

p -divisibility of conjugacy class sizes and normal p -complements

Antonio Beltrán, María José Felipe and Changguo Shao

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Abstract. Let N be a normal subgroup of a group G and let p be a prime. We prove that if the p -part of $|x^G|$ is a constant for every prime-power order element $x \in N \setminus Z(N)$, then N is solvable and has normal p -complement.

1 Introduction

Let G be a finite group. There are numerous analogies between results about irreducible character degrees and results concerning conjugacy class sizes of G , in spite of which the techniques employed in the two subjects usually differ completely. A renowned theorem of Thompson states that if a prime p divides every non-linear irreducible character degree of G , then G possesses a normal p -complement. It is well known, however, that the corresponding result for class sizes fails. If p divides any element of $\text{cs}(G)$ (the set of class sizes of G) distinct from 1, then G need not be solvable or p -solvable, and even when G is p -solvable, it may have arbitrary p -length (see [6]). For instance, $\text{GL}(2, q)$, for any prime-power $q \geq 5$, is not 2-solvable while all the class sizes are divisible by 2. The same happens for odd primes, say, with $\text{GL}(3, 7)$ and the prime 3.

From these facts, we wonder which conditions on the p -part of the numbers in $\text{cs}(G)$ may be sufficient to guarantee the existence, or even the normality, of the p -complements of G . In [5], C. Casolo, S. Dolfi and E. Jabara proved that if all elements of $\text{cs}(G)$ distinct from 1 have the same p -part, then G is solvable and has normal p -complement. In this note, we prove an independent extension of this result in two different directions. On the one hand, recent results have shown that certain structural properties of the whole group can be obtained just from the class sizes of the prime-power order elements, and on the other hand, certain normal

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structure information can be inferred from the G -class sizes, that is, those classes in G of elements lying in normal subgroups of G . This extension is the following.

Theorem A. *Let G be a finite group and let N be a normal subgroup of G . Let p be a fixed prime and let $a > 0$ be an integer. If $|x^G|_p = p^a$ for every prime-power element $x \in N \setminus Z(N)$, then N is solvable and has a normal p -complement.*

Note 1:
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Let us denote by $cs_G(N)$ the set of G -class sizes of the elements in $N \trianglelefteq G$. In particular, we get the following consequence in the context of normal subgroups and G -class sizes.

Corollary B. *Let G be a finite group, let N be a normal subgroup of G , and let p be a fixed prime. If $cs_G(N) = \{1, p^a m_1, \dots, p^a m_t\}$, where m_1, \dots, m_t are p' -numbers, then N is solvable and has a normal p -complement.*

It seems that the class sizes of prime-power order elements still exert a strong influence on the structure of groups. The following result extends [5, Theorem A] for just prime-power order elements.

Corollary C. *Let G be a finite group and let p be a fixed prime. If $|x^G|_p = p^a$, with $a > 0$, for every prime-power element $x \in G \setminus Z(G)$, then G is solvable and has a normal p -complement.*

An example will be given to show that there is no way to recover Theorem A from Corollary C. We would like to stress that the techniques employed here are totally different from those used in the proof of the result by Casolo, **Dolfi and Jabara**. Their proof makes a detailed analysis of the structure of the Sylow p -subgroups of G , and requires, for instance, either the Feit–Thompson Theorem, **the Baer–Suzuki Theorem** or half-transitive group action. However, our approach in Theorem A allows us to work by induction on the order of the normal subgroup N , which is unusual when dealing with class sizes in the whole group. This enables us to reduce the problem to non-abelian simple groups and to rely on an arithmetical property on the order of their Schur multiplier. Our proofs of Lemma 2.3, Lemma 2.4 and Corollary 2.5 use the Classification of the Finite Simple Groups.

All groups are supposed to be finite. If G is a group, then $\pi(G)$ denotes the set of prime divisors of $|G|$, and similarly, if n is an integer, $\pi(n)$ will denote the set of prime divisors of n .

2 Preliminaries

Before taking up the problem, we present here some useful results which will be used in the sequel. First, we recall the well-known Thompson $P \times Q$ -Lemma.

Lemma 2.1. *Let $P \times Q$ be the direct product of a p -group P and a p' -group Q . Suppose that $P \times Q$ acts on a p -group G such that $C_G(P) \leq C_G(Q)$. Then Q acts trivially on G .*

Proof. For instance, see [8, Theorem 4.31]. □

The hypotheses of the main theorems exhibit good behaviour with regard to quotient groups of normal p' -subgroups.

Lemma 2.2. *Let G be a finite group and let N be a normal subgroup of G such that $|x^G|_p = p^e$, with $e > 0$, for every prime-power order element $x \in N \setminus Z(N)$. Let M be a normal p' -subgroup of G contained in N , and denote by $\overline{G} := G/M$. Then we have either*

$$\overline{N} \leq Z(\overline{G})$$

or

$$|\overline{x}^{\overline{G}}|_p = p^e$$

for every prime-power order element $\overline{x} \in \overline{N} \setminus Z(\overline{N})$.

Proof. Suppose that $\overline{N} \not\leq Z(\overline{G})$ and let $\overline{x} \in \overline{N} \setminus Z(\overline{N})$ be a prime-power order element. Notice that we may assume that x is a q -element for some prime q (possibly equal to p) such that $x \notin Z(N)$ and we always have $C_{\overline{G}}(\overline{x}) \geq \overline{C_G(x)}$. Since M is a p' -subgroup, we get $|C_{\overline{G}}(\overline{x})|_p = |C_G(x)|_p$. Therefore, $|C_{\overline{G}}(\overline{x})|_p \geq |C_G(x)|_p$. Now, let P be a p -subgroup of G such that P is a Sylow p -subgroup of $C_{\overline{G}}(\overline{x})$ and consider the action of P on N . By [7, Proposition 14.1], there exists an element $\overline{y} \in \overline{N}$ such that y is centralized by P and $\overline{x} = \overline{y}$. Then, $\overline{y} \notin Z(\overline{N})$ and we notice that y can also be assumed to be a q -element which is non-central in N . Furthermore, note that

$$|C_{\overline{G}}(\overline{x})|_p = |\overline{P}| = |P| \leq |C_G(y)|_p \quad \text{and} \quad |C_G(x)|_p = |C_G(y)|_p$$

by hypothesis, so we deduce that $|C_{\overline{G}}(\overline{x})|_p = |C_G(x)|_p$. Hence

$$|\overline{x}^{\overline{G}}|_p = |x^G|_p$$

and the proof is finished. □

The following result extends [2, Theorem 7] for prime-power order elements. It relates G -class sizes of a non-solvable normal subgroup to the size of its center.

Lemma 2.3. *Let G be a group and let N be a non-solvable normal subgroup of G . If m divides $|x^G|$ for every prime-power order element $x \in N \setminus Z(N)$, then m divides the order of $Z(N)$.*

Proof. First, we claim that we can assume $\pi(N) = \pi(N/Z(N))$. If this does not occur, then N can be factorized as a direct product $N = K \times Q$, with Q a central Sylow q -subgroup of N for some prime q , and K normal in G . Then, if $x \in K \setminus Z(K)$ is a prime-power order element, it follows that $x \in N \setminus Z(N)$. Since K is not solvable either, we can apply induction to get that m divides $|Z(K)|$, which clearly divides $|Z(N)|$. Thus, the lemma is proved.

Now we prove **the inclusion** $\pi(m) \subseteq \pi(N)$. Suppose that $r \in \pi(m) \setminus \pi(N)$. Take $R \in \text{Syl}_r(G)$. If x is a prime-power order element of $C_N(R) \setminus Z(N)$, then $|x^G|$ is not divisible by r . Since r divides m , this means that x has to be in $Z(N)$, a contradiction. So we get $C_N(R) \leq Z(N)$. In particular, $C_N(R)$ is nilpotent. As R acts coprimely on N , by [1, Theorem B], we conclude that N is solvable, a contradiction.

Let $Q \in \text{Syl}_q(G)$ with $q \in \pi(m)$. For every $x \in (Q \cap N) \setminus Z(N)$, there exists **some** $y \in G$ such that $C_{Q^y}(x) \in \text{Syl}_q(C_G(x))$. Moreover, there is **some** $z \in C_G(x)$ such that

$$C_Q(x) \leq (C_{Q^y}(x))^z = C_{Q^{yz}}(x).$$

As a result, m_q divides $|x^G|_q = |Q^{yz} : C_{Q^{yz}}(x)|$, which divides $|x^Q|$. Note that $N_q := Q \cap N \in \text{Syl}_q(N)$ and $N_q \trianglelefteq Q$. By hypothesis and the class equation in N_q , we obtain

$$|N_q| = |N_q \cap Z(N)| + m_q l$$

for some positive integer l . As $q \in \pi(N)$, we deduce that q divides $|N_q \cap Z(N)|$. We can reformulate the above equation as

$$\left| \frac{N_q}{N_q \cap Z(N)} \right| = 1 + \frac{m_q l}{|N_q \cap Z(N)|}.$$

Since the first member of the equation is a non-trivial q -power by the first paragraph, we conclude that m_q divides $|N_q \cap Z(N)|$, and so it divides $|Z(N)|$ for every prime $q \in \pi(m)$. This shows that m divides $|Z(N)|$. \square

In order to prove the following property we employ [9, Theorem 5.1.4]. It describes the Schur multiplier of the non-abelian simple groups, whose orders appear in [9, Tables 5.1.A, B, C and D].

Lemma 2.4. *Let S be a non-abelian simple group and let $M(S)$ denote its Schur multiplier. Then, for every prime $p \in \pi(M(S))$ there exists a conjugacy class x^S where $x \in S$ is a prime-power order element **with the property** that $|x^S|_p$ does not divide $|M(S)|_p$.*

Proof. We divide the proof into three cases according to the Classification of Finite Simple Groups. If $M(S)$ is trivial, there is nothing to prove. So in the following, we are only considering those groups with $|M(S)| \geq 2$.

(1) S is a sporadic simple group. By using [9, Table 5.1.C] and GAP, we can easily check the result.

(2) S is an alternating group of degree n . If $n = 6$ or 7 , we take **some** $x \in S$ with $o(x) = 5$ and then $2^3 \cdot 3^2$ divides $|x^S|$. Since $|M(S)| = 6$, we are finished in this case. If $n = 5$, the class size of a 5-cycle is 12. If $n \geq 8$, we consider again x a 5-cycle in S , so

$$|x^S| = \frac{n(n-1)(n-2)(n-3)(n-4)}{5},$$

which is always divisible by 4. However, in both cases $|M(S)| = 2$, and we also get the desired result.

(3) S is a simple group of Lie type. If S is one of the groups in [9, Table 5.1.D], we can get the result by using GAP. From [9, Tables 5.1.A–B], we easily observe that the characteristic r of S does not divide the order of the Schur multiplier of S . On the other hand, from [4, Section 5.1], we know that there exists an r -element $x \in S$ such that $C_S(x)$ is an r -group. Therefore, for every $p \in \pi(S)$ with $p \neq r$, we obtain $|x^S|_p = |S|_p$. This completes the proof. \square

Corollary 2.5. *Let S be a non-abelian simple group and let $M(S)$ denote its Schur multiplier. Then for every prime $p \in \pi(S)$ we have that $|M(S)|_p$ is strictly less than $|S|_p$.*

Proof. Let $p \in \pi(S)$. If $p \notin \pi(M(S))$, we are finished. If $p \in \pi(M(S))$, then by Lemma 2.4 we know that there exists some prime-power order element $x \in S$ such that $|M(S)|_p < |x^S|_p \leq |S|_p$. \square

3 Proofs

Proof of Theorem A. We show in the first four steps that N is solvable by considering a minimal counterexample and finally we will see that N has a normal p -complement. Let (G, N) be a counterexample with $|N|$ as small as possible.

Step 1. *We may assume that $O_{p'}(N) = 1$.*

Otherwise, let $\overline{G} := G/O_{p'}(N)$ and use the bar notation. If $O_{p'}(N) \leq Z(N)$, then $O_{p'}(N)$ is solvable. On the contrary, if we assume that $O_{p'}(N) \not\leq Z(N)$, since $O_{p'}(N)$ satisfies the hypothesis of the theorem, we deduce that $O_{p'}(N)$ is solvable by minimal counterexample. On the other hand, according to Lemma 2.2, we have that either $\overline{N} \leq Z(\overline{G})$ or \overline{N} satisfies the hypothesis of the theorem. In all cases, we can get that \overline{N} is solvable and, consequently, N is solvable. Therefore, we may assume that $O_{p'}(N) = 1$.

Step 2. *The group $Z(N) = F(N)$ is a p -group.*

By Step 1, we have $Z(N) \leq O_p(N)$. We prove that $Z(N) = O_p(N)$. **To this end**, let $x \in N \setminus Z(N)$ be an r -element, where $r \in \pi(N)$ and $r \neq p$, and let $P \in \text{Syl}_p(C_G(x))$. Let us consider the action of $P \times \langle x \rangle$ on $O_p(N)$; we claim that $C_{O_p(N)}(P) \subseteq C_{O_p(N)}(x)$. For every $v \in C_{O_p(N)}(P)$ we have: if $v \in Z(N)$, then $v \in C_{O_p(N)}(x)$; if $v \notin Z(N)$, then $\langle P, v \rangle \subseteq C_G(v)$. Since $|v^G|_p = |x^G|_p$, it follows that $|P| = |\langle P, v \rangle|$ and thus **we get** $v \in P$. This shows that $v \in C_G(x)$ and then $C_{O_p(N)}(P) \subseteq C_{O_p(N)}(x)$ as claimed. By applying Lemma 2.1, it follows that $x \in C_N(O_p(N))$. So we conclude that $N/C_N(O_p(N))$ is a p -group and, in particular, it is solvable. If $C_N(O_p(N)) < N$, as $C_N(O_p(N))$ satisfies the hypotheses of the theorem, we get that $C_N(O_p(N))$ is solvable too by minimality. This forces N to be solvable, a contradiction, so $O_p(N)$ is central in N as wanted.

Step 3. *The factor $N/Z(N)$ is simple.*

Let N/M be a chief factor of G with $F(N) \leq M$. Notice that $F(M) = F(N)$. We claim that $M = F(N)$. By minimality, M is solvable, and hence **we obtain that** $C_M(F(N)) \leq F(N)$. As $F(N) = Z(N)$, we conclude that $F(N) = M$ as wanted, and also $M = Z(N)$.

Since N is non-solvable and $N/Z(N)$ is a chief factor of G , we can write

$$N/Z(N) = L_1/Z(N) \times \cdots \times L_t/Z(N),$$

where $L_i/Z(N)$ are isomorphic non-abelian simple groups. We prove that $t = 1$. Otherwise, let $L = L_1$ and observe that since $L/Z(N)$ is simple, then

$$L/Z(N) = L'Z(N)/Z(N) \cong L'/Z(L').$$

We consider $N_G(L')$. For every prime-power order element $x \in L' \setminus Z(L')$, we see that $C_G(x) \subseteq N_G(L')$. In fact, if $v \in C_G(x) \setminus N_G(L')$, then **we must have** $x = x^v \in L' \cap L'^v \subseteq L \cap L^v \subseteq Z(N)$, a contradiction. This yields

$$|x^G| = |G : C_G(x)| = |G : N_G(L')| |N_G(L') : C_{N_G(L')}(x)|.$$

If $n = |G : N_G(L')|$, we deduce that $|x^{N_G(L')}|_p = p^a/n_p$. This means that every prime-power order element in $L' \setminus Z(L')$ satisfies that the p -part of its class size in $N_G(L')$ is equal to p^a/n_p . Since $L' < N$, by minimal counterexample, we obtain that L' is solvable, a contradiction. Hence **we have** $t = 1$, as desired, that is, $N/Z(N) = L/Z(N)$ is a simple group.

Step 4. *The group N is solvable.*

Notice that N is certainly a perfect group by minimality, so by Step 3, N is a quasi-simple group. Consequently, $|Z(N)|$ divides the order of the Schur multiplier of $S := N/Z(N)$. By Lemma 2.3, we know that p^a divides $|Z(N)|$, so p^a must divide the order of the Schur multiplier $M(S)$ as well. On the other hand, by Lemma 2.4, we have $|M(S)|_p < |(xZ(N))^S|_p$ for certain prime-power order element $x \in N \setminus Z(N)$. Since $|(xZ(N))^S|$ divides $|x^N|$ and this divides $|x^G|$, we get a contradiction, which completes the proof.

Once we have proved that N is solvable, we show the p -nilpotency.

Step 5. *The group N has a normal p -complement.*

We argue by induction on $|N|$ and we distinguish two cases.

(1) Suppose that $O_{p'}(N) \neq 1$. Let $\bar{G} = G/O_{p'}(N)$. If $\bar{N} \leq Z(\bar{G})$, then \bar{N} is a p -group and hence $O_{p'}(N)$ is the normal p -complement of N , and we are finished. Assume then that $\bar{N} \not\leq Z(\bar{G})$. By Lemma 2.2, we have that \bar{N} satisfies the hypotheses of the theorem. Therefore, by induction we get that \bar{N} has a normal p -complement \bar{A} , so A is a normal p -complement of N .

(2) Assume that $O_{p'}(N) = 1$. By Step 4, N is solvable and hence $O_p(N) \neq 1$. Let $x \in N \setminus Z(N)$ be a prime-power order element, say an r -element for some prime $r \neq p$, and let P be a Sylow p -subgroup of $C_G(x)$. Let us consider the action of $P \times \langle x \rangle$ on $O_p(N)$. For every $v \in C_{O_p(N)}(P)$ we have: if $v \in Z(N)$, then $v \in C_{O_p(N)}(x)$; if $v \notin Z(N)$, then $C_G(v)_p \geq \langle v, P \rangle$. Since $|v^G|_p = |x^G|_p$, we conclude that $v \in P$ and hence $v \in C_{O_p(N)}(x)$. Then, by Lemma 2.1, we get $x \in C_G(O_p(N))$. However, since $O_{p'}(N) = 1$, we have $C_G(O_p(N)) \leq O_p(N)$, which leads to a contradiction. This shows that any r -element, for $r \neq p$, is central in N , so N trivially has a normal p -complement. This completes the proof. \square

Proof of Corollary B. This statement is a particular case (when one considers all elements instead of only prime-power order elements) of Theorem A, taking into account that $Z(G) \cap N \subseteq Z(N)$. \square

Proof of Corollary C. This statement is the particular case in which $N = G$ in Theorem A. \square

Remark 3.1. As we have pointed out in the introduction, Theorem A cannot be recovered from Corollary C in any way. The fact that every element in $cs_G(N)$ distinct from 1 has the same p -part does not imply that $cs(N)$ follows the same pattern. For instance, by using the SmallGroups Library in GAP ([10]), we check that $G = \text{Id}(64, 121)$ has a normal subgroup N , which is isomorphic to $\text{Id}(32, 241)$ and such that $cs(N) = \{1, 2, 4\}$, whereas $cs_G(N) = \{1, 4\}$.

Remark 3.2. We might think that one of the following properties could imply that at least the p -complements of G do exist: the p -part of every class size is either 1 or p^a , for a fixed $a > 0$, or the sizes of the conjugacy classes of non-central p' -elements have the same p -part. However, both questions have a negative answer. For example, if $G = \text{SL}(2, 5)$, then $\text{cs}(G) = \{1, 12, 20, 30\}$, so the 3-part of every class size is either 1 or 3. Also, the class sizes of the non-central 3'-elements of G are 12 and 30, so they have the same 3-part, and nevertheless, G does not possess 3-complements.

Remark 3.3. The authors have proved in [3] a dual variation of Theorem A, that is, when one considers the p' -part instead of the p -part. If N is a normal subgroup of G and $|x^G|_{p'}$ is constant for every prime-power element of $x \in N \setminus Z(N)$, then N has a normal p -complement and this is nilpotent. As Remark 3.2 shows, Theorem A does not hold when considering p' -elements of N , however, the result in [3] still remains true when the hypothesis is restricted to only p' -elements.

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Author information

Antonio Beltrán, Departamento de Matemáticas, Universidad Jaume I,
12071 Castellón, Spain.
E-mail: abeltran@mat.uji.es

María José Felipe, Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, 46022 Valencia, Spain.
E-mail: mfelipe@mat.upv.es

Changguo Shao, School of Mathematical Sciences, University of Jinan,
250022 Shandong, P.R. China.
E-mail: shaoguozi@163.com