

NEW ROTATIONAL INTEGRALS IN SPACE FORMS, WITH AN
APPLICATION TO SURFACE AREA ESTIMATION

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*Cordially dedicated to Professor Vratislav Horálek on his 90th birthday,
and to the memory of Professor Ivan Šaxl*

Abstract. A surface area estimator for three-dimensional convex sets, based on the invariator principle of local stereology, has recently motivated its generalization by means of new rotational Crofton-type formulae using Morse theory. We follow a different route to obtain related formulae which are more manageable and valid for submanifolds in constant curvature spaces. As an application, we obtain a simplified version of the mentioned surface area estimator for non-convex sets of smooth boundary.

Keywords: critical point; height function; submanifold in space forms; invariator principle; local stereology; rotational formulae; surface area estimation

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1. INTRODUCTION

In recent years, new rotational integral formulae have been developed ([1], [7], [8]). This was motivated by the invariator principle of stereology ([5], see also [14], p. 285). In these formulae, integration over all r -planes is replaced with invariant integration of a *measurement function* over all r -planes through a fixed point. In particular, in [5], a convenient method was proposed to estimate the surface area of a convex set from an isotropic plane section (called a pivotal section) through a fixed point in terms of the support function of the pivotal section. Recently, in [15], the pertinent formula ([5], equation (3.2)) has been generalized to non convex

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sets, and new rotational Crofton formulae have thereby been developed using Morse theory. Further, in [16], the mentioned new generalization has been adapted to estimate the surface area in three-dimensional microscopy. In the present paper, we follow a different route to obtain related generalizations which are more manageable and valid for submanifolds in space forms of constant curvature. In particular, we arrive at a surface area representation which constitutes a simplification of the analogous formula given in [15], [16]. The subsequent estimation procedure is thereby simplified—the basic task consists in identifying the local maxima and minima of the height function of the pivotal section.

2. A RELATION BETWEEN DENSITIES OF TOTALLY GEODESIC SUBMANIFOLDS IN SPACE FORMS

Let M_λ^n denote a simply connected Riemannian manifold of sectional curvature λ (namely, the sphere for $\lambda > 0$, the hyperbolic space for $\lambda < 0$ and the Euclidean space \mathbb{R}^n for $\lambda = 0$). Further, let L_r^n denote an r -plane ($r \leq n$), namely a totally geodesic submanifold of dimension r in M_λ^n , and let dL_r^n denote the corresponding density, invariant under the group of Euclidean and non-Euclidean motions.

Let M^q be a compact differentiable manifold of dimension q embedded in M_λ^n . Santaló's formula shows that the volume of M^q can be expressed in terms of the integral of the volume of the intersection of M^q with all the planes L_r^n , assuming that $r + q \geq n$ ([13], equation (14.69)). While the latter formula was obtained for the Euclidean case, in [13], p. 309 (Section 4 of Chapter 17), Santaló shows that the same formula extends to M_λ^n without any change.

An r -plane through a fixed point O in M_λ^n , and its invariant density, are denoted by $L_{r[0]}^n$ and $dL_{r[0]}^n$, respectively ([13], [12]). One of the first problems considered in integral geometry was to determine the densities dL_r^n and $dL_{r[0]}^n$. For lines in \mathbb{R}^2 the invariant density dL_1^2 was proposed by Crofton (see [4]), whereas the invariant densities for lines and planes in M_λ^3 were obtained by Cartan [3] in 1896. In 1935, Blaschke [2] introduced the density dL_r^n for \mathbb{R}^n . Following Blaschke's methodology, Petkantschin [11] and Santaló [13] obtained the corresponding densities for $\lambda > 0$ and $\lambda < 0$, respectively.

In this section, we generalize the expressions obtained in [7] for the density of r -planes in M_λ^n in terms of the density $dL_{p[0]}^n$ of p -planes through a fixed point O , of the density dL_r^p of r -planes in $L_{p[0]}^n$, and of the distance ϱ from O to L_r^p . For the Euclidean case, the result can be deduced from equation (49) in [11]. From the new expression, new rotational versions of Santaló's formula, and equivalences among them, are obtained in Section 3.

Henceforth, the following notation will be used:

$$(2.1) \quad s_\lambda(\varrho) = \begin{cases} \lambda^{-1/2} \sin(\varrho\sqrt{\lambda}), & \lambda > 0, \\ \varrho, & \lambda = 0, \\ |\lambda|^{-1/2} \sinh(\varrho\sqrt{|\lambda|}), & \lambda < 0, \end{cases} \quad c_\lambda(\varrho) = \begin{cases} \cos(\varrho\sqrt{\lambda}), & \lambda > 0, \\ 1, & \lambda = 0, \\ \cosh(\varrho\sqrt{|\lambda|}), & \lambda < 0. \end{cases}$$

The density dL_r^n of r -planes in M_λ^n can be expressed in terms of the distance ϱ from a fixed point O to L_r^n , of the density $dL_{r+1[0]}^n$ of an $(r+1)$ -plane through O containing L_r^n , and of the density dL_r^{r+1} for r -planes contained in $L_{r+1[0]}^n$, as follows ([7], Corollary 3.1):

$$(2.2) \quad dL_r^n = s_\lambda^{n-r-1}(\varrho) dL_r^{r+1} dL_{r+1[0]}^n.$$

Theorem 2.1. For $r \in \{0, 1, \dots, n-2\}$ and $p \in \{r+1, r+2, \dots, n\}$, the following relation between densities is satisfied:

$$(2.3) \quad dL_{p[r+1]}^n dL_r^n = s_\lambda^{n-p}(\varrho) dL_r^p dL_{p[0]}^n,$$

where $dL_{p[r+1]}^n$ denotes the density for p -planes in M_λ^n that contain the fixed $(r+1)$ -plane $L_{r+1[0]}^n$.

Proof. As justified in [13], p. 309, before equation (17.55), from the expressions of the densities of planes in M_λ^n it follows that some density decompositions (such as [13], equation (12.53)) have the same form regardless of the sign of λ . Therefore, multiplying both sides of equation (2.2) by $dL_{p[r+1]}^n$, and bearing equation (12.53) of [13] in mind, we obtain

$$(2.4) \quad dL_{p[r+1]}^n dL_r^n = s_\lambda^{n-p}(\varrho) s_\lambda^{p-r-1}(\varrho) dL_r^{r+1} dL_{r+1[0]}^p dL_{p[0]}^n.$$

Applying again (2.2), we get the result. □

Note that (2.2) is a special case of (2.3) for $p = r+1$.

Corollary 2.2. The density dL_r^n satisfies the following identity:

$$(2.5) \quad dL_r^n = s_\lambda^{n-p}(\varrho) dL_r^p dL_{p[p-r-1]}^n,$$

where $dL_{p[p-r-1]}^n$ denotes the density for p -planes in M_λ^n that contain a fixed plane of dimension $p-r-1$ through O .

Proof. As justified in [13], p. 309, the densities given in (12.26) and (12.27) of [13] also hold in M_λ^n . Therefore,

$$(2.6) \quad dL_{p[0]}^n = dL_{p[r+1]}^n dL_{p[p-r-1]}^n,$$

and the result follows from (2.3). □

3. ROTATIONAL VERSIONS OF SANTALÓ'S FORMULA

Let M^q be a compact differentiable manifold of dimension q embedded in M_λ^n . Assume that $r + q \geq n$ and consider the set of r -planes in M_λ^n . Santaló's formula (equation (14.69) of [13], which is valid in M_λ^n), states that

$$(3.1) \quad \int_{M^q \cap L_r^n \neq \emptyset} \sigma_{r+q-n}(M^q \cap L_r^n) dL_r^n = \frac{O_n \cdots O_{n-r} O_{r+q-n}}{O_r \cdots O_0 O_q} \sigma_q(M^q),$$

where σ_k denotes k -dimensional volume, and $O_k = 2\pi^{(k+1)/2}/\Gamma((k+1)/2)$ is the surface area of the k -dimensional unit sphere.

In this paper, we focus on the special case $r + q = n$. Then equation (14.70) of [13], which is valid in M_λ^n , gives

$$(3.2) \quad \int_{M^{n-r} \cap L_r^n \neq \emptyset} N(M^{n-r} \cap L_r^n) dL_r^n = \frac{O_n \cdots O_{n-r+1}}{O_r \cdots O_1} \sigma_{n-r}(M^{n-r}),$$

where N denotes number, so that $N(M^{n-r} \cap L_r^n)$ denotes the number of points of the intersection $M^{n-r} \cap L_r^n$.

Next we apply Theorem 2.1 to obtain new rotational formulae from (3.2).

Proposition 3.1. *Let $r \in \{0, 1, \dots, n-2\}$ and $p \geq r+1$. Also, let $M^{n-r} \subset M_\lambda^n$ be a compact submanifold of class C^{p-r+1} . Then*

$$(3.3) \quad \begin{aligned} & \sigma_{n-r}(M^{n-r}) \\ &= \frac{1}{c_{n,r,p}} \int_{M^{n-r} \cap L_{p[0]}^n \neq \emptyset} \left[\int_{M_p^{n-r} \cap L_r^p \neq \emptyset} s_\lambda^{n-p}(\varrho) N(M_p^{n-r} \cap L_r^p) dL_r^p \right] dL_{p[0]}^n, \end{aligned}$$

where $M_p^{n-r} = M^{n-r} \cap L_{p[0]}^n$, and

$$(3.4) \quad c_{n,r,p} = \frac{O_n \cdots O_{n-r+1} O_{n-r-2} \cdots O_{n-p}}{O_r \cdots O_1 O_{p-r-2} \cdots O_1 O_0}.$$

Thus, first we fix $L_{p[0]}^n$ and integrate with respect to dL_r^p over all planes in L_r^p which intersect M_p^{n-r} , and then we integrate with respect to $dL_{p[0]}^n$ over all planes $L_{p[0]}^n$ which intersect M^{n-r} . Further, M^{n-r} is assumed to be of class C^k with $k = \max\{1, \dim(M_p^{n-r}) + p - n + 1\} = p - r + 1$. By virtue of Appendix A from [8], this choice ensures that $M_p^{n-r} \cap L_{p[0]}^n$, where $L_{p[0]}^n$ is a generic p -plane, is also of class C^k (and dimension $p - r$). The need to specify the order k of smoothness is explained in [10]. In Theorem 4.1, one has $p = r + 1$, whereby M^{n-r} is assumed to be of class C^2 .

Proof. As explained above, the submanifold M_p^{n-r} is of class C^{p-r+1} and dimension $p-r$, whereby $M_p^{n-r} \cap L_r^p$ is the set of intersection points. Now, bearing in mind equation (12.36) of [13], namely,

$$(3.5) \quad \int_{\text{Total}} dL_{p[r+1]}^n = \frac{O_{n-r-2} \cdots O_{n-p}}{O_{p-r-2} \cdots O_1 O_0},$$

and making use of equation (2.3) in equation (3.2), the result follows. \square

The next corollary shows that for the special case $p = r + 1$, equation (3.3) yields equation (15) from [7] with $q = n - r$.

Corollary 3.2. *Within each p -plane $L_{p[0]}^n$, consider $(r + 1)$ -planes $L_{r+1[0]}^p$. Then $\sigma_{n-r}(M^{n-r})$ admits a rotational expression which may be obtained directly by substituting $p = r + 1$ on the right-hand side of equation (3.3), namely,*

$$(3.6) \quad \sigma_{n-r}(M^{n-r}) = \frac{1}{c_{n,r}} \int_{M^{n-r} \cap L_{r+1[0]}^n \neq \emptyset} \alpha(M^{n-r} \cap L_{r+1[0]}^n) dL_{r+1[0]}^n,$$

where

$$(3.7) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathcal{C} \cap L_r^{r+1} \neq \emptyset} s_\lambda^{n-r-1}(\varrho) N(\mathcal{C} \cap L_r^{r+1}) dL_r^{r+1},$$

with

$$(3.8) \quad c_{n,r} = \frac{O_n \cdots O_{n-r+1}}{O_r \cdots O_1},$$

where \mathcal{C} represents the curve $M_{r+1}^{n-r} = M^{n-r} \cap L_{r+1[0]}^n$ (of class C^2) in $L_{r+1[0]}^n$.

Proof. For $n = p$, equation (2.2) yields

$$(3.9) \quad dL_r^p = s_\lambda^{p-r-1}(\varrho) dL_r^{r+1} dL_{r+1[0]}^p.$$

Therefore, equation (3.3) can be written as

$$(3.10) \quad \sigma_{n-r}(M^{n-r}) = \frac{1}{c_{n,r,p}} \int_{M^{n-r} \cap L_{p[0]}^n \neq \emptyset} \alpha(M^{n-r} \cap L_{p[0]}^n) dL_{p[0]}^n,$$

with

$$(3.11) \quad \alpha(M^{n-r} \cap L_{p[0]}^n) = \int_{M_p^{n-r} \cap L_{r+1[0]}^p \neq \emptyset} \left[\int_{M_{r+1}^{p-r} \cap L_r^{r+1} \neq \emptyset} s_\lambda^{n-r-1}(\varrho) N(\mathcal{C} \cap L_r^{r+1}) dL_r^{r+1} \right] dL_{r+1[0]}^p,$$

where, by virtue of Theorem A.1 from [8], $M_{r+1}^{p-r} = M_p^{n-r} \cap L_{r+1[0]}^p$ is a curve of class C^2 in $L_{r+1[0]}^p$. Now, making use in equation (3.10) of the well-known identity [13]

$$(3.12) \quad dL_{r+1[0]}^p dL_{p[0]}^n = dL_{p[r+1]}^n dL_{r+1[0]}^n,$$

and bearing equation (3.5) in mind, the rotational formula (3.6) is obtained. \square

4. MORSE TYPE REPRESENTATION AND GEOMETRICAL INTERPRETATIONS

In this section, a geometric interpretation of equation (3.7) is given in terms of the critical points of height functions. Of particular interest will be the special case $r = 1$, whereby M^{n-1} is a hypersurface, and the rotational formula is obtained by intersecting M^{n-1} with a 2-plane $L_{2[0]}^n$.

The density dL_r^{r+1} may be decomposed as (see [13])

$$(4.1) \quad dL_r^{r+1} = c_\lambda^r(\varrho) d\varrho du_r,$$

where du_r denotes the surface area element of the r -dimensional unit sphere and $c_\lambda^r(\varrho) = (c_\lambda(\varrho))^r$, where $c_\lambda(\varrho)$ is defined in equation (2.1). Then, for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic), we may write

$$(4.2) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} du_r \int_0^\infty s_\lambda^{n-r-1}(\varrho) c_\lambda^r(\varrho) N(\mathcal{C} \cap L_r^{r+1}) d\varrho,$$

whereas, for the case $\lambda > 0$ (spherical),

$$(4.3) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} du_r \int_0^{\pi/(2\sqrt{\lambda})} s_\lambda^{n-r-1}(\varrho) c_\lambda^r(\varrho) N(\mathcal{C} \cap L_r^{r+1}) d\varrho,$$

where L_r^{r+1} is the r -plane expressed in terms of its distance ϱ from the fixed point O , perpendicular to the geodesic defined from the direction u_r from O , and $N(\mathcal{C} \cap L_r^{r+1}) = 0$ whenever $\mathcal{C} \cap L_r^{r+1} = \emptyset$.

Next we give a geometrical interpretation of $\alpha(M^{n-r} \cap L_{r+1[0]}^n)$, based on the critical points of height functions.

Let u_r denote a unit vector in $\mathbb{S}^r \subset T_O L_{r+1[0]}^n$. The geodesic $\gamma_{u_r}: \mathbb{R} \rightarrow L_{r+1[0]}^n$ with $\gamma_{u_r}(0) = O$ and $\gamma'(0) = u_r$ is given by $\gamma_{u_r}(t) = c_\lambda(t)O + s_\lambda(t)u_r$. Given u_r , let $h_{u_r}: L_{r+1[0]}^n \rightarrow \mathbb{R}$ be the height function whose level hypersurfaces are just the r -planes L_r^{r+1} perpendicular to the geodesic $\gamma_{u_r}(t)$. Note that in the Euclidean case ($\lambda = 0$), this height function coincides with the standard height function considered in [15]. We suppose that the level hypersurface L_r^{r+1} is oriented in such a way that

the unit vector $\nu(p)$, perpendicular to the level set $L_r^{r+1} \subset L_{r+1[0]}^n$ at p , is given by $\nu(p) = \text{grad}(h_{u_r})(p) / \|\text{grad}(h_{u_r})(p)\|$, where $\text{grad}(h_{u_r})$ denotes the gradient of h_{u_r} .

We consider the height function $h_{u_r}|_{\mathcal{C}}: \mathcal{C} \rightarrow \mathbb{R}$, which is generally a Morse function, and apply the Morse theory to $h_{u_r}|_{\mathcal{C}}$ (cf. [10]).

Let $p \in \mathcal{C}$ be a critical point of $h_{u_r}|_{\mathcal{C}}$ along \mathcal{C} . Then there is a level hypersurface L_r^{r+1} of h_{u_r} which satisfies $T_p\mathcal{C} \subset T_pL_r^{r+1}$.

We assume that $h_{u_r}|_{\mathcal{C}}$ is an excellent Morse function for almost all $u_r \in \mathbb{S}^r$, which means that all of the critical points in the direction u_r from O are non-degenerate, and no two of them lie on the same level hypersurface (i.e. they have different critical values).

Note that the critical points of $h_{u_r}|_{\mathcal{C}}$ and $h_{(-u_r)}|_{\mathcal{C}}$ coincide, whereas the corresponding critical values are the same but with opposite signs. Because the integrals in (4.2) and (4.3) are defined for $\varrho \geq 0$, only nonnegative critical values are involved in the relevant integrals.

Let $p_i \in \mathcal{C}$, $i = 1, \dots, m$, be the critical points of the excellent Morse function $h_{u_r}|_{\mathcal{C}}$, and

$$0 \leq \varrho_1 < \varrho_2 < \dots < \varrho_m \quad (\text{with } \varrho_m \leq \frac{1}{2}\pi/\sqrt{\lambda} \text{ for } \lambda > 0)$$

the corresponding nonnegative critical values, where $h_{u_r}|_{\mathcal{C}}(p_i) = \varrho_i$. To each critical point p_i we assign an index $\varepsilon_i = +1$ if p_i is a local maximum of $h_{u_r}|_{\mathcal{C}}$, and an index $\varepsilon_i = -1$ if p_i is a local minimum of $h_{u_r}|_{\mathcal{C}}$. For $r < n \in \{1, 2, \dots\}$ define

$$(4.4) \quad I_{n-r-1,r}(\varrho) = \int s_\lambda^{n-r-1}(\varrho) c_\lambda^r(\varrho) d\varrho.$$

Then for $\lambda = 0$,

$$(4.5) \quad I_{n-r-1,r}(\varrho) = \frac{\varrho^{n-r}}{n-r}.$$

For $\lambda \neq 0$ and for any given pair (n, r) , the integral $I_{n-r-1,r}(\varrho)$ may be evaluated explicitly with the aid of a mathematical software package such as Mathematica[®] or MapleTM. In all cases, we set $I_{n-r-1,r}(0) = 0$.

Theorem 4.1. *Let $M^{n-r} \subset M_\lambda^n$ be a submanifold of class C^2 . Then the intersection $M^{n-r} \cap L_{r+1[0]}^n$ is in general a curve \mathcal{C} of class C^2 in $L_{r+1[0]}^n$. Further, assume that $h_{u_r}|_{\mathcal{C}}$ is an excellent Morse function for almost all $u_r \in \mathbb{S}^r$. Then for $r \in \{0, 1, \dots, n-2\}$ and $\lambda \leq 0$,*

$$(4.6) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = 2 \int_{\mathbb{S}^r} \left(\sum_{k=1}^m \varepsilon_k I_{n-r-1,r}(\varrho_k) \right) du_r,$$

where m represents the number of critical points of the height function $h_{u_r}|_{\mathcal{C}}$ corresponding to the direction u_r ; and for $\lambda > 0$,

$$(4.7) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = 2 \int_{\mathbb{S}^r} \left(\sum_{k=1}^m \varepsilon_k I_{n-r-1,r}(\varrho_k) + I_{n-r-1,r} \left(\frac{\pi}{2\sqrt{\lambda}} \right) N(\mathcal{C} \cap L_r^{r+1}) \right) du_r,$$

where L_r^{r+1} denotes the r -plane perpendicular to the geodesic with direction u_r from O at a distance $\varrho = \frac{1}{2}\pi/\sqrt{\lambda}$.

Proof. The fact that \mathcal{C} is a curve of class C^2 in $L_{r+1[0]}^n$ for a generic $(r+1)$ -subspace $L_{r+1[0]}^n$ follows from Theorem A.1 of [8]. Consider first the case $\lambda \leq 0$. Then (4.2) may be written as

$$(4.8) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} du_r \sum_{k=0}^{m-1} \int_{\varrho_k}^{\varrho_{k+1}} s_\lambda^{n-r-1}(\varrho) c_\lambda^r(\varrho) N(\mathcal{C} \cap L_r^{r+1}) d\varrho,$$

with the convention $\varrho_0 = 0$. Thus,

$$(4.9) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} du_r \sum_{k=0}^{m-1} 2(I_{n-r-1,r}(\varrho_{k+1}) - I_{n-r-1,r}(\varrho_k)) \sum_{j=k+1}^m \varepsilon_j = 2 \int_{\mathbb{S}^r} \left(\sum_{k=1}^m \varepsilon_k I_{n-r-1,r}(\varrho_k) - I_{n-r-1,r}(0) \sum_{k=1}^m \varepsilon_k \right) du_r.$$

As we have set $I_{n-r-1,r}(0) = 0$, the second term of the preceding integrand vanishes, and the proposed result is obtained.

For the case $\lambda > 0$, the proof is similar. Thus,

$$(4.10) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} du_r \sum_{k=0}^{m-1} (I_{n-r-1,r}(\varrho_{k+1}) - I_{n-r-1,r}(\varrho_k)) \left(2 \sum_{j=k+1}^m \varepsilon_j + N(\mathcal{C} \cap L_r^{r+1}) \right) + (I_{n-r-1,r}(\pi/\sqrt{\lambda}) - I_{n-r-1,r}(\varrho_m)) N(\mathcal{C} \cap L_r^{r+1}).$$

Therefore,

$$(4.11) \quad \alpha(M^{n-r} \cap L_{r+1[0]}^n) = \int_{\mathbb{S}^r} \left(2 \sum_{k=1}^m \varepsilon_k I_{n-r-1,r}(\varrho_k) + I_{n-r-1,r} \left(\frac{\pi}{2\sqrt{\lambda}} \right) N(\mathcal{C} \cap L_r^{r+1}) - I_{n-r-1,r}(0) \left(2 \sum_{k=1}^m \varepsilon_k + N(\mathcal{C} \cap L_r^{r+1}) \right) \right) du_r,$$

and because $I_{n-r-1,r}(0) = 0$, the result follows. \square

Remark 4.2. Equations (4.6) and (4.7) are pertinent to the representation of surface area (see the left-hand side of equation (3.6)). Hence, the integrands on the right-hand sides of equations (4.6) and (4.7) depend on the scanning direction u_r in general. Note that the Euler characteristic is of no concern in this paper—we use Morse type concepts, but our route is different.

In order to see the relation between a critical point p and the curvature, we consider a normal coordinate system of $L_{r+1[0]}^n$ at p . Let $\{U; x_1, \dots, x_{r+1}\}$ be a local chart described by the normal coordinate system centered at p ; that is, there exist $U \subset \mathbb{R}^{r+1}$, $V \subset L_{r+1[0]}^n$, and a map $\varphi: U \rightarrow V$ which define the local chart.

Let $\alpha: I \subset \mathbb{R} \rightarrow L_{r+1[0]}^n$ be a parameterization of \mathcal{C} with $\alpha(0) = p$, and set $\alpha := \beta \circ \varphi$, with $\beta(0) = q \in \mathbb{R}^{r+1}$ and $\varphi(q) = p$.

Theorem 4.3. *Given a point $p \in \mathcal{C}$ and a normal chart $\{U; x_i\}$ at p , we have:*

- (1) p is a critical point of $h_{u_r}|_{\mathcal{C}}$ if and only if $\alpha'(0) \perp \nu(p)$, where $\nu(p)$ is the unit vector perpendicular to the level set $L_r^{r+1} \subset L_{r+1[0]}^n$ at p .
- (2) A critical point p of $h_{u_r}|_{\mathcal{C}}$ is nondegenerate if and only if $g(\nu(p), D^2\alpha(0)) \neq 0$, where D is the operator of covariant differentiation with respect to $\alpha'(t)$. Moreover, p is a local maximum (minimum) if $g(\nu(p), D^2\alpha(0)) < 0$ (> 0).

Proof. The point p is a critical point of $h_{u_r}|_{\mathcal{C}}$ if and only if $\left. \frac{d}{dt} \right|_{t=0} (h_{u_r} \circ \varphi \circ \beta) = 0$. Using local coordinates yields

$$(4.12) \quad \left. \frac{d}{dt} \right|_{t=0} (h_{u_r} \circ \varphi \circ \beta) = \sum_{i=1}^{r+1} \left. \frac{\partial (h_{u_r} \circ \varphi)}{\partial x_i} \right|_q x'_i(0) = g(\text{grad}(h_{u_r})(p), \alpha'(0)).$$

Then p is a critical point of $h_{u_r}|_{\mathcal{C}}$ if and only if $\text{grad}(h_{u_r})(p) \perp \alpha'(0)$. Finally, since $\text{grad}(h_{u_r})(p) / \|\text{grad}(h_{u_r})(p)\| = \nu(p)$, the first part of the theorem follows.

To prove the second part we need to determine when $\left. \frac{d^2}{dt^2} \right|_{t=0} (h_{u_r} \circ \varphi \circ \beta)$ vanishes. Using local coordinates yields

$$(4.13) \quad \left. \frac{d^2}{dt^2} \right|_{t=0} (h_{u_r} \circ \varphi \circ \beta) = \sum_{i=1}^{r+1} \left. \frac{\partial^2 (h_{u_r} \circ \varphi)}{\partial x_i \partial x_j} \right|_q x'_i(0) x'_j(0) + \sum_{i=1}^{r+1} \left. \frac{\partial (h_{u_r} \circ \varphi)}{\partial x_i} \right|_q x''_i(0).$$

The first term on the right-hand side of equation (4.13) is

$$(4.14) \quad \begin{aligned} \text{hess}(h_{u_r})(\alpha'(0), \alpha'(0)) &= g(\nabla_{\alpha'(0)} \text{grad}(h_{u_r}), \alpha'(0)) \\ &= -|\varepsilon(p)| II_{L_r}(\alpha'(0)) = 0, \end{aligned}$$

where hess denotes the hessian and II_{L_r} is the second fundamental form of L_r at p , which is zero, because L_r is totally geodesic. As we have considered a normal coordinate system of $L_{r+1[0]}^n$ at p , the Christoffel symbols vanish; thus,

$$(4.15) \quad D^2\alpha(0) = \sum_{k=1}^{r+1} x_k''(0) \frac{\partial\varphi}{\partial x_k} \Big|_q.$$

Further, the coefficients of the First Fundamental Form in normal coordinates are $E(p) = G(p) = 1$, $F(p) = 0$, whereby

$$(4.16) \quad \begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} (h_{u_r} \circ \varphi \circ \beta) &= \sum_{i=1}^{r+1} \frac{\partial(h_{u_r} \circ \varphi)}{\partial x_i} \Big|_q x_i''(0) \\ &= g(\text{grad}(h_{u_r})(p), D^2\alpha(0)) = \|\text{grad}(h_{u_r})(p)\|g(\nu(p), D^2\alpha(0)), \end{aligned}$$

and the second part of the theorem follows. \square

Remark 4.4. If the curve α is parameterized by arc length, then from [9] we have that

$$(4.17) \quad g(\nu(p), D^2\alpha(0)) = \kappa_1 g(\nu(p), D^2\alpha(0)/\|D^2\alpha(0)\|),$$

where $\kappa_1 > 0$ denotes the first curvature of α .

Example 4.5. Hypersurfaces in space forms.

Let M^{n-1} be a hypersurface in M_λ^n . Theorem 4.1 yields

$$(4.18) \quad \sigma_{n-1}(M^{n-1}) = \frac{2O_1}{O_n} \int_{M^{n-1} \cap L_{2[0]}^n \neq \emptyset} \alpha(M^{n-1} \cap L_{2[0]}^n) dL_{2[0]}^n,$$

where

$$(4.19) \quad \alpha(M^{n-1} \cap L_{2[0]}^n) = \begin{cases} \int_{\mathbb{S}^2 \cap L_{2[0]}^n} \sum_{k=1}^m \varepsilon_k \frac{s_\lambda^{n-1}(\varrho_k)}{n-1} du, & \lambda < 0, \\ \int_{\mathbb{S}^2 \cap L_{2[0]}^n} \left(\sum_{k=1}^m \varepsilon_k \frac{s_\lambda^{n-1}(\varrho_k)}{n-1} + \frac{N(\mathcal{C} \cap L_r^{r+1})}{n-1} \right) du, & \lambda > 0. \end{cases}$$

From Theorem 4.3 it follows that p is a critical point of $h_{u_r}|_{\mathcal{C}}$ if and only if $\alpha'(0)$ is perpendicular to the geodesic line L_1^2 . Moreover, if α is parameterized by arc length, then

$$(4.20) \quad g(\nu(p), D^2\alpha(0)) = \kappa_g g(\nu(p), N(0)),$$

where $N(0)$ is the unit vector normal to $\alpha'(0)$ so that the orientation of $\{\alpha'(s), N(s)\}$ agrees with the chosen orientation of $T_p L_{2[0]}^n$. Moreover, the sign of the geodesic curvature κ_g of the curve at a point p is adopted to be positive or negative according to whether the geodesic curvature vector lies to the right or the left of the curve within the geodesic surface $L_{2[0]}^n$. For $\lambda = 0$ the geodesic curvature coincides with the signed curvature κ of the plane curve.

5. APPLICATION: EXTENSION OF THE INVARIATOR METHOD OF STEREOLOGY TO ESTIMATE SURFACE AREA

From equation (4.18) we obtain a simplified version of the Morse-type surface area estimator presented in equation (7) from [16]. We simply associate signed indexes to the critical points of the height function, without resorting to the concept of Euler characteristic. Let Y be a compact set with smooth boundary ∂Y in \mathbb{R}^3 . From equation (4.18) we have that the surface area of ∂Y can be expressed as follows:

$$(5.1) \quad S(\partial Y) = \frac{1}{\pi} \int_{\mathbb{S}_+^2} \alpha(\partial Y \cap L_{2[0]}^3) dL_{2[0]}^3,$$

where, by virtue of equation (4.19),

$$(5.2) \quad \alpha(\partial Y \cap L_{2[0]}^3) = \int_0^\pi du \sum_{k=1}^m \varepsilon_k \varrho_k^2.$$

For each axial direction $u \in [0, \pi)$ in the pivotal plane $L_{2[0]}^3$, the pivotal section is scanned entirely from top to bottom by a sweeping straight line parallel to the axis Ou , in search of critical points. Above the axis Ou , the value of the index ε_k is $+1$, or -1 , according to whether the k th critical point is a local maximum, or a local minimum. Below the axis Ou , it is convenient to imagine the pivotal section rotated by an angle of 180° , and then use the same criterion. As a consequence, the factor $1/(n-1)$ (equal to $1/2$ in this case) on the right-hand side of equation (4.19) does not appear in (5.2). Because the integral of $dL_{2[0]}^3$ over the unit hemisphere \mathbb{S}_+^2 is equal to 2π , the combination of the preceding two equations suggests the unbiased estimator of $S(\partial Y)$ from a single pivotal section, and from a single sweeping direction

$$(5.3) \quad \widehat{S}(\partial Y) = 2\pi \sum_{k=1}^m \varepsilon_k \varrho_k^2,$$

which constitutes the aforementioned modification of equation (7) from [16]. The surface area estimator given by equation (3.2) from [5] corresponds to the case in

which Y is a convex set, whereby $m = 2$. The latter estimator incorporated two mutually perpendicular sweeping directions. In this case, the summation on the right-hand side of equation (5.3) would be replaced with the mean of two summations. In [6], it was shown that two mutually perpendicular sweeping directions yield an accurate surface area estimation for ellipsoidal particles.

Example 5.1. Fig. 1(a) represents a pivotal section through a smooth particle Y , namely a section produced by an isotropically oriented plane $L_{2[0]}^3$ through a fixed pivotal point O previously identifiable in the particle (e.g. a nucleolus of a neuron). The axis Ou has been conveniently oriented as horizontal, but it is supposed to be isotropically oriented about O .

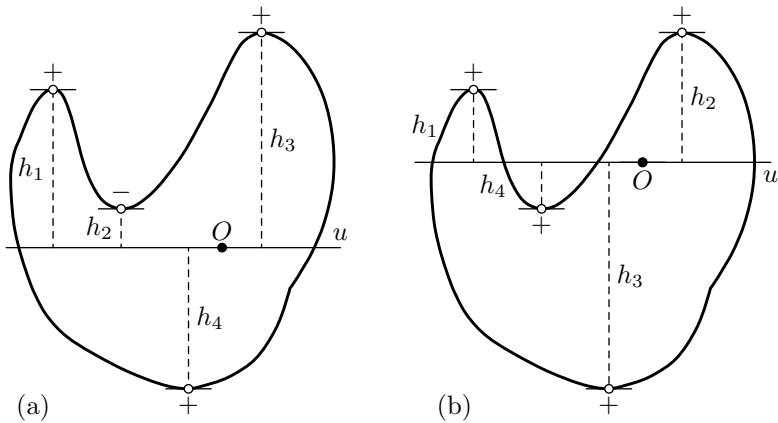


Figure 1. Illustration of the surface area estimator given by equation (5.3). See text, Example 5.1.

In Fig. 1(b), the section is the same, but for the sake of illustration, the pivotal point has a different location relative to the particle. A sweeping line moving parallelly to Ou from top to bottom determines four critical points in each case. In Fig. 1(a), the second critical point is a local minimum, whereby $\varepsilon_2 = -1$. The remaining three critical points are local maxima, hence $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = +1$. Thus, in this case,

$$(5.4) \quad \widehat{S}(\partial X) = 2\pi(h_1^2 - h_2^2 + h_3^2 + h_4^2).$$

The distances to each critical point from the axis Ou have been denoted by $\{h_k\}$ instead of the $\{\varrho_k\}$ used in the rest of the paper, in order to match the notation with that adopted in [5], which evokes ‘height’ measure. In Fig. 1 (b), however, the four critical points are all local maxima, whereby, in this case,

$$(5.5) \quad \widehat{S}(\partial X) = 2\pi(h_1^2 + h_2^2 + h_3^2 + h_4^2).$$

As suggested above, to identify the local maxima and minima below the axis Ou , it is convenient to imagine the section as if turned upside down.

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