Gauss-Bonnet formulae and rotational integrals in constant curvature spaces

S. Barahona\textsuperscript{a}, X. Gual-Arnau\textsuperscript{b}

\textsuperscript{a}Departament de Matemàtiques, Universitat Jaume I. 12071-Castelló, Spain. barahona@uji.es
\textsuperscript{b}Departament de Matemàtiques, Institute of New Imaging Technologies, Universitat Jaume I. 12071-Castelló, Spain. gual@uji.es

Abstract

We obtain generalizations of the main result in [18], and then provide geometric interpretations of linear combinations of the mean curvature integrals that appear in the Gauss-Bonnet formula for hypersurfaces in space forms $M_\lambda^n$. Then, we combine these results with classical Morse theory to obtain new rotational integral formulae for the $k-$th mean curvature integrals of a hypersurface in $M_\lambda^n$.

Keywords: Gauss-Bonnet formula, integral of mean curvature, intrinsic volume, rotational integral formulas, space form.

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1. Introduction

Let $M_\lambda^n$ denote a simply connected Riemannian manifold of constant sectional curvature $\lambda$. Further, let $L^n_r$ denote a $r-$plane, ($r \leq n$), namely a totally geodesic submanifold of dimension $r$ in $M_\lambda^n$, and let $dL^n_r$ denote the corresponding density, invariant under the group of Euclidean and non-Euclidean motions. A $r-$plane through a fixed point $O$ in $M_\lambda^n$, and its invariant density, are denoted by $L^n_{r[0]}$ and $dL^n_{r[0]}$, respectively [16].

In [8] a new expression for the density of $r-$planes in $M_\lambda^n$ has been obtained in terms of the density $dL^n_{r+1[0]}$, of the density $dL^{r+1}_r$ of $r-$planes in $L^n_{r+1[0]}$ and the distance $\rho$ from $O$ to $L^{r+1}_r$. Thus, an invariant $r-$plane in $M_\lambda^n$ may be generated by taking first an isotropic $(r+1)-$plane through a fixed point $O$ and then an invariant $r-$plane within this $(r+1)-$plane, weighted
by a function of $\rho$.

This construction, called the invariator principle in $M_\lambda^n$ ([19]), has opened the way to solve rotational integral equations for different quantities as the volume of a $k$–dimensional submanifold in $M_\lambda^n$ [8], the $k$–th mean curvature integrals or $k$–th intrinsic volumes ([10] and [1], and different curvature measures ([19] for $\lambda = 0$)). The solutions of these equations allow to express these quantities as the integral of some functionals defined in sections produced by isotropic planes through a fixed point. Moreover, in [19], the authors, using classical Morse theory, rewrite the volume of compact submanifolds in $\mathbb{R}^n$ of dimension $n - r$, in terms of critical values of the sectioned object with $(r + 1)$–planes; and in [9] related generalizations valid for submanifolds in space forms of constant curvature are obtained.

On the other hand, in [18] it is proved that the Gauss-Bonnet defect of a hypersurface in $M_\lambda^n$ is the measure of planes $L_{n-2}$ meeting it, counted with multiplicity. From this result an integral-geometric proof of the Gauss-Bonnet theorem for hypersurfaces in $M_\lambda^n$ is given.

The purpose of this paper is twofold: to obtain generalizations of the main result in [18], following a completely different route; and to combine these results with classical Morse theory to obtain new rotational integral formulae for the $k$–th mean curvature integrals of a hypersurface in $M_\lambda^n$.

2. The Gauss-Bonnet theorem in $M_\lambda^n$

Let $Q \subset M_\lambda^n$ be a compact domain with smooth boundary $S = \partial Q$. Let $V$ denote the volume of of $Q$, $F$ the $(n - 1)$–surface area of $S$, $\chi(Q)$ the Euler-Poincaré characteristic of $Q$, and $M_i$ the $i$–th integral of mean curvature of $S$. The Gauss-Bonnet formula for $S$ states that [16]

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-2}{2}}c_1M_1 + \lambda^\frac{n}{2}V = \frac{1}{2}O_n\chi(Q),$$

(1)

for $n$ even, where $O_k = \text{vol}(S^k)$ (surface area of the $k$–dimensional unit sphere), and

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-3}{2}}c_2M_2 + \lambda^\frac{n-1}{2}c_0F = \frac{1}{2}O_n\chi(Q),$$

(2)
for $n$ odd, where
\[
c_h = \left( \frac{n-1}{h} \right) \frac{O_n}{O_h O_{n-1-h}}. \tag{3}
\]

If $n$ is odd, we can use the equality $2\chi(Q) = \chi(S)$, and for $\lambda = 0$, in any case, we obtain $M_{n-1} = O_{n-1}\chi(Q)$.

Let $L_r$ be the space of $r$–dimensional totally geodesic submanifolds of $M^n_\lambda$. Our first result is the following theorem, which is a generalization of the main result in [18].

**Theorem 2.1.** For $n$ and $r$ even, or $n$ and $r$ odd, we have
\[
\frac{1}{2}O_n\chi(Q) - c_{n-1}M_{n-1} - \lambda c_{n-3}M_{n-3} - \cdots - \lambda^{\frac{n-r-2}{2}} c_{r+1}M_{r+1} = \lambda^{\frac{n-r}{2}} \frac{O_r \cdots O_1}{O_{n-1} \cdots O_{n-r}} \int_{L_r} \chi(Q \cap L^n_r) dL^n_r. \tag{4}
\]

**Proof.** We begin assuming that $n$ and $r$ are both even numbers. Given a $r$–plane $L^n_r$ of $M^n_\lambda$, $Q_r = L^n_r \cap Q$ is, in general, a domain of dimension $r$ in $L^n_r$. Applying Eq.(1) to $Q_r$ we obtain
\[
c'_r - c'_{r-3}M'_{r-3} + \cdots + \lambda^{\frac{r-2}{2}} c'_1 M'_{1} + \lambda^{\frac{r}{2}} V(Q_r) = \frac{1}{2} O_r \chi(Q_r), \tag{5}
\]
where $M'_i$ is the $i$–th integral of mean curvature of $\partial Q_r$ and
\[
c'_h = \left( \frac{r-1}{h} \right) \frac{O_r}{O_h O_{r-1-h}}. \tag{6}
\]

Eq.(14.69) for $q = n$ and Eq.(14.78) of [16], which are valid for $M^n_\lambda$, are
\[
\int_{L_r} V(Q_r) dL^n_r = \frac{O_{n-1} \cdots O_{n-r}}{O_{r-1} \cdots O_0} V(Q) \tag{7}
\]
and
\[
\int_{L_r} M'_i dL^n_r = \frac{O_{n-2} \cdots O_{n-r} O_{n-i}}{O_{r-2} \cdots O_0 O_{r-i}} M_i. \tag{8}
\]

Now, having the preceding equalities in mind, we integrate Eq.(5) and we obtain
\[
d_{r-1}M_{r-1} + \lambda d_{r-3}M_{r-3} + \cdots + \lambda^{\frac{r-2}{2}} d_1 M_1 + \lambda^{\frac{r}{2}} d_0 V = \frac{1}{2} O_r \int_{L_r} \chi(Q_r) dL^n_r, \tag{9}
\]
where
\[ d_i = \binom{r-1}{i} \frac{O_r}{O_i O_{r-1-i}} \frac{O_{n-2} \ldots O_{n-r} O_{n-i}}{O_{r-2} \ldots O_0 O_{r-i}}; \quad i = 1, 3, \ldots, r - 1; \]  
\[ d_0 = \frac{O_{n-1} \ldots O_{n-r}}{O_{r-1} \ldots O_0}. \]  
\[ (10) \]

We multiply Eq. (9) by \( \frac{\lambda^{(n-r)/2}}{d_0} \) to obtain
\[ \lambda^{\frac{n-r}{2}} k_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} k_{r-3} M_{r-3} + \cdots + \lambda^{\frac{n-3}{2}} k_2 M_1 + \lambda^\frac{n}{2} V \]
\[ = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{L_r} \chi(Q_r) dL^n, \]  
\[ (12) \]

where
\[ k_i = \binom{r-1}{i} \frac{O_r O_{r-1-i} O_{n-i}}{O_i O_{n-1} O_{r-i} O_{r-i-1}}. \]  
\[ (13) \]

If we compare the constants \( k_i \) and \( c_i \) in Eq. (1), using the equality \( (k-1)O_k = O_1 O_{k-2} \), we have that
\[ k_i = c_i; \]  
\[ (14) \]
then, Eq. (12) can be written as
\[ \lambda^{\frac{n-r}{2}} c_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} c_{r-3} M_{r-3} + \cdots + \lambda^{\frac{n-3}{2}} c_2 M_1 + \lambda^\frac{n}{2} V \]
\[ = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{L_r} \chi(Q_r) dL^n, \]  
\[ (15) \]
and, from Eq. (1) we obtain the result for the case \( n \) and \( r \) even.

If we consider that \( n \) and \( r \) are both odd numbers the proof is similar to the preceding one but considering, instead of Eq. (7), the following equality (Eq. (14.69) of [16] with \( q = n - 1 \)):
\[ \int_{L_r} F(\partial Q_r) dL^n_r = \frac{O_n \ldots O_{n-r} O_{r-1}}{O_r \ldots O_0 O_{n-1}} F, \]  
\[ (16) \]
where \( F(\partial Q_r) \) is the \((r-1)\)-surface area of \( \partial Q \cap L^n_r = \partial (Q \cap L^n_r) \). □

**Remark.** For \( r = n - 2 \), Theorem 2.1 gives Theorem 1 of [18] and, as a result of Theorem 2.1, we obtain the following corollary which is equivalent to Proposition 7 of [18].
Corollary 2.2. Let $Q$ be a compact domain in $M^n_\lambda$ and $L_r \in \mathcal{L}_r$, we have

$$M_r = \frac{(n-r-1)O_r \ldots O_0}{O_{n-2} \ldots O_{n-r-2}} \int_{L_{r+1}} \chi(Q \cap L^n_{r+1})dL^n_{r+1}$$

$$- \lambda^r O_{r-2} \ldots O_0 \int_{L_{r-1}} \chi(Q \cap L^n_{r-1})dL^n_{r-1}. \quad (17)$$

Proof. When $r$ is an even number, Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \ldots + \lambda^{\frac{r-2}{2}} c_1 M_1 + \lambda^{\frac{r}{2}} V = \frac{1}{2} \frac{O_r}{d_0} \int_{L_r} \chi(Q_r) dL^n_r; \quad (18)$$

and the corresponding equation to Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ when $r$ is an odd number is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \ldots + \lambda^{\frac{r-2}{2}} c_2 M_2 + \lambda^{\frac{r-1}{2}} c_0 F = \frac{1}{2} \frac{O_r}{d_0} \int_{L_r} \chi(Q_r) dL^n_r. \quad (19)$$

If $r$ is odd, subtracting each part of Eq.(18), with $r \to r + 1$, minus the corresponding part of $\lambda$ multiplied by Eq.(18) with $r \to r - 1$ we obtain the result. If $r$ is even, we proceed in the same way but using Eq.(19) instead of the Eq.(18). □

Remark. For $\lambda = 0$, Eq.(17) coincides with Eq.(14.79) of [16].

3. Rotational integrals and Morse representations for $M_r$

From rotational integral formulae we obtain quantitative properties (as $M_r$) of differential manifolds in $M^n_\lambda$, from the intersection of the manifold with planes (totally geodesic submanifolds) through a fixed point $O$. In this context, from Eq.(17), we will find measurement functions $\alpha_r$ defined on $L^n_{r+2[0]} \cap Q$ with rotational average equal to $M_r$, that is,

$$M_r = \int_{L^n_{r+2[0]} \cap Q \neq \emptyset} \alpha_r(L^n_{r+2[0]} \cap Q)dL^n_{r+2[0]} \quad (20)$$

Theorem 3.1. Let $Q \subset M^n_\lambda$ be a compact domain with smooth boundary $S = \partial Q$. The measurement functions $\alpha_r$ corresponding to the $r$-th integral
of mean curvature of $S$, $M_r$, can be expressed as

$$\alpha_r(L^n_{r+2[0]} \cap Q) = \frac{O_{r-2} \ldots O_0}{O_{n-2} \ldots O_{n-r-2}} \left[ (n-r-1)O_rO_{r-1} \int \chi((Q \cap L^n_{r+2[0]} \cap L^{r+2}_{r+1})s^{n-r-2}_\lambda(\rho)dL^{r+2}_{r+1} \right. \tag{21}$$

$$- \lambda rO_1O_0 \int \chi(((Q \cap L^n_{r+2[0]} \cap L^{r+2}_{r+1}) \cap L^{r+1}_{r+1}))s^{n-r}_\lambda(\rho)dL^{r+2}_{r+1} \right],$$

where, in both integrals, $\rho$ is the distance from $O$ to the planes $L^n_{r+1}$ and $L^n_{r-1}$, respectively; and

$$s_\lambda(\rho) = \begin{cases} 
\lambda^{-1/2} \sin(\rho \sqrt{\lambda}), & \lambda > 0 \\
\rho, & \lambda = 0 \\
|\lambda|^{-1/2} \sinh(\rho \sqrt{|\lambda|}), & \lambda < 0
\end{cases} \tag{22}$$

**Proof.** The idea of the proof consists in generating the planes $L^n_{r+1}$ and $L^n_{r-1}$, which appear in Eq.(17), by taking first an isotropic plane through $O$ and then an invariant plane within this isotropic plane, weighted by a function of $\rho$; that is, from Corollary 3.1 of [8] we have the identity

$$dL^n_{r+1} = s^{n-r-2}_\lambda(\rho)dL^{r+2}_{r+1}dL^n_{r+2[0]}, \tag{23}$$

and also

$$dL^n_{r-1}dL^n_{r+2[\alpha]} = s^{n-r}_\lambda(\rho)dL^r_{r-1}dL^n_{r+2[\alpha]}dL^n_{r[0]}, \tag{24}$$

where $dL^n_{r+2[\alpha]}$ denotes the density for $(r+2)$--planes about a about a $r$--plane $L^n_r$ (see page 202 of [16]).

As justified in [16], p. 309, before Eq. (17.55), from the expressions of the densities of planes in $M^n_\lambda$ it follows that some density decompositions (such as [16], Eq. (12.53)) have the same form whatever the sign of $\lambda$. Then, from Eq.(12.53) of [16], Eq.(24), can be expressed as

$$dL^n_{r-1}dL^n_{r+2[\alpha]} = s^{n-r}_\lambda(\rho)dL^r_{r-1}dL^{r+2}_{r+1}dL^n_{r+2[0]}, \tag{25}$$

Finally, substituting Eq.(23) and Eq.(25) in Eq.(17), having in mind that

$$\int dL^n_{r+2[\alpha]} = \frac{O_{n-r-1}O_{n-r-2}}{O_1O_0}, \tag{26}$$

we obtain the result. $\square$

**Remark.** For $\lambda = 0$, Eq.(21) coincides, up to a constant factor, with Eq.(18) of [10].

6
3.1. Morse representations for $M_r$

In this section a geometric interpretation is given of Eq.(21) in terms of the critical points of height functions. In particular, and in order to simplify, we will give a geometric interpretation of the function

$$\beta_r = \int \chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) s_{\lambda}^{n-r-1}(\rho) dL_r^{r+1}. \quad (27)$$

The density $dL_r^{r+1}$ may be decomposed as follows,

$$dL_r^{r+1} = c_{\lambda}(\rho) d\rho d(u_r), \quad (28)$$

where $d\rho$ denotes the surface area element of the $r-$dimensional unit sphere and $c_{\lambda}(\rho) = \frac{d}{d\rho}s_{\lambda}(\rho)$. Note that $\rho \geq 0$ for the case $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic); however, for the case $\lambda > 0$ (spherical) $\rho$ varies from 0 (which corresponds to the point $O$) to $\frac{\pi}{\sqrt{\lambda}}$ (which corresponds to the cut locus of $O$ (i.e., the antipodal point of $O$).

Therefore, for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic), we may write,

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{0}^{\infty} s_{\lambda}^{n-r-1}(\rho) c_{\lambda}(\rho) \chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) d\rho, \quad (29)$$

whereas, for the case $\lambda > 0$ (spherical),

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{\sqrt{\pi}^{-\lambda}}^{\sqrt{\pi}^{\lambda}} s_{\lambda}^{n-r-1}(\rho) c_{\lambda}(\rho) \chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) d\rho, \quad (30)$$

where $L_r^{r+1}$ is the $r-$plane expressed in terms of its distance $\rho$ from the fixed point $O$, perpendicular to the geodesic defined from the direction $u_r$ from $O$, and $\chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) = 0$ whenever $(Q \cap L_{r+1}^n) \cap L_r^{r+1} = \emptyset$.

Since we want to give a geometrical interpretation of $\beta_r$, based on critical points of height functions, from now on we will consider that $\rho$ means signed distance and we will rewrite $\beta_r$ as:

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{-\infty}^{\infty} s_{\lambda}^{n-r-1}(|\rho|) c_{\lambda}(\rho) \chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) d\rho, \quad \lambda \leq 0; \quad (31)$$

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{\sqrt{\pi}^{-\lambda}}^{\sqrt{\pi}^{\lambda}} s_{\lambda}^{n-r-1}(\rho) c_{\lambda}(\rho) \chi((Q \cap L_{r+1}^n) \cap L_r^{r+1}) d\rho \quad \lambda > 0. \quad (32)$$
Let \( u_r \) denote a unit vector in \( S^r \subset T_O L^n_{r+1[0]} \). The geodesic \( \gamma_{u_r} : \mathbb{R} \rightarrow L^n_{r+1[0]} \), with \( \gamma_{u_r}(0) = O \) and \( \gamma'(0) = u_r \) is given by \( \gamma_{u_r}(t) = c_\lambda(t)O + s_\lambda(t)u_r \), where \( c_\lambda(t) = \frac{d}{dt}s_\lambda(t) \). Given \( u_r \), let \( h_{u_r} : L^n_{r+1[0]} \rightarrow \mathbb{R} \) be the height function whose level hypersurfaces are just the \( r \)-planes \( L^r_{r+1} \) perpendicular to the geodesic \( \gamma_{u_r}(t) \). Note that in the Euclidean case \( (\lambda = 0) \) this height function coincides with the standard height function considered in [19]. We suppose that the level hypersurface \( L^r_{r+1} \) is oriented in such a way that the unit vector \( \nu(p) \), perpendicular to the level set \( L^r_{r+1} \subset L^n_{r+1[0]} \) at \( p \) is given by \( \nu(p) = \frac{\text{grad}(h_{u_r})(p)}{||\text{grad}(h_{u_r})(p)||} \).

Let us denote \( Q_{r+1} = Q \cap L^n_{r+1[0]} \), which is, in general, a domain with boundary in \( L^n_{r+1[0]} \) (see Appendix A of [10]). In Section 5 (Appendix) we show that in Euclidean and hyperbolic cases; and in the spherical case, if the domain \( Q \) is contained in the hemisphere of \( M^n_\lambda \) with pole \( O \), \( h_{u_r}|_{Q_{r+1}} \) is a strong Morse function for almost all \( u_r \in S^r \), it means that all of the critical points in the direction \( u_r \) from \( O \) are non-degenerate, and no two of them lie on the same level hypersurface (i.e. they have different critical values). In particular, \( h_{u_r}|_{Q_{r+1}} \) has not critical points in \( Q_{r+1} \). Let \( p_i \in \text{Crit}(h_{u_r}|_{\partial Q_{r+1}}), \ i = 1, \ldots, m, \) be the set of critical points, and

\[
\rho_1 < \rho_2 < \cdots < \rho_m, \quad (\text{with} \quad \frac{-\pi}{2\sqrt{\lambda}} \leq \rho_1, \quad \rho_m \leq \frac{\pi}{2\sqrt{\lambda}} \quad \text{for} \quad \lambda > 0)
\]

the corresponding critical values \((h_{u_r}(p_i) = \rho_i)\). To each critical point \( p_i \), we assign an index

\[
\epsilon_i = \chi(Q_{r+1} \cap L^r_{r+1}(\rho_i - \varepsilon)) - \chi(Q_{r+1} \cap L^r_{r+1}(\rho_i + \varepsilon)), \quad (33)
\]

where \( L^r_{r+1}(\rho_i + \varepsilon) \) denotes the \( r \)-plane defined from the direction \( u_r \) at a signed distance \( \rho_i + \varepsilon \) from \( O \); and \( \varepsilon \) is small enough to ensure that there are no critical points of \( \text{Crit}(h_{u_r}|_{\partial Q_{r+1}}) \) whose height function belongs to \((\rho_i - \varepsilon, \rho_i + \varepsilon)\).

For \( r < n \in \{1, 2, \ldots \} \), define:

\[
I_{n-r-1,r}(\rho) = \int s_{\lambda}^{n-r-1}(|\rho|) \ c_\lambda^r(\rho) \ d\rho = \begin{cases} 
\int s_{\lambda}^{n-r-1}(\rho) \ c_\lambda^r(\rho) \ d\rho, & \rho \geq 0, \\
(-1)^{n-r-1} \int s_{\lambda}^{n-r-1}(\rho) \ c_\lambda^r(\rho) \ d\rho, & \rho < 0.
\end{cases} \quad (34)
\]
Then, for \( \lambda = 0 \),

\[
I_{n-r-1,r}(\rho) = \int |\rho|^{n-r-1} \, d\rho = \begin{cases} 
\frac{\rho^{n-r}}{n-r}, & \rho \geq 0, \\
(-1)^{n-r-1} \frac{\rho^{n-r}}{n-r}, & \rho < 0.
\end{cases}
\] (35)

For \( \lambda \neq 0 \), and for any given pair \((n,r)\), the integral \( I_{n-r-1,r}(\rho) \) may be evaluated explicitly from [13], pages 114 and 159, or with the aid of a mathematical software package such as Mathematica®.

**Theorem 3.2.** Let \( O \) be a point in \( M^n_\lambda \) and \( Q \subset M^n_\lambda \) a compact domain which is contained in the hemisphere of \( M^n_\lambda \) with pole \( O \) when \( \lambda > 0 \). Let \( Q_{r+1} = Q \cap L^n_{r+1[0]} \) be the domain with boundary in \( L^n_{r+1[0]} \). Then, for \( r \in \{0, 1, \ldots, n-2\} \),

\[
\beta_r = \frac{1}{2} \int_{S^r} \left( \sum_{k=1}^m \epsilon_k I_{n-r-1,r}(\rho_k) \right) \, du_r,
\] (36)

where \( m \) represents the number of points \( \text{Crit}(h_{u_r}|_{\partial Q_{r+1}}) \) corresponding to the direction \( u_r \).

**Proof.** The fact that \( Q_{r+1} \) will be a domain with boundary in \( L^n_{r+1[0]} \), for a generic \((r+1)-\)space \( L^n_{r+1[0]} \), follows from Theorem A.1 of [10], and the fact that \( h_{u_r}|_{Q_{r+1}} \) will in general be a strong Morse function for almost all \( u_r \in S^r \) follows from the appendix, having in mind that \( Q_{r+1} \) is contained in the hemisphere of \( L^n_{r+1[0]} \) with pole \( O \).

Then Eq.(31) and Eq.(32) may be written as follows,

\[
\beta_r = \frac{1}{2} \int_{S^r} du_r \sum_{k=1}^{m-1} \int_{\rho_k}^{\rho_{k+1}} s_\lambda^{n-r-1}(|\rho|) \epsilon_k^{e^r}(\rho) \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_{r+1}) \, d\rho,
\] (37)

Thus,

\[
\beta_r = \frac{1}{2} \int_{S^r} du_r \sum_{k=1}^{m-1} \left( I_{n-r-1,r}(\rho_{k+1}) - I_{n-r-1,r}(\rho_k) \right) \sum_{j=k+1}^m \epsilon_j
\]

\[
= \frac{1}{2} \int_{S^r} \left( \sum_{k=2}^m \epsilon_k I_{n-r-1,r}(\rho_k) - I_{n-r-1,r}(\rho_1) \sum_{k=2}^m \epsilon_k \right) \, du_r.
\] (38)
Finally, since $\sum_{k=1}^{m} \epsilon_k = 0$, it means $\sum_{k=2}^{m} \epsilon_k = -\epsilon_1$, and the proposed result is obtained. □

4. Applications

Let $Q \subset M^3_\lambda (\lambda \neq 0)$ be a compact domain with smooth boundary $S = \partial Q$; then, from Theorem 2.1 with $n = 3$ and $r = 1$, we have

$$2\pi \chi(S) - \int_{S} K(x)dx = \frac{2\lambda}{\pi} \int_{\mathcal{L}} \chi(Q \cap L^3) dL^3_1,$$

where $K(x)$ is the Gauss curvature of $S$ at $x$, and $\chi$ denotes Euler characteristic.

Now, from Eq.(23) and the definition of $\beta_1$ (Eq.(27)), a rotational formula of the defect of the surface in $M^3(\lambda)$ is given by

$$2\pi \chi(S) - \int_{S} K(x)dx = \frac{2\lambda}{\pi} \int_{Q \cap L^3_2[0] \neq \emptyset} \beta_1(Q \cap L^3_2[0]) dL^3_2[0];$$

where, using Theorem 3.2,

$$\beta_1(Q \cap L^3_2[0]) = \frac{1}{2} \int_{S^2 \cap L^3_2[0]} \sum_{k=1}^{m} \epsilon_k I_{1,1}(\rho_k) du.$$

**Example.** Let $S$ be a geodesic sphere of radius $\rho$ centered at $O$ in $M^3(\lambda)$; then, $\chi(S) = 2$, and $\int_{M^2} K(x)dx = 4\pi\epsilon^2_\lambda(\rho)$.

On the other hand, $S \cap L^3_2[0]$ is a geodesic circle (boundary of a geodesic ball) in $L^3_2[0]$; that is, all the points in $S \cap L^3_2[0]$ are a distance $\rho$ apart from $O$. Then, for all directions $u \in S^1$, $m = 2$, $\epsilon_1 = 1$, $\epsilon_2 = -1$, $I_{1,1}(\rho_1) = I_{1,1}(\rho) = \frac{1}{2}s^2_\lambda(\rho)$ and $I_{1,1}(\rho_2) = I_{1,1}(-\rho) = -\frac{1}{2}s^2_\lambda(\rho)$, $\beta_1(S \cap L^3_2[0]) = \pi s^2_\lambda(\rho)$; and Eq.(40) is satisfied.

If we consider a domain $Q$ in $\mathbb{R}^3 (\lambda = 0)$, Corollary 2.2, with $r = 1$ and $n = 3$, coincides with Eq.(12) of [6], Theorem 2.1 coincides with Eq.(12) of [6], and, since

$$2\chi(Q_2 \cap L^2_1) = N(\partial Q_2 \cap L^2_1),$$

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where $N$ denotes number, Theorem 3.2 coincides with the integrand of Eq.(50) in [6]; but now, for each axial direction $u \in [0, 2\pi)$ in the pivotal plane $L^2_{2[0]}$, the pivotal section is scanned entirely from top to bottom by a sweeping straight line parallel to the axis $Ou$, in search of critical points.

5. Appendix

Let $X$ be a smooth manifold with boundary. We say that a smooth function $f : X \to \mathbb{R}$ is a strong Morse function if

1. all critical points of $f : X \to \mathbb{R}$ are non-degenerate and are contained in the interior of $X$,
2. all critical points of the restriction $f : \partial X \to \mathbb{R}$ are also non-degenerate,
3. if $x, y \in X$ are distinct critical points of either $f : X \to \mathbb{R}$ or $f : \partial X \to \mathbb{R}$, then $f(x) \neq f(y)$.

5.1. Preliminary results for the Euclidean case ($\lambda = 0$)

Assume now that $X \subseteq \mathbb{R}^n$ is a submanifold with boundary and for each unit vector $v \in S^{n-1}$, let us denote by $h_v : X \to \mathbb{R}$ the height function defined as $h_v(x) = \langle x, v \rangle$.

Theorem 5.1. Let $X \subseteq \mathbb{R}^n$ be a compact submanifold with boundary. For almost any $v \in S^{n-1}$, $h_v : X \to \mathbb{R}$ is a strong Morse function.

Proof. We consider $S = X$ or $S = \partial X$ which are compact spaces in $\mathbb{R}^n$. From Theorem 3 of [14], since $(1, p)$ is in the nice range for all $p = dim(S)$, the linear map $h_a : S \to \mathbb{R}$ given by $h_a(x) = \sum_i a_i x_i$ is stable for almost any $a \in \mathbb{R}^n \setminus \{0\}$.

Let $W \subseteq \mathbb{R}^n \setminus \{0\}$ be the set of points $a$ such that $h_a : S \to \mathbb{R}$ is not stable. Since $W$ is a null set in $\mathbb{R}^n \setminus \{0\}$, $p(W)$ is a null set in $S^{n-1}$, where $p : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the normalization map. Then, for any $v \in S^{n-1} \setminus p(W)$, $h_v : S \to \mathbb{R}$ is stable.

In the case of functions, it is well known that stability is equivalent to that all critical points are non-degenerate with distinct critical values (see [4]). Therefore $h_v : X \to \mathbb{R}$ and $h_v : \partial X \to \mathbb{R}$ are Morse functions with distinct critical values for almost any $v \in S^{n-1}$. Since $h_v : X \to \mathbb{R}$ has not critical points, critical values of $h_v : \partial X \to \mathbb{R}$ cannot coincide with critical values of $h_v : X \to \mathbb{R}$. Then, $h_v : X \to \mathbb{R}$ is a strong Morse function for almost any $v \in S^{n-1}$. □
Corollary 5.2. Let $Q \subset \mathbb{R}^n$ be a compact domain with boundary. For almost any $v \in S^{n-1}$, $h_v : Q \to \mathbb{R}$ is a strong Morse function.

5.2. General case $M^n_\lambda$ ($\lambda \neq 0$)

Lemma 5.3. Let $X \subset M^n_\lambda$ be a submanifold and let $\psi : I \to \mathbb{R}$ be a diffeomorphism, where $I$ is an open interval in $\mathbb{R}$. If $f : X \to I$ is a strong Morse function, then $g := \psi \circ f$ is a strong Morse function.

Proof. Since $\psi$ is a diffeomorphism and $f$ is a strong Morse function, it is deduced that $g$ is also a strong Morse function. Note that the critical points of $f$ coincide with the critical points of $g$. □

Let $Q \subset M^n_\lambda$ be a compact domain with boundary, $O \in M^n_\lambda$ and $v$ denote a unit vector in $S^{n-1} \subset T_O Q$. The geodesic $\gamma_v : I \subset \mathbb{R} \to Q$ is given by $\gamma_v = c_\lambda(t)O + s_\lambda(t)v$, where $I = [-\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}}]$ for $\lambda > 0$ and $I = \mathbb{R}$ for $\lambda < 0$.

Then, given $v$, let $h_v : Q \subset M^n_\lambda \to \mathbb{R}$ be the height function in $M^n_\lambda$, whose level hypersurfaces are perpendicular to the geodesic $\gamma_v$.

Theorem 5.4. Let $Q \subset M^n_\lambda$ be a compact domain with boundary which, for $\lambda > 0$, it is contained in the hemisphere of $M^n_\lambda$ with pole $O$. Then, for almost any $v \in S^{n-1}$, $h_v : Q \to \mathbb{R}$ is a strong Morse function.

Proof. It is useful to consider the embedding of the space form $M^n_\lambda$ into $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_\lambda)$ as follows:

\[
\begin{cases}
  x_0 = 1, & \lambda = 0, \\
  x_0^2 + x_1^2 + \ldots + x_n^2 = \frac{1}{\lambda}, & \lambda > 0, \\
  -x_0^2 + x_1^2 + \ldots + x_n^2 = \frac{1}{\lambda}, & x_0 > 0, \lambda < 0,
\end{cases}
\]

(43)

where $(x_0, x_1, \ldots, x_n)$ denote the coordinates of a point in $\mathbb{R}^{n+1}$, and $\langle \cdot, \cdot \rangle_\lambda$ is the appropriate metric to the embedding, which depends on the sign of $\lambda$.

Using this embedding, $Q \subset M^n_\lambda \subset \mathbb{R}^{n+1}$ can be considered as a compact submanifold with boundary in $\mathbb{R}^{n+1}$. Then, the height function of $\mathbb{R}^{n+1}$ with respect to the direction $v$, restricted to $Q$ is:
\[ h^{\mathbb{R}^{n+1}}_{v,\lambda} : \quad Q \rightarrow \mathbb{R} \]
\[ x \rightarrow \langle x, v \rangle_{\lambda} \quad (44) \]

From Theorem 5.1, \( h^{\mathbb{R}^{n+1}}_{v,\lambda} \) is a strong Morse function for almost any \( v \in \mathbb{S}^{n-1} \). Moreover, we note \( h^{\mathbb{R}^{n+1}}_{v,\lambda}(Q) \subset \overline{I} \).

Since \( \langle v, O \rangle_{\lambda} = 0 \), we have that,

\[ h^{\mathbb{R}^{n+1}}_{v,\lambda}(\gamma_v(\rho)) = \langle \gamma_v(\rho), v \rangle_{\lambda} = s_{\lambda}(\rho) = \begin{cases} \lambda^{-1/2} \sin(\rho \sqrt{\lambda}), & \lambda > 0, \\ |\lambda|^{-1/2} \sinh(\rho \sqrt{|\lambda|}), & \lambda < 0. \end{cases} \quad (45) \]

Eq.(45) gives a relation between the height function \( h_v(\gamma_v(\rho)) = \rho \) of \( Q \) in \( M^n_{\lambda} \) and the height function \( h^{\mathbb{R}^{n+1}}_{v,\lambda} \) of \( Q \) in \( \mathbb{R}^{n+1} \). That is,

\[ h_v(x) = \psi(h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)) = \begin{cases} \frac{1}{\sqrt{\lambda}} \arcsin(\sqrt{\lambda} h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)), & \lambda > 0, \\ \frac{1}{\sqrt{-\lambda}} \arcsinh(\sqrt{-\lambda} h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)), & \lambda < 0. \end{cases} \quad (46) \]

Finally, since \( Q \) is contained in the hemisphere of \( M^n_{\lambda} \) with pole \( O \) for \( \lambda > 0 \), we have that \( \psi \) is a diffeomorphism from \( I \) to \( \mathbb{R} \) when \( I = \left[ -\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}} \right] \) for \( \lambda > 0 \) and when \( I = \mathbb{R} \) for \( \lambda < 0 \); therefore from Lemma 5.3 we obtain the result. \( \square \)

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