OPTIMAL EXTENSIONS OF NARROW OPERATORS

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Abstract. Compactness type properties for operators acting in Banach function spaces are not preserved when the operator is extended to a bigger space. Moreover, it is known that there exists a maximal (weakly) compact linear extension of an operator (weakly) compact if and only if its maximal linear continuous extension to its optimal domain is (weakly) compact. We show that the same happens if we consider AM-compactness for the operator. We also give some partial results regarding Dunford-Pettis operators. In the positive, we show that there is a property weaker than these compactness properties that extends always to the maximal extension of the operator: narrow operators from Banach function spaces extend to narrow operators. Some applications of this result are shown.

1. Introduction

Consider a Banach space valued operator —that is a bounded linear map— $T : X(\mu) \to E$ acting in a σ-order continuous Banach function space $X(\mu)$ over the finite positive measure space (Ω, Σ, μ). Suppose that T is also compact. Then it is well known —and easy to find an example for it— that a continuous extension of T to any other σ-order continuous Banach function space $Y(\mu)$ containing $X(\mu)$ is not necessarily compact. Actually, a recent paper by S. Okada [11] shows that more is true. Let us say that *the operator T allows a maximal compact linear extension* if there is a σ-order continuous Banach function space $Y(\mu)$ containing $X(\mu)$ such that $Y(\mu)$ is the bigger space to which T can be extended preserving compactness. Assume that T is μ-determined, i.e. the null sets for μ are the same that for m_T —see the definition below—. Then the compact operator T allows a maximal compact extension if and only $Y(\mu)$ coincides with the optimal domain of the operator, the space $L^1(m_T)$. This is the space of integrable functions with respect to the vector measure $m_T : \Sigma \to E$, that is given by $m_T(A) := T(\chi_A)$, $A \in \Sigma$. This space $L^1(m_T)$ plays the role of the optimal domain of T, that is, T

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always factors as

$$
\begin{array}{ccc}
X(\mu) & \xrightarrow{\ T\ } & E, \\
{\scriptstyle i}\downarrow & \nearrow_{I_{m_T}} & \\
L^1(m_T) & &
\end{array}
\tag{1.1}
$$

where i is the inclusion map, I_{m_T} is the integration operator associated to m_T and $L^1(m_T)$ is the biggest σ-order continuous Banach function space with a weak unit to which T can be extended —see [12, Theorem 4.14] and the references therein—. This is the so called Optimal Domain Theorem by G. P. Curbera and W. J. Ricker. The same happens regarding for instance weak compactness: the inclusion map $i : L^2[0,1] \to L^1[0,1]$ is weakly compact, but its extension to the identity map $i : L^1[0,1] \to L^1[0,1]$ is not. In the same paper [11], S. Okada shows that there exists and optimal weakly compact extension if and only if the integration map I_{m_T} —the maximal linear extension— is weakly compact.

In this paper we analyze three properties more, namely being AM-compact, Dunford-Pettis or narrow. In the first part —section 3– we show that regarding AM-compactness the answer is the same: in general, the property of being AM-compact for an operator $T : X(\mu) \to E$ cannot be extended to the optimal domain, and there is a maximal extension of T preserving the property if and only if the associated integration map I_{m_T} satisfies this property —this is Theorem 3.2—. In section 4 we study the Dunford-Pettis property. Although we do not solve the question with full generality we give some results and provide some examples to illustrate the difficulties. However, in the last part of the paper —section 5— we show a positive result, that provides a weaker property associated to compactness that is always preserved. Motivated in part by some comments of V. Kadets, we analyze the case of the narrow operators. As we will show in Theorem 5.2, if T is a μ-determined narrow operator, then the integration map I_{m_T} —and so, the maximal continuous linear extension of T— is narrow. Since all the above mentioned properties for T imply that T is narrow, we can say that whenever T has any compactness type property, it admits a maximal narrow extension. Using the numerous recent results obtained on narrow operators, we also show some applications that provide information and examples of narrow extensions of operators and narrow integration maps.

2. Preliminaries

Let (Ω, Σ, μ) be a positive finite measure space. We denote by $L^0(\mu)$ the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. Endowed with the μ-a.e. pointwise order, that is, $f \le g$ if and only if $f \le g$ μ-a.e., $L^0(\mu)$ is a vector lattice. By a *Banach function space* (briefly, B.f.s.) associated to μ we mean a Banach space $X(\mu) \subseteq L^0(\mu)$ containing the set of all simple functions, $\mathrm{sim}(\Sigma)$, and satisfying that if $|f| \le |g|$ with $f \in L^0(\mu)$ and $g \in X(\mu)$ then $f \in X(\mu)$ and $\|f\| \le \|g\|$. We say that $X(\mu)$ is *σ-order continuous* if for every sequence $(f_n)_n \subseteq X(\mu)$ with $f_n \downarrow 0$ it follows that $\|f_n\|_{X(\mu)} \to 0$. Note that sim(Σ) is always dense in any σ-order continuous B.f.s.. A B.f.s. $X(\mu)$ has *absolutely continuous norm* if $\lim_{\mu(A)\to 0}\|f\chi_A\| = 0$ for each $f \in X(\mu)$. We denote by $\mathbf{B}[X(\mu)]$ the closed unit ball of $X(\mu)$.

Throughout the paper $m : \Sigma \to E$ will be a countably additive vector measure, namely $m(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ in the norm topology of the Banach space E for all sequences $\{A_n\}_n$ of pairwise disjoint sets of Σ. Let E' be the (topological) dual space of E. For each element $x' \in E'$ the formula $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$, $A \in \Sigma$, defines a (countably additive) scalar measure. We write $|\langle m, x'\rangle|$ for its variation, i.e. $|\langle m, x'\rangle|(A) := \sup \sum_{B\in\Pi} |\langle m(B), x'\rangle|$, for $A \in \Sigma$ —where the supremum is computed over all finite measurable partitions Π of A—. The nonnegative set function $\|m\|$ whose value on a set $A \in \Sigma$ is given by $\|m\|(A) = \sup\{|\langle m, x'\rangle|(A) : x' \in X', \|x'\| \leq 1\}$ is called the semivariation of m. The measure m is absolutely continuous with respect to μ if $\lim_{\mu(A)\to 0}\|m\|(A) = 0$; we say that μ is a control measure for m and we write $m \ll \mu$. It is well-known that there always exists $x' \in E'$ such that $m \ll |\langle m, x'\rangle|$. Such kind of measures are called Rybakov measures for m —see [4, Ch.IX,2]—.

A measurable function $f : \Omega \to \mathbb{R}$ is said to be *integrable with respect to* m if: (i) it is integrable with respect to each scalar measure $\langle m, x'\rangle$, for every $x' \in E'$ and, (ii) for every $A \in \Sigma$ there is a unique element $\int_A f dm \in E$ such that $\langle \int_A f dm, x'\rangle = \int_A f d\langle m, x'\rangle$, for all $x' \in E'$. The set consisting of equivalence classes of such functions —identifying functions that are $\|m\|$-a.e. equal— is denoted by $L^1(m)$, and it is a σ-order continuous Banach function space —over any Rybakov measure for m— endowed with the norm

$$\|f\|_m = \sup\{\int_\Omega |f| d|\langle m, x'\rangle| : \|x'\| \leq 1\}, \quad f \in L^1(m).$$

For $1 < p < \infty$, the set consisting of —equivalence classes— of measurable functions $f : \Omega \to \mathbb{R}$ such that $|f|^p \in L^1(m)$ is denoted by $L^p(m)$. It is also a σ-order continuous B.f.s. over any Rybakov measure for m when endowed with the norm

$$\|f\|_{m,p} = \big\||f|^p\big\|_m^{1/p} = \sup\{(\int_\Omega |f|^p d|\langle m, x'\rangle|)^{1/p} : \|x'\| \leq 1\}, \quad f \in L^p(m).$$

We write $L(X(\mu), E)$ for the set of all linear and continuous maps from $X(\mu)$ into E. If $X(\mu)$ is a σ-order continuous B.f.s. then T defines a vector measure $m_T : \Sigma \to E$ by the formula $m_T(A) := T(\chi_A)$, $A \in \Sigma$. The operator T is said to be *μ-determined* if the semivariation $\|m_T\|$ of this measure is equivalent to μ, i.e. μ-null sets and $\|m_T\|$-null sets coincide. It is well-known that such an operator can be extended with continuity to the space $L^1(m_T)$. This extension is given by the *integration map* $I_{m_T} : L^1(m_T) \to E$ defined by $I_{m_T}(f) = \int_\Omega f dm_T$, for each $f \in L^1(m_T)$. Actually, by the Optimal Domain Theorem this extension satisfies the optimality property. Namely $L^1(m_T)$ is the bigger σ-order continuous Banach function space to which T can be extended —see Corollary 3.3 in [3]—. We have the following diagram:

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\ T\ } & E. \\ {\scriptstyle i}\big\downarrow & \nearrow_{I_{m_T}} & \\ L^1(m_T) & & \end{array} \tag{2.1}$$

An operator $T : X(\mu) \to E$ is called *narrow* if for every $0 \leq f \in X(\mu)$ and every $\varepsilon > 0$ there exists $g \in X(\mu)$ such that $|g| = f$ and $\|T(g)\| < \varepsilon$. Here we have to pointed out that there is another definition for narrow operators —see Definition 1.5 in [14]—. The one that we use is Definition 10.1 in [14]. Although it is an open problem if, in the general case, both definitions are equivalent —see Open problem 10.3 in [14]— it is well known that this is the case for B.f.s. having absolutely continuous norm —see [14, Proposition 10.2]—. Since the B.f.s $X(\mu)$ that we use in this work needs to be σ-order continuous —and then $X(\mu)$ has absolutely continuous norm *cf.* [12, Lemma 2.37 (ii)]— then for our purposes both definitions are equivalent.

The reader is referred to our standard references [12] for the study of the theory of integrable functions with respect to vector measures, [14] for the study of narrow operators and [10] for Banach lattices.

3. AM-compact linear extension

Recall that an operator T from a B.f.s. $X(\mu)$ into a Banach space E is said to be *AM-compact* if it transforms order bounded subsets of $X(\mu)$ into relatively compact subsets of E. In [13], some results are provided for determining when this operator admits a maximal extension preserving compactness, concluding that this is only possible in case that the associated integration map I_{m_T} is compact, which is not in general the case —in fact, this is a rather unusual case—. The question that arise now is the following:

When a given AM-compact operator admits a maximal AM-compact extension?

We will see that the answer to this question is the same as for the case of compact and weakly compact operators study by S. Okada in [11]. Namely, a μ-determinated AM-compact operator admits a maximal AM-compact extension if, and only if, the integration operator defined in the corresponding space L^1 of the vector associated to the operator is AM-compact. The main construction in order to prove our result where developed in [11]. For the sake of completeness we include a brief summary of the definitions and facts needed to our proofs.

Given $1 < p < \infty$, the conjugate index q is defined to be the real number that satisfies that $1/p + 1/q = 1$. It is well known that if $f \in L^p(m_T)$ and $g \in L^q(m_T)$ then the pointwise product fg belongs to $L^1(m_T)$ —see [12, Ch.3]—. If we fix $g \in L^q(m_T)$, we define the set

$$g \cdot L^p(m_T) := \{gf \,:\, f \in L^p(m_T)\} \subseteq L^1(m_T), \tag{3.1}$$

where the inclusion is continuous as a consequence of the Hölder's type inequality $\|gf\|_{m_T} \leq \|f\|_{m_T,p}\|g\|_{m_T,q}$, for each $f \in L^p(m_T)$ and $g \in L^q(m_T)$. Moreover, the space $g \cdot L^p(m_T)$ is an order ideal of $L^1(m_T)$ with the lattice norm given by the formula $\|h\|_{g\cdot L^p(m_T)} := \|h/g\|_{m_T,p}$ for all $h \in g \cdot L^p(m_T)$ and understanding that $0/0 = 0$. If, in addition, we assume that $g \geq c\chi_\Omega$ for some $c > 0$ then $\mathrm{sim}\Sigma \subseteq g \cdot L^p(m_T)$. Furthermore, the linear operator $\phi_p^{(g)} : g \cdot L^p(m_T) \to L^p(m_T)$ given by $\phi_p^{(g)}(h) = h/g$ is a linear isomorphism that preserves the norm and the order. Therefore $g \cdot L^p(m_T)$ is an σ-order continuous B.f.s. over (Ω, Σ, μ). Consider now the inclusion map $\alpha_p^{(g)} : g \cdot L^p(m_T) \to L^1(m_T)$ and define the restriction of the integration operator $I_{m_T} : L^1(m_T) \to E$ to $g \cdot L^p(m_T)$ by $I_{m_T}^{(g,p)} = I_{m_T} \circ \alpha_p^{(g)}$ as the following diagram:

$$\begin{array}{ccc} g \cdot L^p(m_T) & \xrightarrow{\quad I_{m_T}^{(g,p)} \quad} & E \\ & {\scriptstyle \alpha_p^{(g)}} \searrow \qquad \nearrow {\scriptstyle I_{m_T}} & \\ & L^1(m_T) & \end{array} \tag{3.2}$$

In [11, Lemma 2.2 (ii-b)] is proved that $I_{m_T}^{(g,p)}$ is compact if, and only if, the range of the vector measure m_T is relatively compact in E. In our next lemma we prove that, actually, this facts are also equivalent to $I_{m_T}^{(g,p)}$ being AM-compact.

Lemma 3.1. *Let T be a μ-determined bounded linear map defined from a σ-order continuous B.f.s. $X(\mu)$ into the Banach space E. For $1 < p < \infty$ take q the conjugate exponent and $g \in L^q(m_T)$ such that $g \geq c\chi_\Omega$ for some $c > 0$. Then the following assertions are equivalent:*

1. *$I_{m_T}^{(g,p)}$ is compact,*
2. *$I_{m_T}^{(g,p)}$ is AM-compact, and*
3. *The range of the vector measure $m_T : \Sigma \to E$,*

$$\mathbf{R}(m_T) = \{m_T(A) : A \in \Sigma\} = \{T(\chi_A) : A \in \Sigma\},$$

is relatively compact.

Proof. Since a compact operator is always AM-compact then (1)⇒(2). Let us see now (2)⇒(3). If $I_{m_T}^{(g,p)}$ is AM-compact then $I_{m_T}^{(g,p)}$ transforms order bounded subsets of $g \cdot L^p(m_T)$ into relatively compact subsets of $L^1(m_T)$. The set

$$K := \{\chi_A : A \in \Sigma\},$$

satisfies that $K \subseteq [\frac{-1}{c}\chi_\Omega, \frac{1}{c}\chi_\Omega]$ so it is an order bounded subset in $g \cdot L^p(m_T)$. Therefore $I_{m_T}^{(g,p)}(K) = \mathbf{R}(m_T)$ is then relatively compact in E. Finally (3)⇒(1) is just [11, Lemma 2.2 (ii-b)]. □

We continue the construction by considering now a σ-order continuous B.f.s. $Y(\mu)$ over (Ω, Σ, μ) such that $Y(\mu) \subseteq L^1(m_T)$ and $g \cdot L^p(m_T)$ is not contained in $Y(\mu)$. We define the order ideal of $L^1(m_T)$:

$$Z(\mu) := Y(\mu) + g \cdot L^p(m_T), \tag{3.3}$$

with the lattice norm

$$\|f\|_{Z(\mu)} := \inf\{\|\phi\|_{Y(\mu)} + \|\psi\|_{g\cdot L^p(m_T)}\}, \tag{3.4}$$

that is defined for each $f \in Z(\mu)$; where the infimum is computed for all decomposition $f = \phi + \psi$, for $\phi \in Y(\mu)$ and $\psi \in g \cdot L^p(m_T)$. Then $Z(\mu)$ is also a σ-order continuous B.f.s. over (Ω, Σ, μ).

Before to state and to prove our first result let us adopt the following classical notation. If $0 \le h \in X(\mu)$, we define the order interval $[-h, h]$ in $X(\mu)$ as

$$[-h, h] := \{f \in X(\mu) : |f| \le h\}.$$

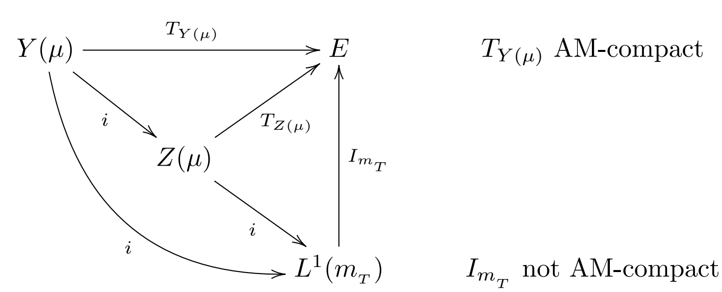
Theorem 3.2. *Let $X(\mu)$ be a σ-order continuous B.f.s, E a Banach space and let $T : X(\mu) \to E$ a μ-determined AM-compact operator. Then T admits a maximal AM-compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is AM-compact.*

Proof. Suppose that I_{m_T} is an AM-compact extension of T. Since $L^1(m_T)$ is the largest σ-order continuous B.f.s. into which $X(\mu)$ is continuously embedded and $I_{m_T} : L^1(m_T) \to E$ is an extension of T, this extension must be maximal and AM-compact.

For the converse, assume that I_{m_T} is not AM-compact and let us see that in such case T does not admit a maximal AM-compact linear extension. For this aim consider any σ-order continuous B.f.s. $Y(\mu)$ over (Ω, Σ, μ) such that $X(\mu)$ is continuously embedded in $Y(\mu)$ and for which $T_{Y(\mu)} : Y(\mu) \to E$ is an AM-compact linear extension. Note that this extension always exists since T is AM-compact so we can take $Y(\mu) = X(\mu)$. Let us see that there exists a proper AM-compact linear extension of $T_{Y(\mu)}$.

Due to the Optimal Domain Theorem (2.1) the continuity of $T_{Y(\mu)}$ implies that $Y(\mu)$ is continuously embedded into $L^1(m_T)$ and in fact, the restriction to $Y(\mu)$ of I_{m_T} is $T_{Y(\mu)}$ —see also [11, Lemma 2.1]— . On the other hand, since I_{m_T} is not AM-compact and $T_{Y(\mu)}$ is AM-compact we have that $Y(\mu) \subsetneq L^1(m_T)$. Hence there is a function $g \in L^q(m_T)$ such that $g \cdot L^p(m_T)$ is not contained in $Y(\mu)$ —see the final part of page 319 in [11]—. Therefore we can consider the Banach function space $Z(\mu)$ defined as was explained above. Note that since $g \cdot L^p(m_T)$ is not contained in $Y(\mu)$ then $Y(\mu) \subsetneq Z(\mu)$. We have then the following diagram:

(3.5)



Claim. $T_{Z(\mu)}$ is AM-compact. Let B be an order bounded subset contained in $Z(\mu)$. Then there exists an interval $[-u, u]$ in $Z(\mu)$ such that $B \subseteq [-u, u]$ where

$0 \le u \in Z(\mu)$. We have to show that $I_{m_T}(B)$ is a relatively compact subset of the Banach space E. First, consider a decomposition of u as $u = u_0 + u_1$, $u_0 \in Y(\mu)$ and $u_1 \in g \cdot L^p(m_T)$. Then we have that $u \le |u_0| + |u_1|$, so we can assume, without loss of generality, that $0 \le u_0$ and $0 \le u_1$. Let us see that $[-u, u] \subseteq [-2u_0, 2u_0] + [-2u_1, 2u_1]$. Indeed, take a function $f \in [-u, u]$ and define the measurable set

$$C = \{w \in \Omega : u_0(w) \ge u_1(w)\}.$$

Note that the complement C^c is $\{w : u_0(w) < u_1(w)\}$, and clearly $u\chi_C \le 2u_0$ and $u\chi_{C^c} \le 2u_1$ so we can write a decomposition of f as $f = f\chi_C + f\chi_{C^c}$. Therefore, $|f\chi_C| \le u\chi_C \le 2u_0$ and $|f\chi_{C^c}| \le u\chi_{C^c} \le 2u_1$. Thus $f\chi_C \in Y(\mu)$ and $f\chi_{C^c} \in g \cdot L^p(m_T)$ with $f\chi_C \in [-2u_0, 2u_0]$ and $f\chi_{C^c} \in [-2u_1, 2u_1]$, so

$$f = f\chi_C + f\chi_{C^c} \in [-2u_0, 2u_0] + [-2u_1, 2u_1].$$

Hence, we obtain that

$$\begin{aligned} T_{Z(\mu)}(B) &= I_{m_T}(B) \subseteq I_{m_T}([-u, u]) \subseteq I_{m_T}([-2u_0, 2u_0]) + I^{(g,p)}_{m_T}([-2u_1, 2u_1]) \\ &\subseteq T_{Y(\mu)}([-2u_0, 2u_0]) + I^{(g,p)}_{m_T}([-2u_1, 2u_1]), \end{aligned}$$

where the last inclusion follows from the fact that I_{m_T} coincides with $T_{Y(\mu)}$ on $Y(\mu)$. The set $K = \{\chi_A : A \in \Sigma\}$ satisfies that $K \subseteq [-\chi_\Omega, \chi_\Omega]$ in $X(\mu)$. Due to the fact that the operator T is AM-compact, the set $\{T(\chi_A) : A \in \Sigma\} = \{m_T(A) : A \in \Sigma\} = \mathbf{R}(m_T)$ is relatively compact. Then, by using Lemma 3.1, $I^{(g,p)}_{m_T}$ is AM-compact so $I^{(g,p)}_{m_T}([-2u_1, 2u_1])$ is a relatively compact subset in E. According that $T_{Y(\mu)}$ is AM-compact, $T_{Y(\mu)}([-2u_0, 2u_0])$ is a relatively compact subset of E. Therefore $T_{Z(\mu)}(B)$ is relatively compact in E so $T_{Z(\mu)}$ is AM-compact and the claim is proved.

Finally, since for all $B \subseteq Y(\mu)$ one has

$$T_{Y(\mu)}(B) = I_{m_T}(B) = T_{Z(\mu)}(B),$$

then $T_{Z(\mu)}$ provides a *proper* AM-compact linear extension of $T_{Y(\mu)}$ and T does not have a maximal AM-compact linear extension. Hence the proof is complete. □

4. Dunford-Pettis linear extension

In what follows we analyze maximal linear extensions of Dunford-Pettis operators. Recall that a linear operator $T : E \to F$ between two Banach spaces E, F is called *Dunford-Pettis* if it sends weakly compact sets to relatively compact sets. By the *Eberlein-Šmulian Theorem* this is equivalent to the fact that T sends weakly null sequences from E to norm null sequences in F. These operators are often called *completely continuous*. Compact operators are always Dunford-Pettis; however the converse is not true unless the domain of the operator is reflexive. For instance, let $\lambda : 2^{\mathbb{N}} \to [0, \infty]$ be the counting measure that it is a purely atomic scalar measure, then $L^1(\lambda)$ coincides with ℓ^1. The canonic inclusion map $i : \ell^1 \to \ell^2$ is Dunford-Pettis by the Schur property of ℓ^1. This inclusion is not compact. Indeed, the set $\{i(\chi_A) : A \in 2^{\mathbb{N}}, i(A) < \infty\}$ contains all units basis vectors of ℓ^2 and so cannot be relatively compact.

Although in general the integration operator $I_{m_T} : L^1(m_T) \to E$ is not Dunford-Pettis, we can also find some positive examples. In the case that $L^1(m_T)$ is lattice isomorphic to an abstract L^1-space then we have $L^1(|m_T|) = L^1(m_T)$ with their norms being equivalent (see [12, Lemma 3.14]). Recall that a Banach lattice E is said to be an *abstract L^1-space* if $\|x + y\|_E = \|x\|_E + \|y\|_E$ whenever $x \wedge y = 0$, $0 \le x, y \in E$. Then if we apply [12, Proposition 3.56] we obtain that I_{m_T} is a Dunford-Pettis integration operator. Let us write in the next remark some known facts on Dunford-Pettis integration operators.

Remark 4.1. (1) In general for a Dunford-Pettis operator T from a σ-order continuous B.f.s. $X(\mu)$ into a Banach space E, the subset $\{T(\chi_A) : A \in \Sigma\}$ is a relatively compact set in E: indeed, if $T : X(\mu) \to E$ is a Dunford-Pettis operator from the B.f.s. $X(\mu)$ to a Banach space E, due to $X(\mu)$ is σ-order continuous B.f.s., the subset $\{\chi_A : A \in \Sigma\}$ is uniform μ-absolutely continuous —see [12, Lemma 2.37]—. According to Proposition 2.39 in [12], the subset $\{\chi_A : A \in \Sigma\}$ is a relatively weakly compact subset of $X(\mu)$. Therefore, the subset $\{T(\chi_A) : A \in \Sigma\}$ is a relatively compact subset in E. However the converse is false —see for instance Example 2.36 in [12]—.
(2) It is well know that if $T : X(\mu) \to Y(\mu)$ is a Dunford-Pettis operator where $X(\mu)$ is a B.f.s. with σ-order continuous norm, and $Y(\mu)$ is a B.f.s. then the operator T is AM-compact. Furthermore, if $Y(\mu)$ is also an L-spacethen the converse is also true —see Proposition 3.7.11 and Theorem 3.7.20 in [10]—.
(3) By (1) if T is a Dunford-Pettis operator then $\mathbf{R}(m_T)$ is relatively compact in E. Hence by using Lemma 3.1 the restriction of the integration operator $I_{m_T} : L^1(m_T) \to E$ to $g \cdot L^p(m_T)$ given by $I_{m_T}^{(g,p)} = I_{m_T} \circ \alpha_p^{(g)}$ is compact —actually AM-compact— and so Dunford-Pettis.

In the following results we give some properties regarding the maximal linear extension of Dunford-Pettis operators.

Proposition 4.2. *Let $T : X(\mu) \to E$ be a μ-determined Dunford-Pettis operator where $X(\mu)$ is a σ-order continuous B.f.s. and E is a Banach space. If there exists a Dunford-Pettis maximal linear extension of T given by an operator $T_{Y(\mu)} : Y(\mu) \to E$, being $Y(\mu)$ a σ-order continuous B.f.s., then $Y(\mu)$ is not reflexive.*

Proof. To see this, assume first that the maximal Dunford-Pettis linear extension is exactly the integration map I_{m_T}. This means that $Y(\mu) = L^1(m_T)$ and $T_{Y(\mu)} = I_{m_T}$. In this case if $L^1(m_T)$ is reflexive then I_{m_T} is compact and therefore $L^1(m_T)$ is isomorphic to an L^1-space of a positive scalar measure —in fact $L^1(m_T) = L^1(|m_T|)$, see [12, Proposition 3.48]—. This contradicts the reflexivity of $L^1(m_T)$. Therefore, the maximal Dunford-Pettis linear extension $T_{Y(\mu)}$ must be defined in a σ-order continuous Banach function space $Y(\mu)$ —strictly smaller than the space $L^1(m_T)$—. The Dunford-Pettis maximality of the extension $T_{Y(\mu)}$ gives that I_{m_T} is not compact —since otherwise $T_{Y(\mu)} = I_{m_T}$—. But if we assume now that $Y(\mu)$ is reflexive then again the operator $T_{Y(\mu)}$ is compact, and so the result by S. Okada [11, Theorem 1.1] gives a compact —and so Dunford-Pettis— linear extension to a strictly bigger Banach function space. This contradicts the fact that $T_{Y(\mu)}$ has no longer the maximal Dunford-Pettis linear extension. $\square$

In fact, the same argument gives a stronger result. Using the well-known result by H. P. Rosenthal on copies of ℓ^1 in Banach spaces —see [5, 16]—, it can be easily proved that if X is a Banach space not containing a copy of ℓ^1, then a Banach space valued operator is compact if and only if it is Dunford-Pettis. Thus, we can prove a stronger result than the one above: if there exists a Dunford-Pettis maximal extension of T to an σ-order continuous Banach function space $Y(\mu)$, then $Y(\mu)$ cannot contain a copy of ℓ^1; otherwise, the extension would be compact, and then the argument above applies to get a contradiction again.

Proposition 4.3. *Let $T : X(\mu) \to E$ be a μ-determined Dunford-Pettis linear operator where $X(\mu)$ is an σ-order continuous B.f.s. and E is a Banach space. If there exists a Dunford-Pettis maximal linear extension of T given by an operator $T_{Y(\mu)} : Y(\mu) \to E$, being $Y(\mu)$ a σ-order continuous B.f.s., then $Y(\mu)$ cannot contain a copy of ℓ^1.*

Unfortunately, the argument that proves the non-existence of optimal domain for the case of the compactness properties that are known (compactness, weak compactness and AM-compactness) cannot be applied in this case. The technical reason is easy to understand. For getting a contradiction in the proof, we need to find an inclusion of any weakly compact subset V of a suitable bigger space Z containing the optimal domain Y in a sum of a weakly compact set W of Y and a multiple of the ball of $\mathbf{B}[g \cdot L^p(m_T)]$, i.e.

$$V \subseteq W + k\mathbf{B}[g \cdot L^p(m_T)].$$

However, it is no easy to find such a decomposition for any weakly compact set of Y, and so the procedure does not work in this case. So we let this question as an

Open problem: *Is there a maximal linear extension for every Dunford-Pettis operator from a σ-order continuous B.f.s. preserving the property of being Dunford-Pettis?*

In order to center this question, we finish the section with an example that illustrates the fact that the optimal domain for continuity of the operator —the space of integrable functions $L^1(m_T)$—, is not in general Dunford-Pettis, even if the original operator is.

Example 4.4. Let μ be the Lebesgue measure on the Borel σ-algebra $\mathcal{B}([0,1])$. We consider the Volterra operator $V_r : L^r([0,1]) \to L^r([0,1])$ such that

$$(V_r f)(t) := \int_0^t f(u)du, \quad t \in [0,1], \quad f \in L^r([0,1]), \quad 1 \leq r \leq \infty.$$

In this case $X(\mu) := L^r([0,1])$ and $E := L^r([0,1])$. It is clear that $m_r(A) := V_r(\chi_A)$ for $A \in \mathcal{B}([0,1])$ define a vector measure $m_r : \mathcal{B}([0,1]) \to L^r([0,1])$ and $L^1(m_r) = L^1((1-t)dt)$ is the maximal σ-order continuous domain. For each $1 \leq r \leq \infty$ the operator V_r is μ-determined because it is injective on the subset $\{\chi_A : A \in \mathcal{B}([0,1])\}$ of its domain $L^r([0,1])$ —see [12, Lemma 4.5 (iii)]—. Furthermore, for each $1 \leq r \leq \infty$ the operator V_r is compact and weakly compact. Since compact operators between Banach spaces are always Dunford-Pettis then V_r is Dunford-Pettis. However, for the case $r = 1$, due to Example in page 320 in [11] the maximal continuous linear extension I_{m_1} is not compact and it is not even weakly compact. Moreover for $r > 1$ the Volterra integral operator I_{m_r} is not Dunford-Pettis —see [12, Proposition 3.52]—.

5. Narrow maximal extension and applications

The spaces E for which every operator $T : L^p(\mu) \to E$ is narrow has been largely studied in several papers —see for example [6, 7, 8, 17]—. In this section we analyze the extension of the property of being narrow to the optimal domain of a μ-determined operator $T : X(\mu) \to E$, where $X(\mu)$ is a σ-order continuous Banach function space.

Remark 5.1. The definition of Banach function space that is adopted in this paper is relevant due to the following technical reason. In general, it is known that $L^1(m)$ of a Banach space valued measure m is a Banach function space in the most restrictive sense of [9, p.28]. However, note that in case the vector measure is equivalent to any other (finite positive) measure μ, $L^1(m_T)$ is also a Banach function space over *the same* μ if the definition that is considered is the one that we gave in Section 2.

The result regarding the optimal extension of a narrow operator is in this case true —narrow operators extend to narrow operators— and easy to prove.

Theorem 5.2. *Let $X(\mu)$ be a σ-order continuous B.f.s. and let E be a Banach space. Let $T : X(\mu) \to E$ be a μ-determined operator. Then T is narrow if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is narrow.*

Proof. Assume that T is a μ-determined narrow operator. Recall that we use the definition of narrow operator acting in an σ-order continuous B.f.s. that is given in [14, Definition 10.1] and has been explained in Section 2.

Let $0 \leq f \in L^1(m_T)$ and let $\varepsilon > 0$. Then there is a positive simple function s_ε in $X(\mu)$ such that $\|f - s_\varepsilon\|_{L^1(m_T)} < \varepsilon$. Since T is narrow, there is a function $g_\varepsilon \in X(\mu)$ such that $|g_\varepsilon| = s_\varepsilon$, and $\|T(g_\varepsilon)\|_E < \varepsilon$. Define $g = f sgn(g_\varepsilon)$, where $sgn(g_\varepsilon)$ is the sign of g_ε, and note that $g \in L^1(m_T)$ since $f \in L^1(m_T)$. Observe also that $|g| = f$, and

$$\begin{aligned}
\|I_{m_T}(g)\|_{L^1(m_T)} &= \|I_{m_T}(g - g_\varepsilon)\| + \|I_{m_T}(g_\varepsilon)\| \leq \|I_{m_T}\| \cdot \|g - g_\varepsilon\|_{L^1(m_T)} + \varepsilon \\
&\leq \|I_{m_T}\| \cdot \|(f - s_\varepsilon) sgn(g_\varepsilon)\|_{L^1(m_T)} + \varepsilon \\
&= \|I_{m_T}\| \cdot \|f - s_\varepsilon\|_{L^1(m_T)} + \varepsilon \leq (\|I_{m_T}\| + 1)\varepsilon.
\end{aligned}$$

This shows that I_{m_T} is narrow.

Conversely, assume that the integration operator I_{m_T} is narrow. For each function $0 \leq f \in X(\mu)$ and a given $\varepsilon > 0$ there exists a function $g \in L^1(m_T)$ with $|g| = f$ and such that $\|I_{m_T}(g)\|_E < \varepsilon$. Since T is μ-determined and the function f is in $X(\mu)$ then g is also in $X(\mu)$. Finally from the Optimal Domain theorem (2.1) it follows that

$$\|T(g)\|_E = \|I_{m_T} \circ i(g)\|_E = \|I_{m_T}(g)\| < \varepsilon,$$

and the proof is done. □

We finish this section with some applications regarding maximality linear extensions of narrow operators. First, if $X(\mu)$ is a B.f.s having absolutely continuous norm and E is a Banach space then each AM-compact operator $T : X(\mu) \to E$ is narrow —see Proposition 2.1 in [14]—. On the other hand if (Ω, Σ, μ) is a nonatomic probability measure space with $L^\infty(\mu) \subseteq X(\mu) \subseteq L^1(\mu)$ then each Dunford-Pettis operator $T : X(\mu) \to E$ is narrow —see Theorem 11.57 in [1]—. Then:

Corollary 5.3. *Let (Ω, Σ, μ) be a positive finite measure space. Let $X(\mu)$ be a σ-order continuous B.f.s. over (Ω, Σ, μ) and let E a Banach space.*

(1) *If $T : X(\mu) \to E$ is a μ-determined AM-compact operator then T is narrow and the integration operator $I_{m_T} : L^1(m_T) \to E$ is the maximal narrow linear extension.*
(2) *If $T : X(\mu) \to E$ is a μ-determined Dunford-Pettis operator with (Ω, Σ, μ) a nonatomic probability measure space and $L^\infty(\mu) \subseteq X(\mu) \subseteq L^1(\mu)$ then T is narrow and the integration operator $I_{m_T} : L^1(m_T) \to E$ is the maximal narrow linear extension.*

Now, we study the particular case when $X(\mu) = L^1(\mu)$ for a finite positive measure μ. On the one hand, each representable operator $T : L^1(\mu) \to E$ is narrow —see Proposition 2.4 in [14]—. In particular if E has the Radon-Nikodým property, the operator $T : L^1(\mu) \to E$ is representable and hence T is narrow. Therefore

Corollary 5.4. *Let $T : L^1(\mu) \to E$ be a μ-determined continuous linear operator. Let E be a Banach space with the Radon-Nikodým property. Then the integration operator I_{m_T} is narrow and it is the maximal narrow linear extension.*

Another application comes from the connection between the convexity of the range of a vector measure and the narrow operators. The classical *Lyapunov theorem* states that if E is finite dimensional then the range of any E-valued (countable) additive vector measure convex —in fact, the converse is also true—. Nevertheless if $\dim(E) = \infty$ then there is a (countable) additive vector $m : \Sigma \to E$ having bounded variation and such that $\mathbf{R}(m)$ is non convex. However things are different if we think about the notion of if $\mathbf{R}(m)$ has convex closure. In fact, if $\mathbf{R}(m)$ has convex closure for each (countable) additive vector $m : \Sigma \to E$ having bounded variation then each $T \in L(L^1(\mu), E)$ is narrow, and reciprocally —see Theorem 1 in [7]—. Following the lines of the proof of the previous result we have:

Corollary 5.5. *Let $X(\mu)$ be a σ-order continuous B.f.s. over the Lebesgue measure μ on the σ-algebra $\Sigma = \mathcal{B}([0,1])$ of the Borel subsets of $[0,1]$. Let E be a Banach space and $T : X(\mu) \to E$ a μ-determined linear operator. If the range $\mathbf{R}(m_T)$ has convex closure then T is narrow and I_{m_T} is the maximal narrow linear extension.*

Proof. Let A be a Borel subset of $[0,1]$ and we consider the restriction of m_T over the subsets of A —write Σ_A to the corresponding σ-algebra—. Let $\varepsilon > 0$ and consider

$$\frac{1}{2}T(\chi_A) = \frac{1}{2}m_T(A) = \frac{1}{2}m_T(A) + \frac{1}{2}m_T(\emptyset).$$

Since $\mathbf{R}(m_T)$ has convex closure we can find $A_\varepsilon \in \Sigma_A$ such that $\|(1/2)T(\chi_A) - T(\chi_{A_\varepsilon})\| < \varepsilon$. Consider now the sign on A given by $x = \chi_{A \setminus A_\varepsilon} - \chi_{A_\varepsilon}$. Therefore

$$\|T(x)\| = \|T(\chi_{A\setminus A_\varepsilon}) - T(\chi_{A_\varepsilon})\| = 2\Big\|\frac{1}{2}T(\chi_A) - T(\chi_{A_\varepsilon})\Big\| < 2\varepsilon,$$

and the operator T is narrow. □

Remark 5.6. In the previous result we have use the following equivalent definition for a narrow operator T —see [15, Proposition 1.9]— defined in a B.f.s. having absolutely continuous norm and with values in the Banach space E: *T is narrow if for all $A \in \Sigma$ and each $\varepsilon > 0$ there is a sign x on A such that $\|T(x)\| < \varepsilon$.* Recall that a *sign* function is just a function whose values are $-1, 0$ or 1.

Remark 5.7. Again in the previous result note that the E-valued measure m_T defined by $m_T(A) = T(\chi_A), A \in \Sigma$ has bounded variation. Indeed, since $X(\mu)$ is a σ-order continuous B.f.s. then m_T is countable additive. On the other hand the operator T is μ-determined so, by using Lemma 4.5 (i) in [12], μ is control measure for m_T. Hence m_T has bounded variation.

Finally, a result by J. Bourgain and H. P. Rosenthal in [2] states that if (Ω, Σ, μ) is a finite atomless measure space and E is a Banach space that does not contain copies of ℓ^1 then every $T \in L(L^1(\mu), E)$ is narrow. Therefore we finish this paper with the following result:

Corollary 5.8. *Let (Ω, Σ, μ) be a finite atomless measure space and E is a Banach space that does not contain copies of ℓ^1. If $T : L^1(\mu) \to E$ is a μ-determined operator then T is narrow and the integration operator $I_{m_T} : L^1(m_T) \to E$ is the maximal narrow linear extension.*

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