

SENSITIVE DEPENDENCE FOR NONAUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT. Given a nonautonomous discrete dynamical system (NDS) $(X, f_{1,\infty})$ we show that transitivity and density of periodic points do not imply sensitivity in general, i.e., in the definition of Devaney chaos there are no redundant conditions for NDS. In addition, we show that if we also assume uniform convergence of the sequence (f_n) that induces the NDS, then sensitivity follows. Furthermore, in contrast to the autonomous case, we show that there exist minimal NDS which are neither equicontinuous nor sensitive.

1. INTRODUCTION

Nonautonomous discrete dynamical systems (NDS for short) were introduced by S. Kolyada and L. Snoha in [11]. A very readable account of some recent developments on the theory of nonautonomous discrete dynamical systems has been given in [3].

Let (X, d) a metric space. An NDS is a pair $(X, f_{1,\infty})$ where $f_{1,\infty} = (f_n : X \rightarrow X)_{n \in \mathbb{N}}$ is a sequence of continuous functions. The composition

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$$

is said to be the n^{th} -iterate of $f_{1,\infty}$ for all $n \in \mathbb{N}$ and, as usual, the symbol f_1^0 will stand for Id_X , the identity map on X . The *orbit* of a point $x \in X$ is the set

$$\mathcal{O}_{f_{1,\infty}}(x) := \{x, f_1^1(x), f_1^2(x), \dots, f_1^n(x), \dots\}$$

which can also be described by the difference equation $x_1 = x$ and $x_{n+1} = f_n(x_n)$ for each $n \in \mathbb{N}$.

Recall that given a compact metric space (X, d) and a sequence of continuous functions $(f_n : X \rightarrow X)_{n \in \mathbb{N}}$, a general form of a nonautonomous difference equation is the following:

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$$\begin{cases} x_0 = x, \\ x_{n+1} = f_n(x_n) \end{cases}$$

for each $x \in X$. This kind of nonautonomous difference equation has been considered by several mathematicians (see for instance [17], [18] among others). Classical examples deal with $X = [0, 1]$ the unit interval endowed with the usual metric.

NDS generalize autonomous discrete systems (ADS for short). An ADS is given by (X, f) , where (X, d) is a metric space and $f : X \rightarrow X$ is continuous. These systems can be seen as a particular case of NDS just by considering $f_n = f$ for all $n \in \mathbb{N}$.

In this paper we address some questions related to transitivity and sensitivity on NDS. Our motivation is Devaney's definition of chaos [7]. Devaney defined an ADS (X, f) to be chaotic if it satisfies the following three conditions: (i) (X, f) is transitive, (ii) the set of periodic points of (X, f) is a dense subset of X and (iii) (X, f) is sensitive. Banks et al. [5] proved that conditions (i) and (ii) imply condition (iii) (to avoid degenerate cases X is assumed to be infinite).

These results suggest the question whether the two former conditions, i.e. transitivity and density of periodic points, in an NDS imply the last one, i.e. sensitivity. This was proposed by Lan [14] in full generality:

In nonautonomous dynamical systems, does transitivity together with density of periodic points imply sensitivity?

Under additional conditions, considering the density of k -periodic points of X , Zhu et al. [18, Theorems 3.1 and 3.2] answered this question positively. However, in [16, Example 4.4] the authors provided a negative answer for an NDS on the interval $[0, 1]$. In the papers by Zhu et al. [18], Lan [14] and Sanchez et al. [16] the authors considered a periodic point for the NDS to be $x \in X$ satisfying $f_1^n(x) = x$ for some $n \in \mathbb{N}$. In our paper we will consider periodic points to be those $x \in X$ satisfying that $f_1^{nk}(x) = x$ for some $n \in \mathbb{N}$ and any $k \in \mathbb{N}$. This definition coincides with the classical one of periodic point in the ADS case.

The paper is organized as follows. Section 2 is devoted to Devaney's chaos for NDS. We will provide new examples which give a negative answer to the question proposed by Lan [14]. Indeed, for each transitive non-sensitive ADS we show the existence of an NDS $(X, f_{1,\infty})$ which is transitive, has a dense subset of periodic points but fails to be sensitive.

The main result of this section gives a positive answer to Lan's question and states that if (X, d) is a metric space without isolated points and $(X, f_{1,\infty})$ is an NDS such that $f_n : X \rightarrow X$ converges to f uniformly, then transitivity of the NDS and density of periodic points in X imply sensitivity.

In Section 3 we deal with equicontinuity and sensitivity. We show that if a transitive NDS $(X, f_{1,\infty})$ is equicontinuous at x_0 , then x_0 is a transitive

point. As a consequence, if an NDS $(X, f_{1,\infty})$ is topologically transitive and equicontinuous, then it is minimal. We finish with an example of a minimal NDS which is neither equicontinuous nor sensitive. This fact is a significant difference between the theory of ADS and the theory of NDS. Further references in ADS can be found in [9]. In [2, 3] the reader can find further information about NDS. The interested reader in chaos, transitivity and sensitivity for NDS might consult [4, 8, 10, 15].

2. DEVANEY CHAOS ON NDS

An NDS $(X, f_{1,\infty})$ is said to be *topologically transitive* (transitive for short) if for any pair of non-empty open sets $U, V \subset X$ there exists $N > 0$ such that $f_1^N(U) \cap V \neq \emptyset$. It is said to have sensitive dependence on initial conditions (sensitive for short) if there exists a constant $\delta > 0$ such that for any $x \in X$ and $\epsilon > 0$ there exists $y \in X$ with $d(x, y) < \epsilon$ such that $d(f_1^n(x), f_1^n(y)) > \delta$ for some $n \geq 0$.

We will say that a point p is *periodic* for the NDS $(X, f_{1,\infty})$ if there exists $n \in \mathbb{N}$ such that $f_1^{nk}(p) = p$ for any $k \in \mathbb{N}$.

Definition 2.1. *An NDS $(X, f_{1,\infty})$ is Devaney chaotic if it satisfies the following conditions:*

- i) $(X, f_{1,\infty})$ is topologically transitive;
- ii) $(X, f_{1,\infty})$ has a dense set of periodic points;
- iii) $(X, f_{1,\infty})$ has sensitive dependence on initial conditions.

As we have mentioned in Section 1, ADS are particular cases of NDS. Indeed, all these definitions can also be considered for ADS by considering $f_n = f$ for all $n \in \mathbb{N}$. In particular, $x \in X$ is periodic for the ADS (X, f) if $f^n(x) = x$ for some $n \in \mathbb{N}$. This shows that the definition of a periodic point for NDS coincides with the classical one for ADS. We will now show that sensitivity for NDS does not follow from transitivity and density of periodic points even if all the points are periodic. We first recall the following well-known result.

Proposition 2.2. *There exist ADS which are transitive but they fail to be sensitive.*

Proof. It is sufficient to consider the ADS (\mathbb{T}, f) where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and f is an irrational rotation, that is, $f : \mathbb{T} \rightarrow \mathbb{T}$ is given by $f(z) = e^{i\alpha}z$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. It is well-known that the autonomous system (\mathbb{T}, f) is topologically transitive [9, Example 1.12] but it fails to be sensitive since $d(f^n(z), f^n(w)) = d(z, w)$. \square

Hence we obtain the main result of this section.

Proposition 2.3. *For any transitive non-sensitive ADS (X, f) there exists an NDS $(X, f_{1,\infty})$ which is transitive, all its points are periodic but fails to be sensitive.*

Proof. Let (X, f) be a topologically transitive ADS that fails to be sensitive which exists by Proposition 2.2. Suppose that f has inverse f^{-1} . Denote

by f^{-n} the n^{th} -iterate $\overbrace{f^{-1} \circ \dots \circ f^{-1}}^{n\text{-fold}}$ and define the NDS $(X, f_{1,\infty})$ where the sequence (f_n) is given by

$$(f_1, f_2, f_3, f_4, f_5, f_6, f_7, \dots) = (f, f^{-1}, f^2, f^{-2}, f^3, f^{-3}, \dots)$$

so that the sequence

$$(f_1^1, f_1^2, f_1^3, f_1^4, f_1^5, f_1^6, f_1^7, \dots) = (f, Id_X, f^2, Id_X, f^3, Id_X, \dots).$$

It is clear that every $x \in X$ is periodic since $f_1^{2k} = Id_X$ for any $k \in \mathbb{N}$.

We show that the NDS is topologically transitive. Indeed, let $U, V \neq \emptyset$ be open sets of X . Since (X, f) is topologically transitive, there exists n such that $f^n(U) \cap V \neq \emptyset$. It is clear that there exists $m \in \mathbb{N}$ such that $f_1^m = f^n$ and we are done.

Now we show that $(X, f_{1,\infty})$ is not sensitive. Since (X, f) is not sensitive, for any $k \in \mathbb{N}$, there exists $x_k \in X$ and $0 < \varepsilon_k < 1/k$ such that $\sup_{n \in \mathbb{N}} d(f^n(x_k), f^n(y)) < 1/k$ for all $y \in X$ satisfying $d(x_k, y) < \varepsilon_k$.

Notice that for any $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}_0$ such that $f_1^{m_n} = f^n$. So for any $k \in \mathbb{N}$, we have that

$$\sup_{n \in \mathbb{N}} d(f_1^{m_n}(x_k), f_1^{m_n}(y)) = \sup_{n \in \mathbb{N}} d(f^n(x_k), f^n(y)) < \frac{1}{k}$$

for any $y \in X$ such that $d(x_k, y) < \varepsilon_k$ so $(X, f_{1,\infty})$ is not sensitive. \square

Although we provide a negative answer for the question proposed in [14], the answer becomes true if we assume an extra condition on the NDS.

Theorem 2.4. *Let (X, d) be a metric space without isolated points. Suppose that $f_n : X \rightarrow X$ converges uniformly to f . If*

- a) $(X, f_{1,\infty})$ is topologically transitive and
- b) there exists a dense subset $P \subset X$ of periodic points for $(X, f_{1,\infty})$,

then $(X, f_{1,\infty})$ is sensitive.

In order to show this result we need the following lemmas.

Lemma 2.5. *Let (X, d) be a metric space without isolated points. Suppose that $f_n : X \rightarrow X$ converges pointwise to f . Then,*

- a) If p is a periodic point for $(X, f_{1,\infty})$ then p is a periodic point for (X, f) .
- b) If there exists an infinite set of periodic points of f , then there exists $\eta > 0$ such that for any $x \in X$ there is a periodic point p of f such that $d(x, f^n(p)) \geq \eta$ for all $n \in \mathbb{N}$.

Proof. First we prove a). Since p is periodic, there exists $N \in \mathbb{N}$ such that $f_1^{jN}(p) = p$ for all $j \in \mathbb{N}$. Notice that for any $0 \leq i \leq N$ we have that $f_1^{jN+i}(p) = f_{jN+i} \circ \dots \circ f_{jN+1}(p)$ which is convergent to $f^i(p)$ when $j \rightarrow \infty$.

In particular, for $i = N$, we have that $f_1^{jN+N}(p) = p$ is convergent to $f^N(p)$ so $f^N(p) = p$ and we are done.

Now we show b). Since X has no isolated points it is an infinite set, then we can find two periodic points p_1, p_2 whose orbits are disjoint (since the orbits are finite). Hence,

$$\eta := \inf_{m, n \in \mathbb{N}_0} d(f^m(p_1), f^n(p_2))/2 > 0.$$

It then follows from the triangle inequality that, for any $x \in X$, either for $j = 1$ or for $j = 2$ we have that $d(x, f^n(p_j)) \geq \eta$ for all $n \in \mathbb{N}_0$. \square

For any pair of non-empty open sets U and V of X , let $N(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$.

Lemma 2.6. *Let (X, d) be a metric space without isolated points. If $(X, f_{1, \infty})$ is topologically transitive, then for any pair of non-empty open subsets U, V of X , the set $N(U, V)$ is infinite.*

Proof. Let U, V be non-empty open sets. Since the NDS is topologically transitive, $N(U, V)$ is non-empty. Let us suppose that it is finite and let

$$m = \max\{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}.$$

Since X has no isolated points, V contains infinitely many points, thus we can fix a collection $V_j \subset V$, $j = 1, \dots, m+1$ of mutually disjoint open sets. Since $m \in N(U, V)$, there exists $x \in U$ such that $f_1^m(x) \in V$. Now, there is $j_0 \in \{1, \dots, m+1\}$ such that $f_1^i(x) \notin V_{j_0}$ for all $i = 1, \dots, m$ due to the choice of $(V_j)_{j=1}^{m+1}$. By continuity, there is an open neighbourhood U' of x contained in U such that

$$(1) \quad f_1^i(U) \cap V_{j_0} = \emptyset, \quad i = 1, \dots, m.$$

Since $(X, f_{1, \infty})$ is transitive, there exists $k \in \mathbb{N}$ such that $f_1^k(U') \cap V_{j_0} \neq \emptyset$. By (1), we have $k > m$ which is a contradiction because $N(U', V_{j_0}) \subset N(U, V)$. \square

We now provide the proof of our main theorem.

Proof of Theorem 2.4. We fix a metric d defining the topology of X .

Let $x \in X$ and $\varepsilon > 0$. By Lemma 2.5, there is a periodic point p such that

$$(2) \quad d(x, f^n(p)) \geq \eta \quad \text{for } n \in \mathbb{N}_0.$$

We now claim that $(X, f_{1, \infty})$ is sensitive with sensitivity constant $\delta := \eta/4$. By assumption there is a periodic point q such that

$$d(x, q) < \min\{\varepsilon, \frac{\eta}{4}\}.$$

Let q have period N , that is, $f_1^{jN}(q) = q$ for any $j \in \mathbb{N}$. Since the functions f^j are continuous for $j = 1, 2, \dots, N$, there is some neighbourhood V of p such that

$$(3) \quad d(f^j(p), f^j(y)) < \delta/2 \quad \text{for } j = 0, 1, \dots, N \quad \text{and } y \in V.$$

Consider $U = B_\varepsilon(x)$ and V the neighbourhood of p . Since f_n converges to f uniformly, we find $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$ we have for any $0 \leq l < N$,

$$d((f_{jN} \circ f_{jN-1} \circ \dots \circ f_{(j-1)N+l+1})(v), f^{N-l}(v)) < \delta/2 \quad \text{for any } v \in V.$$

Since $(X, f_{1,\infty})$ is topologically transitive, there exists an infinite set of positive integers m such that $f_1^m(U) \cap V \neq \emptyset$ by Lemma 2.6. Hence we can consider $k = (j-1)N+i$ for $0 \leq i < N$ such that $j \geq j_0$ and $f_1^k(U) \cap V \neq \emptyset$. Take $z \in U$ (that is, $d(x, z) < \varepsilon$) such that $f_1^k(z) \in V$.

Let $S := f_{jN} \circ f_{jN-1} \circ \dots \circ f_{(j-1)N+i+1}$. By hypothesis we will obtain that

$$d(S(f_1^k(z)), f^{N-i}(f_1^k(z))) < \delta/2.$$

By the triangle inequality we obtain using inequality (3) that

$$\begin{aligned} & d(f^{N-i}(p), S(f_1^k(z))) \leq \\ & d(S(f_1^k(z)), f^{N-i}(f_1^k(z))) + d(f^{N-i}(p), f^{N-i}(f_1^k(z))) < \delta/2 + \delta/2 = \delta \end{aligned}$$

and together with (2) then yields that

$$\begin{aligned} d(f_1^{jN}(q), f_1^{jN}(z)) &= d(f_1^{jN}(q), S(f_1^k(z))) = d(q, S(f_1^k(z))) \geq \\ & d(x, f^{N-i}(p)) - d(f^{N-i}(p), S(f_1^k(z))) - d(x, q) > 4\delta - \delta - \delta = 2\delta. \end{aligned}$$

Hence, either $d(f_1^{jN}(x), f_1^{jN}(q)) > \delta$ or $d(f_1^{jN}(x), f_1^{jN}(z)) > \delta$ and then the theorem holds. \square

3. EQUICONTINUITY AND SENSITIVITY ON NDS

An NDS $(X, f_{1,\infty})$ is said to be *equicontinuous at the point* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f_1^n(y), f_1^n(x_0)) < \varepsilon$ for all $n \geq 0$ whenever $d(y, x_0) < \delta$. An NDS $(X, f_{1,\infty})$ is said to be *equicontinuous* if it is equicontinuous at every point $x_0 \in X$. It is a well-known fact that an equicontinuous NDS $(X, f_{1,\infty})$ defined on a compact metric space X is *uniformly equicontinuous*, that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(f_1^n(y), f_1^n(x)) < \varepsilon$ for all $n \geq 0$ whenever $d(y, x) < \delta$. Recall also that a point $x_0 \in X$ is called *transitive* if the orbit of x_0 is dense in X . If all the points of X are transitive, then the NDS is called *minimal*.

An interesting aspect of the theory of discrete dynamical systems lies in the fact that if a topologically transitive ADS (X, f) is equicontinuous at least at a point of X , then the set of equicontinuous points coincides with the set of transitive points. In particular, a minimal ADS is either equicontinuous or sensitive (see [1] for details and, for a generalization to semigroup actions, see [12]). As the example below shows, the situation is

quite different if we replace ADS by NDS. We first prove an easy connection between equicontinuity and transitive points.

Theorem 3.1. *Let $(X, f_{1,\infty})$ be a topologically transitive NDS. If $(X, f_{1,\infty})$ is equicontinuous at x_0 , then x_0 is a transitive point.*

Proof. Let $y \in X$ and $\varepsilon > 0$. Since the NDS is equicontinuous at x_0 , there exists an open ball $B_\delta(x_0)$ such that $d(f_1^n(x_0), f_1^n(z)) < \varepsilon$ for any $z \in B_\delta(x_0)$ and all $n \in \mathbb{N}$. Since $(X, f_{1,\infty})$ is topologically transitive, we can find $n \geq 0$ such that $f_1^n(B_\delta(x_0)) \cap B_\varepsilon(y) \neq \emptyset$. Then, if $f_1^n(z) \in f_1^n(B_\delta(x_0)) \cap B_\varepsilon(y)$, we have

$$d(f_1^n(x_0), y) \leq d(f_1^n(x_0), f_1^n(z)) + d(f_1^n(z), y) < 2\varepsilon,$$

that is, $f_1^n(x_0)$ belongs to $B_{2\varepsilon}(y)$. This completes the proof. \square

A straightforward consequence of the previous result is the following corollary.

Corollary 3.2. *If an NDS $(X, f_{1,\infty})$ is topologically transitive and equicontinuous, then $(X, f_{1,\infty})$ is minimal.*

We now present the promised result. Recall that the Cantor set \mathcal{C} has minimal equicontinuous systems. Indeed, minimal equicontinuous systems on the Cantor set \mathcal{C} are conjugate to an odometer, that is, they are Kronecker systems on \mathcal{C} (see [13]) and, consequently, they are isometries. We will take advantage of the following useful property of the Cantor set: two clopen subsets of \mathcal{C} are homeomorphic (this is a particular case of the well-known Brouwer's theorem [6] which states that every non-empty, perfect, compact, totally disconnected and metrizable space is homeomorphic to the Cantor set).

Theorem 3.3. *There exists a minimal NDS which is neither equicontinuous nor sensitive.*

Proof. Let $\{M_1, M_2\}$ be a partition of the Cantor set with M_i a clopen subset for each $i = 1, 2$ and consider a minimal equicontinuous isometry T on \mathcal{C} . Now, given the clopen sets $C_i = T(M_i)$ for each $i = 1, 2$, our first step is to define a sequence of homeomorphisms $\{f_n : n \in \mathbb{N}\}$ on \mathcal{C} that will allow us to construct a non-equicontinuous NDS. For this, choose a point x_1 in C_2 and consider a sequence $\{F_n : n \in \mathbb{N}\}$ which satisfies, for each $n \in \mathbb{N}$, the following properties:

- (1) F_n is a clopen subset of \mathcal{C} whose diameter is $1/n$;
- (2) $F_{n+1} \subsetneq F_n$;
- (3) $x_1 \in F_{n+1}$.

Now, choose clopen subsets S_1, S_2 of C_2 such that $d(S_1, S_2) = \varepsilon_0 > 0$. Select, for each $n \in \mathbb{N}$, homeomorphisms $g_n : C_2 \setminus F_n \rightarrow C_2 \setminus (S_1 \cup S_2)$, $h_n : F_n \setminus F_{n+1} \rightarrow S_1$ and $q_n : F_{n+1} \rightarrow S_2$. Given $n \in \mathbb{N}$, we define f_n as follows:

$$f_n(x) = \begin{cases} x & \text{if } x \in C_1, \\ g_n(x) & \text{if } x \in C_2 \setminus F_n, \\ h_n(x) & \text{if } x \in F_n \setminus F_{n+1}, \\ q_n(x) & \text{if } x \in F_{n+1}. \end{cases}$$

Now let $(\mathcal{C}, f_{1,\infty})$ be the NDS defined by the sequence

$$\{T, f_1, f_1^{-1}, T, T^{-1}, f_2, f_2^{-1}, T^2, T^{-2}, f_3, f_3^{-1}, T^3, T^{-3}, \dots\}.$$

It is an easy matter to check that

$$\begin{aligned} f_1^1 &= T, & f_1^2 &= f_1 \circ T \\ f_1^3 &= T, & f_1^4 &= T^2 \\ f_1^5 &= T, & f_1^6 &= f_2 \circ T \\ f_1^7 &= T, & f_1^8 &= T^3 \\ f_1^9 &= T, & f_1^{10} &= f_3 \circ T \\ &\vdots & &\vdots \end{aligned}$$

Notice that the sequence $\{T^n : n \in \mathbb{N}\}$ is a subsequence of $\{f_1^n : n \in \mathbb{N}\}$. Thus, transitivity and minimality of $(\mathcal{C}, f_{1,\infty})$ follow from transitivity and minimality of (\mathcal{C}, T) . Moreover, since the restriction of f_n to C_1 is the identity for all $n \in \mathbb{N}$, the NDS $(\mathcal{C}, f_{1,\infty})$ is equicontinuous at every $x \in C_1$ (since T is equicontinuous). Thus, $(\mathcal{C}, f_{1,\infty})$ is nonsensitive.

We will prove that $(\mathcal{C}, f_{1,\infty})$ is nonequicontinuous. Indeed, for fixed $\delta > 0$, we can choose $n \in \mathbb{N}$ with $1/n < \delta$. If $x \in F_n \setminus F_{n+1}$, then $f_n(x) = h_n(x) \in S_1$. Since $f_n(x_1) = q_n(x_1) \in S_2$, we have

$$d((f_n \circ T)(T^{-1}(x)), (f_n \circ T)(T^{-1}(x_1))) \geq \varepsilon_0.$$

Since T^{-1} is an isometry, this proves that $(\mathcal{C}, f_{1,\infty})$ is not equicontinuous at $T^{-1}(x_1)$. \square

Remark 3.4. *As the point $T^{-1}(x_0)$ of Theorem 3.3 shows, for transitive NDS the existence of equicontinuous points does not imply that every transitive point is equicontinuous. This is an important difference with the case of ADS [1].*

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