On the isotropic constant of random polytopes

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Abstract

Let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on an isotropic convex body $K \subset \mathbb{R}^N$, and let K_N be the symmetric convex hull of X_i's. We show that with high probability $L_{K_N} \le C\sqrt{\log(2N/n)}$, where C is an absolute constant. This result closes the gap in known estimates in the range $Cn \le N \le n^{1+\delta}$. Furthermore, we extend our estimates to the symmetric convex hulls of vectors $y_1X_1, \ldots, y_NX_N$, where $y = (y_1, \ldots, y_N)$ is a vector in $\mathbb{R}^N$.

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1 Introduction

In this paper we estimate the isotropic constant of some random polytopes (for the definitions and notations see Section 2). It is known (see e.g. [24]) that among all the convex bodies in $\mathbb{R}^n$ the Euclidean ball is the one with the smallest isotropic constant, that is $L_K \ge L_{B_2^n} \ge c$, where c is an absolute positive constant. However, it is still an open problem to determine whether

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there exists or not an absolute constant C such that for every convex body $K \subset \mathbb{R}^n$ one has $L_K \leq C$. The boundness of L_K by an absolute constant is equivalent to the long standing hyperplane conjecture ([9]). The best general upper bound known up to now is $L_K \leq Cn^{1/4}$ ([18]). This estimate slightly improves (by a logarithmic factor) the earlier Bourgain's upper bound ([10]).

Since remarkable Gluskin's result [14] random polytopes are known to provide many examples of convex bodies (and related normed spaces) with a "patologically bad" behaviour of various parameters of a linear and geometric nature (we refer to the survey [23] and references therein; see also recent examples in [15] and [17]). Not surprisingly, they were also a natural candidate for a potential counterexample for the hyperplane conjecture. This was resolved in [19], where it was shown that the convex hull or the symmetric convex hull of independent Gaussian random vectors in $\mathbb{R}^n$ with high probability has the bounded isotropic constant. Some other distributions for vertices were also considered. In all of them the vertices had independent coordinates.

Following the ideas in [19], the problem of estimating of the isotropic constant of random polytopes was considered in [3], for independent random vectors distributed uniformly on the sphere S^{n-1}, and in [11], for independent random vectors uniformly distributed on an isotropic unconditional convex body. Also in these cases the isotropic constant of random polytopes generated by these vectors is bounded with high probability. One can check that the same method works for independent random vectors uniformly distributed on a ψ_2 isotropic convex body as well.

In this paper we estimate the isotropic constant of a random polytope in an isotropic convex body (see Section 2 for the definitions). It is known (see [5], [16] or [4]) that if K_N is a polytope in $\mathbb{R}^n$ with N vertices then

$$L_{K_N} \leq C \min\left\{\sqrt{\frac{N}{n}}, \log N\right\},$$

where C is an absolute constant.

In [12, 13], the authors provided a lower estimate for the volume of a random polytope K_N obtained as the convex hull of $N \leq e^{\sqrt{n}}$ random points, namely

$$|K_N|^{\frac{1}{n}} \geq c\sqrt{\frac{\log \frac{N}{n}}{n}} L_K$$

(see end of Section 2 for more precise formulation and details). On the other hand, the proof of the estimate $L_{K_N} \le C \log N$ in [4] passes through showing that if $X_1, \dots, X_N$ are the vertices of K_N, then for any affine transformation T we have

$$L_{K_N} \le \frac{C \max_{1\le i\le N} |TX_i| \log N}{n|TK_N|^{\frac{1}{n}}}.$$

Consequently, taking T to be the identity operator and using the concentration of measure result proved by Paouris [26], we obtain that if K_N is the convex hull or the symmetric convex hull of $n+1 \le N \le e^{\sqrt{n}}$ independent random vectors uniformly distributed on an isotropic convex body, then with high probability

$$L_{K_N} \le \frac{C \log N}{\sqrt{\log \frac{N}{n}}}. \tag{1.1}$$

Notice that if $N \ge n^{1+\delta}$, $\delta \in (0,1)$, this estimate does not exceed $(C/\delta)\sqrt{\log \frac{N}{n}}$. However, the constant C/δ tends to infinity as δ tends to 0. On the other hand, if N is proportional to n the isotropic constant of K_N is bounded (by an absolute constant), while the upper bound in (1.1) is not. The following theorem closes the the gap between $N \le cn$ and $N \ge n^{1+\delta}$.

Theorem 1.1. *There exist absolute positive constants c, C such that if $n \le N$, and $X_1, \dots, X_N$ are independent random vectors uniformly distributed on an isotropic convex body K, and K_N is their symmetric convex hull, then*

$$\mathbb{P}\left(\left\{L_{K_N} \le C\sqrt{\log \frac{2N}{n}}\right\}\right) \ge 1 - \exp(-c\sqrt{n}).$$

Furthermore, we extend Theorem 1.1 to a much more general setting, namely to a family of perturbations of a random polytope. To desribe our extension we need more notations.

For a vector $y = (y_1, ..., y_N) \in \mathbb{R}^N$, we denote by $y^* = (y_1^*, ..., y_N^*)$ the vector, whose coordinates are the decreasing rearrangement of $\{|y_i|\}_i$. Given $k \le N$ we consider the following norm

$$\|y\|_{k,2} := \left(\sum_{i=1}^{k} y_i^{*2}\right)^{1/2} = \max |P_E y|,$$

where the maximum is taken over all coordinate subspaces E of dimension k. The ball of radius a in this norm we denote by

$$B(a) := \left\{ y \in \mathbb{R}^N \, : \, \|y\|_{k,2} \le a \right\}.$$

For $n \le m \le N$ and $y \in \mathbb{R}^N$ denote

$$\alpha_{y,m} := \left(\prod_{i=m-n+1}^{m} |y_i^*| \right)^{\frac{1}{n}} \ge y_m^*.$$

Let $X_1, \ldots, X_N \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$. Denote

$$K_{N,y} = \mathrm{conv}\{\pm y_1 X_1, \ldots, \pm y_N X_N\}.$$

Theorem 1.2. *There exist absolute positive constants c and C such that the following holds. Let $Cn \le N$ and let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on an isotropic convex body K. Then for every $y \in \mathbb{R}^N$ the event*

$$L_{K_{N,y}} \le C \, \frac{\|y\|_{n,2}}{\sqrt{n}} \, \inf \left\{ \alpha_{y,m}^{-1} \, \frac{\log(2N/n)}{\sqrt{\log(2m/n)}} \;\; | \;\; Cn \le m \le N \right\}$$

occurs with probability greater than $1 - e^{-c\sqrt{n}}$. Moreover, for every $a > 0$ the event

$$\sup_{y \in B(a)} L_{K_{N,y}} \le C \, \frac{a}{\sqrt{n}} \, \inf \left\{ \alpha_{y,m}^{-1} \, \frac{\log(2N/n)}{\sqrt{\log(2m/n)}} \;\; | \;\; N - \frac{cN}{\log N} \le m \le N \right\}$$

occurs with probability greater than $1 - e^{-c\sqrt{n}}$.

Remark 1. Note that if $n \le N \le Cn$ then $L_K \le C$ for any symmetric polytope K generated by N vectors ([5]).

Remark 2. Clearly Theorem 1.2 applied to the vector $y = (1, \ldots, 1)$ implies Theorem 1.1.

Finally, in the last section, we consider the case when the vector describing the perturbation is also random. Such a setting has been recently considered in [6]. In Theorem 4.2 we show that for the Gaussian vector y in $\mathbb{R}^N$, under ceratin conditions on the ψ_2 behavior of linear functionals on K, with high probability we have

$$L_{K_{N,y}} \le C \sqrt{\log \frac{2N}{n}}.$$

2 Preliminaries

Along the paper, the letters $c, C, c_1, C_1, \dots$ will always denote absolute positive constants, whose values may change from line to line. Given two functions f and g we say that they are equivalent and write $f \approx g$ if $c_1 f \le g \le c_2 f$.

By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$. The (unit) Euclidean ball and sphere are denoted by B_2^n and S^{n-1}. Let K be a symmetric convex body in $\mathbb{R}^n$ and let $\|\cdot\|_K$ be its associated norm

$$\|x\|_K = \inf\{\lambda > 0 \, : \, x \in \lambda K\}.$$

The support function of K is $h_K(y) = \max_{x \in K} \langle x, y \rangle$ and it is the norm associated to the polar body of K,

$$K^\circ = \{y \in \mathbb{R}^n \, : \, \langle x, y \rangle \le 1 \ \forall x \in K\}.$$

Given convex body K we denote by $|K|$ its volume. We also denote by $|E|$ the cardinality of a finite set E. For $E \subset \{1, \dots, N\}$ the coordinate projection on $\mathbb{R}^E$ is denoted by P_E.

We say that a convex body $K \subseteq \mathbb{R}^n$ is isotropic if it has volume $|K| = 1$, its center of mass is at 0 (i.e. $\int_K x dx = 0$) and for every $\theta \in S^{n-1}$ one has

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2,$$

where L_K is a constant independent of θ. L_K is called the isotropic constant of K.

It is known that every convex body has a unique (up to an orthogonal transformation) affine image that is isotropic. This allows to define the isotropic constant of any convex body as the isotropic constant of its isotropic image. It is also known (see e.g. [24]) that

$$nL_K^2 = \inf\left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} |x|^2 dx \, : \, T \in GL(n), a \in \mathbb{R}^n \right\}. \tag{2.1}$$

We need two more results on the distribution of Euclidean norms of random vectors and their sums. Let X_i, $i \le N$, be independent random vectors uniformly distributed in an isotropic convex body $K \in \mathbb{R}^n$. Let A be a random $n \times N$ matrix, whose columns are X_i's. For $m \leq N$ denote

$$A_m = \sup\{|Ay| \ \mid \ y \in B_2^N, \ |\operatorname{supp} y| \leq m\}$$

(supp denotes the support of y). Theorem 3.13 in [1] (note the different normalization) implies the following estimate.

Theorem 2.1. *There is an absolute positive constant C such that for every $\gamma \geq 1$ and every $m \leq N$*

$$\mathbb{P}\left(\left\{ A_m \geq CL_K \gamma \sqrt{m} \log \frac{2N}{m} + 6 \max_i |X_i| \right\}\right) \leq \exp\left(-\gamma \sqrt{m} \log \frac{2N}{m}\right).$$

The following theorem is a combination of Paouris' theorem ([26], see also [2] for a short proof) with the union bound (cf. Lemma 3.1 in [1]).

Theorem 2.2. *There exists an absolute positive constant C such that for any $N \leq \exp(\sqrt{n})$ and for every $\lambda \geq 1$ one has*

$$\mathbb{P}\left(\left\{ \max_{i \leq N} |X_i| \geq C \lambda \sqrt{n}\, L_K \right\}\right) \leq \exp\left(-\lambda \sqrt{n}\right).$$

Finally we need the estimate on the volume of the random polytope

$$K_N = \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\},$$

where X_i, $i \leq N$, are independent random vectors uniformly distributed in an isotropic convex body $K \subset \mathbb{R}^n$. The estimates of the following theorem were observed in [12] (see Fact 3.2, the remarks following it, and Fact 3.3 there).

Theorem 2.3. *There are absolute positive constants C, c_1, c_2 such that for $Cn \leq N \leq e^{\sqrt{n}}$,*

$$\mathbb{P}\left(\left\{ |K_N|^{1/n} \geq c_1 \sqrt{\frac{\log(N/n)}{n}}\, L_K \right\}\right) \geq 1 - \exp(-c_2 \sqrt{N}).$$

In fact this theorem is a combination of three results. The first says that K_N contains the centroid body. Recall that for $p \geq 1$ the p-centroid body

$Z_p(K)$ was introduced in [22] (with a different normalization) as the convex body, whose support function is

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}}. \tag{2.2}$$

In [13] (Theorem 1.1) the authors proved that for every parameters $\beta \in (0, 1/2)$ and $\gamma > 1$ one has the inclusion $K_N \supset c_1 Z_p(K)$ for $p = c_2 \beta \log(2N/n)$ and $N \geq c_3 \gamma n$ with the probability greater than

$$1 - \exp(-c_4 N^{1-\beta} n^{\beta}) - \mathbb{P}\left(\left\{ \|A\| > \gamma L_K \sqrt{N} \right\} \right),$$

where A is the random matrix whose columns are $X_1, \ldots, X_N$. The probability that norm of A (note $\|A\| = A_N$) is large was estimated in [1] (combine Theorems 2.1 and 2.2 above). Finally, from results of [20] and [26] the bound

$$|Z_p(K)|^{1/n} \approx \sqrt{p/n}\, L_K \tag{2.3}$$

more details?

follows provided that $p \leq \sqrt{n}$ (it improves the bound provided in [21]).

We also will need the definition of ψ_α norm. For a real random variable z and $\alpha \in [1, 2]$ we define the ψ_α-norm by

$$\|z\|_{\psi_\alpha} = \inf \left\{ C > 0 \mid \mathbb{E} \exp\left(|Y|/C \right)^{\alpha} \leq 2 \right\}.$$

It is well known that the the condition $\|z\|_{\psi_\alpha} \leq c_1$ is equivalent to the condition

$$\forall p > 1 : \quad (\mathbb{E}|z|^p)^{1/p} \leq c_2\, p^{1/\alpha}\, \mathbb{E}|z|.$$

Let X be a centered random vector in $\mathbb{R}^n$ and $\alpha > 0$. We say that X is ψ_α or a ψ_α vector, if

$$\|X\|_{\psi_\alpha} := \sup_{y \in S^{n-1}} \| \langle X, y \rangle \|_{\psi_\alpha} < \infty. \tag{2.4}$$

3 Proofs

In this section we prove Theorem 1.2. The proof consists of two propositions.

Proposition 3.1. *Let $n \le N \le e^{\sqrt{n}}$ and $X_1, \dots, X_N$ are independent random vectors distributed uniformly on an isotropic convex body K. Let $a > 0$. Then the event*

$$\sup \left\{ \frac{1}{|K_{N,y}|} \int_{K_{N,y}} |x|^2 dx \;\mid\; y \in B(a),\, y_n^* > 0 \right\} \le C\, \frac{a^2}{n}\, L_K^2 \log^2 \frac{2N}{n}$$

occurs with probability greater than $1 - \exp(-\sqrt{n}\, \log(2N/n))$, where C is an absolute constant.

To prove this proposition we need the following lemma.

Lemma 3.2. *Let $1 \le n \le N$ be integers and $P = \mathrm{conv}\{P_1, \dots, P_N\}$ be a non-degenerated symmetric polytope in $\mathbb{R}^n$. Then*

$$\frac{1}{|P|} \int_P |x|^2 dx \le \frac{1}{(n+1)(n+2)} \sup_E \left(\sum_{i \in E} |P_i|^2 + \left| \sum_{i \in E} P_i \right|^2 \right),$$

where the supremum is taken over all subsets $E \subset \{1, \dots, N\}$ of cardinality n.

Proof. We can decompose P as a disjoint union of simplices (up to sets of measure 0), say $P = \cup_{i=1}^{\ell} C_i$, where each C_i is of the form $\mathrm{conv}\{0, P_{i_1}, \dots, P_{i_n}\}$ for some choice of P_{i_j}'s. For every such C_i, denote $F_i := \mathrm{conv}\{P_{i_1}, \dots, P_{i_n}\}$. Then for any integrable function f we have

$$\int_{C_i} f(x)dx = \int_{F_i} \int_0^1 r^{n-1} f(ry) |\langle y, \nu(y) \rangle| dy dr = d(0, F_i) \int_{F_i} \int_0^1 r^{n-1} f(ry) dy dr,$$

where $\nu(y)$ is the outer normal vector to P at the point y and $d(0, F_i)$ is the distance from the origin to the affine subspace spanned by F_i. Thus, as in [19], for every $i \le \ell$ one has

$$|C_i| = n^{-1} |F_i| d(0, F_i) \qquad \text{and} \qquad \int_{C_i} |x|^2 = \frac{d(0, F_i)}{n+2} \int_{F_i} |y|^2 dy.$$

In particular,

$$|P| = \sum_{i=1}^{\ell} |C_i| = \frac{1}{n} \sum_{i=1}^{\ell} |F_i| d(0, F_i).$$

Therefore,

$$
\begin{aligned}
\frac{1}{|P|}\int_P |x|^2 dx &= \frac{1}{|P|}\sum_{i=1}^{\ell} \frac{d(0,F_i)}{n+2}\int_{F_i}|y|^2 dy \\
&\le \frac{1}{|P|}\sum_{i=1}^{\ell} \frac{d(0,F_i)|F_i|}{n+2}\sup_{1\le i\le \ell}\frac{1}{|F_i|}\int_{F_i}|y|^2 dy \\
&\le \frac{n}{n+2}\sup_F \frac{1}{|F|}\int_F |y|^2 dy,
\end{aligned}
$$

where the supremum is taken over all $F = \mathrm{conv}\{P_{i_1},\dots,P_{i_n}\}$. Note that any such F can be presented as $F = T\Delta^{n-1}$, where $\Delta^{n-1} = \mathrm{conv}\{e_1,\dots,e_n\}$ denotes the regular $n-1$ dimensional simplex in $\mathbb{R}^n$ and T is the matrix whose columns are the vectors P_{i_j}. Since

$$
\frac{1}{|\Delta^{n-1}|}\int_{\Delta^{n-1}} y_i y_j dy = \frac{1+\delta_{ij}}{n(n+1)},
$$

where δ_{ij} is the Kronecker delta, for every $F = \mathrm{conv}\{P_{i_1},\dots,P_{i_n}\}$ we obtain

$$
\frac{1}{|F|}\int_F |y|^2 dy = \frac{1}{n(n+1)}\left(\sum_{j=1}^{n}|P_{i_j}|^2 + \left|\sum_{j=1}^{n} P_{i_j}\right|^2\right).
$$

This implies the desire estimate. □

Proof of Proposition 3.1. Note that if $y_N^* > 0$ then the cardinality of support of y is at least n, so $K_{N,y}$ is not degenerated with probability one. Therefore, with probability one $K_{N,y}$ is non-degenerated for any countable dense set in $B_0(a) := \{y \in B(a) \mid y_n^* > 0\}$. Clearly, the supremum under question is the same over $y \in B_0(a)$ and over such a dense set.

Now, by Lemma 3.2 we have that $\sup_{y\in B_0(a)} |K_{N,y}|^{-1}\int_{K_{N,y}} |x|^2 dx$ is bounded from above by

$$
\frac{1}{(n+1)(n+2)}\sup_{y\in B_0(a)}\sup_{|E|=n}\left(\sum_{i\in E}|y_i X_i|^2 + \left|\sum_{i\in E} y_i X_i\right|^2\right)
$$

(formally, we should additionally take supremum over $\varepsilon_i = \pm 1$ and to have $y_i\varepsilon_i X_i$ in the formula under suprema, but, since $B_0(a)$ is unconditional, the supremum over ε_i's can be omitted).

Note that

$$\sum_{i\in E} |y_i X_i|^2 \le \|y\|_{n,2}^2 \max_{i\le N} |X_i|^2$$

and

$$\left|\sum_{i\in E} y_i X_i\right| = |AP_E y| \le A_n \|y\|_{n,2},$$

where A is the matrix whose columns are $X_1, \ldots, X_N$. Therefore, applying Theorem 2.1 and Theorem 2.2 (with $m = n$ and $\lambda = 2\log(2N/n)$) we obtain that

$$\sup_{y\in B_0(a)} \sup_{|E|=n} \left(\sum_{i\in E} |y_i X_i|^2 + \left|\sum_{i\in E} y_i X_i\right|^2 \right) \le a^2 n L_K^2 \log^2 \frac{2N}{n}$$

with probability greater than $1 - \exp(-\sqrt{n}\, \log(2N/n))$. □

Proposition 3.3. *There exist absolute positive constants c_1, c_2, C such that if $Cn \le N \le e^{\sqrt{n}}$ and $X_1, \ldots, X_N$ are independent random vectors distributed uniformly on an isotropic convex body K, then for every $y \in \mathbb{R}^N$,*

$$\mathbb{P}\left(\left\{\forall m \ge Cn: \quad |K_{N,y}|^{\frac{1}{n}} \ge c_1 \alpha_{y,m} L_K \sqrt{\frac{\log(2m/n)}{n}}\right)\right\} \ge 1 - \exp\left(-c_2\sqrt{n}\right).$$

Moreover, the event

$$\forall m \ge N - \frac{c_1 N}{\log N} \;\; \forall y \in \mathbb{R}^N \qquad |K_{N,y}|^{\frac{1}{n}} \ge c_1 \alpha_{y,m} L_K \sqrt{\frac{\log(2m/n)}{n}}$$

occurs with probability greater than $1 - \exp\left(-c_2\sqrt{N}\right)$.

The probability estimates in Proposition 3.3 are based on an estimate of corresponding probability for a fixed y and the union bound. We start with the following lemma.

Lemma 3.4. *There exist absolute positive constants c_1, c_2, C such that the following holds. Let $Cn \le m \le N \le e^{\sqrt{n}}$ and $X_1, \ldots, X_N$ are independent random vectors distributed uniformly on an isotropic convex body K. Then for every $y \in \mathbb{R}^N$ with $y_m^* > 0$ there exists $v = v(y) \in \mathbb{R}^N$ having 0/1 coordinates with exactly m ones such that*

$$|K_{N,y}| \ge \alpha_{y,m} |K_{N,v}|$$

and

$$\mathbb{P}\left(\left\{|K_{N,v}|^{\frac{1}{n}} \geq c_1\, L_K \sqrt{\frac{\log(2m/n)}{n}}\right\}\right) \geq 1 - \exp\left(-c_2\sqrt{m}\right).$$

Proof. Fix $y \in \mathbb{R}^N$ with $y_m^* > 0$ (i.e. $|\operatorname{supp} y| \geq m$). Let $i_1, \ldots, i_m$ be the indices such that $y_{i_j} = y_j^*$ and let $v = v(y) \in \mathbb{R}^N$ be the vector with $v_k = 1$ if $k = i_j$ and 0 otherwise. Decompose the polytope $K_{N,v}$ into a disjoint union of simplices (up to a set of zero measure)

$$K_{N,v} = \bigcup_{k=1}^{\ell} C_k,$$

where $C_k = \operatorname{conv}\{0, \varepsilon_{k_1} X_{k_1}, \ldots, \varepsilon_{k_n} X_{k_n}\}$ for some $\varepsilon_{k_j} = \pm 1$ and some vectors X_{k_j}, given by the simplicial decomposition of the facets of $K_{N,v}$. Denote

$$C_{k,y} = \operatorname{conv}\{0, \varepsilon_{k_1}|y_{k_1}|X_{k_1}, \ldots, \varepsilon_{k_n}|y_{k_n}|X_{k_n}\} \subset K_{N,y}.$$

Clearly, $C_{k,y}$'s are also disjoint up to a set of zero measure and

$$\begin{aligned} |C_k| &= |\det\left(\varepsilon_{k_1} X_{k_1}, \ldots, \varepsilon_{k_n} X_{k_n}\right| |\operatorname{conv}\{0, e_1, \ldots, e_n\})| \\ &\leq \frac{1}{\alpha_{y,m}} |\det\left(\varepsilon_{k_1}|y_{k_1}|X_{k_1}, \ldots, \varepsilon_{k_n}|y_{k_n}|X_{k_n}\right)| |\operatorname{conv}\{0, e_1, \ldots, e_n\}|. \end{aligned}$$

This implies

$$|K_{N,v}| = \sum_{k=1}^{\ell} C_k \leq \alpha_{y,m}^{-1} \sum_{k=1}^{\ell} C_{k,y} \leq \alpha_{y,m}^{-1} |K_{N,y}|.$$

This proves the first estimate. The second one follows by Theorem 2.3, since $K_{N,v}$ is a symmetric random polytope in an isotropic convex body generated by $m \geq Cn$ random points. □

Proof of Proposition 3.3. Without loss of generality we only consider y's satisfying $y_n^* > 0$ (otherwise estimates are trivial).

The first estimate follows from Lemma 3.4 and the union bound, since

$$\sum_{m \geq Cn} e^{-c_2\sqrt{m}} \leq e^{-c_2\sqrt{n}}, \tag{3.1}$$

provided that C is large enough.

To prove the second bound note that the set $\{v(y)\}_{y\in\mathbb{R}^N}$ ($v(y)$ is from Lemma 3.4) has cadinality $\binom{N}{m}$ and that denoting $k = N - m$

$$\binom{N}{m} \exp\left(-c_2\sqrt{m}\right) \le \exp(-c_2\sqrt{m} + k\log(eN/k)) \le \exp\left(-c_2\sqrt{m}/2\right),$$

provided that $k \le c\sqrt{N}/\log N$. Lemma 3.4 and the union bound imply

$$\mathbb{P}\left(\left\{\forall y \in \mathbb{R}^N : \ |K_{N,v}|^{\frac{1}{n}} \ge c_1\, \alpha_{y,m}\, L_K \sqrt{\frac{\log(2m/n)}{n}}\right\}\right) \ge 1-\exp\left(-c_2\sqrt{m}/2\right).$$

The result follows by the union bound and (3.1). □

Proof of Theorem 1.2. For $Cn \le N \le e^{\sqrt{N}}$ Propositions 3.1 and 3.3 imply the result, since, by (2.1),

$$nL^2_{K_{N,y}} \le \frac{1}{|K_{N,y}|^{1+\frac{2}{n}}} \int_{K_{N,y}} |x|^2 dx.$$

For $N \ge e^{\sqrt{n}}$ the theorem follows from the general estimate $L_K \le Cn^{1/4}$ for any n-dimensional convex body ([18]). □

4 Random perturbations of random polytopes

Recall that a Z_p body and ψ_α norm were defined in (2.2) and (2.4). In this section $G = (g_1, \dots, g_N)$ denotes a standard Gaussian random vector in $\mathbb{R}^N$, independent of any other random variables. We also denote

$$\gamma_p := \left(\mathbb{E}|g_1|^p\right)^{1/p} \approx \sqrt{p}.$$

Proposition 4.1. *There are absolute positive constants C, c_1 and c_2 such that the following holds. Let $\beta > 2$ and $N \ge C(\log\beta)^2 n$. Let $X_1, \dots, X_N$ be independent copies of a random vector uniformly distributed on an isotropic convex body K. Assume that $\|X_1\|_{\psi_2} \le \beta\, L_K \sqrt{\log \frac{N}{n}}$. Then*

$$\mathbb{P}_{G,X_1,\dots,X_N}\left(\left\{K_{N,G} \supseteq c_1\sqrt{\log\frac{N}{n}}\, Z_{\log(N/n)}(K)\right\}\right) \ge 1 - \exp\left(-c_2\sqrt{N}\right).$$

Proof. First note that for any $p \geq 1$, $i \leq N$ and $\theta \in S^{n-1}$, one has

$$(\mathbb{E}|\langle g_i X_i, \theta\rangle|^p)^{1/p} = (\mathbb{E}|g_i|^p)^{1/p} (\mathbb{E}|\langle X_i, \theta\rangle|^p)^{1/p} \leq c_1 \sqrt{p}\, (\mathbb{E}|\langle X_i, \theta\rangle|^p)^{1/p}.$$

Thus,

$$\sup_{i\leq N} \sup_{\theta\in S^{n-1}} \|\langle g_i X_i, \theta\rangle\|_{\psi_1} \leq c_2 \sup_{i\leq N} \sup_{\theta\in S^{n-1}} \|\langle X_i, \theta\rangle\|_{\psi_2} \leq c_3 \beta\, L_K \sqrt{\log \frac{N}{n}}.$$

Denote by A the $n \times N$ random matrix whose columns are the vectors $g_i X_i$. By Theorem 3.13 in [1] (cf. Theorem 2.1),

$$\mathbb{P}_{G,X_1,\ldots,X_N}\left(\left\{\|A\| \geq c_4 \beta L_K \sqrt{N \log \frac{N}{n}} + 6 \max_{i\leq N} |X_i|\right\}\right) \leq \exp\left(-2\sqrt{N}\right).$$

Together with Theorem 2.2 (applied with $\lambda = 2\sqrt{n}$), we have that

$$\mathbb{P}_{G,X_1,\ldots,X_N}\left(\left\{\|A\| \geq c_5 \beta L_K \sqrt{N \log \frac{N}{n}}\right\}\right) \leq e^{-\sqrt{N}}. \tag{4.1}$$

On the other hand, for every $\sigma \subseteq \{1, \ldots, N\}$, $q \geq 1$ and $\theta \in S^{n-1}$, by Paley-Zygmund inequality,

$$\begin{aligned} & \mathbb{P}_{G,X_1,\ldots,X_N}\left(\left\{\max_{i\in\sigma} |\langle g_i X_i, \theta\rangle| \leq \frac{1}{2} (\mathbb{E}|g_1|^q)^{\frac{1}{q}} (\mathbb{E}|\langle X_1, \theta\rangle|^q)^{\frac{1}{q}}\right\}\right) \\ = & \prod_{i\in\sigma} \mathbb{P}_{G,X_1,\ldots,X_N}\left(\left\{|\langle g_i X_i, \theta\rangle| \leq \frac{1}{2} (\mathbb{E}|g_1|^q)^{\frac{1}{q}} (\mathbb{E}|\langle X_1, \theta\rangle|^q)^{\frac{1}{q}}\right\}\right) \\ \leq & \left(1 - \left(1 - \left(\frac{1}{2}\right)^q\right)^2 \frac{(\mathbb{E}|g_1|^q \mathbb{E}|\langle X_1, \theta\rangle|^q)^2}{\mathbb{E}|g_1|^{2q} \mathbb{E}|\langle X_1, \theta\rangle|^{2q}}\right)^{|\sigma|}. \end{aligned}$$

Since $\gamma_p \approx \sqrt{p}$, and from Borell's lemma ([8], see also Appendix III in [25]),

$$\mathbb{E}|\langle X_1, \theta\rangle|^{2q} \leq c_6^q\, \mathbb{E}|\langle X_1, \theta\rangle|^q,$$

the quantity above is bounded by

$$\left(1 - \frac{1}{4C^q}\right)^{|\sigma|} \leq \exp\left(-\frac{|\sigma|}{4C^q}\right).$$

Set $m = [N/n]$. Let $\sigma_1, \dots, \sigma_n$ be a partition of $\{1, \dots, N\}$ with $m \le |\sigma_i|$ for every i and $\| \cdot \|_0$ be the norm

$$\|u\|_0 = \frac{1}{n} \sum_{i=1}^{n} \max_{j \in \sigma_i} |u_j|.$$

Note that $\| \cdot \| \le n^{-1/2} | \cdot |$. Since for all $1 \le i \le k$ and every $z \in \mathbb{R}^n$

$$h_{K_{N,G}}(z) = \max_{1 \le j \le N} |\langle g_j X_j, z \rangle| \ge \max_{j \in \sigma_i} |\langle g_j X_j, z \rangle|,$$

then for every $z \in \mathbb{R}^n$

$$h_{K_{N,G}}(z) \ge \|Az\|_0.$$

Cleary, if $z \in \mathbb{R}^n$ verifies $\|Az\|_0 \le \frac{1}{4} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}}$, then there exists a set $I \subseteq \{1, \dots, k\}$ with $|I| \ge \frac{k}{2}$ such that

$$\max_{j \in \sigma_i} |\langle g_j X_j, z \rangle| \le \frac{1}{2} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}}$$

for every $i \in I$. Thus, for every $z \in \mathbb{R}^n$,

$$\begin{aligned}
& \mathbb{P}_{G, X_1, \dots, X_N} \left(\left\{ \|Az\|_0 \le \frac{1}{4} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}} \right\} \right) \\
\le & \sum_{|I| = \lceil \frac{n}{2} \rceil} \mathbb{P}_{G, X_1, \dots, X_N} \left(\left\{ \forall i \in I \ : \ \max_{j \in \sigma_i} |\langle g_j X_j, z \rangle| \le \frac{1}{2} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}} \right\} \right) \\
\le & \sum_{|I| = \lceil \frac{n}{2} \rceil} \prod_{i \in I} \mathbb{P}_{G, X_1, \dots, X_N} \left(\left\{ \max_{j \in \sigma_i} |\langle g_j X_j, z \rangle| \le \frac{1}{2} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}} \right\} \right) \\
\le & \sum_{|I| = \lceil \frac{n}{2} \rceil} \prod_{i \in I} \exp \left(- \frac{|\sigma_i|}{4 C^q} \right) \le 2^n \exp \left(- \frac{nm}{4 C^q} \right) \le 2^n \exp \left(- \frac{N}{8 C^q} \right) \\
\le & \exp \left(- \frac{\sqrt{Nn}}{16} \right)
\end{aligned}$$

provided that $q := (1/2) \log(N/n)$ and $N > 125n$.

Now, let

$$S = \left\{ z \in \mathbb{R}^n \ | \ \frac{1}{2} \gamma_q \left(\mathbb{E} |\langle X_1, z \rangle|^q \right)^{\frac{1}{q}} = 1 \right\}$$

and let $U \subset S$ be a δ-net (in metric defined by S) with cardinality $|U| \le \left(\frac{3}{\delta}\right)^n$, i.e., for every $z \in S$ there is $u \in U$ such that $\frac{1}{2}\gamma_q \left(\mathbb{E}|\langle X_1, z-u\rangle|^q\right)^{\frac{1}{q}} \le \delta$. Then

$$\mathbb{P}\left(\left\{\exists u \in U \,:\, \|Au\|_0 \le \frac{1}{2}\right\}\right) \le \exp\left(n\log\frac{3}{\delta} - \sqrt{Nn}/16\right).$$

By isotropicity we have that $(\mathbb{E}|\langle X_1, z\rangle|^q)^{\frac{1}{q}} \ge L_k|z|$ (because we have chosen $q = (1/2)\log(N/n) > 2$). Thus, assuming $\|A\| \le c_5\beta L_K\sqrt{N\,\log\frac{N}{n}}$, we have

$$\begin{aligned}\|Az\|_0 \le \frac{1}{\sqrt{n}}|Az| &\le c_5\beta L_K\sqrt{\frac{N}{n}}\sqrt{\log\frac{N}{n}}\,|z| \le c_5\beta\sqrt{\frac{N}{n}}\sqrt{2q}\,\left(\mathbb{E}|\langle X_1, z\rangle|^q\right)^{\frac{1}{q}}\\ &\le c_6\beta\,\gamma_q\sqrt{\frac{N}{n}}\,\left(\mathbb{E}|\langle X_1, z\rangle|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Therefore, if $u \in U$ approximates $z \in S$, that is if $\frac{1}{2}\gamma_q \left(\mathbb{E}|\langle X_1, z-u\rangle|^q\right)^{\frac{1}{q}} \le \delta$, then u also satisfies

$$\|Au\|_0 \le \|Az\|_0 + c_7\beta\sqrt{\frac{N}{n}}\,\delta.$$

Choosing $\delta = \sqrt{n}/(4\beta c_7\sqrt{N})$ and denoting the event

$$\Omega_0 := \left\{\|A\| \le c_5\beta L_K\sqrt{N\,\log\frac{N}{n}}\right\}$$

we obtain

$$\begin{aligned}&\mathbb{P}_{G,X_1,\dots,X_N}\left(\left\{\omega \in \Omega_0 \;\middle|\; \exists z \in \mathbb{R}^n \,:\, \|Az\|_0 \le \frac{1}{8}\gamma_q\left(\mathbb{E}|\langle X_1, z\rangle|^q\right)^{\frac{1}{q}}\right\}\right)\\ =\;& \mathbb{P}_{G,X_1,\dots,X_N}\left(\left\{\omega \in \Omega_0 \;\middle|\; \exists z \in S \,:\, \|Az\|_0 \le \frac{1}{4}\right\}\right)\\ \le\;& \mathbb{P}_{G,X_1,\dots,X_N}\left(\left\{\omega \in \Omega_0 \;\middle|\; \exists u \in U \,:\, \|Au\|_0 \le \frac{1}{2}\right\}\right)\\ \le\;& \exp\left(n\log\frac{12c_7\,\beta\sqrt{N}}{\sqrt{n}} - \sqrt{Nn}/16\right) \le \exp\left(-\sqrt{Nn}/20\right)\end{aligned}$$

provided $N \ge C(\log\beta)^2 n$ for a big enough absolute constant C. Since $h_{K_{N,G}}(z) = \|Az\|_\infty \ge \|Az\|_0$, this together with (4.1) and the definition (2.2), implies that with probability at least $1 - \exp\left(-\sqrt{N}\right) - \exp\left(-\sqrt{Nn}/20\right)$,

$$K_{N,G} \supseteq \frac{1}{8}\,\gamma_q\, Z_q(K) \supseteq c\sqrt{\log\frac{N}{n}}\, Z_{\log(N/n)}(K).$$

□

Proposition 4.1 implies the following theorem.

Theorem 4.2. *There are absolute positive constants C, c_1 and c_2 such that the following holds. Let $\beta > 2$ and $N \geq C(\log\beta)^2 n$. Let $X_1, \ldots, X_N$ be independent copies of a random vector uniformly distributed on an isotropic convex body and assume that $\|X_1\|_{\psi_2} \leq \beta\sqrt{\log\frac{N}{n}}$. Then*

$$\mathbb{P}_{G,X_1,\ldots,X_N}\left(\left\{L_{K_{N,G}} \leq c_1\sqrt{\log\frac{2N}{n}}\right\}\right) \geq 1 - \exp\left(-c_2\sqrt{n}\log\frac{2N}{n}\right).$$

Proof. By Proposition 3.1, the probability (with respect to X_i's) of the event

$$\forall y \in \mathbb{R}^n \text{ with } y_n^* > 0 \,:\, \frac{1}{|K_{N,y}|}\int_{K_{N,y}} |x|^2 dx \leq CL_K^2 \frac{\|y\|_{n,2}^2}{n}\log^2\frac{2N}{n}$$

is at least $1 - \exp\left(-\sqrt{n}\log(2N/n)\right)$.

It is well know (and can be directly calculated) that for the Gaussian vector $G = (g_1, \ldots, g_N)$ one has

$$\mathbb{E}\|G\|_{n,2} \approx \sqrt{n\,\log\frac{N}{n}}.$$

Using concentration (see e.g. Theorem 1.5 in [27]), we observe that for some absolute constant $C_1 > 0$,

$$\mathbb{P}_G\left(\|G\|_{n,2} \geq C_1\sqrt{n\,\log\frac{N}{n}}\right) \leq \exp\left(-n\log(N/n)\right).$$

Therefore, the probability (with respect to G and X_i's) of the event

$$\frac{1}{|K_{N,G}|}\int_{K_{N,G}} |x|^2 dx \leq CL_K^2 \log^3\frac{N}{n}$$

is at least $1 - \exp\left(-c\sqrt{n}\log(N/n)\right)$. On the other hand by the previous proposition and (2.3)

$$\mathbb{P}_{G,X_1,\dots,X_N}\left(\left\{|K_{N,G}|^{1/n} \geq \frac{cL_K\log(N/n)}{\sqrt{n}}\right\}\right) \geq 1 - \exp\left(-c'\sqrt{N}\right).$$

Since

$$nL_{K_{N,G}}^2 \leq \frac{1}{|K_{N,G}|^{1+\frac{2}{n}}}\int_{K_{N,G}}|x|^2dx,$$

we obtain the desired result. □

Remark 4.3. *Finally we would like to note that for $N \geq n^2$ with high probability $|K_{N,G}|^{1/n} \geq (cL_K \log N)/\sqrt{n}$, even if the convex body K is not ψ_2 (see Lemma 4 in [6]). Therefore in this case, using*

$$\max_{i\leq N}|g_iX_i| = \max_{i\leq N}|g_i| \max_{i\leq N}|X_i| \leq C_1\sqrt{\log N}\sqrt{n}$$

with high probability one has

$$L_{K_{N,G}} \leq c\,\frac{\max_{i\leq N}|g_iX_i|\log N}{n|K_{N,G}|^{\frac{1}{n}}} \leq C\sqrt{\log N}.$$

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