\lim_{|z|\to\infty}|w(z)|e^{m(|az+b|-|z|)}=0. \tag{3.3}$$

Observe that (3.2) implies that the multiplier w must belong to $H^0_{v_m}(\mathbb{C})$ and the compactness on $H_{v_m}(\mathbb{C})$ yields $C_{w,\varphi}(H_{v_m}(\mathbb{C}))\subseteq H^0_{v_m}(\mathbb{C})$. As a consequence we get:

(i) If $|a|>1$, $C_{w,\varphi}$ can never be continuous on $H_{v_m}(\mathbb{C})$, neither on $H^0_{v_m}(\mathbb{C})$.

(ii) if $|a| = 1$, $C_{w,\varphi}$ is continuous on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ if and only if $w \equiv \lambda$ for some $\lambda \in \mathbb{C}$. In this case, $\|C_{w,\varphi}\| = |\lambda|\|C_\varphi\| = |\lambda|e^{m|b|}$ and $C_{w,\varphi}$ is never compact.

(iii) If $|a| < 1$, the continuity and compactness depends on the multiplier. For instance:

a) $C_{w,\varphi}$ is continuous and compact if w is a polynomial. In particular, λC_φ is compact for every $\lambda \in \mathbb{C}$.

b) If $w(z) = p_N(z)e^{q_M(z)}$, with p_N and q_M polynomials of degrees $N \geq 0$ and $M > 0$, respectively, we get:

- *$C_{w,\varphi}$ is continuous if and only if $w(z) = p_N(z)e^{b_1 z + b_0}$ and $|b_1| < m(1 - |a|)$, or $w(z) = \lambda e^{b_1 z + b_0}$ and $|b_1| = m(1 - |a|)$.*
- *$C_{w,\varphi}$ is compact if and only if $w(z) = p_N(z)e^{b_1 z + b_0}$ and $|b_1| < m(1 - |a|)$.*

Proof. Lemmata 1 and 2 yield the characterization of continuity, since (3.2) implies $w \in H^0_{v_m}(\mathbb{C})$. $\overline{B^0_{v_m}}^{\tau_{co}} = B_{v_m}$ because the weights are rapidly decreasing, so Lemma 3 yields the characterization of compactness of $C_{w,\varphi} : H^0_{v_m}(\mathbb{C}) \to H^0_{v_m}(\mathbb{C})$. As $C_{w,\varphi} : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$ is its bitranspose (see [10, Corollary 1.2 and Example 2.2]), we also get the characterization holds on $H_{v_m}(\mathbb{C})$. Moreover, compactness yields $C_{w,\varphi}(H_{v_m}(\mathbb{C})) \subseteq H^0_{v_m}(\mathbb{C})$.
Let us see first the continuity in (i) and (ii). Assume the operator is continuous and $|a| \geq 1$. We have

$$\|C_{w,\varphi}\| = \sup_{z\in\mathbb{C}} |w(z)|e^{m|az+b|-m|z|} \geq \sup_{z\in\mathbb{C}} |w(z)|e^{m|az|-m|b|-m|z|} \geq e^{-m|b|} \sup_{z\in\mathbb{C}} |w(z)|,$$

so w must be constant, that is, there exists $\lambda \in \mathbb{C}$ such that $C_{w,\varphi} = \lambda C_\varphi$. Now, (3.2) yields the conclusions. The compactness of (ii) follows by (3.3), since for $|a| = 1$, $|\lambda|e^{m(|az+b|-|z|)} \geq |\lambda|e^{-m|b|}$ for every $z \in \mathbb{C}$.
(iii) a) If $w(z) = p_N(z)$, that is, a polynomial of degree N, it is easy to see that

$$\lim_{|z|\to\infty} |p_N(z)|e^{m(|az+b|-|z|)} \leq e^{m|b|} \lim_{|z|\to\infty} |p_N(z)|e^{m|z|(|a|-1)} = 0.$$

b) Given $p_N(z) = \sum_{j=0}^N a_j z^j, a_j \in \mathbb{C}$, and $q_M(z) = \sum_{j=0}^M b_j z^j, b_j \in \mathbb{C}$, consider $\widetilde{p}_N(z) = \sum_{j=0}^N |a_j| z^j, a_j \in \mathbb{C}$, and $\widetilde{q}_M(z) = \sum_{j=0}^M |b_j| z^j, b_j \in \mathbb{C}$. Then for every $z \in \mathbb{C}$ we have

$$|p_N(z)e^{q_M(z)}|e^{m(|az+b|-|z|)} \leq \widetilde{p}_N(|z|)e^{\widetilde{q}_M(|z|)-m|z|(1-|a|)}e^{m|b|},$$

which yields the conditions for the continuity. Moreover, as the last inequality implies $\lim_{|z|\to\infty} |p_N(z)e^{q_M(z)}|e^{m(|az+b|-|z|)} = 0$ if $M = 1$ and $|b_1| < m(1 - |a|)$, we get compactness in this case. On the other hand,

$$\sup_{z\in\mathbb{C}} |p_N(z)e^{q_M(z)}|e^{m(|az+b|-|z|)} \geq e^{-m|b|} \sup_{z\in\mathbb{C}} |p_N(z)||e^{q_M(z)-m|z|(1-|a|)}|.$$

So, for $M \geq 2$ the operator can not be continuous. If $M = 1$, then $q_M(z) = b_1 z + b_0$ and we get that there exists $c \in \mathbb{C}$, $|c| = 1$, such that

$$\sup_{z\in\mathbb{C}} |p_N(z)e^{q_M(z)}|e^{m(|az+b|-|z|)} \geq e^{-m|b|}|e^{b_0}| \sup_{r\geq 0} |p_N(cr)|e^{|b_1|r-mr(1-|a|)},$$

which yields the assertion about the lack of continuity. When $|b_1| = m(1-|a|)$ and $N = 0$, the last inequality yields

$$\lim_{|z|\to\infty} |w(z)|e^{m(|az+b|-|z|)} \geq e^{-m|b|}|\lambda e^{b_0}| \neq 0, \tag{3.4}$$

and so the operator can not be compact. $\square$

4 Dynamics of weighted composition operators

In this section we study the dynamics of $C_{w,\varphi}$ on $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$. The iterates have the expression

$$C^k_{w,\varphi} = \left(\prod_{j=0}^{k-1} w(\varphi^j(z))\right) f(\varphi^k(z)),\ k \in \mathbb{N}, f \in H_{v_m}(\mathbb{C}).$$

In what follows, denote $w_{[k]}(z) := \prod_{j=0}^{k-1} w(\varphi^j(z))$, $z \in \mathbb{C}$. Observe that the symbol $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, has a fixed point $z_0 = \frac{b}{1-a}$ if and only if $a \neq 1$, and for $k \in \mathbb{N}$, we get

$$\varphi^k(z) = \begin{cases} a^k z + b\frac{1-a^k}{1-a} = a^k(z - \frac{b}{1-a}) + \frac{b}{1-a} \text{ if } a \neq 1 \\ z + bk \text{ if } a = 1. \end{cases} \tag{4.1}$$

Let us study first some general preliminary results.

Proposition 9 *Consider a weight u on $\mathbb{C}$ such that $C_{w,\varphi} : H_u(\mathbb{C}) \to H_u(\mathbb{C})$ is continuous. Then:*

(i) If $|w(z_0)| > 1$ for $z_0 \in \mathbb{C}$ a fixed point of φ, then $C_{w,\varphi}$ is neither power bounded nor mean ergodic.

(ii) If $C_{w,\varphi}$ is power bounded, then there exists $C > 0$ such that $\|w_{[k]}\|_u \leq C$ for every $k \in \mathbb{N}$.

(iii) If $C_{w,\varphi}$ is mean ergodic, then $\lim_k \frac{\|w_{[k]}\|_u}{k} = 0$.

Proof. (i) follows easily from the fact that, for $z_0 \in \mathbb{C}$ a fixed point of φ, then

$$\frac{\|C^k_{w,\varphi}(1)\|_u}{k} \geq \frac{|C^k_{w,\varphi}(1)(z_0)|}{k} u(z_0) = \frac{|w(z_0)|^k}{k} u(z_0).$$

For (ii) and (iii), observe that there exists $C > 0$ such that $\|C^k_{w,\varphi}(1)\|_u = \|w_{[k]}\|_u \leq C$ for every $k \in \mathbb{N}$ if $C_{w,\varphi}$ is power bounded, and $\lim_k \frac{\|C^k_{w,\varphi}(1)\|_u}{k} = \lim_k \frac{\|w_{[k]}\|_u}{k} = 0$ if the operator is mean ergodic. $\square$

Since $C^k_{w,\varphi}$, $k \in \mathbb{N}$, is a weighted composition operator associated to the symbol φ^k and the multiplier $w_{[k]}(z) = \prod_{j=0}^{k-1} w(\varphi^j(z))$, Lemmata 1 and 2 provide a characterization for the power boundedness of $C_{w,\varphi}$:

Proposition 10 *Given a weight u, the operator $C_{w,\varphi} : H_u(\mathbb{C}) \to H_u(\mathbb{C})$ is power bounded if and only if there exists $C > 0$ such that, for every $k \in \mathbb{N}$,*

$$\|C_{w,\varphi}^k\| = \sup_{z\in\mathbb{C}} \left(\prod_{j=0}^{k-1} |w(\varphi^j(z))|\right) \frac{u(z)}{\widetilde{u}(\varphi^k(z))} < C. \tag{4.2}$$

If u is essential, we replace $\widetilde{u}$ by u in (4.2). In the case of $C_{w,\varphi} : H_u^0(\mathbb{C}) \to H_u^0(\mathbb{C})$, we also need $w \in H_u^0(\mathbb{C})$ and u rapidly decreasing.

Corollary 11 *The operator $C_{w,\varphi}$ is power bounded on $H_{v_m}(\mathbb{C})$ and on $H_{v_m}^0(\mathbb{C})$ if and only if there exists $C > 0$ such that, for all $k \in \mathbb{N}$,*

$$\|C_{w,\varphi}^k\| = \sup_{z\in\mathbb{C}} \left(\prod_{j=0}^{k-1} |w(\varphi^j(z))|\right) e^{m(|\varphi^k(z)|-|z|)} < C. \tag{4.3}$$

The next theorem relates the spectrum of the operator to uniform mean ergodicity. The necessary condition is due to Dunford [19, Proposition 3.1] and the sufficiency is proved by Lin [24].

Theorem 12 (Dunford-Lin) *An operator T on a Banach space X is uniformly mean ergodic if and only if $(\frac{\|T^n\|}{n})_n$ converges to 0 and, either $1 \in \mathbb{C} \setminus \sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C} \setminus \sigma(T) \to L(X)$, $R_T(\lambda) := (T - \lambda I)^{-1}$. Consequently, if 1 is an accumulation point of $\sigma(T)$, then T is not uniformly mean ergodic.*

In the following result we calculate the spectrum of the composition operator. It will be useful in order to study the dynamics.

Proposition 13 *Given $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, and $C_\varphi : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$ or $C_\varphi : H_{v_m}^0(\mathbb{C}) \to H_{v_m}^0(\mathbb{C})$, we get:*

(i) If $|a| \leq 1,\ a \neq 1$, $\sigma(C_\varphi) = \overline{\{a^n,\ n = 0, 1, \dots\}}$.

(ii) If $a = 1$, $\sigma(C_\varphi) = \{e^\delta, |\delta| \leq m|b|\}$.

Proof. As the bitranspose of $C_\varphi : H_{v_m}^0(\mathbb{C}) \to H_{v_m}^0(\mathbb{C})$ is $C_\varphi : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$ (see [10, Corollary 1.2 and Example 2.2]), we get that the spectrum is the same in both spaces.
(i) If $a \neq 1$, then $C_\varphi \left(z - \frac{b}{1-a}\right)^n = a^n \left(z - \frac{b}{1-a}\right)^n$, $n \in \mathbb{N}_0$, so $\overline{\{a^n,\ n = 0, 1, \dots\}} \subseteq \sigma(C_\varphi)$. Moreover, by (4.3), $r(C_\varphi) = \lim_k \|C_\varphi^k\|^{1/k} \leq \lim_k \exp(\frac{2m|b|}{k|1-a|}) = 1$, thus, $\sigma(C_\varphi) \subseteq \overline{\mathbb{D}}$. Proceeding as in the proof of [21, Proposition 3.3(i)], we get that a nonzero eigenvalue in $\mathbb{C}$ must be of the form a^n for some positive integer $n \in \mathbb{N}_0$. If $|a| < 1$, C_φ is compact by Theorem 8, then its spectrum contains only zero and eigenvalues, then the conclusion holds. Now, consider the case $|a| = 1$. If there exists $n \in \mathbb{N}$ such that $a^n = 1$, then $C_\varphi^n = I$, therefore $(\sigma(C_\varphi))^n = \sigma(C_\varphi^n) = \{1\}$ by the spectral mapping theorem and so, $\sigma(C_\varphi) \subseteq \overline{\{a^n,\ n = 0, 1, \dots\}}$. Otherwise, if $|a| = 1$ and $a^n \neq 1$ for every $n \in \mathbb{N}$,

we get $\mathbb{T} \subseteq \sigma(C_\varphi)$. C_φ has $C_{\varphi^{-1}}$ as a continuous inverse, where $\varphi^{-1}(z) = \frac{1}{a}z - b$ and $r(C_{\varphi^{-1}}) = \lim_k \|C^k_{\varphi^{-1}}\|^{1/k} \le \lim_k \exp(\frac{2m|b|}{k|1-1/a|}) = 1$. From this, together with $r(C_\varphi) \le 1$, we get $\sigma(C_\varphi) \subseteq \mathbb{T}$, as we wanted to see.
(ii) If $a = 1$, then C_φ is the translation operator $T_b : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C}), f(z) \mapsto f(z+b)$. Observe that it is indeed the differential operator $\phi(D)$ associated to the exponential function $\phi(z) = e^{bz}$. So, again by the spectral mapping theorem and [3, Proposition 5.10], we get $\sigma(C_\varphi) = \phi(\sigma(D)) = \phi(m\overline{\mathbb{D}}) = \{e^{bw},\ |w| \le m\}$. □

In what follows we study the power boundedness, (uniform) mean ergodicity and hypercyclicity of $C_{w,\varphi}$, associated to the symbol $\varphi(z) = az+b$, by distinguishing the following three possible cases for the parameter a: (1) if $|a| < 1$, that is, if $z_0 = \frac{b}{1-a}$ is an attractive fixed point of φ; (2) if $|a| = 1, a \neq 1$, that is, if φ is a rotation around $z_0 = \frac{b}{1-a}$; or (3) if $a = 1$, that is, if φ is a translation. Hypercyclicity is considered only in the case $a = 1$, since otherwise φ has a fixed point and [7, Proposition 2.1] yields $C_{w,\varphi}$ can not be weakly supercyclic. In the case $a \neq 1$, by the next remark we can consider without loss of generality that $b = 0$.

Remark 14 If $a \neq 1$, and $X = H_v(\mathbb{C})$ or $X = H^0_v(\mathbb{C})$, it is easy to see that the dynamical systems $C_\varphi : X \to X,\ f \mapsto f(az+b)$ and $C_{az} : X \to X,\ f \mapsto f(az)$ are conjugated through the homeomorphism $T : X \to X,\ f(z) \mapsto f(z - \frac{b}{1-a})$, since $T^{-1} \circ C_{az} \circ T = C_\varphi$.

4.1 Case $|a| < 1$

Proposition 15 *Given $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, $|a| < 1$, the operator $C_{w,\varphi} : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$ and $C_{w,\varphi} : H^0_{v_m}(\mathbb{C}) \to H^0_{v_m}(\mathbb{C})$ satisfies:*

(i) If $w(z) = p_N(z)e^{b_1 z + b_0}$, where p_N is a polynomial of degree $N \ge 0$ and $b_0, b_1 \in \mathbb{C}$ are such that $C_{w,\varphi}$ is continuous, it is not power bounded if $|p_N(\frac{b}{1-a})||e^{\frac{b_1 b}{1-a}+b_0}| > 1$. In particular, this is satisfied if $w(z) = \lambda z^N e^{b_1 z + b_0}$ and $|\lambda| \left|\frac{b}{1-a}\right|^N > |e^{\frac{b_1 b}{a-1}-b_0}|$.

(ii) If $w(z) = \lambda e^{b_1 z + b_0}$, $|b_1| \le m(1-|a|)$, it is power bounded if and only if $|\lambda| \le |e^{\frac{b_1 b}{a-1}-b_0}|$. If $|\lambda| < |e^{\frac{b_1 b}{a-1}-b_0}|$, then $\|C^k_{w,\varphi}\| \to 0$ as $k \to \infty$.

(iii) If $w(z) = \lambda(z - \frac{b}{1-a})^N e^{b_1 z + b_0}$, $|b_1| < m(1-|a|)$, $N \neq 0$, it is always power bounded. Even more, $\|C^k_{w,\varphi}\| \to 0$ as $k \to \infty$.

Proof. (i) follows directly by Proposition 9(i).

For the cases (ii) and (iii) we need some estimates. If $w(z)=\lambda(z-\frac{b}{1-a})^N e^{b_1 z+b_0}$, $N\geq 0$,

$$
\begin{aligned}
\|C_{w,\varphi}^k\| &= |\lambda|^k \left(\prod_{j=0}^{k-1} |a|^{Nj} |e^{b_1(a^j z+b\frac{1-a^j}{1-a})+b_0}|\right) \sup_{z\in\mathbb{C}} \left|z-\frac{b}{1-a}\right|^{Nk} e^{m(|a^k z+b\frac{1-a^k}{1-a}|-|z|)} \\
&\leq e^{2m\frac{|b|}{|1-a|}} \left|e^{-bb_1\frac{1-a^k}{(1-a)^2}}\right| |a|^{N(k-1)k/2} \left(|\lambda||e^{b_1\frac{b}{1-a}+b_0}|\right)^k \sup_{z\in\mathbb{C}} \left|z-\frac{b}{1-a}\right|^{Nk} e^{|z|(1-|a|^k)(\frac{|b_1|}{1-|a|}-m)} \\
&\leq e^{2\frac{|b|}{|1-a|}(m+\frac{|b_1|}{|1-a|})} |a|^{N(k-1)k/2} \left(|\lambda||e^{b_1\frac{b}{1-a}+b_0}|\right)^k \sup_{z\in\mathbb{C}} \left|z-\frac{b}{1-a}\right|^{Nk} e^{|z|(\frac{|b_1|}{1-|a|}-m)}. \qquad (4.4)
\end{aligned}
$$

(ii) follows by Proposition 9 and because if we put $N=0$ in (4.4), we get

$$\|C_{w,\varphi}^k\| \leq e^{2\frac{|b|}{|1-a|}(m+\frac{|b_1|}{|1-a|})} \left(|\lambda||e^{b_1\frac{b}{1-a}+b_0}|\right)^k.$$

(iii) For $|b_1|<m(1-|a|)$ and k big enough, we get

$$
\begin{aligned}
\sup_{z\in\mathbb{C}} \left|z-\frac{b}{1-a}\right|^{Nk} e^{|z|(\frac{|b_1|}{1-|a|}-m)} &\leq \max\left(\sup_{|z|\leq\frac{|b|}{|1-a|}} \left|z-\frac{b}{1-a}\right|^{Nk}, 2^{Nk} \sup_{|z|\geq\frac{|b|}{|1-a|}} |z|^{Nk} e^{-|z|(m-\frac{|b_1|}{1-|a|})}\right) \\
&\leq \max\left(\left|\frac{2b}{1-a}\right|, \frac{2Nk}{e(m-\frac{|b_1|}{1-|a|})}\right)^{Nk}. \qquad (4.5)
\end{aligned}
$$

Therefore, by (4.4) and (4.5), for k big enough we obtain

$$\|C_{w,\varphi}^k\| \leq e^{2\frac{|b|}{|1-a|}(m+\frac{|b_1|}{|1-a|})} |a|^{N(k-1)k/2} \left(|\lambda||e^{b_1\frac{b}{1-a}+b_0}| \left(\frac{2Nk}{e(m-\frac{|b_1|}{1-|a|})}\right)^N\right)^k \overset{k\to\infty}{\longrightarrow} 0,$$

which yields the conclusion. □

In the last assertion of the next theorem we provide an example of a power bounded but not mean ergodic operator on $H_{v_m}^0(\mathbb{C})$. This differs from the corresponding result obtained when considering the space $H(\mathbb{C})$. In this case, the operator is power bounded, and thus, uniformly mean ergodic (see [6, Theorem 3.10]).

Theorem 16 *Consider $\varphi(z)=az+b$, $z\in\mathbb{C}$, $a,b\in\mathbb{C}$, $|a|<1$, and $w(z)=p_N(z)e^{b_1 z+b_0}$, $b_0,b_1\in\mathbb{C}$, where p_N is a polynomial of degree $N\geq 0$. The operator $C_{w,\varphi}:H_{v_m}(\mathbb{C})\to H_{v_m}(\mathbb{C})$ and $C_{w,\varphi}:H_{v_m}^0(\mathbb{C})\to H_{v_m}^0(\mathbb{C})$ satisfies:*

a) If $|b_1|<m(1-|a|)$, then $C_{w,\varphi}$ is uniformly mean ergodic whenever it is power bounded. This is satisfied, for instance, if w is a polynomial.

- *If $w(z)=\lambda e^{b_1 z+b_0}$, $\lambda\in\mathbb{C}$, power boundedness, mean ergodicity and uniformly mean ergodicity are equivalent to $|\lambda|\leq|e^{\frac{b_1 b}{a-1}-b_0}|$. If $|\lambda|<|e^{\frac{b_1 b}{a-1}-b_0}|$ we even get $\|C_{w,\varphi}^k\|\to 0$ as $k\to\infty$.*

- *If* $w(z) = \lambda(z - \frac{b}{1-a})^N e^{b_1 z + b_0}$, $N \neq 0$, $C_{w,\varphi}$ *is always uniformly mean ergodic. Even more,* $\|C_{w,\varphi}^k\| \to 0$ *as* $k \to \infty$.

b) If $|b_1| = m(1 - |a|)$ *and* $w(z) = \lambda e^{b_1 z + b_0}$, $\lambda \in \mathbb{C}$, *we have:*

- *If* $|\lambda| < \left| e^{\frac{b_1 b}{a-1} - b_0} \right|$, *then* $C_{w,\varphi}$ *is uniformly mean ergodic and* $\|C_{w,\varphi}^k\| \to 0$ *as* $k \to \infty$.
- *If* $|\lambda| > \left| e^{\frac{b_1 b}{a-1} - b_0} \right|$, *then* $C_{w,\varphi}$ *is not mean ergodic.*
- *If* $\lambda = e^{\frac{b_1 b}{a-1} - b_0}$ *and* $a \in \mathbb{R}$, $a > 0$, *then* $C_{w,\varphi}$ *is power bounded but not mean ergodic on* $H_{v_m}^0(\mathbb{C})$.

Proof. a) follows by Theorem 8, since $C_{w,\varphi}$ is compact, and thus, uniformly mean ergodic whenever it is power bounded. The examples follow by Propositions 9(i) and 15.
b) The case $|\lambda| < \left| e^{\frac{b_1 b}{a-1} - b_0} \right|$ follows by Proposition 15 and the case $|\lambda| > \left| e^{\frac{b_1 b}{a-1} - b_0} \right|$ by Proposition 9(i). Consider $\lambda = e^{\frac{b_1 b}{a-1} - b_0}$ and $a \in \mathbb{R}$, $a > 0$. By Remark 14 we can assume without loss of generality that $b = 0$, and thus, that $w(z) = e^{b_1 z}$ and $\varphi(z) = az$. Observe that for $f \equiv 1$ and a fix $z \in \mathbb{C}$, we obtain $C_{w,\varphi}^k(f)(z) = e^{b_1 z \frac{1-a^k}{1-a}} \to e^{\frac{b_1 z}{1-a}}$ as k tends to ∞. So, if we assume the operator is mean ergodic on $H_{v_m}^0(\mathbb{C})$, then the Cesàro means of f must converge to $e^{\frac{b_1 z}{1-a}} \in H_{v_m}^0(\mathbb{C})$. But this is a contradiction, since $\lim_{|z| \to \infty} \left| e^{\frac{b_1 z}{1-a}} \right| e^{-m|z|} \neq 0$. Indeed, as $|b_1| = m(1 - |a|)$, we can find $c \in \mathbb{C}, |c| = 1$ such that

$$\left| e^{\frac{b_1 c r}{1-a}} \right| e^{-mr} = e^{\frac{|b_1| r}{|1-a|}} e^{-mr} = 1.$$

$\square$

By Proposition 15 and Theorem 16, in the case of multiples of composition operators, we get:

Corollary 17 *Consider* $\varphi(z) = az + b$, $z \in \mathbb{C}$, $|a| < 1$, *and the operator* λC_φ, $\lambda \in \mathbb{C}$, *on* $H_{v_m}(\mathbb{C})$ *and on* $H_{v_m}^0(\mathbb{C})$. *The following are equivalent:*

(i) λC_φ *is power bounded.*

(ii) λC_φ *is mean ergodic on* $H_{v_m}(\mathbb{C})$ *and on* $H_{v_m}^0(\mathbb{C})$.

(iii) λC_φ *is uniformly mean ergodic.*

(iv) $|\lambda| \leq 1$.

In the case $|\lambda| < 1$, *we get* $\|(\lambda C_\varphi)^k\| \to 0$ *as* $k \to \infty$.

4.2 Case $|a| = 1, a \neq 1$

In Theorem 8 we have seen that continuous weighted composition operators when $|a| = 1, a \neq 1$, are just those of the form λC_φ, $\lambda \in \mathbb{C}$. In the next theorem we characterize power boundedness and mean ergodicity.

Theorem 18 *Let $\varphi(z) = az + b$, $|a| = 1, a \neq 1$, and consider the operator λC_φ, $\lambda \in \mathbb{C}$. The following is satisfied:*

(i) If $|\lambda| < 1$, the sequence $(\lambda C_\varphi)^k$ converges to 0 on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$. Thus, the operator is power bounded and uniformly mean ergodic.

(ii) If $|\lambda| > 1$, the operator is neither power bounded nor mean ergodic on $H_{v_m}(\mathbb{C})$ neither on $H^0_{v_m}(\mathbb{C})$.

(iii) If $|\lambda| = 1$, the operator is power bounded and mean ergodic on $H^0_{v_m}(\mathbb{C})$. Moreover, it satisfies:

a) If $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$ (consider the smallest n_0) the operator is uniformly mean ergodic on $H^0_{v_m}(\mathbb{C})$ and on $H_{v_m}(\mathbb{C})$.

- $\lim_k \frac{1}{k}\sum_{j=1}^k (\lambda C_\varphi)^j = 0$ *if $\lambda^s \neq 1$ for every $s \in \mathbb{N}$ or if $\lambda^{s_0} = 1$ for some $s_0 \in \mathbb{N}$ and $a^j \neq \frac{1}{\lambda}$ for every $j \in \mathbb{N}$.*
- *Otherwise, if $\lambda^{s_0} = 1$ for some $s_0 \in \mathbb{N}$ and there exists $j_0 \in \mathbb{N}_0$ (consider the smallest s_0 and j_0) such that $a^{j_0} = \frac{1}{\lambda}$, then $\lim_k \frac{1}{k}\sum_{j=1}^k (\lambda C_\varphi)^j f(z) = \sum_{l=0}^\infty a_{ln_0+j_0}(z-z_0)^{ln_0+j_0}$ for every $f(z) = \sum_{l=0}^\infty a_l (z-z_0)^l \in H_{v_m}(\mathbb{C})$.*

When $\lambda^{s_0} = 1$ for some $s_0 \in \mathbb{N}$, the operator is periodic with period $m.c.m(n_0, s_0)$.

b) If $a^n \neq 1$ for every $n \in \mathbb{N}$, the operator is not uniformly mean ergodic, neither mean ergodic on $H_{v_m}(\mathbb{C})$.

- *If there exists $j_0 \in \mathbb{N}_0$ such that $a^{j_0} = \frac{1}{\lambda}$, then $\lim_k \frac{1}{k}\sum_{j=1}^k (\lambda C_\varphi)^j f(z) = a_{j_0}(z-z_0)^{j_0}$ for every $f(z) = \sum_{l=0}^\infty a_l(z-z_0)^l \in H^0_{v_m}(\mathbb{C})$.*
- *Otherwise, $\lim_k \frac{1}{k}\sum_{j=1}^k (\lambda C_\varphi)^j f = 0$ for every $f \in H^0_{v_m}(\mathbb{C})$.*

Proof. (i) and the power boundedness of (iii) follow by (4.3), since $\|(\lambda C_\varphi)^k\| \leq |\lambda|^k e^{2m\frac{|b|}{|1-a|}}$ for every $k \in \mathbb{N}$. For (ii) apply Proposition 9 (ii) and (iii).
(iii) Let us study first the uniform mean ergodicity. By Proposition 13(i), we have that $\sigma(\lambda C_\varphi) = \overline{\{\lambda a^n,\ n = 0, 1, \dots\}}$. So, if $a^n \neq 1$ for every $n \in \mathbb{N}$, 1 is an accumulation point of $\sigma(\lambda C_\varphi)$ and then, by Theorem 12, the operator can not be uniformly mean ergodic, neither mean ergodic on $H_{v_m}(\mathbb{C})$. If $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$, the spectrum is finite and we have the following situation. In the case $\lambda^s \neq 1$ for every $s \in \mathbb{N}$, we have $1 \notin \sigma(\lambda C_\varphi)$, so the operator is uniformly mean ergodic by Theorem 12. Otherwise, if $\lambda^{s_0} = 1$ for some $s_0 \in \mathbb{N}$, the operator is periodic with period m.c.m(n_0, s_0), thus, uniformly mean ergodic.

Let us calculate now the Cesàro means of the monomials. Put $z_0 = \frac{b}{1-a}$ and observe that, for $j_0 \in \mathbb{N}_0$,

$$\lim_k \frac{1}{k}\sum_{j=1}^{k}(\lambda C_\varphi)^j(z-z_0)^{j_0} = \lim_k \frac{(z-z_0)^{j_0}}{k}\sum_{j=1}^{k}(\lambda a^{j_0})^j = \frac{\lambda a^{j_0}(z-z_0)^{j_0}}{1-\lambda a^{j_0}}\lim_k \frac{1-(\lambda a^{j_0})^k}{k} \quad (4.6)$$

So, we get

$$\lim_k \frac{1}{k}\sum_{j=1}^{k}(\lambda C_\varphi)^j(z-z_0)^{j_0} = \begin{cases} 0 \text{ if } a^{j_0} \neq \frac{1}{\lambda} \\ (z-z_0)^{j_0} \text{ if } a^{j_0} = \frac{1}{\lambda} \end{cases} \quad (4.7)$$

(a) Assume $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$ (consider the smallest). If $a^j \neq \frac{1}{\lambda}$ for every $j \in \mathbb{N}_0$ (this is always satisfied if $\lambda^s \neq 1$ for every $s \in \mathbb{N}$), we get $\lim_k \frac{1}{k}\sum_{j=1}^{k}(\lambda C_\varphi)^j f = 0$ for every $f \in H_{v_m}(\mathbb{C})$ by (4.7). If $\lambda^{s_0} = 1$ for some $s_0 \in \mathbb{N}$ (consider the smallest) and there exists $j_0 \in \mathbb{N}_0$ (consider the smallest) such that $a^{j_0} = \frac{1}{\lambda}$ (this yields $s_0|n_0$) then (4.7) implies $\lim_k \frac{1}{k}\sum_{j=1}^{k}(\lambda C_\varphi)^j f(z) = \sum_{l=0}^{\infty} a_{ln_0+j_0}(z-z_0)^{ln_0+j_0}$ for every $f(z) = \sum_{l=0}^{\infty} a_l(z-z_0)^l \in H_{v_m}(\mathbb{C})$, since $a^j = \frac{1}{\lambda}$ if and only if $j = ln_0 + j_0$, $l \in \mathbb{N}$.
(b) Assume now $a^n \neq 1$ for every $n \in \mathbb{N}$. If there exists $j_0 \in \mathbb{N}_0$ such that $a^{j_0} = \frac{1}{\lambda}$, this j_0 is unique since a is not a root of unity. Thus, (4.7) yields that $(\frac{1}{k}\sum_{j=1}^{k}(\lambda C_\varphi)^j f)_k$ either converges to $\frac{f^{(j_0)}(z_0)}{j_0!}(z-z_0)^{j_0}$ or to 0 in $H_{v_m}(\mathbb{C})$ for every polynomial f. As the polynomials are dense in $H^0_{v_m}(\mathbb{C})$ and λC_φ is power bounded, we obtain the mean ergodicity on $H^0_v(\mathbb{C})$. $\square$

4.3 Case $a = 1$

In Theorem 8 we have seen that continuous weighted composition operators when $a = 1$ are just those of the form λC_φ, $\lambda \in \mathbb{C}$, where C_φ is the translation operator $T_b : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C})$, $f(z) \mapsto f(z+b)$. Observe that λC_φ is the differential operator $\phi(D)$ associated to the exponential function $\phi(z) = \lambda e^{bz}$.

In the next theorem we study the power boundedness and (uniform) mean ergodicity of the operator. In assertion (iii) we provide an example of a uniformly mean ergodic operator whose iterates converge to 0 in the strong operator topology but not in the operator norm.

Theorem 19 *Let $\varphi(z) = z + b$, $b \in \mathbb{C}$, $b \neq 0$, and consider the operator λC_φ, $\lambda \in \mathbb{C}$. We get $\|(\lambda C_\varphi)^k\| = (|\lambda|e^{m|b|})^k$, $k \in \mathbb{N}$ and the following assertions:*

(i) If $|\lambda| < e^{-m|b|}$, the sequence $(\lambda C_\varphi)^k$ is norm convergent to 0. Thus, the operator is power bounded and uniformly mean ergodic on both spaces.

(ii) If $|\lambda| > e^{-m|b|}$, the operator is neither power bounded nor mean ergodic on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$.

(iii) If $|\lambda| = e^{-m|b|}$, then $\|(\lambda C_\varphi)^k\| = 1$ for every k, and so the operator is power bounded and $(\lambda C_\varphi)^k$ does not converge to 0. It is mean ergodic on $H^0_{v_m}(\mathbb{C})$ with $\lim_k (\lambda C_\varphi)^k f = 0$ for every $f \in H^0_{v_m}(\mathbb{C})$. If $\lambda \neq e^{-m|b|}$, it is uniformly mean ergodic. However, if $\lambda = e^{-m|b|}$, it is not uniformly mean ergodic and not mean ergodic on $H_{v_m}(\mathbb{C})$.

Proof. (i) and the power boundedness of (ii) and (iii) follow by (4.3), since it implies $\|(\lambda C_\varphi)^k\| = (|\lambda| e^{m|b|})^k$, $k \in \mathbb{N}$.
(ii) Given $\lambda \in \mathbb{C}$ such that $|\lambda| > e^{-m|b|}$, consider $\alpha \in \mathbb{C}$ such that $|\alpha| < m$, $\alpha b = |\alpha||b|$ and $|\lambda| > e^{-|\alpha||b|} > e^{-m|b|}$. Then, $e_\alpha(z) := e^{\alpha z} \in H^0_{v_m}(\mathbb{C})$ and $\frac{1}{k}(\lambda C_\varphi)^k(e_\alpha(z)) = e_\alpha(z)\frac{(\lambda e^{\alpha b})^k}{k}$. As $|\lambda e^{\alpha b}| > 1$, we get the operator can not be mean ergodic on $H^0_{v_m}(\mathbb{C})$, neither on $H_{v_m}(\mathbb{C})$.
(iii) By Proposition 13(ii), we get $\sigma(\lambda C_\varphi) = \{\lambda e^\delta,\ |\delta| \leq m|b|\}$. For $|\delta| \leq m|b|$, $\lambda e^\delta = 1$ yields $e^{\Re(\delta)} = |1/\lambda| = e^{m|b|}$, and thus, $\lambda = e^{-m|b|}$. Therefore, if $\lambda \neq e^{-m|b|}$, $1 \notin \sigma(\lambda C_\varphi)$ and so, Theorem 12 yields the operator is uniformly mean ergodic. If $\lambda = e^{-m|b|}$, then 1 is an accumulation point of $\sigma(\lambda C_\varphi)$. So, the operator is not uniformly mean ergodic by Theorem 12 and not mean ergodic on $H_{v_m}(\mathbb{C})$, as it is a Grothendieck Dunford-Pettis space. Let us see the mean ergodicity in this case. Consider $\alpha \in \mathbb{C}, |\alpha| < m$. Observe that $e_\alpha \in H^0_{v_m}(\mathbb{C})$ and $(\lambda C_\varphi)^k(e_\alpha(z)) = e_\alpha(z)(\lambda e^{\alpha b})^k$. As by hypothesis $|\lambda e^{\alpha b}| < 1$, we get $(\lambda C_\varphi)^k(e_\alpha) \xrightarrow{k} 0$. As the set $\mathrm{span}(\{e_\alpha, |\alpha| < m\})$ is dense in $H^0_{v_m}(\mathbb{C})$ (see [3, Lemma 5.4]) and λC_φ is power bounded, then it is mean ergodic on $H^0_{v_m}(\mathbb{C})$ with $\lim_k (\lambda C_\varphi)^k f = 0$ for every $f \in H^0_{v_m}(\mathbb{C})$. □

Theorem 20 *Given $\varphi(z) = z + b$, $b \in \mathbb{C}$, $b \neq 0$, the weighted composition operator λC_φ, $\lambda \in \mathbb{C}$, satisfies:*

(i) It is not hypercyclic if $|\lambda| \leq e^{-m|b|}$ or $|\lambda| > e^{m|b|}$.

(ii) It is hypercyclic if $e^{-m|b|} < |\lambda| < e^{m|b|}$. In this case, it is topologically mixing and chaotic.

Proof. (i) For $|\lambda| \leq e^{-m|b|}$, the operator is power bounded by Theorem 19, hence not hypercyclic. Let us study the case $|\lambda| > e^{m|b|}$. By Proposition 13(ii), we get $\sigma(\lambda C_\varphi) = \{\lambda e^\delta,\ |\delta| \leq m|b|\}$, then $\sigma(\lambda C_\varphi)$ does not intersect the unit circle $\mathbb{T}$, as $|\lambda||e^\delta| = |\lambda| e^{Re(\delta)} \geq |\lambda| e^{-m|b|} > 1$ for every $|\delta| \leq m|b|$. Therefore, by Kitai's criterion [20, Proposition 5.3], the operator can not be hypercyclic.
(ii) Consider now the path $\{\alpha(t) = \lambda e^{t|b|}, -m \leq t \leq m\} \subseteq \sigma(\lambda C_\varphi)$. If $e^{-m|b|} < |\lambda| < e^{m|b|}$, as $|\alpha(-m)| < 1$ and $|\alpha(m)| > 1$, $\sigma(\lambda C_\varphi) \cap \mathbb{T} \neq \emptyset$ and $\min\{|z| : |\lambda||e^{bz}| = 1\} < m$. Indeed, there exists $|\delta| \leq m|b|$ such that $|\lambda||e^\delta| = 1$ and $e^{-m|b|} < e^{\Re(\delta)} < e^{m|b|}$ by the hypothesis on λ. Thus, $\min\{|z| : |\lambda||e^{bz}| = 1\} = 1/|b| \min\{|\alpha| : |\lambda| e^{\Re(\alpha)} = 1\} = 1/|b| \min\{|t| : |\lambda| e^t = 1, t \in \mathbb{R}\} < m$. As $\lim_{r\to\infty} e^{-mr} e^{\beta r} = 0$ for $\min\{|z| : |\lambda||e^{bz}| = 1\} < \beta < m$, the operator is topologically mixing, chaotic and not mean ergodic by [4, Theorem 3.1]. □

5 Composition operators on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$

In this concluding section, we compile the results for the relevant case $w \equiv 1$, that is, for composition operators C_φ, $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. The dynamics in this case is completely characterized. Multiplication operators are not considered, since they are trivial on these spaces (see Theorem 8(ii)).

Proposition 21 *Let $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. The composition operator C_φ satisfies:*

(i) C_φ is continuous on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ if and only if $|a| \leq 1$. If it is continuous, then $\|C_\varphi\| = e^{m|b|}$.

(ii) $C_\varphi : H_{v_m}(\mathbb{C}) \to H^0_{v_m}(\mathbb{C})$ is compact if and only if $C_\varphi : H^0_{v_m}(\mathbb{C}) \to H^0_{v_m}(\mathbb{C})$ is so, if and only if $|a| < 1$. If C_φ is compact, it can be approximated by finite-rank operators.

Proof. By Theorem 8, it only remains to prove that for $|a| < 1$, C_φ can be approximated by finite-rank operators. As $H_{v_m}(\mathbb{C})$ and $H^0_{v_m}(\mathbb{C})$ are isomorphic to ℓ_∞ and c_0, respectively [26], the spaces have the (bounded) approximation property, so as the operator is compact on them in this case, it can be approximated by finite-rank operators. Indeed, given $f(z) = \sum_{j=0}^\infty c_j(z - \frac{b}{1-a})^j \in H_{v_m}(\mathbb{C})$, we get $C_\varphi f(z) = \sum_{j=0}^\infty a^j c_j(z - \frac{b}{1-a})^j$. By the Cauchy inequalities, $|c_j|\|(z - \frac{b}{1-a})^j\|_m \leq \|f\|_m$ for every $j \in \mathbb{N}$. So,

$$\|C_\varphi f - \sum_{j=0}^k a^j c_j(z - \frac{b}{1-a})^j\|_m \leq \sum_{j=k+1}^\infty |a|^j \|f\|_m = \|f\|_m \frac{|a|^{k+1}}{1-|a|} \xrightarrow{k} 0$$

and we obtain that $C_\varphi f$ belongs to the closure of the polynomials, that is, to $H^0_{v_m}(\mathbb{C})$. The argument above also shows that the finite-rank operators $(C_\varphi)_N(\sum_{j=0}^\infty c_j(z - \frac{b}{1-a})^j) := \sum_{j=0}^N a^j c_j(z - \frac{b}{1-a})^j$ are bounded on $H_{v_m}(\mathbb{C})$ and that $\|C_\varphi - (C_\varphi)_N\| \leq \frac{|a|^{N+1}}{1-|a|} \to 0$ as N tends to ∞. □

Theorem 22 *Let $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, $|a| \leq 1, a \neq 1$. The composition operator C_φ is always power bounded on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$, hence not hypercyclic, with*

$$\|C_\varphi^k\| = e^{m|b|\left|\frac{1-a^k}{1-a}\right|} \leq e^{2m\frac{|b|}{|1-a|}} \text{ for every } k \in \mathbb{N}. \tag{5.1}$$

Moreover, we get:

(i) If $|a| < 1$, C_φ is uniformly mean ergodic with $\lim_k \|\frac{1}{k}\sum_{j=0}^k C_\varphi^j - C_{\frac{b}{1-a}}\| = 0$, where $C_{\frac{b}{1-a}}$ is the evaluation at the fixed point $\frac{b}{1-a}$.

(ii) If $|a| = 1, a^n = 1$ for some $n \in \mathbb{N}$, C_φ is periodic and then uniformly mean ergodic. In this case, for every $f(z) = \sum_{j=0}^\infty a_j(z - z_0)^j$ in the space,

$$\lim_k \frac{1}{k}\sum_{j=1}^k C_\varphi^j f(z) = \frac{1}{k}\sum_{j=1}^k f(a^j z + b\frac{1-a^j}{1-a}) = \sum_{l=0}^\infty a_{ln}(z - z_0)^{ln}. \tag{5.2}$$

(iii) If $|a| = 1, a^n \neq 1$ for every $n \in \mathbb{N}$, C_φ is not uniformly mean ergodic, it is mean ergodic on $H^0_{v_m}(\mathbb{C})$ with $\lim_k \frac{1}{k}\sum_{j=1}^k C^j_\varphi f = C_{\frac{b}{1-a}} f$ but not mean ergodic on $H_{v_m}(\mathbb{C})$.

Proof. Proceeding as in the proofs of [31, Theorems 4 and 5] we get $\lim_k \|\frac{1}{k}\sum_{j=0}^k C^j_\varphi - C_{\frac{b}{1-a}}\| = 0$ if $|a| < 1$. The other assertions are a consequence of (4.3) and Theorem 18. Statements (ii) and (iii) can also be obtained by [22, Proposition 2.3] and Remark 14. □

In the case $a = 1$, C_φ is the translation operator $T_b : H_{v_m}(\mathbb{C}) \to H_{v_m}(\mathbb{C}), f(z) \mapsto f(z + b)$. It is the differential operator $\phi(D)$ associated to the exponential function $\phi(z) = e^{bz}$. Theorems 19 and 20 yield the following:

Theorem 23 *Given $\varphi(z) = z + b$, $b \in \mathbb{C}$, the translation operator C_φ on $H_{v_m}(\mathbb{C})$ and on $H^0_{v_m}(\mathbb{C})$ satisfies $\|C^k_\varphi\| = e^{m|b|k}$, $k \in \mathbb{N}$, and:*

(i) It is topologically mixing and chaotic on $H^0_{v_m}(\mathbb{C})$.

(ii) It is neither power bounded nor mean ergodic on $H^0_{v_m}(\mathbb{C})$ and on $H_{v_m}(\mathbb{C})$.

Acknowledgments

The research of the author was supported by the project MTM2016-76647-P.

References

[1] F. Bayart, E. Matheron, Dynamics of linear operators. Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.

[2] M.J. Beltrán, Spectra of weighted (LB)-algebras of entire functions on Banach spaces, J. Math. Anal. Appl. 387 (2012) 604–617.

[3] M.J. Beltrán, Dynamics of differentiation and integration operators on weighted spaces of entire functions, Studia Math. 221 (1) (2014) 35–60.

[4] M.J. Beltrán, J. Bonet, C. Fernández, Classical Operators on the Hörmander Algebras, Discrete and Continuous Dynamical Systems 35 (2) (2015) 637–652.

[5] M.J. Beltrán-Meneu, M.C. Gómez-Callado, E. Jordá, D. Jornet, Mean ergodic composition operators on Banach spaces of holomorphic functions, J. Funct. Anal. 270 (12) (2016) 43694385.

[6] M.J. Beltrán-Meneu, M.C. Gómez-Collado, E. Jordá, D. Jornet, Mean ergodicity of weighted composition operators on spaces of holomorphic functions, J. Math. Anal. Appl. 444 (2016) 1640–1651.

[7] J. Bès, Dynamics of weighted composition operators, Complex Anal. Oper. Theory 8 (2014) 159–176.

[8] K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on balanced domains, Michigan Math. J. 40 (1993) 271-297.

[9] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, Studia Math. 127(2) (1998) 137–168.

[10] K.D. Bierstedt, W.H. Summers, Biduals of weighted Banach spaces of analytic functions, J. Austral. Math. Soc. (Series A) 175 (1993) 70–79.

[11] J. Bonet, P. Domański, A note on mean ergodic composition operators on spaces of holomorphic functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 105(2) (2011) 389–396.

[12] J. Bonet, P. Domański, M. Lindström, J. Taskinen, Composition operators between weighted Banach spaces of analytic functions, J. Austral. Math. Soc. 64 (1998) 101–118.

[13] J. Bonet, M. Friz, E. Jordá, Composition operators between weighted inductive limits of spaces of holomorphic functions, Publ. Math. 67(3) (2005) 333–348.

[14] J. Bonet, E.M. Mangino, Associated weights for spaces of p-integrable entire functions, Quaestiones Mathematicae (2019), DOI: 10.2989/16073606.2019.1605420.

[15] J. Bonet, W.J. Ricker, Mean ergodicity of multiplication operators on weighted spaces of holomorphic functions, Arch. Math. 92 (2009) 428–437.

[16] T. Carroll, C. Gilmore, Weighted composition operators on Fock Spaces and their dynamics, arXiv:1911.07254v1.

[17] M.D. Contreras, A.G. Hernández-Díaz, Weighted composition operators in weighted Banach spaces of analytic functions, J. Austral. Math. Soc. 69(1) (2000) 41–60.

[18] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.

[19] N. Dunford, Spectral Theory I. Convergence to projections, Trans. Amer. Math. Soc. 54 (1943) 185–217.

[20] K.G. Grosse-Erdmann, A. Peris, Linear Chaos, Springer, London, 2011.

[21] K. Guo, K. Izuchi, Composition operators on Fock type spaces, Acta Sci. Math. 74 (2008) 807–828.

[22] E. Jordá, A. Rodríguez, Ergodic properties of composition operators on Banach spaces of analytic functions, J. Math. Anal. Appl. 486 (2020), 14 pp. 123891, https://doi.org/10.1016/j.jmaa.2020.123891.

[23] Y.X. Liang, Z.H. Zhou, Hypercyclic behaviour of multiples of composition operators on the weighted Banach space, Bull. Belg. Math. Soc. Simon Stevin 21 (2014) 385–401.

[24] M. Lin, On the uniform ergodic theorem, Proc. Amer. Math. Soc. 43 (1974) 337–340.

[25] H.P. Lotz, Uniform convergence of operators on L^∞ and similar spaces, Math. Z. 190 (1985) 207–220.

[26] W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomophic functions, Studia Math. 175 (2006) 19–45.

[27] T. Mengestie, Cyclic and supercyclic weighted composition operators on the Fock space, arXiv:1901.01697.

[28] A. Miralles, E. Wolf, Hypercyclic composition operators on H_v^0-spaces, Math. Nachr. 286 (1) 34–41 (2013).

[29] A. Montes-Rodríguez, Weighted composition operators on weighted Banach spaces of analytic functions, J. London Math. Soc. 61 (2000) 872–884.

[30] K. Petersen, Ergodic Theory, Cambridge University Press, Cambridge, 1983.

[31] W. Seyoum, T. Mengestie, J. Bonet, Mean ergodic composition operators on generalized Fock spaces, RACSAM 11 (6) (2020), https://doi.org/10.1007/s13398-019-00738-w.

[32] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer, Berlin, 1993.

[33] E. Wolf, Power bounded weighted composition operators, New York J. Math. 18 (2012) 201–212.

[34] K. Yosida, S. Kakutani, Operator-theoretical treatment of Markoff's Process and Mean Ergodic Theorem. Ann. Math. 42 (1941) 188–228.